



# Lateral buckling of beams with shear deformations – A hyperelastic formulation

Mario M. Attard<sup>a,\*</sup>, Moon-Young Kim<sup>b</sup>

<sup>a</sup> School of Civil and Environmental Engineering, The University of New South Wales, Sydney 2052, Australia

<sup>b</sup> Department of Civil and Environmental Engineering, Sungkyunkwan University, Suwon, Republic of Korea

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## ABSTRACT

The equilibrium and buckling equations are derived for the lateral buckling of a prismatic straight beam. A consistent finite strain constitutive law is used, which is based on a hyperelastic model for an isotropic material. The kinematics of the cross-sectional deformations are based on a Timoshenko type beam displacement of the cross-sectional plane using Euler angles and two shear finite rotations coupled with warping taken normal to the displaced plane. Also derived are the second order approximations to the displacements, curvatures, twist and internal actions. The constitutive relationships for the internal actions reveal new coupling terms between the bending moments, torsion and bimoment, which are functions of the cross-sectional warping and shear deformations. New Wagner type nonlinear torsion terms are derived which are functions of the warping of the cross-sectional plane, and are coupled to the twisting and shear deformations of the cross-section. Solutions are determined for the lateral buckling of a prismatic monosymmetric beam under pure bending and the flexural–torsional buckling under axial compression. For the flexural–torsional buckling problem it is found that the Euler type column buckling formula is consistent with Haringx's column buckling formula while the torsional buckling formula is different to conventional equations. The second variation of the total potential is also derived. The effects of shear deformations are explored by examining the non-dimensional lateral buckling equation for a simply supported beam.

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## 1. Introduction

Thin-walled structural components are widely used in many industries including aerospace, building, aircraft and shipbuilding. It is important that the nonlinear and stability response of these structural components, be accurately modelled and estimated by Mohareb and Dabbas (2003) and later Emre Erkmen and Mohareb (2008) showed that shear deformations can be very important when estimating the lateral buckling of a long column supporting a short wide flange cantilever. The inclusion of shear deformations in the buckling analysis of axially loaded columns and sandwich type columns with a soft shear core has received attention in the literature. The major issue of contention involves the definition of the constitutive relationship for the axial force and shear force for one-dimensional beam theory. Two distinct methods for column buckling with shear have been presented, one essentially by Engesser (1891) and the other by Haringx (1942). The controversy has been discussed extensively by Zielger (1982), Reissner (1982), Kardomateas and Dancila (1997), Bažant (2003), Bažant and Beghini (2004, 2006), Bažant and Cedolin (1991), Beghini et al. (2008), Aristizabal-Ochoa (2008), Blaauwendraad (2008). Attard

(2003a,b) and Attard and Hunt (2008a,b) applied a finite strain constitutive model derived from a hyperelastic formulation to the problem of column and sandwich column buckling and was shown to be consistent with Haringx's approach and Reissner's proposal for beam actions Reissner (1972).

Several authors have extended the theory of lateral buckling and nonlinear analysis of thin-walled beams to include shear deformations, including Chang et al. (1996), Lee et al. (submitted for publication), Machado (2008), Machado and Cortínez (2005a,b), Reissner (1989), Sapountzakis and Mokos (2008) and Simo et al. (1984). Lee et al. (submitted for publication) provided spatial stability theory for shear-flexible thin-walled beams using both the Engesser's and the Haringx's buckling approach and compared the predictions. Analytical solutions in dimensionless form were presented for the spatial lateral buckling of simply supported thin-walled beams having monosymmetric and non-symmetric cross-sections, and a parametric study undertaken to compare the prediction using the different approaches.

An outline of the contents of this paper is as follows. In Section 2, the kinematics for the lateral buckling problem are developed using the Timoshenko beam analogy coupled with warping taken to occur in a normal direction to the displaced cross-sectional plane. The cross-sectional displacements without warping are first defined using three finite Euler angles and two shear finite

\* Corresponding author. Tel.: +61 2 9385 5075; fax: +61 2 9385 6139.

E-mail address: [m.attard@unsw.edu.au](mailto:m.attard@unsw.edu.au) (M.M. Attard).

rotations. Many nonlinear and stability analyses for the lateral torsional behaviour of beams are often based on assuming a Hookean constitutive relationship between Green's strain tensor and its conjugate stress, the second Piola–Kirchhoff stress tensor. Attard (2003a) showed that such a finite strain constitutive relationship is incorrect when including shear deformations for an isotropic material and that the constitutive relationship should be derived from a consistent finite strain hyperelastic formulation. In Section 3, the hyperelastic constitutive relations for stresses and internal actions are developed using the strain energy density for a compressible isotropic neo-Hookean material proposed in Attard and Hunt (2004). The constitutive relationships are developed using Lagrangian physical stresses defined with respect to the unwarped displaced cross-sectional plane. Section 4 details the derivation of the equations of equilibrium for elastic initially straight isotropic prismatic beams under conservative loads, taking into account the nonlinear effects of changes in geometry and bending shear. The equilibrium equations derived are consistent with the equations presented by Reissner (1989) and the well-known vector representation of the beam equilibrium equations. Terms normally associated with the Wagner effect emerge from a consideration of warping, and are coupled to the twisting and shear deformations of the cross-section. The nonlinear theory presented can be used to predict the static and stability response of thin-walled beams when displacements are considered to be finite. Warping shear is ignored here. A closed form solution for the problem of the lateral buckling of a prismatic monosymmetric beam under pure bending is presented in Section 5. Section 6 details the derivation for the flexural–torsional buckling of a simply supported prismatic beam under axial compression. Expressions for the second variation of the total potential are developed in Section 7. Generalised non-dimensional lateral buckling equations for a simply supported beam are developed in Section 8. Shear deformations are shown to be important for short spans and beams with soft flexible shear rigidity. Finally, Section 9 provides a summary of the work in this paper. Further detailed derivations needed to support the work in this paper are presented in Appendices A,B,C,D,E,F.

## 2. Kinematics

Consider a straight prismatic beam with the materials lines aligned with a Cartesian rectangular coordinate system with origin at the centroid. Since the initial axis system is a Cartesian rectangular system there is no distinction between covariant and contravariant tensor components. The longitudinal axis of centroids of the undeformed beam is taken as the  $x$  or 1 axis. The cross-sectional centroidal principal axes are taken as the  $y$  or 2 axis and the  $z$  or

3 axis (refer to Fig. 1). Unit vectors in the direction of the coordinate axes  $x$ ,  $y$  &  $z$  are denoted by  $\mathbf{i}_1$ ,  $\mathbf{i}_2$  &  $\mathbf{i}_3$ . Under bending, axial, shear and torsion deformations excluding warping, it is assumed that the cross-sectional shape remains unchanged (undergoes no strain within the cross-sectional plane). The Timoshenko beam hypothesis is adopted with two shear angles of rotation.

We will firstly consider the case of no warping. The directions of the tangent base vectors in the deformed state can be described using Euler angles defined as  $\psi$ ,  $\theta$  &  $\phi$  as shown in Fig. 1. The Euler angles are taken as functions of  $x$  only. The angle  $\psi$  is related to the tortuosity of the deformed beam axis, see Love (1944). The angle  $\theta$  is a bending rotation while  $\phi$  is a twisting angle taken here about the shear centre with coordinates  $(y_s, z_s)$  and identified in Fig. 1. The covariant tangent base vectors in the deformed state denoted by  $\hat{\mathbf{g}}_i$  are defined by:

$$\hat{\mathbf{g}}_i = (\delta_i^j + u_i^j) \mathbf{g}_j \quad (1)$$

Here,  $i$  and  $j$  are indices which take values of 1, 2 or 3 and are associated with the  $x$ -,  $y$ - and  $z$ -axes, respectively,  $\delta_i^j$  is the Kronecker delta,  $\mathbf{g}_j$  are the covariant initial tangent base vectors in the undeformed state ( $\mathbf{g}_1 = \mathbf{i}_1$ ,  $\mathbf{g}_2 = \mathbf{i}_2$  &  $\mathbf{g}_3 = \mathbf{i}_3$ ) and  $u_i^j$  represents the derivatives of the  $u^j$  displacement vector components with respect to the coordinate corresponding to the index  $i$ . The convention is adopted where a repeated index is used to imply summation. A bracketed index indicates that the summation convention is suppressed. A scalar quantity is denoted by a lowercase italic light symbol such as  $c$  or  $\phi$ . A bold lower case symbol such as  $\mathbf{u}$  is used to represent a vector while a bold capital symbol such as  $\mathbf{R}$  is used to distinguish a second order tensor.

