



On the dynamics of periodically restrained flexural structures under moving loads

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ABSTRACT

The dynamic response of periodically restrained flexural structures, modeled as Euler–Bernoulli beams supported by flexible and dissipative supports at equally spaced points, to moving loads is investigated. Bloch–Floquet theorem and modal analysis are used to examine the dispersion relation and band structure of the infinitely long periodic beam and the frequency spectrum of the finite beam, respectively. It is shown that the natural frequencies of the finite beam fall within the propagation bands of the corresponding infinite beam and get closer to each other as the number of spans increases ultimately covering the propagation bands. The dynamic response to moving load is obtained via a novel method based on Floquet transformation in the frequency domain for infinite periodic beams and through modal summation for finite periodic beams. Through numerical examples, the effect of boundary conditions, e.g. support stiffness and damping, and the impact of moving load speed on the flexural response as well as the energy stored and dissipated within the supports are discussed.

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1. Introduction

Inspired by phononic crystals (Hussein et al., 2014)—periodic composites composed of phases with different elastic properties and densities—the use of periodic structures for vibration isolation and control has gained considerable attention in recent years, particularly over the past decade; see (Cheng et al., 2013; Trainiti et al., 2015; Fallah et al., 2015; Taniker and Yilmaz, 2017; Piccolroaz et al., 2017) and references therein. The advantageous property of periodic structures is that they can give rise to band gaps which are frequency bands within which waves cannot propagate. While these band gaps can emerge as a result of either Bragg scattering (Brillouin, 2003) or local resonance (Liu et al., 2000), the advantage of locally resonant periodic structures is that they can be designed with band gaps at significantly lower frequencies (as compared to the Bragg phononic crystals), and therefore are applicable to vibration mitigation in civil structures (Xiang et al., 2012; Dertimanis et al., 2016; Beli et al., 2018). The idea has recently been used for seismic base isolation purposes by designing periodic foundations in such a way that the dominant earthquake frequencies fall within the band gaps of foundation (Cheng and Shi, 2014; Huang and Shi, 2015; Yan et al., 2014; 2015; Shi and Huang, 2013).

In addition, and from an analysis perspective, many civil structures are modeled as one-dimensional periodic and/or periodically supported structures, composed of array of identical unit cells. Examples include high-rise buildings, when high frequency vibrations are of interest, as well as railway tracks, pipelines and multi-span bridges, under the simplifying assumption that inertia effects are secondary (Ozmutlu et al., 2018; Gry and Gontier, 1997; Belotserkovskiy, 1998; García-Palacios et al., 2012; Li et al., 2016), Liu et al., (Brun et al., 2011). Dynamic characteristics of these structures are of interest for reducing their vibration response and the corresponding failure, fatigue and damage as well as mitigating the transmitted noise to the surrounding environment. Ever since Fryba's pioneering work on dynamic response of solids and structures under moving loads (Fryba, 1972), many researchers have attempted analytical and numerical solutions for vibration response of beam-type structures to moving loads (Thambiratnam and Zhuge, 1996; Hayashikawa and Watanabe, 1981; Wu and Dai, 1987; Zheng et al., 1998; Sun, 2001; Shi and Selvadurai, 2016; Dimitrovova, 2017). Most of the existing literature, however, focuses on either finite or homogeneous infinite structures and either lack the generality when it comes to support conditions or fail to critically examine the connection between finite beam's frequency spectrum and the band structure of infinite beam and the associated intricacies such as dissipation (transfer) of energy within (into) the supports. In this paper, we focus on those structures for which, due to structure's geometry and support conditions, a predominately flexural (e.g. Euler–Bernoulli) model with periodically placed supports

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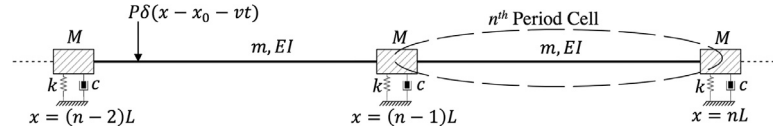


Fig. 1. Periodic beam subject to moving load.

constitutes a good approximation. This includes railway tracks, pipelines on equally spaced supports and long narrow multi-span bridges. We critically examine the dynamic characteristics (irrespective of the loading) as well as dynamic response of these structures to moving loads, disregarding the load inertia effect. To this end, we first reduce the complexity of the problem by formulating the dynamics of periodic beam in a non-dimensional but general setting. The dispersion relation and band structure of the infinite periodic beam as well as its response to a moving load are then determined using Floquet–Bloch theorem and a semi-analytical approach based on Floquet transformation. This is followed by a careful examination of the dynamic characteristics and response of the finite periodic beam with different numbers of spans and support conditions through modal summation. It is shown that the natural frequencies of the finite beam with many spans create a set of densely packed numbers that fall within the propagation bands of its infinite beam counterpart and fully cover them as the number of spans increases. It is also shown that the first natural frequency of the finite beam is independent (or asymptotically independent) of the number of spans and marks the starting point for the first dispersion branch of the infinite beam. Numerical results indicate the existence of “master curve” for maximum displacement, independent of the dimensionless mass per unit length for infinitely long periodic beams on rigid supports, a “distance lag” between the location of the moving load and the maximum displacement for periodic beams supported by damped flexible supports, and an upper bound for the total dissipated energy within supports at a speed-dependent characteristic damping ratio. The impact of moving load speed as well as the role of support conditions (including the impact of supports’ damping ratio) on these observations are also critically examined.

2. Dynamics of periodic beams under moving loads

Consider an Euler–Bernouli beam with periodic mass M at equal distance L , subject to moving load P . The periodic masses are supported by Kelvin–Voigt supports with stiffness k and damping coefficient c as illustrated in Fig. 1. The functional relation for the beam transverse displacement u can be expressed as $u = f(EI, m, P, v, L, c, k, M)$, with EI and m the flexural rigidity and the mass per unit length of the beam, respectively, and v the speed of the moving load. It is useful to perform a dimensional analysis to explore the dimensional invariants and reduce the complexity of the system. Using an LMT-base dimension system where $\mathbf{L}$, $\mathbf{M}$ and $\mathbf{T}$ respectively denote the dimensions of length, mass and time, the exponent matrix of dimensions reads:

	$[EI]$	$[u]$	$[m]$	$[P]$	$[v]$	$[c_p]$	$[L]$	$[c]$	$[k]$	$[M]$	$[T]$	$[A]$
$\mathbf{L}$	3	1	-1	0	1	1	1	0	0	0	0	2
$\mathbf{M}$	1	0	1	1	0	0	0	1	1	1	0	0
$\mathbf{T}$	-2	0	0	-2	-1	-1	0	-1	-2	0	1	0

(1)

which is a rank 3 matrix. Therefore, based on Buckingham Π -Theorem (Buckingham, 1914), three independent variables, e.g. flexural rigidity, EI , length of the period cell, L , and wave speed, $c_p = \sqrt{AE/m}$ (with A the cross section area and E the Young’s modulus) are used to express the remaining variables in a dimension-

less form as:

$$w = u/L, \quad \bar{m} = mc_p^2 L^2 / EI, \quad \bar{P} = PL^3 / EI, \quad \bar{v} = v/c_p$$

$$\bar{c} = cc_p L^2 / EI, \quad \bar{k} = kL^3 / EI, \quad \bar{M} = Mc_p^2 L / EI \quad (2)$$