The rotation tensor denoted by the symbol  $\mathbf{R}$ , rotates a given vector  $\mathbf{v}$  about an axis parallel to a normalised axis vector  $\mathbf{u}$  ( $\mathbf{u} \cdot \mathbf{u} = 1$ ) through an angle  $0 \leq \omega \leq 2\pi$  as depicted in Fig. 2. The angle is defined as positive using the right hand screw rule where the thumb of the right hand is extended in the direction of the axis vector  $\mathbf{u}$  and the closing fingers define a positive rotation. The rotation vector  $\mathbf{R}$  is given by:

$$\mathbf{R} = \cos \omega \mathbf{I} + [1 - \cos \omega] \mathbf{u} \otimes \mathbf{u} + \sin \omega \mathbf{u} \times \quad (2)$$

To describe the displacement of the cross-section, first consider a rotation of  $\psi$  about the unit vector  $\mathbf{i}_1$  in the 1 or  $x$ -axis, such that

$$\mathbf{R}_\psi = \cos \psi \mathbf{I} + [1 - \cos \psi] \mathbf{i}_1 \otimes \mathbf{i}_1 + \sin \psi \mathbf{i}_1 \times \quad (3)$$

Applying this rotation to the  $y$ - and  $z$ -axis gives:

$$\begin{aligned} \hat{\mathbf{b}} &= \mathbf{R}_\psi \mathbf{i}_3 = -\sin \psi \mathbf{i}_2 + \cos \psi \mathbf{i}_3 \\ \hat{\mathbf{h}} &= \mathbf{R}_\psi \mathbf{i}_2 = \cos \psi \mathbf{i}_2 + \sin \psi \mathbf{i}_3 \end{aligned} \quad (4)$$

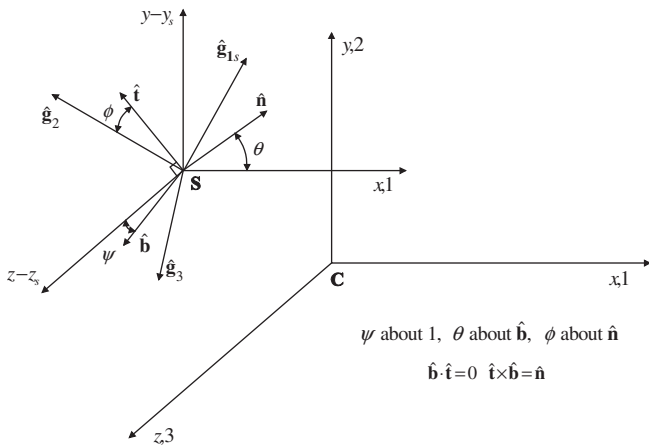


Fig. 1. Tangent base vectors at the shear centre.

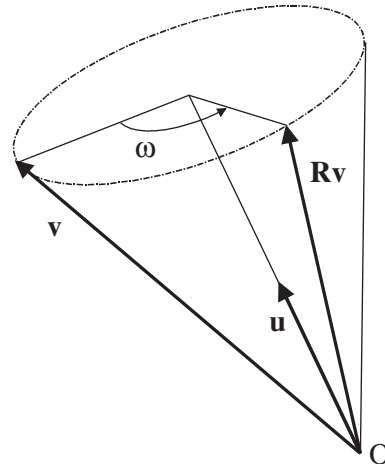


Fig. 2. Rotation tensor  $\mathbf{R}$ .

Here,  $\hat{\mathbf{h}}$  is orthonormal to  $\hat{\mathbf{b}}$  and both are in the original  $\{\mathbf{i}_2, \mathbf{i}_3\}$  plane. The vectors  $\{\hat{\mathbf{h}}, \hat{\mathbf{b}}\}$  are the  $\{\mathbf{i}_2, \mathbf{i}_3\}$  vectors rotated about the  $\mathbf{i}_1$  through an angle of  $\psi$ . Consider next a bending rotation  $\theta$  about the vector  $\hat{\mathbf{b}}$  such that

$$\mathbf{R}_\theta = \cos \theta \mathbf{I} + [1 - \cos \theta] \hat{\mathbf{b}} \otimes \hat{\mathbf{b}} + \sin \theta \hat{\mathbf{b}} \times \quad (5)$$

The unit vector  $\mathbf{i}_1$  is rotated to  $\hat{\mathbf{n}}$  and the unit vector  $\mathbf{i}_2$  rotated to  $\hat{\mathbf{t}}$ , such that

$$\hat{\mathbf{n}} = \mathbf{R}_\theta \mathbf{i}_1 = \cos \theta \mathbf{i}_1 + \sin \theta \hat{\mathbf{h}} \quad \hat{\mathbf{t}} = \mathbf{R}_\theta \mathbf{i}_2 = -\sin \theta \mathbf{i}_1 + \cos \theta \hat{\mathbf{h}} \quad (6)$$

Finally, a twist rotation  $\phi$  about the vector  $\hat{\mathbf{n}}$  taken through the shear centre is applied, defined by:

$$\mathbf{R}_\phi = \cos \phi \mathbf{I} + [1 - \cos \phi] \hat{\mathbf{n}} \otimes \hat{\mathbf{n}} + \sin \phi \hat{\mathbf{n}} \times \quad (7)$$

so that the tangent base vectors within the displaced cross-sectional plane ( $\hat{\mathbf{g}}_2$  &  $\hat{\mathbf{g}}_3$ ) are expressed by:

$$\hat{\mathbf{g}}_2 = \mathbf{R}_\phi \hat{\mathbf{t}} = \sin \phi \hat{\mathbf{b}} + \cos \phi \hat{\mathbf{t}} \quad \hat{\mathbf{g}}_3 = \mathbf{R}_\phi \hat{\mathbf{b}} = \cos \phi \hat{\mathbf{b}} - \sin \phi \hat{\mathbf{t}} \quad (8)$$

The vectors  $\hat{\mathbf{n}}, \hat{\mathbf{t}}$  &  $\hat{\mathbf{b}}$  form a moving orthonormal triad frame. The beam shear deformations can be characterised by two rotations of the normal vector  $\hat{\mathbf{n}}$  about the vectors  $\hat{\mathbf{t}}$  &  $\hat{\mathbf{b}}$  that lie within the plane of the cross-section. We select the first shear rotation as  $\varphi_0$  about the vector  $\hat{\mathbf{b}}$  and then a rotation  $-\alpha_0$  about the vector  $\hat{\mathbf{t}}$ . The shear rotation tensors are then:

$$\mathbf{R}_{\varphi_0} = \cos \varphi_0 \mathbf{I} + [1 - \cos \varphi_0] \hat{\mathbf{b}} \otimes \hat{\mathbf{b}} + \sin \varphi_0 \hat{\mathbf{b}} \times \quad (9)$$

$$\mathbf{R}_{-\alpha_0} = \cos \alpha_0 \mathbf{I} + [1 - \cos \alpha_0] \hat{\mathbf{t}} \otimes \hat{\mathbf{t}} - \sin \alpha_0 \hat{\mathbf{t}} \times \quad (10)$$

Applying the shear rotations, the tangent base vector  $\hat{\mathbf{g}}_1$  taken at the shear centre ( $y = y_s, z = z_s$ ) is:

$$\begin{aligned} \hat{\mathbf{g}}_1 &= \lambda_{1s} \mathbf{R}_{-\alpha_0} \mathbf{R}_{\varphi_0} \hat{\mathbf{n}} \\ &= \lambda_{1s} (\cos \varphi_0 \cos \alpha_0 \hat{\mathbf{n}} + \sin \varphi_0 \hat{\mathbf{t}} + \sin \alpha_0 \cos \varphi_0 \hat{\mathbf{b}}) \\ &= \hat{\gamma}_1 \hat{\mathbf{n}} + \hat{\gamma}_2 \hat{\mathbf{g}}_2 + \hat{\gamma}_3 \hat{\mathbf{g}}_3 \end{aligned} \quad (11)$$

Here  $\lambda_{1s}$  is the stretch of the shear centre axis and  $\hat{\gamma}_1, \hat{\gamma}_2, \hat{\gamma}_3$  are defined by

$$\begin{aligned} \hat{\gamma}_1 &= \lambda_{1s} \cos \varphi_0 \cos \alpha_0 \\ \hat{\gamma}_2 &= \lambda_{1s} \sin \varphi_0 \cos \phi + \lambda_{1s} \sin \alpha_0 \cos \varphi_0 \sin \phi \\ \hat{\gamma}_3 &= -\lambda_{1s} \sin \varphi_0 \sin \phi + \lambda_{1s} \sin \alpha_0 \cos \varphi_0 \cos \phi \end{aligned} \quad (12)$$

The term  $\hat{\gamma}_1$  represents the axial stretch component in the normal direction  $\hat{\mathbf{n}}$ , while  $\hat{\gamma}_2$  &  $\hat{\gamma}_3$  are measures of the shear deformation components taken in the tangent base vector directions in the deformed state,  $\hat{\mathbf{g}}_2$  &  $\hat{\mathbf{g}}_3$ , respectively. At the shear centre axis, the direction cosines ( $l_1, l_2, l_3, m_1, m_2, m_3, n_1, n_2$  and  $n_3$ ) and the derivatives of the displacement functions, can be derived from the following equations:

$$\begin{aligned} \hat{\mathbf{g}}_{1s} \cdot \mathbf{i}_1 &= 1 + u_{,1}^1 = \lambda_{1s} l_1 = \lambda_{1s} (\cos \theta \cos \varphi_0 \cos \alpha_0 - \sin \theta \sin \varphi_0) \\ \hat{\mathbf{g}}_{1s} \cdot \mathbf{i}_2 &= u_{,1}^2 = \lambda_{1s} m_1 = \lambda_{1s} (\sin \theta \cos \psi \cos \varphi_0 \cos \alpha_0 \\ &\quad + \cos \psi \cos \theta \sin \varphi_0 - \sin \psi \cos \varphi_0 \sin \alpha_0) \\ \hat{\mathbf{g}}_{1s} \cdot \mathbf{i}_3 &= u_{,1}^3 = \lambda_{1s} n_1 = \lambda_{1s} (\sin \theta \sin \psi \cos \varphi_0 \cos \alpha_0 \\ &\quad + \sin \psi \cos \theta \sin \varphi_0 + \cos \psi \cos \varphi_0 \sin \alpha_0) \end{aligned} \quad (13)$$

$$\begin{aligned} \hat{\mathbf{g}}_2 \cdot \mathbf{i}_1 &= u_{,2}^1 = l_2 = -\sin \theta \cos \phi \\ \hat{\mathbf{g}}_2 \cdot \mathbf{i}_2 &= 1 + u_{,2}^2 = m_2 = -\sin \psi \sin \phi + \cos \psi \cos \phi \cos \theta \\ \hat{\mathbf{g}}_2 \cdot \mathbf{i}_3 &= u_{,2}^3 = n_2 = \cos \psi \sin \phi + \sin \psi \cos \phi \cos \theta \end{aligned} \quad (14)$$

$$\begin{aligned} \hat{\mathbf{g}}_3 \cdot \mathbf{i}_1 &= u_{,3}^1 = l_3 = \sin \theta \sin \phi \\ \hat{\mathbf{g}}_3 \cdot \mathbf{i}_2 &= u_{,3}^2 = m_3 = -\sin \psi \cos \phi - \cos \psi \sin \phi \cos \theta \\ \hat{\mathbf{g}}_3 \cdot \mathbf{i}_3 &= 1 + u_{,3}^3 = n_3 = \cos \psi \cos \phi - \sin \psi \sin \phi \cos \theta \end{aligned} \quad (15)$$

In the above,  $u^1, u^2$  &  $u^3$  are displacement functions in the  $x$ -,  $y$ - and  $z$ -directions, respectively. The comma notation symbolises differentiation with respect to the proceeding variable. In Eqs. (13)–(15), the displacement components are taken at the shear centre ( $y = y_s, z = z_s$ ). Eq. (13) leads to the following:

$$\begin{aligned} 1 + u_{s,x} &= \lambda_{1s} (\cos \theta \cos \varphi_0 \cos \alpha_0 - \sin \theta \sin \varphi_0) \\ v_{s,x} &= \lambda_{1s} (\sin \theta \cos \psi \cos \varphi_0 \cos \alpha_0 \\ &\quad + \cos \psi \cos \theta \sin \varphi_0 - \sin \psi \cos \varphi_0 \sin \alpha_0) \\ w_{s,x} &= \lambda_{1s} (\sin \theta \sin \psi \cos \varphi_0 \cos \alpha_0 \\ &\quad + \sin \psi \cos \theta \sin \varphi_0 + \cos \psi \cos \varphi_0 \sin \alpha_0) \end{aligned} \quad (16)$$

where  $u_s(x)$  is the axial displacement of the shear centre and,  $v_s(x)$  and  $w_s(x)$  are the displacements of the shear centre in the  $y$ - and  $z$ -directions, respectively. The stretch of the shear centre axis  $\lambda_{1s}$  can be written as:

$$\lambda_{1s} = \sqrt{(1 + u_{s,x})^2 + v_{s,x}^2 + w_{s,x}^2} = \sqrt{(\hat{\gamma}_1)^2 + (\hat{\gamma}_2)^2 + (\hat{\gamma}_3)^2} \quad (17)$$

Eq. (16) can be rearranged thus

$$\begin{aligned} \lambda_{1s} \cos \varphi_0 \cos \alpha_0 &= (1 + u_{s,x}) \cos \theta + (v_{s,x} \cos \psi + w_{s,x} \sin \psi) \sin \theta \\ \lambda_{1s} \sin \varphi_0 &= -(1 + u_{s,x}) \sin \theta + (v_{s,x} \cos \psi + w_{s,x} \sin \psi) \cos \theta \\ \lambda_{1s} \cos \varphi_0 \sin \alpha_0 &= -v_{s,x} \sin \psi + w_{s,x} \cos \psi \end{aligned} \quad (18)$$

with the shear angles defined by

$$\begin{aligned} \sin \varphi_0 &= \frac{-(1 + u_{s,x}) \sin \theta + (v_{s,x} \cos \psi + w_{s,x} \sin \psi) \cos \theta}{\sqrt{(1 + u_{s,x})^2 + v_{s,x}^2 + w_{s,x}^2}} \\ \tan \alpha_0 &= \frac{-v_{s,x} \sin \psi + w_{s,x} \cos \psi}{(1 + u_{s,x}) \cos \theta + (v_{s,x} \cos \psi + w_{s,x} \sin \psi) \sin \theta} \end{aligned} \quad (19)$$

Expressions (18) and (19) give the normal and shear components of the longitudinal stretch of the shear centre axis  $\lambda_{1s}$ , and the shear angles in terms of the displacement derivatives  $u_{s,x}, v_{s,x}$  &  $w_{s,x}$  and the Euler angles. The vector components in the  $x$ -,  $y$ - and  $z$ -directions, of the tangent base vector taken without shear deformations (taken normal to the plane of the cross-section, see Eq. (6)) can also be used to characterise the shear deformations by noting that

$$\begin{aligned} 1 + \bar{u}_{s,x} &= 1 + u_{s,x} - \gamma_1 = \bar{\lambda}_{1s} \cos \theta \\ \bar{v}_{s,x} &= v_{s,x} - \gamma_2 = \bar{\lambda}_{1s} \sin \theta \cos \psi \bar{w}_{s,x} = w_{s,x} - \gamma_3 = \bar{\lambda}_{1s} \sin \theta \sin \psi \end{aligned} \quad (20)$$

where  $\bar{u}_{s,x}, \bar{v}_{s,x}$  &  $\bar{w}_{s,x}$  are the deflection derivatives taken without shear deformations,  $\gamma_1, \gamma_2$  &  $\gamma_3$  are shear deformations and  $\bar{\lambda}_{1s}$  is a modified stretch defined by:

$$\bar{\lambda}_{1s} = \sqrt{(1 + \bar{u}_{s,x})^2 + \bar{v}_{s,x}^2 + \bar{w}_{s,x}^2} \quad (21)$$

Eq. (20) leads to the following equations, which relate the bending angle and tortuosity angle at the shear centre:

$$\cos \theta = \frac{1 + \bar{u}_{s,x}}{\bar{\lambda}_{1s}} \quad \sin \theta = \frac{\sqrt{\bar{w}_{s,x}^2 + \bar{v}_{s,x}^2}}{\bar{\lambda}_{1s}} \quad \tan \psi = \frac{\bar{w}_{s,x}}{\bar{v}_{s,x}} \quad (22)$$

The geometric torsion or tortuosity of the shear centre axis is related to the derivative of the angle  $\psi$  with respect to  $x$ , and can be derived from the above equations, hence:

$$\frac{d\psi}{dx} = \frac{d}{dx} \tan^{-1} \left( \frac{\bar{w}_{s,x}}{\bar{v}_{s,x}} \right) = \frac{\bar{v}_{s,x} \bar{w}_{s,xx} - \bar{w}_{s,x} \bar{v}_{s,xx}}{\bar{w}_{s,x}^2 + \bar{v}_{s,x}^2} \quad (23)$$

Also from Eq. (22) we have for the curvature:

$$\begin{aligned} \frac{d\theta}{dx} &= \frac{d}{dx} \tan^{-1} \left( \frac{\sqrt{\bar{w}_{s,x}^2 + \bar{v}_{s,x}^2}}{1 + \bar{u}_{s,x}} \right) \\ &= \frac{(\bar{v}_{s,x} \bar{v}_{s,xx} + \bar{w}_{s,x} \bar{w}_{s,xx})(1 + \bar{u}_{s,x}) - \bar{u}_{s,xx}(\bar{w}_{s,x}^2 + \bar{v}_{s,x}^2)}{\bar{\lambda}_{1s}^2 \sqrt{\bar{w}_{s,x}^2 + \bar{v}_{s,x}^2}} \end{aligned} \quad (24)$$

Integrating Eqs. (14) and (15) leads to expressions for the displacement functions, that is:

$$\begin{aligned} u^1 &= u_s(x) + (y - y_s)l_2 + (z - z_s)l_3 \\ u^2 &= v_s(x) + (y - y_s)(m_2 - 1) + (z - z_s)m_3 \\ u^3 &= w_s(x) + (y - y_s)n_2 + (z - z_s)(n_3 - 1) \end{aligned} \quad (25)$$

The displacement function in vector form is:

$$\begin{aligned} \mathbf{u} &= \mathbf{u}_s + (y - y_s)(\hat{\mathbf{g}}_2 - \mathbf{i}_2) + (z - z_s)(\hat{\mathbf{g}}_3 - \mathbf{i}_3) \\ &= \mathbf{u}_s + (y - y_s)(\sin \phi \hat{\mathbf{b}} + \cos \phi \hat{\mathbf{t}} - \mathbf{i}_2) + (z - z_s) \\ &\quad \times (\cos \phi \hat{\mathbf{b}} - \sin \phi \hat{\mathbf{t}} - \mathbf{i}_3) = \mathbf{u}_s + \tilde{\mathbf{z}}\hat{\mathbf{b}} + \tilde{\mathbf{y}}\hat{\mathbf{t}} - (y - y_s)(\mathbf{i}_2) - (z - z_s)(\mathbf{i}_3) \end{aligned} \quad (26)$$

in which,

$$\begin{aligned} \tilde{\mathbf{z}} &= (y - y_s) \sin \phi + (z - z_s) \cos \phi \\ \tilde{\mathbf{y}} &= (y - y_s) \cos \phi - (z - z_s) \sin \phi \end{aligned} \quad (27)$$