Using the above dimensionless variables and disregarding the rotational inertia of the masses and the inertial effect of the moving load, the classical equation of motion reads:

$$\frac{\partial^4 w}{\partial \bar{x}^4} + \bar{m} \frac{\partial^2 w}{\partial \bar{t}^2} = \bar{P} \delta(\bar{x} - \bar{v} \bar{t}) \quad (3)$$

where $\bar{x} = x/L$, $\bar{t} = tc_p/L$ and $\delta(\cdot)$ is the Dirac delta function. The right hand side of Eq. (3) represents a moving load with amplitude P traveling at speed $\bar{v}$. For an infinite beam, the boundary conditions at the n^{th} support ($\bar{x} = n$) shown in Fig. 1—the continuity of displacement and slope along with the equilibrium of bending moments and shear forces—can be expressed in terms of the dimensionless variables:

$$w(n^-, \bar{t}) = w(n^+, \bar{t})$$

$$\frac{\partial w}{\partial \bar{x}} \Big|_{\bar{x}=n^-} = \frac{\partial w}{\partial \bar{x}} \Big|_{\bar{x}=n^+}$$

$$\frac{\partial^2 w}{\partial \bar{x}^2} \Big|_{\bar{x}=n^-} = \frac{\partial^2 w}{\partial \bar{x}^2} \Big|_{\bar{x}=n^+}$$

$$\frac{\partial^3 w}{\partial \bar{x}^3} \Big|_{\bar{x}=n^-} - \frac{\partial^3 w}{\partial \bar{x}^3} \Big|_{\bar{x}=n^+} = \bar{k} w(n, \bar{t}) + \bar{c} \frac{\partial w}{\partial \bar{t}} \Big|_{\bar{x}=n} + \bar{M} \frac{\partial^2 w}{\partial \bar{t}^2} \Big|_{\bar{x}=n} \quad (4)$$

Here, n^+ and n^- denote the right and the left of n^{th} support, respectively. For the case of a finite beam with N period cells, in addition to equations in (4), the boundary conditions at the two ends need to be satisfied:

$$\frac{\partial^2 w}{\partial \bar{x}^2} \Big|_{\bar{x}=0} = \frac{\partial^2 w}{\partial \bar{x}^2} \Big|_{\bar{x}=N} = 0$$

$$\frac{\partial^3 w}{\partial \bar{x}^3} \Big|_{\bar{x}=0} = -\bar{k} w(0, \bar{t}) - \bar{c} \frac{\partial w}{\partial \bar{t}} \Big|_{\bar{x}=0} - \bar{M} \frac{\partial^2 w}{\partial \bar{t}^2} \Big|_{\bar{x}=0}$$

$$\frac{\partial^3 w}{\partial \bar{x}^3} \Big|_{\bar{x}=N} = \bar{k} w(N, \bar{t}) + \bar{c} \frac{\partial w}{\partial \bar{t}} \Big|_{\bar{x}=N} + \bar{M} \frac{\partial^2 w}{\partial \bar{t}^2} \Big|_{\bar{x}=N} \quad (5)$$

In what follows, vibration of finite and infinite periodic beams with Kelvin–Voigt supports traversed by a constant moving load is studied. It is noted that a variety of boundary conditions can be recovered from Eqs. (4,5). For example, the boundary conditions for a periodic beam supported by rollers can be replicated as $\bar{k} \rightarrow \infty$ and $\bar{c} \rightarrow 0$. However, it is more convenient to directly use Dirichlet boundary conditions at the two ends $\bar{x} = 0, N$.

2.1. Infinite periodic beam

2.1.1. Floquet transformation

Consider a three-dimensional domain Ω , with Cartesian base $\{e_1, e_2, e_3\}$ that is periodic in e_1 direction with period L . Any func-

tion $f(x)$ at $x = x_1 e_1 + x_2 e_2 + x_3 e_3$ defined in Ω can be transformed to a function at $\tilde{x}$ in generic cell $\tilde{\Omega}$, where $\tilde{\Omega} = \{x \in \Omega | 0 < x_1 < L\}$, by Floquet transformation as follows (Floquet, 1883; Chebli et al., 2006):

$$\tilde{f}(\tilde{x}, k) = \sum_{n=-\infty}^{\infty} f(\tilde{x} + nLe_1) e^{inkL} \quad (6)$$

where $i = \sqrt{-1}$ and k denotes the wavenumber in the first Brillouin zone, i.e. $k \in [-\frac{\pi}{L}, \frac{\pi}{L}]$. By definition the Floquet transform of a scalar function at any point can be expressed in terms of its Floquet transform at the corresponding point within the generic cell:

$$\tilde{f}(\tilde{x} + mLe_1, k) = e^{-imkL} \tilde{f}(\tilde{x}, k) \quad (7)$$

This is known as the second kind periodicity of $\tilde{f}(\tilde{x}, k)$. Finally, f may be recovered through inverse Floquet transformation as:

$$f(\tilde{x} + mLe_1) = \frac{L}{2\pi} \int_{-\pi/L}^{\pi/L} \tilde{f}(\tilde{x}, k) e^{-imkL} dk \quad (8)$$

where the integral on the right-hand side is evaluated over the first Brillouin zone.

2.1.2. Dispersion relation and band structure

In the absence of any external loading, the general solution to Eq. (3) is expressed as a harmonic function for each unit cell:

$$w^n(\tilde{x}, \tilde{t}) = \phi^n(\tilde{x}) e^{i\tilde{\omega}\tilde{t}} \quad \text{for } n-1 \leq \tilde{x} \leq n \quad (9)$$

with $\tilde{\omega} = \omega L/c_p$ the dimensionless frequency. Substituting the above solution in the equation of motion, the function ϕ^n for span n is given by:

$$\phi^n(\tilde{x}) = C_1^n \sinh(\lambda\tilde{x}) + C_2^n \cosh(\lambda\tilde{x}) + C_3^n \sin(\lambda\tilde{x}) + C_4^n \cos(\lambda\tilde{x}) \quad (10)$$

where $\lambda^4 = \tilde{m}\tilde{\omega}^2$ and C_i^n are unknown constants for period cell n . Enforcing the boundary conditions in Eq. (4) now leads to the following relationship between the unknown constants corresponding to adjacent spans:

$$\{C^{n+1}\} = \mathbf{T}\{C^n\} \quad (11)$$

where $\{C^n\}^T = [C_1^n \ C_2^n \ C_3^n \ C_4^n]$ and $\mathbf{T}$ is a transfer matrix for which the elements are summarized in Appendix A. In addition, the periodicity of the system allows for the use of Floquet–Bloch theorem (Kittel, 2005):

$$\{C^{n+1}\} = e^{i\tilde{q}} \{C^n\} \quad (12)$$

where $\tilde{q} = kL \in [-\pi, \pi]$ is the dimensionless wavenumber. Subtracting Eq. (12) from (11) yields the following homogenous system of equations:

$$(\mathbf{T} - e^{i\tilde{q}} \mathbf{I})\{C^n\} = \{0\} \quad (13)$$

with $\mathbf{I}$ the identity matrix. The null-space of the coefficient matrix in Eq. (13) includes nontrivial solutions which are obtained by setting the determinant of $\mathbf{T} - e^{i\tilde{q}} \mathbf{I}$ equal to zero. This leads to the dispersion relation, thus linking the Floquet dimensionless wavenumber $\tilde{q}$ to dimensionless frequency $\tilde{\omega}$:

$$\frac{4\lambda^3 (\cos \tilde{q} - \cos \lambda) (\cos \tilde{q} - \cosh \lambda) + \Gamma \cosh \lambda \sin \lambda}{\Gamma (\cos \tilde{q} \sin \lambda + (\cos \lambda - \cos \tilde{q}) \sinh \lambda)} = 1 \quad (14)$$

where $\Gamma = -\tilde{M}\tilde{\omega}^2 + i\tilde{c}\tilde{\omega} + \tilde{k}$. Fig. 2(a) shows the relation between $\tilde{q}$ and $\tilde{\omega}$ for a roller-supported beam and Figs. 2(b) and (c) respectively depict the effect of $\tilde{k}$ (for $\tilde{M} = \tilde{c} = 0$ and $\tilde{m} = 1$) and $\tilde{M}$ (for $\tilde{k} = \tilde{c} = 0$ and $\tilde{m} = 1$) on the dispersion relation. For an undamped system, the Floquet wavenumber $\tilde{q}$ is either pure real or pure imaginary. The white regions of the contour plots in Fig. 2(a)–(c) are band gaps (attenuation zone) where $\tilde{q}$ is pure imaginary,

indicating the ranges of dimensionless frequency $\tilde{\omega}$ where waves cannot propagate, whereas the colored regions are the propagation bands which correspond to real values of $\tilde{q}$. Fig. 2(d) illustrates the dispersion curves for a homogeneous beam ($\tilde{k} = \tilde{M} = 0$) with no band gap and a periodic roller-supported beam with gray regions representing band gaps. The dispersion curves correspond to the dashed lines in Fig. 2(a) and (b).