In the above,  $\mathbf{u}_s$  is the displacement of the shear centre axis in space, given by:

$$\mathbf{u}_s = u_s \mathbf{i}_1 + v_s \mathbf{i}_2 + w_s \mathbf{i}_3 \quad (28)$$

## 2.1. Curvatures and torsion

The curvatures will be calculated with respect to the moving triad orthonormal vector system  $\hat{\mathbf{n}}, \hat{\mathbf{t}} \& \hat{\mathbf{b}}$  defined at the shear centre, as well as, the orthonormal frame  $\hat{\mathbf{n}}, \hat{\mathbf{g}}_2 \& \hat{\mathbf{g}}_3$ . The moving triad orthonormal frame  $\hat{\mathbf{n}}, \hat{\mathbf{t}} \& \hat{\mathbf{b}}$  is analogous to a Darboux frame used in the differential geometry of surfaces. The deformation curvatures with respect to the unit normal  $\hat{\mathbf{n}}$  at the shear centre are given by:

$$\kappa = \hat{\mathbf{n}}_x \cdot \hat{\mathbf{g}}_2 = \frac{d\theta}{dx} \cos \phi + \frac{d\psi}{dx} \sin \theta \sin \phi \quad (29)$$

$$\kappa' = -\hat{\mathbf{n}}_x \cdot \hat{\mathbf{g}}_3 = \frac{d\theta}{dx} \sin \phi - \frac{d\psi}{dx} \sin \theta \cos \phi \quad (30)$$

The deformation torsion about the unit normal to the cross-section with respect to the triad vectors  $\hat{\mathbf{n}}, \hat{\mathbf{g}}_2 \& \hat{\mathbf{g}}_3$  is defined by:

$$\tau = -\hat{\mathbf{g}}_{3,x} \cdot \hat{\mathbf{g}}_2 = \frac{d\phi}{dx} + \frac{d\psi}{dx} \cos \theta \quad (31)$$

The curvatures can also be written with respect to the triad vectors  $\hat{\mathbf{n}}, \hat{\mathbf{t}} \& \hat{\mathbf{b}}$ . That is, the normal curvature is given by:

$$\tilde{\kappa} = \hat{\mathbf{n}}_x \cdot \hat{\mathbf{t}} = \kappa \cos \phi + \kappa' \sin \phi = \frac{d\theta}{dx} \quad (32)$$

While the geodesic curvature is:

$$\tilde{\kappa}' = \hat{\mathbf{n}}_x \cdot \hat{\mathbf{b}} = \kappa \sin \phi - \kappa' \cos \phi = \sin \theta \frac{d\psi}{dx} \quad (33)$$

With respect to the triad vectors  $\hat{\mathbf{n}}, \hat{\mathbf{t}} \& \hat{\mathbf{b}}$ , the relative or geodesic torsion is defined by

$$\hat{\mathbf{b}}_x \cdot \hat{\mathbf{t}} = \tau - \frac{d\phi}{dx} \quad (34)$$

Eqs. (31)–(33) produce for the rate of change of the Euler angles:

$$\frac{d\theta}{dx} = \tilde{\kappa} \quad \frac{d\psi}{dx} = \frac{\tilde{\kappa}'}{\sin \theta} \quad \frac{d\phi}{dx} = \tau - \frac{\tilde{\kappa}'}{\tan \theta} \quad (35)$$

The evolution of the Darboux type triad sets  $\hat{\mathbf{n}}, \hat{\mathbf{t}} \& \hat{\mathbf{b}}$  and  $\hat{\mathbf{n}}, \hat{\mathbf{g}}_2 \& \hat{\mathbf{g}}_3$  along the axis of the beam is governed by the equations:

$$\begin{aligned} \hat{\mathbf{n}}_x &= \tilde{\kappa} \hat{\mathbf{t}} + \tilde{\kappa}' \hat{\mathbf{b}} \quad \hat{\mathbf{t}}_x = -\tilde{\kappa} \hat{\mathbf{n}} + \left( \tau - \frac{d\phi}{dx} \right) \hat{\mathbf{b}} \\ \hat{\mathbf{b}}_x &= -\tilde{\kappa}' \hat{\mathbf{n}} - \left( \tau - \frac{d\phi}{dx} \right) \hat{\mathbf{t}} \end{aligned} \quad (36)$$

$$\hat{\mathbf{n}}_x = \kappa \hat{\mathbf{g}}_2 - \kappa' \hat{\mathbf{g}}_3 \quad \hat{\mathbf{g}}_{2,x} = -\kappa \hat{\mathbf{n}} + \tau \hat{\mathbf{g}}_3 \quad \hat{\mathbf{g}}_{3,x} = \kappa' \hat{\mathbf{n}} - \tau \hat{\mathbf{g}}_2$$

These are associated with rotation vectors given by:

$$\left( \tau - \frac{d\phi}{dx} \right) \hat{\mathbf{n}} - \tilde{\kappa} \hat{\mathbf{t}} + \tilde{\kappa}' \hat{\mathbf{b}} \quad \tau \hat{\mathbf{n}} + \kappa' \hat{\mathbf{g}}_2 + \kappa \hat{\mathbf{g}}_3 \quad (37)$$

## 2.2. Covariant tangent base vectors

Using Eqs. (1) and (26), the covariant tangent base vector  $\hat{\mathbf{g}}_1$  can now be written in terms of the curvatures and torsion of the shear centre axis as:

$$\begin{aligned} \hat{\mathbf{g}}_1 &= [\lambda_{1s} \cos \phi_0 \cos \alpha_0 - (y - y_s)\kappa + (z - z_s)\kappa'] \hat{\mathbf{n}} \\ &\quad + (\lambda_{1s} \sin \phi_0 - \tau \tilde{\mathbf{z}}) \hat{\mathbf{t}} + (\lambda_{1s} \sin \alpha_0 \cos \phi_0 + \tau \tilde{\mathbf{y}}) \hat{\mathbf{b}} \\ &= [\hat{\gamma}_1 - (y - y_s)\kappa + (z - z_s)\kappa'] \hat{\mathbf{n}} + [\hat{\gamma}_2 - \tau(z - z_s)] \hat{\mathbf{g}}_2 \\ &\quad + [\hat{\gamma}_3 + \tau(y - y_s)] \hat{\mathbf{g}}_3 \end{aligned} \quad (38)$$

It is convenient to split the longitudinal stretch into its normal and shear components. These components of longitudinal stretch can be derived from Eq. (38) and are:

$$\hat{\mathbf{g}}_1 \cdot \hat{\mathbf{n}} = \lambda_n = \lambda_{1s} \cos \phi_0 \cos \alpha_0 - (y - y_s)\kappa + (z - z_s)\kappa' \quad (39)$$

$$\begin{aligned} \hat{\mathbf{g}}_1 \cdot \hat{\mathbf{t}} &= \lambda_{st} = \lambda_{1s} \sin \phi_0 - \tau \tilde{\mathbf{z}} \quad \hat{\mathbf{g}}_1 \cdot \hat{\mathbf{b}} = \lambda_{sb} \\ &= \lambda_{1s} \sin \alpha_0 \cos \phi_0 + \tau \tilde{\mathbf{y}} \end{aligned} \quad (40)$$

where  $\lambda_n$  is the normal component of longitudinal stretch while  $\lambda_{st}$  &  $\lambda_{sb}$  are the tangential components of the longitudinal stretch, taken in  $\hat{\mathbf{t}} \& \hat{\mathbf{b}}$  directions, respectively. The covariant tangent base vectors within the plane of the displaced cross-section were defined in Eq. (8). Because we have assumed the cross-sectional shape does not change during deformation, the following holds:

$$\hat{\mathbf{g}}_2 \cdot \hat{\mathbf{g}}_2 = 1 \quad \hat{\mathbf{g}}_3 \cdot \hat{\mathbf{g}}_3 = 1 \quad \hat{\mathbf{g}}_2 \cdot \hat{\mathbf{g}}_3 = 0 \quad (41)$$

The first invariant  $I_\lambda$  and the Jacobian  $J$  of the system are therefore:

$$\begin{aligned} I_\lambda - 2 &= (\lambda_n)^2 + (\lambda_{1s} \sin \phi_0 - \tau \tilde{\mathbf{z}})^2 + (\lambda_{1s} \sin \alpha_0 \cos \phi_0 + \tau \tilde{\mathbf{y}})^2 \\ J &= \lambda_n \end{aligned} \quad (42)$$

The Jacobian  $J$  is related to the ratio of the volume change. Here the Jacobian is a function of the normal component of the longitudinal stretch only.