2.1.3. Dynamic response to moving load

Taking the Fourier transformation of Eq. (3), we get:

$$\frac{\partial^4 \hat{w}}{\partial \tilde{x}^4} - \lambda^4 \hat{w} = \frac{\tilde{P}}{\tilde{v}} e^{-i\tilde{\omega}\tilde{x}/\tilde{v}} \quad (15)$$

where $\hat{w}$, the Fourier transform of w , can be expressed in terms of the associated Green's function $G(\tilde{x}, \tilde{\xi}, \tilde{\omega})$ as:

$$\hat{w}(\tilde{x}, \tilde{\omega}) = \frac{\tilde{P}}{\tilde{v}} \int_{-\infty}^{\infty} G(\tilde{x}, \tilde{\xi}, \tilde{\omega}) e^{-i\tilde{\omega}\tilde{\xi}/\tilde{v}} d\tilde{\xi} \quad (16)$$

Here, we use Floquet decomposition to transform the above integral to an integral over the generic cell in the frequency domain (Chebli et al., 2006). The frequency domain response of the periodic beam in the generic cell then reads:

$$\hat{w}(\tilde{x}, \tilde{\omega}) = \frac{\tilde{P}}{2\pi} \int_0^1 \tilde{G}(\tilde{x}, \tilde{\xi}, \tilde{\omega}) e^{-i\tilde{\omega}\tilde{\xi}/\tilde{v}} d\tilde{\xi} \quad (17)$$

where $\tilde{G}(\tilde{x}, \tilde{\xi}, \tilde{\omega})$ is the Floquet transformation of the Green's function $G(\tilde{x}, \tilde{\xi}, \tilde{\omega})$ with $\tilde{x}, \tilde{\xi} \in [0, 1]$ and $\tilde{\omega}/\tilde{v} \in [-\pi, \pi]$ to ensure wavenumber is in the first Brillouin zone. It should be noted that obtaining the Green's function directly is computationally expensive due to the complexity of the boundary conditions. We thus use the second kind periodicity of $\tilde{G}(\tilde{x}, \tilde{\xi}, \tilde{\omega})$ to determine the frequency domain response:

$$\hat{w}(\tilde{x} + n, \tilde{\omega}) = e^{in\tilde{\omega}/\tilde{v}} \hat{w}(\tilde{x}, \tilde{\omega}) \quad (18)$$

Taking the inverse Fourier transformation of the above, $w(\tilde{x} + m, \tilde{t} + mT) = w(\tilde{x}, \tilde{t})$, the boundary conditions in (4) are expressed in terms of those of the generic cell ($\tilde{x} \in [0, 1]$):

$$\begin{aligned} \hat{w}(0, \tilde{\omega}) &= e^{-i\tilde{\omega}/\tilde{v}} \hat{w}(1, \tilde{\omega}) \\ \frac{\partial \hat{w}}{\partial \tilde{x}} \Big|_{\tilde{x}=0} &= e^{-i\tilde{\omega}/\tilde{v}} \frac{\partial \hat{w}}{\partial \tilde{x}} \Big|_{\tilde{x}=1} \\ \frac{\partial^2 \hat{w}}{\partial \tilde{x}^2} \Big|_{\tilde{x}=0} &= e^{-i\tilde{\omega}/\tilde{v}} \frac{\partial^2 \hat{w}}{\partial \tilde{x}^2} \Big|_{\tilde{x}=1} \\ -\frac{\partial^3 \hat{w}}{\partial \tilde{x}^3} \Big|_{\tilde{x}=0} + e^{-i\tilde{\omega}/\tilde{v}} \frac{\partial^3 \hat{w}}{\partial \tilde{x}^3} \Big|_{\tilde{x}=1} &= (-\tilde{\omega}^2 \tilde{M} + i\tilde{\omega} \tilde{c} + \tilde{k}) \hat{w}(0, \tilde{\omega}) \end{aligned} \quad (19)$$

The Fourier transform of displacement $\hat{w}$ is the sum of particular solution to (15):

$$\hat{w}_p(\tilde{x}, \tilde{\omega}) = \frac{\tilde{P}\tilde{v}^3 e^{-i\tilde{\omega}\tilde{x}/\tilde{v}}}{\tilde{\omega}^4 - \tilde{m}\tilde{v}^4 \tilde{\omega}^2}; \quad (20)$$

and the solution to the homogenous equation:

$$\hat{w}_h(\tilde{x}, \tilde{\omega}) = C_1 \sinh(\lambda\tilde{x}) + C_2 \cosh(\lambda\tilde{x}) + C_3 \sin(\lambda\tilde{x}) + C_4 \cos(\lambda\tilde{x}) \quad (21)$$

where the frequency dependent constants C_i 's ($i = 1, 2, 3, 4$) are determined from (19). Subsequently, the frequency domain response of the generic cell is expanded according to Eq. (18) to arrive at full beam response:

$$w(\tilde{x}, \tilde{t}) = \frac{1}{2\pi} \int_{-\pi/\tilde{v}}^{\pi/\tilde{v}} \hat{w}(\tilde{x} - [\tilde{x}], \tilde{\omega}) e^{i\tilde{\omega}([\tilde{x}]/\tilde{v} + \tilde{t})} d\tilde{\omega} \quad (22)$$

In the above, $[\tilde{x}]$ denotes the greatest integer less than or equal to $\tilde{x}$.

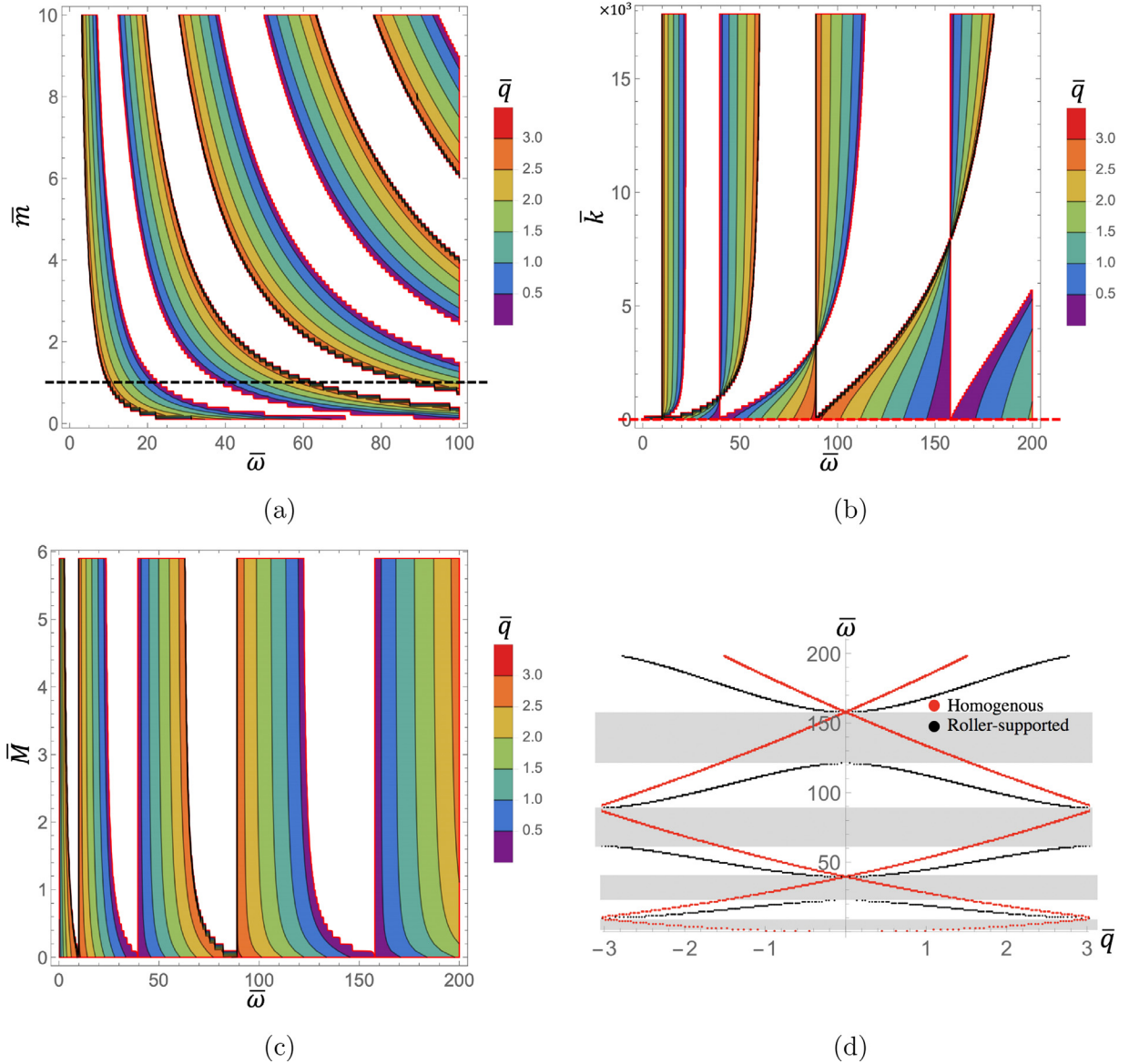


Fig. 2. (a) Contour plot of Floquet wavenumber $\bar{q}$ versus dimensionless frequency $\bar{\omega}$ and $\bar{m}$ for roller-supported beam, white regions represent the band gaps; (b) Same contour plot versus dimensionless frequency and $\bar{k}$ for $\bar{M} = 0$ and unit $\bar{m}$ for a beam supported by periodic elastic foundations; (c) Same contour plot versus dimensionless frequency and $\bar{M}$ for $\bar{k} = \bar{c} = 0$ and unit $\bar{m}$; (d) Dispersion curves for homogeneous beam with no band gap and roller-supported beam with $\bar{m} = 1$.

2.2. Finite periodic beam

2.2.1. Free vibration

Consider now a finite beam with N equal spans resting on spring-damper supports similar to infinite beam in the preceding section. Using modal superposition, the harmonic response for the free vibration of n^{th} span can be written as:

$$w^n(\bar{x}, \bar{t}) = \sum_{j=1}^{\infty} \phi_j^n(\bar{x}) e^{i\bar{\omega}_j \bar{t}} \quad \text{for } n-1 \leq \bar{x} \leq n \quad (23)$$

with

$$\phi_j^n(\bar{x}) = C_{1,j}^n \sinh(\lambda_j \bar{x}) + C_{2,j}^n \cosh(\lambda_j \bar{x}) + C_{3,j}^n \sin(\lambda_j \bar{x}) + C_{4,j}^n \cos(\lambda_j \bar{x}) \quad (24)$$

The natural frequencies $\bar{\omega}_j = \lambda_j^2 / \sqrt{\bar{m}}$ and unknown constants $C_{i,j}^n$, ($i = 1, 2, \dots, 4$) will be determined from boundary conditions. This includes enforcing the continuity and equilibrium relationships in Eq. (4) that relate the constants for two adjacent spans

through the transfer matrix (see Eq. (11)). It also includes the imposition of end conditions in equation Eq. (5) which translate to $\mathbf{A}^1 \{C^1\} = \{0\}$ for $\bar{x} = 0$ and $\mathbf{A}^N \{C^N\} = \{0\}$ for $\bar{x} = N$, where $\mathbf{A}^1$ and $\mathbf{A}^N$ are 4×4 matrices with elements given in Appendix B. The natural frequencies $\bar{\omega}_j$ as well as the unknown constants C_{ij}^n corresponding to the first span are determined after seeking the non-zero elements of the null-space of the coefficient matrix $\mathbf{B}$ in the following system of equations:

$$\underbrace{(\mathbf{A}^1 + \mathbf{A}^N \mathbf{T}^{N-1})}_{\mathbf{B}} \{C^1\} = \{0\} \quad (25)$$

The remaining constants (C_{ij}^n , $n > 1$) are obtained via Eq. (12) and the mode shapes from the following equations:

$$\Phi_j(\bar{x}) = \sum_{n=1}^N \phi_j^n(\bar{x} - n)(H(\bar{x} - n + 1) - H(\bar{x} - n)) \quad (26)$$

where $H(\cdot)$ is the Heaviside function.

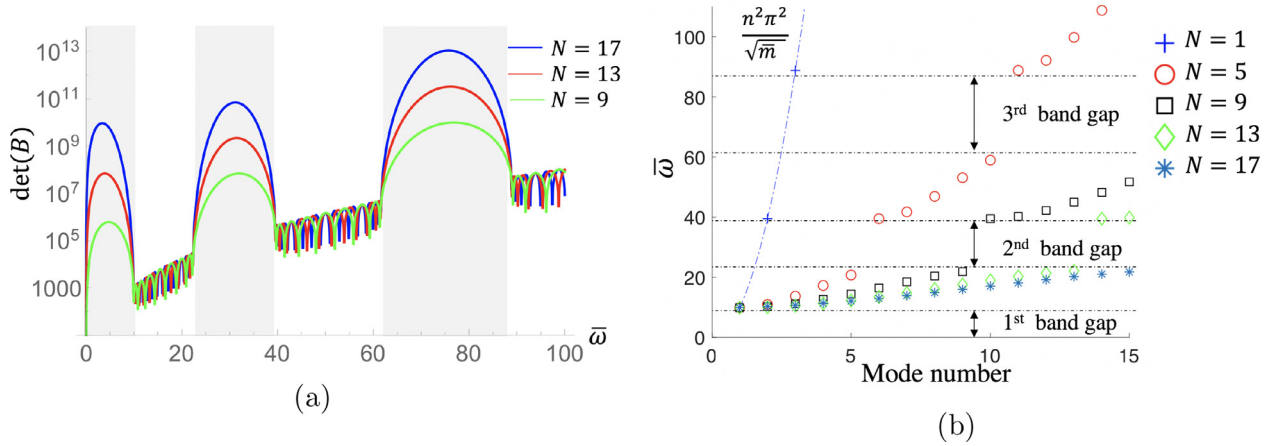


Fig. 3. (a) Determinant of matrix B for roller-supported beam versus dimensionless frequency $\bar{\omega}$ for various span numbers N , the gray regions represent band gaps (b) natural frequencies of finite span beams versus mode number.