## 2.3. Warping

Here we assume that warping of the cross-sectional plane is a function of cross-sectional geometry, proportional to the deformation torsion and occurs about the unit normal to the cross-sectional plane. Thus, we augment Eq. (26) to give the displacement vector as:

$$\begin{aligned} \mathbf{u} &= \mathbf{u}_s - (y - y_s)(\mathbf{i}_2 - \sin \phi \hat{\mathbf{b}} - \cos \phi \hat{\mathbf{t}}) \\ &\quad - (z - z_s)(\mathbf{i}_3 - \cos \phi \hat{\mathbf{b}} + \sin \phi \hat{\mathbf{t}}) - \tau \omega(y, z) \hat{\mathbf{n}} \end{aligned} \quad (43)$$

where  $\omega(y, z)$  is a cross-sectional warping function. Note in the above,  $\mathbf{u}_s$  is now the displacement of the shear centre axis only if the warping function is zero at the shear centre (the displacement of the shear centre is  $\mathbf{u}_s - \tau \omega(y_s, z_s) \hat{\mathbf{n}}$ ). In Appendix A, expressions

for the displacements are truncated to second order so that comparisons can be made to other lateral buckling formulations which are based on second order approximations. The covariant tangent base vectors, which include warping, can now be derived from Eq. (43) and are given by:

$$\hat{\mathbf{g}}_{1\omega} = (\lambda_n - \omega\tau_x)\hat{\mathbf{n}}[\hat{\gamma}_2 - \tau(z - z_s + \omega\kappa)]\hat{\mathbf{g}}_2 + [\hat{\gamma}_3 + \tau(y - y_s + \omega\kappa')]\hat{\mathbf{g}}_3 \quad (44)$$

$$\hat{\mathbf{g}}_{2\omega} = \sin\phi\hat{\mathbf{b}} + \cos\phi\hat{\mathbf{t}} - \tau\omega_y\hat{\mathbf{n}} = \hat{\mathbf{g}}_2 - \tau\omega_y\hat{\mathbf{n}} \quad (45)$$

$$\hat{\mathbf{g}}_{3\omega} = \cos\phi\hat{\mathbf{b}} - \sin\phi\hat{\mathbf{t}} - \tau\omega_z\hat{\mathbf{n}} = \hat{\mathbf{g}}_3 - \tau\omega_z\hat{\mathbf{n}} \quad (46)$$

The contravariant base vectors are:

$$J\hat{\mathbf{g}}^{1\omega} = \hat{\mathbf{n}} + \tau\bar{\omega}_2\hat{\mathbf{t}} + \tau\bar{\omega}_3\hat{\mathbf{b}} = \hat{\mathbf{n}} + \tau\omega_y\hat{\mathbf{g}}_2 + \tau\omega_z\hat{\mathbf{g}}_3 \quad (47)$$

$$J\hat{\mathbf{g}}^{2\omega} = (-\hat{\gamma}_2 + \tau(z - z_s))\hat{\mathbf{n}} + (\lambda_n - \omega\tau_x)\hat{\mathbf{g}}_2 + (\hat{\gamma}_3\tau + (y - y_s + \omega\kappa')\tau^2)\hat{\mathbf{g}}_2\omega_z + (-\hat{\gamma}_2\tau + (z - z_s + \omega\kappa)\tau^2)\hat{\mathbf{g}}_3\omega_z \quad (48)$$

$$J\hat{\mathbf{g}}^{3\omega} = (-\hat{\gamma}_3 - \tau(y - y_s))\hat{\mathbf{n}} + (\lambda_n - \omega\tau_x)\hat{\mathbf{g}}_3 + (\hat{\gamma}_2\tau - (z - z_s + \omega\kappa)\tau^2)\hat{\mathbf{g}}_3\omega_y + (-\hat{\gamma}_3\tau - (y - y_s + \omega\kappa')\tau^2)\hat{\mathbf{g}}_2\omega_y \quad (49)$$

in which  $J$  is the volume invariant or Jacobian. The stretch of the beam and the shear deformations are thus calculated as:

$$\hat{\mathbf{g}}_{1\omega} \cdot \hat{\mathbf{g}}_{1\omega} = (\lambda_1)^2 = (\lambda_n - \omega\tau_x)^2 + [\hat{\gamma}_2 - \tau(z - z_s + \omega\kappa)]^2 + [\hat{\gamma}_3 + \tau(y - y_s + \omega\kappa')]^2 \quad (50)$$

$$\hat{\mathbf{g}}_{2\omega} \cdot \hat{\mathbf{g}}_{2\omega} = 1 + (\tau\omega_y)^2 \quad \hat{\mathbf{g}}_{3\omega} \cdot \hat{\mathbf{g}}_{3\omega} = 1 + (\tau\omega_z)^2 \quad \hat{\mathbf{g}}_{2\omega} \cdot \hat{\mathbf{g}}_{3\omega} = \tau^2\omega_y\omega_z \quad (51)$$

The cross-section no-longer remains planar because of warping. The first invariant  $I_\lambda$  and the volume invariant  $J$  are updated to including warping and are related to the deformations by:

$$I_\lambda - 2 = (\lambda_n - \omega\tau_x)^2 + [\hat{\gamma}_2 - \tau(z - z_s + \omega\kappa)]^2 + [\hat{\gamma}_3 + \tau(y - y_s + \omega\kappa')]^2 + \tau^2(\omega_y^2 + \omega_z^2) \quad (52)$$

$$J = \lambda_n - \omega\tau_x + \tau(\hat{\gamma}_2\omega_y + \hat{\gamma}_3\omega_z) + \tau^2\bar{\omega} \quad (53)$$

in which

$$\bar{\omega} = \omega_z(y - y_s + \omega\kappa') - \omega_y(z - z_s + \omega\kappa) \quad (54)$$

### 3. Hyperelastic constitutive relations for stresses and internal actions

The strain energy density function  $U$  for a compressible isotropic neo-Hookean material (see Attard and Hunt, 2004) is given by:

$$U = \frac{1}{2}G(\text{tr}(\mathbf{C} - \mathbf{I}) - 2\ln J) + \frac{1}{2}\Lambda(\ln J)^2 \quad (55)$$

Here,  $\text{tr}$  symbolizes the trace of a tensor and  $\mathbf{C}$  is right Cauchy–Green deformation tensor. The constitutive relationship for a hyperelastic material can be established for the second Piola–Kirchhoff stress tensor  $\mathbf{\Pi}$  by (see Attard and Hunt, 2004):

$$\mathbf{\Pi} = 2\frac{\partial U}{\partial \mathbf{C}} = \mathbf{G}\mathbf{I} - p_h\mathbf{C}^{-1} \quad p_h = G - \Lambda\ln J \quad (56)$$

In the above,  $p_h$  represents a hydrostatic stress.

A physical Lagrangian stress system is used here, defined with respect to the cross-sectional plane before warping and directions

defined by the moving orthonormal triad frame  $\hat{\mathbf{n}}, \hat{\mathbf{t}}$  &  $\hat{\mathbf{b}}$ . The transformation between the second Piola–Kirchhoff stress tensor components and the Lagrangian physical stresses can be established using vector transformation. The normal stresses  $S^{nn}$  are taken normal to the cross-sectional plane (in the direction  $\hat{\mathbf{n}}$ ) while the tangential shear stresses  $S^{nb}$  &  $S^{nt}$  are taken within the cross-sectional plane in the directions  $\hat{\mathbf{t}}$  &  $\hat{\mathbf{b}}$ , respectively. The transformation between the second Piola–Kirchhoff stress tensor components and Lagrangian stresses are therefore:

$$\Pi^{11}\hat{\mathbf{g}}_{1\omega} + \Pi^{12}\hat{\mathbf{g}}_{2\omega} + \Pi^{13}\hat{\mathbf{g}}_{3\omega} = S^{nn}\hat{\mathbf{n}} + S^{nt}\hat{\mathbf{t}} + S^{nb}\hat{\mathbf{b}} \quad (57)$$

$$\Pi^{12} = (S^{nn}\hat{\mathbf{n}} + S^{nt}\hat{\mathbf{t}} + S^{nb}\hat{\mathbf{b}}) \cdot \hat{\mathbf{g}}^{2\omega} \quad (58)$$

$$\Pi^{13} = (S^{nn}\hat{\mathbf{n}} + S^{nt}\hat{\mathbf{t}} + S^{nb}\hat{\mathbf{b}}) \cdot \hat{\mathbf{g}}^{3\omega} \quad (59)$$

and hence:

$$S^{nn} = \Pi^{11}\hat{\mathbf{g}}_{1\omega} \cdot \hat{\mathbf{n}} + \Pi^{12}\hat{\mathbf{g}}_{2\omega} \cdot \hat{\mathbf{n}} + \Pi^{13}\hat{\mathbf{g}}_{3\omega} \cdot \hat{\mathbf{n}} = \Pi^{11}(\lambda_n - \omega\tau_x) - \Pi^{12}\tau\omega_y - \Pi^{13}\tau\omega_z$$

$$S^{nt} = \Pi^{11}\hat{\mathbf{g}}_{1\omega} \cdot \hat{\mathbf{t}} + \Pi^{12}\hat{\mathbf{g}}_{2\omega} \cdot \hat{\mathbf{t}} + \Pi^{13}\hat{\mathbf{g}}_{3\omega} \cdot \hat{\mathbf{t}} = \Pi^{11}(\lambda_{1s} \sin\phi_0 - \tau\bar{z}) + \Pi^{12} \cos\phi - \Pi^{13} \sin\phi \quad (60)$$

$$S^{nb} = \Pi^{11}\hat{\mathbf{g}}_{1\omega} \cdot \hat{\mathbf{b}} + \Pi^{12}\hat{\mathbf{g}}_{2\omega} \cdot \hat{\mathbf{b}} + \Pi^{13}\hat{\mathbf{g}}_{3\omega} \cdot \hat{\mathbf{b}} = \Pi^{11}(\lambda_{1s} \sin\alpha_0 \cos\phi_0 + \tau\bar{y}) + \Pi^{12} \sin\phi + \Pi^{13} \cos\phi$$

in which:

$$\bar{y} = \bar{y} - \omega\kappa' \quad \bar{z} = \bar{z} + \omega\kappa \quad (61)$$