2.2.2. Vibration under moving load

To obtain the forced vibration response for the finite periodic beam we use the method of modal summation, that is we write the solution in general coordinates as:

$$w(\bar{x}, \bar{t}) = \sum_{i=1}^{\infty} \Phi_i(\bar{x}) q_i(\bar{t}) \quad (27)$$

Substituting the above into the equation of motion, one can write:

$$\sum_{i=1}^{\infty} \frac{d^4 \Phi_i(\bar{x})}{d\bar{x}^4} q_i(\bar{t}) + \bar{m} \Phi_i(\bar{x}) \ddot{q}_i(\bar{t}) = \bar{P} \delta(\bar{x} - \bar{v}\bar{t}) \quad (28)$$

It can be shown that the mode shapes Φ_i are orthogonal when the end conditions are either clamped, free or simply supported. For the general case of Kelvin-Voigt supports at the ends, the modes are not orthogonal. Multiplying the two sides of Eq. (28) by Φ_j leads to the following coupled system of equations:

$$a_{ij} \ddot{q}_i + b_{ij} \dot{q}_i = \bar{P} \Phi_i(\bar{v}\bar{t}) \quad (29)$$

where we have used Einstein's notation and a_{ij} and b_{ij} are given by:

$$a_{ij} = \bar{m} \int_0^N \Phi_i(\bar{x}) \Phi_j(\bar{x}) d\bar{x} \quad b_{ij} = \bar{m} \bar{\omega}_i^2 \int_0^N \Phi_i(\bar{x}) \Phi_j(\bar{x}) d\bar{x} \quad (30)$$

Eq. (29) is solved in state-space using a Runge-Kutta scheme.

3. Numerical results and discussion

In what follows, we examine the dynamic characteristics and response of periodic beams numerically. We investigate the dy-

namic behavior of roller-supported periodic beams, an undamped system that is mainly used to model multi-span bridges and pipelines. In addition, we examine beams periodically supported by Kelvin-Voigt supports often used as a simplified model for railway tracks where the rail is connected to equally spaced massive sleepers that rest on the ballast. In both these cases we assume the inertial effects are secondary. Our results are in agreement with the results reported in the literature; the natural frequencies of finite beams and the dispersion relation of the infinite beams are in agreement with the results reported in Frýba (1972), Wu and Dai (1987), Mead (1970) and Mead (1996) and the dynamic response of the finite and infinite beams are in agreement with those reported in Kumar et al. (2015a). We provide further validation through comparison with the results obtained from finite element analysis of finite and infinite beams with the latter modeled as a finite beam with “many” spans.

3.1. Roller-supported beams

Periodically supported infinite beams, in contrast to homogeneous beams, have band gaps. The frequency ranges associated with stopping bands depend on beam dimensionless mass (see Fig. 2(a)). It is instructive to examine how the band gap frequencies in an infinite periodic beam are related to the natural frequencies of the associated finite periodic beam.

Fig. 3 (a) depicts the determinant of B in Eq. (25) versus dimensionless frequency for a roller-supported finite beam with $\bar{m} = 1$. It can be observed that the determinant does not have roots within the band gap regions (the gray regions), indicating that natural

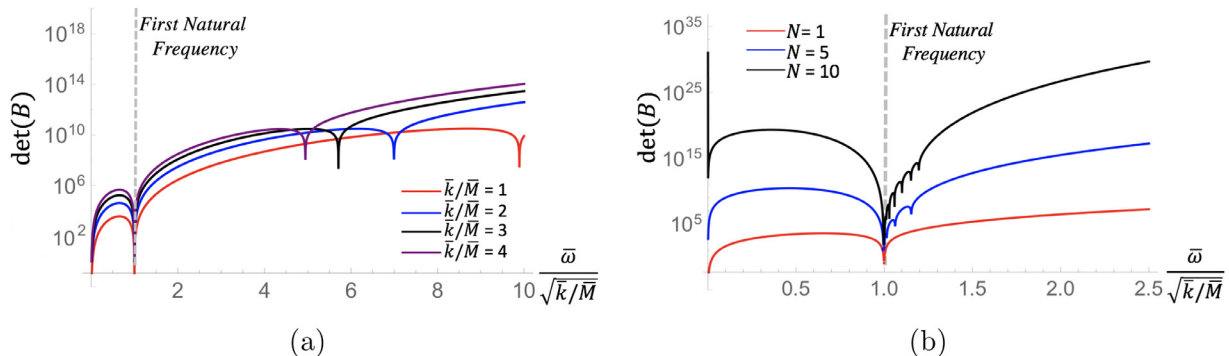


Fig. 4. The determinant of matrix B versus dimensionless frequency at $\bar{m} = 0.01\bar{M} = 1$ for (a) $N=1$ and different values of $\bar{k}/\bar{M}$ and (b) $\bar{k}/\bar{M} = 1$ and different number of spans.

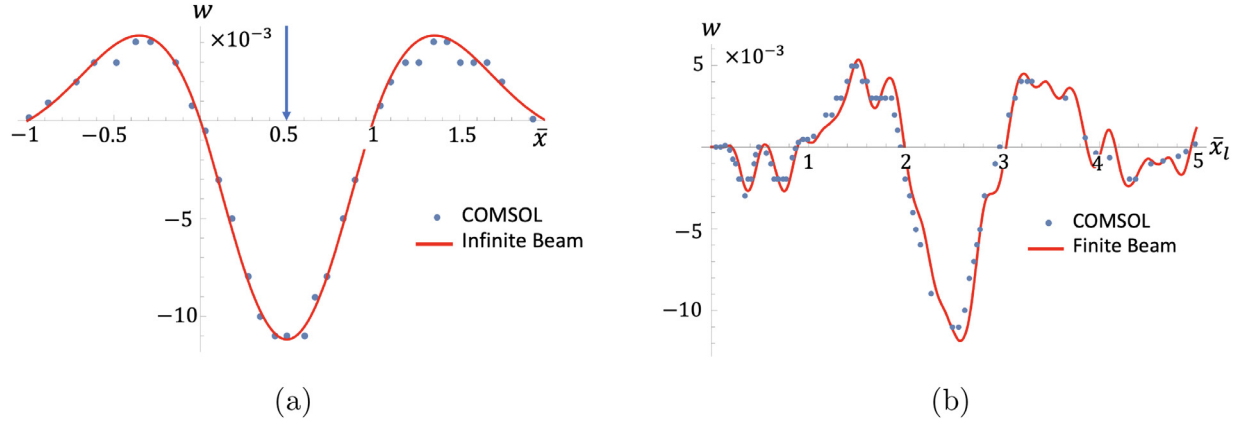


Fig. 5. (a) Displacement of an infinite roller-supported beam when the load arrives at the mid-span (when using COMSOL a finite beam with large number of spans ($N = 50$) was used to model the infinite beam); (b) displacement at the center of a finite roller-supported beam with five spans ($N = 5$) as a function of moving load location; $\bar{v} = 0.23\bar{v}_{cr}$.

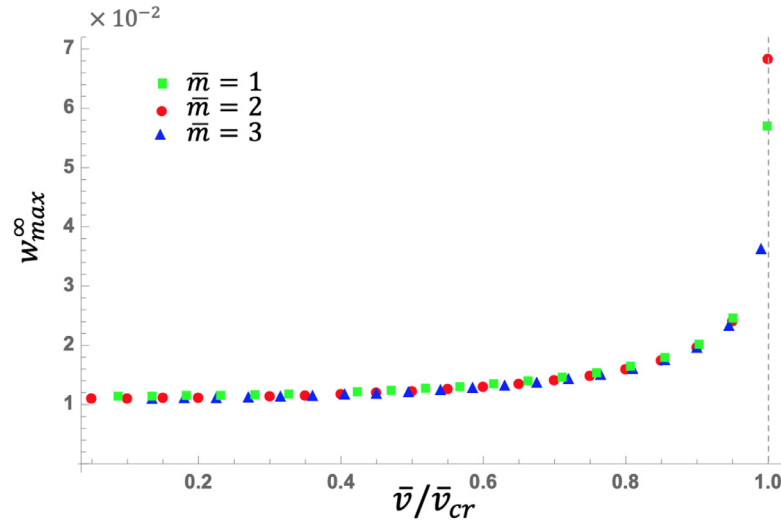


Fig. 6. Maximum dimensionless displacement of infinite periodic roller-supported beam subject to unit $\bar{P}$.

frequencies of the finite beam fall outside the band gaps of the corresponding infinite beam. Furthermore, as the number of spans increases no new frequencies emerge over frequency ranges that correspond to band gaps for the infinite periodic beam. In fact, as can be seen in Fig. 3(b) which depicts the first 15 natural frequencies of the finite beam with $\bar{m} = 1$, the new and more densely packed frequencies appear “only” within propagation bands for the infinite periodic beam. It is also seen that the first natural frequency of the finite beam is independent of the number of spans (Wang, 1997); rather, it is equal to the natural frequency of a simply supported beam, i.e. $\bar{\omega}_1 = \pi^2/\sqrt{\bar{m}}$. For a finite beam supported by undamped flexible supports the same statement is true asymptotically, i.e. assuming $\bar{M}$, $\bar{k} \gg \bar{m}$ and considering only the leading terms in the expression for the determinant, the first natural frequency can be approximated by $\bar{\omega}_1 \approx \sqrt{\bar{k}/\bar{M}}$ which corresponds to the natural frequency of a single-degree-of-freedom (SDOF) system; see Fig. 4(a) and (b) where the determinant of matrix $\mathbf{B}$ is plotted in function of $\bar{\omega}\sqrt{\bar{M}/\bar{k}}$ for beams with various $\bar{k}/\bar{M}$ values and different number of spans.

Fig. 5 illustrates the displacement of an infinite roller-supported beam when the load arrives at the mid-span as well as the displacement at the center of a finite roller-supported beam with five spans ($N = 5$) as a function of moving load location against the results obtained using COMSOL multiphysics COMSOL where a

close agreement is observed. The relationship between maximum dimensionless displacement of an infinite beam $w_{\max}^{\infty}$ subject to unit $\bar{P}$, which occurs at $\bar{x} = 1/2$, as a function of the speed of the moving load, is illustrated in Fig. 6 for different values of $\bar{m}$. The

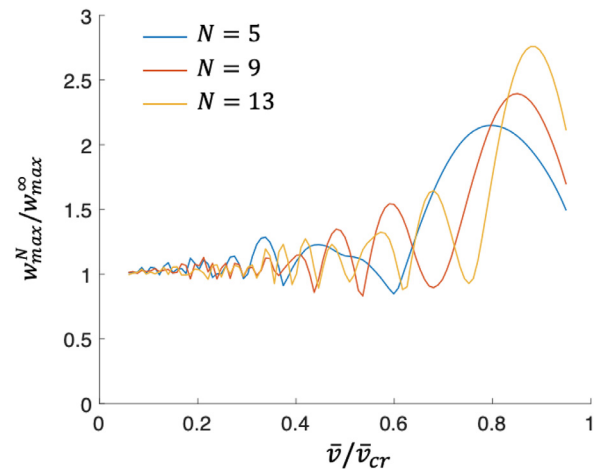


Fig. 7. Maximum displacement of finite periodic beam normalized by that of the infinite beam.

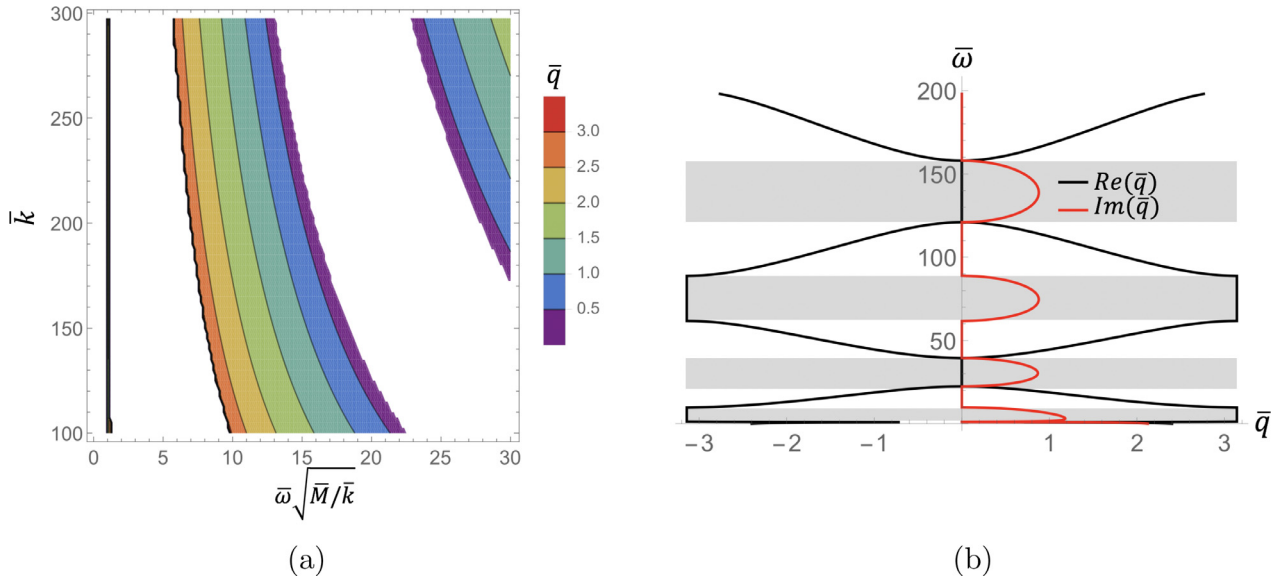


Fig. 8. (a) Contour plot of Floquet wavenumber $\bar{q}$ versus dimensionless support stiffness $\bar{k}$, white regions represent the band gaps; (b) dispersion curve, gray regions denote the band gaps.

results indicate that by rescaling the speed in the form of $\bar{v}/\bar{v}_{cr}$ with $\bar{v}_{cr} = \omega_1/\pi$ (Kumar et al., 2015b), all the curves collapse onto a “master” curve that is independent of the dimensionless mass per unit length of the beam. It is observed that $w_{\max}^{\infty}$ increases with moving load speed to 6.8×10^{-2} when $\bar{v}\sqrt{\bar{m}}/\pi \rightarrow 1$. For finite beams, the maximum displacement at the mid span ($\bar{x} = 1/2$) is calculated assuming zero initial conditions. The results are obtained by retaining 40 modes and subsequently normalized by the maximum displacement of the infinite beam and summarized in Fig. 7. In contrast to the infinite beams, where no cancellation phenomenon for speeds below the critical speed is observed, in multi-span periodic beams cancellation results in significant reduction of maximum displacement at many speeds (see the local minima at different $\bar{v}/\bar{v}_{cr}$ values). For a very large number of spans, the solution of the finite beam converges to that of the infinite beam. This can be readily observed in Fig. 5(a) which compares the response of an infinite beam and a beam with large number of spans ($N = 50$) under moving load. This can also be observed in Fig. 7 where it is seen that as the number of spans increases, the oscillation of normalized dynamic maximum displacement around unity is shifted to higher values of moving load speed. Asymptotically, as the number of spans approaches infinity this oscillation would be negligible even at very large values of moving load speed, indicating the convergence of the response of the finite beam to that of the infinite beam. It is also evident from Fig. 7 that the relation between the number of spans and the maximum displacement depends significantly on the speed of the moving load which is in agreement with other results in literature (Wang, 1997; Cheung et al., 1999).