Incorporating the constitutive relations given in Eq. (56), expanding to second order in terms of the deformations, we can write the constitutive relations as:

$$S^{nn} = \bar{E}(\lambda_n - \omega\tau_x - 1) + (G + \Lambda)(\tau(\hat{\gamma}_2\omega_y + \hat{\gamma}_3\omega_z) + \tau^2\bar{\omega})$$

$$S^{n2} = G(\hat{\gamma}_2 - \tau(z - z_s + \omega\kappa))S^{n3} = G(\hat{\gamma}_3 + \tau(y - y_s - \omega_z + \omega\kappa')) \quad (62)$$

where  $G = \frac{E}{2(1+\eta)}$  is the shear modulus,  $\bar{E} = 2G + \Lambda$ ,  $\Lambda = \frac{2G\eta}{(1-2\eta)}$  is the Lamé constant,  $E$  is the elastic modulus and  $\eta$  is the Poisson's ratio. The material parameter governing the normal stress is not the elastic modulus  $E$  as would be expected for a uniaxial stress state, as the displacements assumed restrain the dilation of the cross-section shape (see Attard, 2003a; Attard and Hunt, 2008b). An approximation in beam theory is to replace  $\bar{E}$  by  $E$ . The alternate shear stresses  $S^{n2}$  &  $S^{n3}$  are taken in the  $\hat{\mathbf{g}}_2$  &  $\hat{\mathbf{g}}_3$  directions, respectively, as indicated in Fig. 4. The  $\tau^2\bar{\omega}$  term in Eq. (62) gives rise to terms which are analogous to the Wagner effect although unlike conventional beam theory, the Wagner effect here is related to the warping of the section (see Alwis and Wang, 1996; Attard, 1986).

Integrating over the cross-sectional plane gives for the internal actions:

$$N = \int_A S^{nn} dA = EA(\hat{\gamma}_1 + y_s\kappa - z_s\kappa' - 1) + (G + \Lambda)A\tau(z_s\hat{\gamma}_2 - y_s\hat{\gamma}_3) + (G + \Lambda)(I_{p0} - J_t)\tau^2 \quad (63)$$

$$Q_t = \int_A S^{nt} dA = GA\lambda_{1s} \sin\phi_0 \quad Q_b = \int_A S^{nb} dA = GA\lambda_{1s} \sin\alpha_0 \cos\phi_0$$

$$Q_2 = GA\hat{\gamma}_2 \quad Q_3 = GA\hat{\gamma}_3 \quad (64)$$

$$M_3 = \int_A -yS^{nn} dA = EI_{zz}(\kappa - \beta_1^*\tau^2) - (G + \Lambda)\tau(C_{yy}\hat{\gamma}_2 + C_{yz}\hat{\gamma}_3) \quad (65)$$

$$M_2 = \int_A zS^{nn} dA = EI_{yy}(\kappa' - \beta_2^*\tau^2) + (G + \Lambda)\tau(C_{zy}\hat{\gamma}_2 + C_{zz}\hat{\gamma}_3) \quad (66)$$

$$M_t = \int_A [S^{n3}(y - y_s - \omega_z + \omega\kappa') - S^{n2}(z - z_s + \omega_y - \omega\kappa)] dA \\ = GJ_t \tau + 2G\tau(C_{\omega z}\kappa' + C_{\omega y}\kappa) + GJ_\omega \tau((\kappa')^2 + \kappa^2) \quad (67)$$

$$M_\omega = \int_A -\omega S^{nn} dA = EI_\omega(\tau_x - U_\omega^* \tau^2) - (G + A)\tau(C_{\omega y}\hat{\gamma}_2 + C_{\omega z}\hat{\gamma}_3) \quad (68)$$

where  $I_{zz}$  and  $I_{yy}$  are the second moment of areas about the  $y$  and  $z$  axes, respectively, and  $I_{po}$  is the polar second moment of area about the shear centre, such that

$$I_{zz} = \int_A y^2 dA \quad I_{yy} = \int_A z^2 dA \quad I_{po} \\ = \int_A [(y - y_s)^2 + (z - z_s)^2] dA \quad (69)$$

In the above equations,  $M_3$  and  $M_2$  are the resultant bending moments about the deformed  $\hat{\mathbf{g}}_3$  &  $\hat{\mathbf{g}}_2$  axes passing through the centroid, respectively;  $M_t$  is the Saint Venant twisting moment component about the unit normal to the cross-section;  $M_\omega$  is the bimoment;  $J_t$  is the Saint Venant torsion constant, defined by:

$$J_t = \int_A (\omega_y + z - z_s)^2 + (\omega_z - y + y_s)^2 dA \quad (70)$$

$N$  is the axial force acting at the centroid and normal to the deformed cross-sectional plane in the direction  $\hat{\mathbf{n}}$ , while  $Q_t$  and  $Q_b$  are the shear force resultants acting at the shear centre in the directions  $\hat{\mathbf{t}}$  &  $\hat{\mathbf{b}}$ , respectively. The shear forces  $Q_2$  &  $Q_3$  act in the directions,  $\hat{\mathbf{g}}_2$  &  $\hat{\mathbf{g}}_3$ , respectively. Hence

$$8Q_2 = Q_t \cos \phi + Q_b \sin \phi \quad Q_3 = -Q_t \sin \phi + Q_b \cos \phi \quad (71)$$

The warping related geometric parameters are defined by:

$$C_{yy} = \int_A y\omega_y dA \quad C_{yz} = \int_A y\omega_z dA \quad C_{zy} = \int_A z\omega_y dA \\ C_{zz} = \int_A z\omega_z dA \quad C_{\omega y} = \int_A \omega\omega_y dA \quad C_{\omega z} = \int_A \omega\omega_z dA \quad (72)$$

Using the [de Saint-Venant \(1855\)](#) theory of torsion as detailed in [Love \(1944\)](#), the following integral can be evaluated as (see also [Appendix B](#)):

$$\int_A \bar{\omega} dA = \int_A [\omega_z(y - y_s) - \omega_y(z - z_s)] dA = I_{po} - J_t \quad (73)$$

The other parameters are defined by:

$$\beta_1^* = \frac{\int_A y\bar{\omega} dA}{E^* I_{zz}} \quad \beta_2^* = \frac{\int_A -z\bar{\omega} dA}{E^* I_{yy}} \quad U_\omega^* = \frac{\int_A -\omega\bar{\omega} dA}{E^* I_\omega} \quad E^* = \frac{E}{G + A} \quad (74)$$

$$\int_A \omega_y dA = z_s A \int_A \omega_z dA = -y_s A \int_A \omega dA = 0 \int_A y\omega dA = 0 \int_A z\omega dA = 0 \quad (75)$$

[Appendix B](#) examines the expressions for the parameters in Eq. (74) for thin-walled open beams. The geometric parameters in Eq. (72) also appear in Chapter 10 of [Vlasov \(1959\)](#). We can give physical meaning to these warping related geometric parameters by considering the expressions for the shear stresses in Eq. (62) and restricting our attention to torsion only, hence we have for  $S^{n2}$  &  $S^{n3}$ :

$$S^{n2} = -G\tau(z - z_s + \omega_y - \omega\kappa)S^{n3} = G\tau(y - y_s - \omega_z + \omega\kappa') \quad (76)$$

The following resultants of the product of the shear stresses  $S^{n2}$  with the coordinates  $y, z, \omega$  are then

$$\frac{\int_A zS^{n2} dA}{-G\tau} = \int_A z(z - z_s + \omega_y - \omega\kappa) dA = I_{yy} + C_{zy} \quad (77)$$

$$\frac{\int_A yS^{n2} dA}{-G\tau} = \int_A y(z - z_s + \omega_y - \omega\kappa) dA = C_{yy} \quad (78)$$

$$\frac{\int_A \omega S^{n2} dA}{-G\tau} = \int_A \omega(z - z_s + \omega_y - \omega\kappa) dA = C_{\omega y} - I_\omega \kappa \quad (79)$$

We see that the warping related geometrical parameters are related to the distribution of the shear stresses due to torsion. By examining a state of loading which consists of only an axial force and bending moment internal actions (no twisting moment), the following expressions are derived in [Appendix C](#):

$$C_{yy} = \int_A y\omega_y dA = 0 \quad C_{yz} = \int_A y\omega_z dA = I_{zz} \\ C_{zy} = \int_A z\omega_y dA = -I_{yy} \quad C_{zz} = \int_A z\omega_z dA = 0 \quad (80)$$

For a loading state that only involves torsion, the nonlinear constitutive equations, (63)–(68) and a consideration of equilibrium, indicate that there would be nonlinear axial shortening and bending curvature under torsion (see [Attard, 1986](#)). [Appendix D](#) contains the derivation of the nonlinear axial shortening and bending curvature effects under a loading of only torsion.