3.2. Beams supported by Kelvin–Voigt supports

Beams with periodic masses supported by flexible and damped supports can be used to model railway tracks. Assuming the dimensionless mass of sleepers $\bar{M}$ and the ballast stiffness $\bar{k}$ are orders of magnitude larger than the mass per unit length of the rail $\bar{m}$ (Thompson, 2005), we report here the results for dimensionless parameters $\bar{m} = 0.01\bar{M} = 0.01\bar{k} = 1$.

As was mentioned before, for the case of undamped support ($\bar{c} = 0$), the first natural frequency is approximately $\sqrt{\bar{k}/\bar{M}}$ (see Fig. 4) for finite beams independent of the number of spans; that

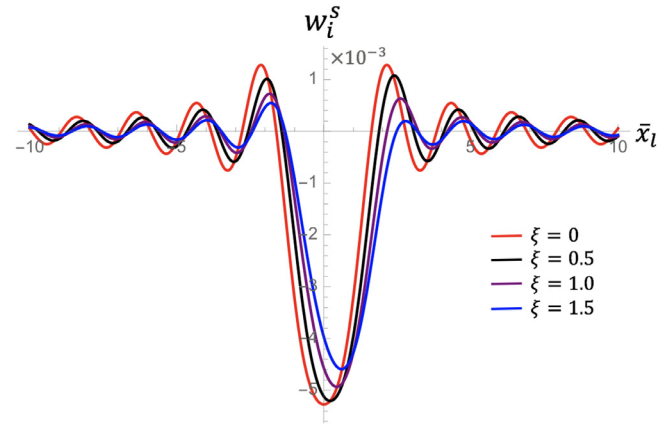


Fig. 9. Displacement of a support versus the location of moving load for speed $\bar{v} = 0.9\bar{v}_{cr}$ for various damping ratios.

is in the absence of heavy beams the system’s first mode of vibration is similar to that of a series of supported masses connected to each other by massless elements.

For the infinite beam, Fig. 8(a) depicts the Floquet wavenumber versus the dimensionless stiffness of the supports $\bar{k}$. Analogous to the case of roller-supported beam, the first dispersion branch starts at $\bar{\omega}_1 = \sqrt{\bar{k}/\bar{M}}$. Fig. 8(b) shows the dispersion curves for $\bar{k} = 100$, where black curves (the real part of $\bar{q}$) are propagation branches and red curves (the imaginary part of $\bar{q}$) represent attenuation branches.

The presence of damping in periodic supports changes the band structure of the system. The wavenumbers become complex rather than pure real or pure imaginary and the system does no longer possess band gaps. The dimensionless displacement of the i th support w_i^s as a function of moving load position $\bar{x}_l$ is illustrated in Fig. 9 for $\bar{v} = 0.9\bar{v}_{cr}$ and various damping ratios $\xi = \bar{c}/\sqrt{2\bar{M}\bar{k}}$. The results indicate reduction in maximum deflection as the damping ratio increases. Interestingly, for an undamped system ($\xi = 0$), the maximum deflection occurs at the instant when moving load reaches the support, whereas for damped systems, maximum deflection has a distance lag with the load position which

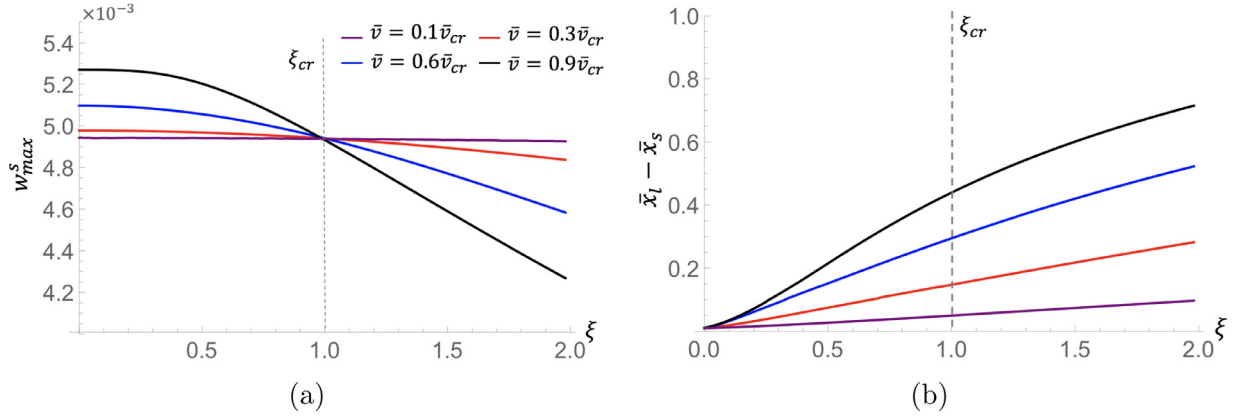


Fig. 10. (a) Maximum displacement of a support and (b) distance lag between the locations of moving load and maximum displacement at the support.

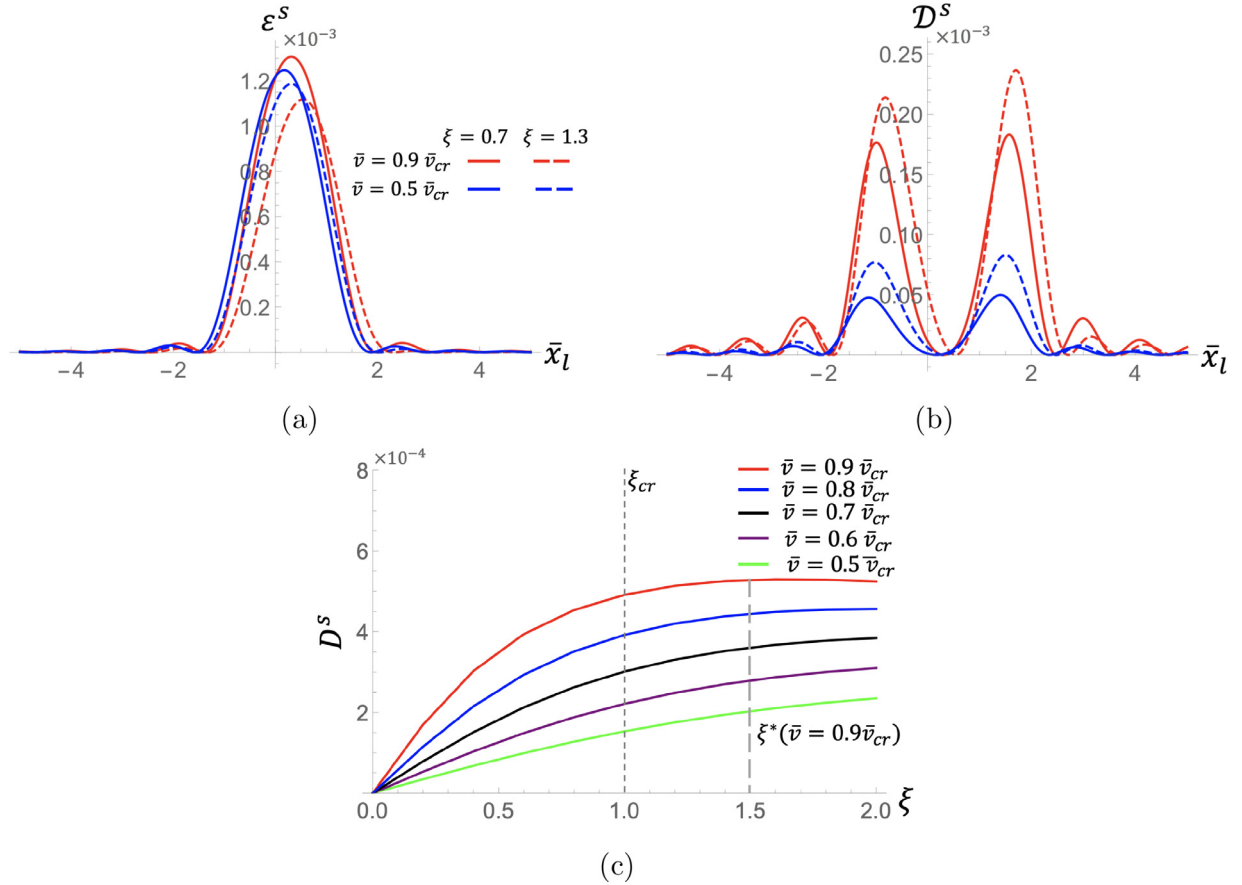


Fig. 11. (a) Instantaneous stored energy in the support as a function of the location of moving load for an under-damped ($\xi = 0.7$) and an over-damped ($\xi = 1.3$) system, (b) the same plot for the rate of dissipation, (c) dimensionless dissipated energy in one support for various damping ratios and speeds.

increases with damping ratio. This phenomenon has recently been observed in dynamics of viscoelastic beams under moving loads (see Louhghalam et al., 2014). Fig. 10 summarizes the maximum displacement and the distance lag between the locations of maximum displacement and moving load for different speeds and damping ratios. The results indicate that higher speeds lead to higher support displacements for under-damped systems ($\xi < \xi_{cr}$) whereas for over-damped systems ($\xi > \xi_{cr}$) it is the reverse. Furthermore, the maximum displacement is independent of moving load speed at critical damping ratio ($\xi = \xi_{cr}$) and the distance between the location of the moving load and the support, when the support displacement is maximum, increases both with damping ratio and moving load speed.

To study the energy dissipated in and transmitted to the bal-
last, we examine the instantaneous dimensionless energy stored
in the spring ($\mathcal{E}^s(\bar{t}) = k\bar{w}(0, \bar{t})^2/2$) and the rate of dimensionless
dissipation within the dashpot ($\mathcal{D}^s(\bar{t}) = \bar{c}\bar{w}(0, \bar{t})^2$) supporting mass
 $\bar{M}$ as the moving load travels over the periodic beam. The max-
imum stored energy in the support, $\mathcal{E}^s$, corresponds to the max-
imum displacement, which for undamped systems, occurs when
the load is exactly at the support. For damped systems, however,
this maximum stored energy occurs after the load passes the sup-
port, with a distance lag that depends on the damping ratio and
speed (see Fig. 11(a)). On the other hand, the dissipation rate, $\mathcal{D}^s$,
is maximum when the vertical speed of the beam at the location
of support reaches its maximum (see Figure (b)). Finally, the total

dimensionless energy dissipated in the damper in one support, D^s , as the load travels over the periodic beam ($-\infty < \bar{x}_l < +\infty$) is illustrated in Fig. 11(c) for various moving load speeds and damping ratios. As can be seen, for a given moving load speed, D^s converges to an upper bound at a characteristic damping ratio.

4. Conclusions

The flexural wave propagation and the dynamic response within periodically restrained beams subject to moving loads were investigated for both infinite and finite beams. The Bloch–Floquet theorem and Floquet transform along with modal analysis were used to study the band structure and dynamic characteristics. The following points deserve attention:

1. The natural frequencies of a finite periodic beam fall within the propagation bands of its corresponding infinite beam. As the number of spans increases (the finite beam approaches an infinite beam), these natural frequencies get closer to each other and create a set of densely packed numbers that finally covers the propagation band of the infinite beam.
2. The first natural frequency of a finite roller-supported periodic beam is independent of the number of spans and is equal to the fundamental frequency of a simply supported beam. For periodic beams supported by flexible supports, this is true “asymptotically” when the dimensionless mass of the beam is negligible compared to dimensionless periodic mass and support stiffness. The results also indicate that the first dispersion branch of the infinite beam starts at the first natural frequency of its corresponding finite beam.
3. The maximum displacement increases monotonically with the moving load speed to an asymptotic line at critical speed for infinite roller-supported beam. A “master curve” of maximum displacement, independent of the dimensionless mass per unit length of the beam, is obtained by scaling the speed in the form of $\bar{v}\sqrt{\bar{m}}/\pi$. For finite beams, however, this dependence is not monotonic and the relationship with the number of spans and maximum displacement is strongly affected by the moving load speed. This is due to the fact that in multi-span periodic finite beams, in contrast to infinite beams, cancellation results in significant reductions of the maximum displacement at many speeds below the critical speed.
4. For periodic beams supported by damped flexible supports, a higher damping ratio reduces the magnitude of maximum support displacement but increases the total dissipated energy within the support. Interestingly, for non-zero damping ratios, there exists a distance lag between the location of the moving load and the maximum displacement which increases with increasing damping ratio. In addition, for under-damped systems ($\xi < \xi_{cr}$) higher moving load speeds result in greater displacement and more energy stored within the support while for overdamped systems it is the reverse.
5. The total dissipated energy at the supports increases with the moving load speed and damping ratio but saturates to a maximum value at a characteristic damping ratio $\xi^*(\bar{v})$ that depends on the speed of the moving load.

Appendix A. Components of the transfer matrix T

The components of transfer matrix, T , are calculated by satisfying the boundary conditions at the n^{th} support and are given by:

$$\begin{aligned} T_{11} &= \gamma \cosh \lambda - \gamma \sinh \lambda & T_{12} &= \sinh \lambda - \gamma \cosh \lambda \\ T_{13} &= -\gamma \sin \lambda & T_{14} &= -\gamma \cos \lambda \\ T_{21} &= \sinh \lambda & T_{22} &= \cosh \lambda \\ T_{23} &= T_{24} = 0 & T_{31} &= \gamma \sinh \lambda \end{aligned} \quad (31)$$

$$\begin{aligned} T_{32} &= \gamma \cosh \lambda & T_{33} &= \cos \lambda + \gamma \sin \lambda \\ T_{34} &= \sin \lambda + \gamma \cos \lambda & T_{41} &= T_{42} = 0 \\ T_{43} &= \sin \lambda & T_{44} &= \cos \lambda \end{aligned}$$

with $\gamma = \Gamma/2\lambda^3$.

Appendix B. Components of matrices A^1 and A^N

The components of matrix A^1 are determined via satisfying the boundary conditions at $\bar{x}^1 = 0$ and are given by:

$$A^1 = \begin{bmatrix} 0 & \lambda_j^2 & 0 & -\lambda_j^2 \\ \lambda_j^3 & \Gamma_j & -\lambda_j^3 & \Gamma_j \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad (32)$$

with $\Gamma_j = -\bar{M}\bar{\omega}_j^2 + i\bar{c}\bar{\omega}_j + \bar{k}$. Similarly the boundary conditions at $\bar{x}^1 = 0$ lead to the components of A^N as follows:

$$\begin{aligned} A_{11}^N &= A_{12}^N = A_{13}^N = A_{14}^N = A_{21}^N = A_{22}^N = A_{23}^N = A_{24}^N = 0 \\ A_{31}^N &= \lambda_j^2 \sinh \lambda_j \\ A_{32}^N &= \lambda_j^2 \cosh \lambda_j \\ A_{41}^N &= \lambda_j^3 \cosh \lambda_j - \Gamma_j \sinh \lambda_j \\ A_{42}^N &= \lambda_j^3 \sinh \lambda_j - \Gamma_j \cosh \lambda_j \\ A_{33}^N &= -\lambda_j^2 \sin \lambda_j \\ A_{34}^N &= -\lambda_j^2 \cos \lambda_j \\ A_{43}^N &= -\lambda_j^3 \cos \lambda_j - \Gamma_j \sin \lambda_j \\ A_{44}^N &= \lambda_j^3 \sin \lambda_j - \Gamma_j \cos \lambda_j \end{aligned} \quad (33)$$

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