#### 4. Virtual work

The virtual work  $\delta W$  in terms of the second Piola–Kirchhoff stress tensor is given by:

$$\delta W = \int_V \frac{1}{2} \text{tr}(\Pi \delta \mathbf{C}) dV + \delta W_E \quad (81)$$

with  $V$  being the volume in the undeformed state, kinematically admissible variations denoted by the symbol  $\delta$  and the variation of the external work denoted by  $\delta W_E$ . In component form, the trace term in Eq. (81) is

$$\frac{1}{2} \text{tr}(\Pi \delta \mathbf{C}) = \frac{1}{2} \Pi^{ij} \delta \hat{\mathbf{g}}_{ij} = \Pi^{ij} \hat{\mathbf{g}}_i \cdot \delta \hat{\mathbf{g}}_j \\ = (\Pi^{11} \hat{\mathbf{g}}_{1\omega} + \Pi^{12} \hat{\mathbf{g}}_{2\omega} + \Pi^{13} \hat{\mathbf{g}}_{3\omega}) \cdot \delta \hat{\mathbf{g}}_{1\omega} \\ + (\Pi^{21} \hat{\mathbf{g}}_{1\omega} + \Pi^{22} \hat{\mathbf{g}}_{2\omega} + \Pi^{23} \hat{\mathbf{g}}_{3\omega}) \cdot \delta \hat{\mathbf{g}}_{2\omega} \\ + (\Pi^{31} \hat{\mathbf{g}}_{1\omega} + \Pi^{32} \hat{\mathbf{g}}_{2\omega} + \Pi^{33} \hat{\mathbf{g}}_{3\omega}) \cdot \delta \hat{\mathbf{g}}_{3\omega} \quad (82)$$

For beam theory, we ignore the terms associated with  $\Pi^{22}$ ,  $\Pi^{23}$  &  $\Pi^{33}$ , incorporating Eqs. (57)–(59) and taking account of the symmetry of the stress tensor gives:

$$\frac{1}{2} \text{tr}(\Pi \delta \mathbf{C}) = (\Pi^{11} \hat{\mathbf{g}}_{1\omega} + \Pi^{12} \hat{\mathbf{g}}_{2\omega} + \Pi^{13} \hat{\mathbf{g}}_{3\omega}) \cdot \delta \hat{\mathbf{g}}_{1\omega} \\ + \Pi^{12} \hat{\mathbf{g}}_{1\omega} \cdot \delta \hat{\mathbf{g}}_{2\omega} + \Pi^{13} \hat{\mathbf{g}}_{1\omega} \cdot \delta \hat{\mathbf{g}}_{3\omega} \\ = (\mathbf{S}^{nn} \hat{\mathbf{n}} + \mathbf{S}^{nt} \hat{\mathbf{t}} + \mathbf{S}^{nb} \hat{\mathbf{b}}) \cdot \delta \hat{\mathbf{g}}_{1\omega} \\ + [(\mathbf{S}^{nn} \hat{\mathbf{n}} + \mathbf{S}^{nt} \hat{\mathbf{t}} + \mathbf{S}^{nb} \hat{\mathbf{b}}) \cdot \hat{\mathbf{g}}^{2\omega}] \hat{\mathbf{g}}_{1\omega} \cdot \delta \hat{\mathbf{g}}_{2\omega} \\ + [(\mathbf{S}^{nn} \hat{\mathbf{n}} + \mathbf{S}^{nt} \hat{\mathbf{t}} + \mathbf{S}^{nb} \hat{\mathbf{b}}) \cdot \hat{\mathbf{g}}^{3\omega}] \hat{\mathbf{g}}_{1\omega} \cdot \delta \hat{\mathbf{g}}_{3\omega} \quad (83)$$

Substituting Eqs. (44), (45), (46), (47), (48), (49) and (53), taking the variations, assuming that  $\frac{(\lambda_n - \omega\tau_x)}{1} \approx 1$ , integrating over the area, incorporating Eqs. (63)–(68) and using the condition that

$$S^{nt} \lambda_{1s} \sin \alpha_0 \cos \phi_0 = S^{nb} \lambda_{1s} \sin \phi_0 \quad (84)$$

reduces the virtual work equation to:

$$\int_0^L \left[ N\delta(\hat{\gamma}_1 + y_s \kappa - z_s \kappa') + Q_2 \delta \hat{\gamma}_2 + Q_3 \delta \hat{\gamma}_3 \right. \\ \left. + M_2 \delta \kappa' + M_3 \delta \kappa + (M_t + M_{\text{wagner}}) \delta \tau + M_\omega \delta \tau_x \right] dx + \delta W_E \quad (85)$$

with

$$M_{\text{wagner}} = \int_A [S^{nm}(\hat{\gamma}_2 \omega_y + \hat{\gamma}_3 \omega_z + \tau \bar{\omega})] dA \quad (86)$$

where  $M_{\text{wagner}}$  denotes what has been traditionally been associated with the Wagner effect although here there are added shear terms and the physical derivations have depended on the presence of cross-section warping. Eq. (85) results in the following equilibrium equations:

$$Q_t \lambda_{1s} \sin \alpha_0 \cos \varphi_0 = Q_b \lambda_{1s} \sin \varphi_0 \quad (87)$$

$$\frac{dN}{dx} - Q_t \tilde{\kappa} - Q_b \tilde{\kappa}' + p_n = 0 \quad (88)$$

$$\frac{dQ_t}{dx} + N \tilde{\kappa} - Q_b \left( \tau - \frac{d\phi}{dx} \right) + p_t = 0 \quad (89)$$

$$\frac{dQ_b}{dx} + N \tilde{\kappa}' + Q_t \left( \tau - \frac{d\phi}{dx} \right) + p_b = 0 \quad (90)$$

$$\frac{d\tilde{M}_1}{dx} - \tilde{M}_2 \tilde{\kappa} - \tilde{M}_3 \tilde{\kappa}' + m_n = 0 \quad (91)$$

$$\begin{aligned} \frac{d\tilde{M}_3}{dx} + \tilde{M}_1 \tilde{\kappa}' + \tilde{M}_2 \left( \tau - \frac{d\phi}{dx} \right) + m_b \\ = N \lambda_{1s} \sin \varphi_0 - Q_t \lambda_{1s} \cos \alpha_0 \cos \varphi_0 \\ = -(1 + u_{s,x})(P_y \cos \psi + P_z \sin \psi) + P_x(v_{s,x} \cos \psi + w_{s,x} \sin \psi) \end{aligned} \quad (92)$$

$$\begin{aligned} \frac{d\tilde{M}_2}{dx} + \tilde{M}_1 \tilde{\kappa} - \tilde{M}_3 \left( \tau - \frac{d\phi}{dx} \right) + m_t \\ = -N \lambda_{1s} \sin \alpha_0 \cos \varphi_0 + Q_b \lambda_{1s} \cos \alpha_0 \cos \varphi_0 \\ = \{ (1 + u_{s,x})(-P_y \sin \psi + P_z \cos \psi) + P_x(-v_{s,x} \sin \psi + w_{s,x} \cos \psi) \} \sec \theta \end{aligned} \quad (93)$$

in which

$$\tilde{M}_1 = M_t + M_{\text{wagner}} - \frac{dM_{\omega}}{dx} \quad (94)$$

The terms  $P_x$ ,  $P_y$  &  $P_z$  are internal force results taken in the  $x$ ,  $y$  &  $z$ , respectively. Eq. (87) is always zero if the constitutive relation (64) is used. There are, therefore six equations of equilibrium. Eqs. (88)–(90) represent the equilibrium of the rate change of the forces along the member while Eqs. (91)–(93) represent the moment increment equilibrium equations.

The internal action force vector  $\mathbf{p}$  and resultant moment vector  $\mathbf{m}$  are defined with components given in Eq. (63)–(68) and are

$$\begin{aligned} \mathbf{p} = N\hat{\mathbf{n}} + Q_t\hat{\mathbf{t}} + Q_b\hat{\mathbf{b}} \quad \mathbf{m} = \tilde{M}_1\hat{\mathbf{n}} + \tilde{M}_2\hat{\mathbf{t}} + \tilde{M}_3\hat{\mathbf{b}} \\ = N\hat{\mathbf{n}} + Q_2\hat{\mathbf{g}}_2 + Q_3\hat{\mathbf{g}}_3 \quad = \tilde{M}_1\hat{\mathbf{n}} + M_2\hat{\mathbf{g}}_2 + M_3\hat{\mathbf{g}}_3 \end{aligned} \quad (95)$$

in which

$$\begin{aligned} \tilde{M}_2 = M_2 \cos \phi - M_3 \sin \phi - N \bar{z}_s \\ \tilde{M}_3 = M_2 \sin \phi + M_3 \cos \phi + N \bar{y}_s \end{aligned} \quad (96)$$

where

$$\bar{z}_s = y_s \sin \phi + z_s \cos \phi \quad \bar{y}_s = y_s \cos \phi - z_s \sin \phi \quad (97)$$

The distributed force and moment intensity vectors are depicted by  $\mathbf{dp}$  and  $\mathbf{dm}$ , respectively, with components:

$$\begin{aligned} \mathbf{dp} = p_n\hat{\mathbf{n}} + p_t\hat{\mathbf{t}} + p_b\hat{\mathbf{b}} \quad \mathbf{dm} = m_n\hat{\mathbf{n}} + m_t\hat{\mathbf{t}} + m_b\hat{\mathbf{b}} \\ = p_n\hat{\mathbf{n}} + p_2\hat{\mathbf{g}}_2 + p_3\hat{\mathbf{g}}_3 \quad = m_n\hat{\mathbf{n}} + m_2\hat{\mathbf{g}}_2 + m_3\hat{\mathbf{g}}_3 \end{aligned} \quad (98)$$

The well-known vector representation of the beam equilibrium equations are:

$$\begin{aligned} \mathbf{p}_x + \mathbf{dp} = 0 \\ \mathbf{m}_x + \hat{\mathbf{g}}_{1\omega}(y_s, z_s) \times \mathbf{p} + \mathbf{dm} = 0 \end{aligned} \quad (99)$$

The equilibrium equations developed in Eqs. (88)–(93) are consistent with Eq. (99) and the equations presented by Reissner (1989). In Eq. (99),  $\hat{\mathbf{g}}_{1\omega}(y_s, z_s)$  is taken at the shear centre and the warping of the shear centre has been assumed to be zero. The equilibrium equations (87)–(93) can also be recast, in terms of the internal force and bending moment components taken about the  $\hat{\mathbf{g}}_2$  &  $\hat{\mathbf{g}}_3$ , hence:

$$Q_2 \hat{\gamma}_3 = Q_3 \hat{\gamma}_2 \quad (100)$$

$$\frac{dN}{dx} - Q_2 \kappa + Q_3 \kappa' + p_n = 0 \quad (101)$$

$$\frac{dQ_2}{dx} + N \kappa - Q_3 \tau + p_2 = 0 \quad (102)$$

$$\frac{dQ_3}{dx} - N \kappa' + Q_2 \tau + p_3 = 0 \quad (103)$$

$$\frac{d\tilde{M}_1}{dx} - (M_2 - N z_s) \kappa + (M_3 + N y_s) \kappa' + m_n = 0 \quad (104)$$

$$\frac{d(M_3 + N y_s)}{dx} - \tilde{M}_1 \kappa' + (M_2 - N z_s) \tau + m_2 = N \hat{\gamma}_2 - Q_2 \hat{\gamma}_1 \quad (105)$$

$$\frac{d(M_2 - N z_s)}{dx} + \tilde{M}_1 \kappa - (M_3 + N y_s) \tau + m_3 = -N \hat{\gamma}_3 + Q_3 \hat{\gamma}_1 \quad (106)$$

## 5. Lateral buckling of a simply supported prismatic monosymmetric beam under pure bending

Consider a simply supported statically determinate beam of length  $L$  under pure bending denoted by  $M_3^0$  taken about the  $z$ -axis. The initial conditions are that  $N$ ,  $Q_t$ ,  $Q_b = 0$ , the bending curvature is constant given by  $\kappa^0 = \frac{M_3^0}{EI_{zz}}$ , the initial stress state is  $S_o^{nm} = \frac{-M_3^0 y}{I_{zz}}$  and the initial stretch of the shear centre axis derived from Eq. (63) is  $\lambda_{1s}^0 = 1 - y_s \kappa^0$ . The buckling load can be established by looking at the equilibrium equations (88)–(93) under small perturbations or variations about the initial loaded state. The variation symbol  $\delta$  will be used to indicate small perturbations. Firstly, since there are no resultant internal forces at any cross-section under pure bending if the problem is statically determinate, then under perturbations we have

$$\therefore \delta N = 0 \quad \delta Q_t = 0 \Rightarrow \delta \varphi_0 = 0 \quad \delta Q_b = 0 \Rightarrow \delta \alpha_0 = 0 \quad (107)$$

An important conclusion here is that there are no shear deformations and no shear forces, involved with the pure bending buckling problem. Using the constitutive relations in Eqs. (66) and (67), Eqs. (91) and (93) become:

$$\frac{d\delta \tilde{M}_1}{dx} = (EI_{yy} \kappa^0 - M_3^0) \delta \kappa' \quad (108)$$

$$EI_{yy} \frac{d\delta \kappa'}{dx} + \delta \tilde{M}_1 \kappa^0 - M_3^0 \delta \tau = 0 \quad (109)$$

where

$$\begin{aligned} \delta \tilde{M}_1 = GJ_t \delta \tau - EI_{\omega} \delta \tau_{xx} - \int_A \left[ \frac{M_3^0}{I_{zz}} (y \bar{\omega}) \right] dA \delta \tau \\ = (GJ_t - M_3^0 E^* \beta_1^*) \delta \tau - EI_{\omega} \delta \tau_{xx} \end{aligned} \quad (110)$$

Differentiating Eq. (108) with respect to  $x$  and substituting into Eq. (109) gives:

$$\begin{aligned} \frac{EI_{yy}}{(EI_{yy} - 1) M_3^0} \frac{d^2 \delta \tilde{M}_1}{dx^2} + \frac{M_3^0}{EI_{zz}} \delta \tilde{M}_1 - M_3^0 \delta \tau \\ = 0 \quad \text{Provided } EI_{yy} \neq EI_{zz} \end{aligned} \quad (111)$$

Using Eq. (110) we have the following uncoupled fourth order differential equation:

$$\delta\tau_{xx} \left[ \frac{EI_{yy}EI_{zz}}{\left(1 - \frac{EI_{yy}}{EI_{zz}}\right)M_3^0} \left( \frac{M_3^0 E^* \beta_1^*}{EI_{zz}} - \frac{GJ_t}{EI_{zz}} \right) - M_3^0 \frac{EI_{\omega}}{EI_{zz}} \right] + \delta\tau_{xxx} \frac{EI_{yy}EI_{\omega}}{\left(1 - \frac{EI_{yy}}{EI_{zz}}\right)M_3^0} - \delta\tau M_3^0 \left( 1 - \frac{GJ_t}{EI_{zz}} + \frac{M_3^0 E^* \beta_1^*}{EI_{zz}} \right) = 0 \quad (112)$$

The associated equation for the lateral curvature is:

$$\delta\kappa' = \frac{(M_3^0 E^* \beta_1^* - GJ_t)\delta\tau_x + EI_{\omega}\delta\tau_{xxx}}{M_3^0 \left( 1 - \frac{EI_{yy}}{EI_{zz}} \right)} \quad (113)$$

The solution to the standard differential equation Eq. (112), is then:

$$\delta\tau = C_1 \sin(k_1 x) + C_2 \cos(k_1 x) + C_3 \sinh(k_2 x) + C_4 \cosh(k_2 x) \quad (114)$$

$$\delta\kappa' = [C_1 \cos(k_1 x) - C_2 \sin(k_1 x)]k_1(-fk_1^2 + e) + [C_3 \cosh(k_2 x) + C_4 \sinh(k_2 x)]k_2(fk_2^2 + e) \quad (115)$$

$$(k_1)^2 = \frac{b + \sqrt{b^2 - 4ac}}{2c} \quad (k_2)^2 = \frac{-b + \sqrt{b^2 - 4ac}}{2c} \quad (116)$$

$$\begin{aligned} a &= (M_3^0)^2 \left\{ \frac{GJ_t}{EI_{zz}} - M_3^0 \frac{E^* \beta_1^*}{EI_{zz}} - 1 \right\} \\ b &= \frac{EI_{yy}EI_{zz}}{\left(1 - \frac{EI_{yy}}{EI_{zz}}\right)} \left\{ M_3^0 \frac{E^* \beta_1^*}{EI_{zz}} - \frac{GJ_t}{EI_{zz}} \right\} - (M_3^0)^2 \frac{EI_{\omega}}{EI_{zz}} \\ c &= \frac{EI_{yy}EI_{\omega}}{\left(1 - \frac{EI_{yy}}{EI_{zz}}\right)} \quad e = \frac{M_3^0 E^* \beta_1^* - GJ_t}{M_3^0 \left( 1 - \frac{EI_{yy}}{EI_{zz}} \right)} \quad f = \frac{EI_{\omega}}{M_3^0 \left( 1 - \frac{EI_{yy}}{EI_{zz}} \right)} \end{aligned} \quad (117)$$

The boundary conditions for the simply supported beam are that  $\delta\kappa' = 0$  &  $\delta\tau_x = 0$  at  $x = 0$  &  $L$ . Using these boundary conditions, the determinant of the system of resulting equations is then:

$$\sinh(k_2 L) \sin(k_1 L) f^2 (k_1)^2 (k_2)^2 [(k_1)^2 + (k_2)^2]^2 = 0 \quad (118)$$

The solutions to the above equation are:

$$k_1 = \frac{n\pi}{L} \quad k_2 = \frac{in\pi}{L} \quad b^2 - 4ac = 0 \quad k_1 = 0 \quad k_2 = 0 \quad (119)$$

where  $n$  is an integer. The appropriate equation for the buckling moment ( $EI_{yy} \neq EI_{zz}$ ) is therefore:

$$\begin{aligned} (k_1)^2 &= \frac{b + \sqrt{b^2 - 4ac}}{2c} = \left( \frac{n\pi}{L} \right)^2 \quad \therefore c \left( \frac{n\pi}{L} \right)^4 - b \left( \frac{n\pi}{L} \right)^2 + a = 0 \\ \therefore & \frac{EI_{yy}EI_{\omega}}{\left(1 - \frac{EI_{yy}}{EI_{zz}}\right)} \left( \frac{n\pi}{L} \right)^4 + (M_3^0)^2 \left\{ \frac{GJ_t}{EI_{zz}} - M_3^0 \frac{E^* \beta_1^*}{EI_{zz}} - 1 \right\} \\ & - \left\{ \frac{EI_{yy}EI_{zz}}{\left(1 - \frac{EI_{yy}}{EI_{zz}}\right)} \left\{ M_3^0 \frac{E^* \beta_1^*}{EI_{zz}} - \frac{GJ_t}{EI_{zz}} \right\} - (M_3^0)^2 \frac{EI_{\omega}}{EI_{zz}} \right\} \left( \frac{n\pi}{L} \right)^2 = 0 \end{aligned} \quad (120)$$

Eq. (120) is a cubic equation for the buckling moment and is independent of any shear rigidity. Standard solutions are available for the cubic problem. For the case of a cross-section with  $\beta_1^* = 0$ , Eq. (120) reduces to a quadratic equation with the classical solution.

$$M_{cr} = \frac{\frac{n\pi}{L} \sqrt{\frac{EI_{yy}}{\left(1 - \frac{EI_{yy}}{EI_{zz}}\right)} \left( EI_{\omega} \left( \frac{n\pi}{L} \right)^2 + GJ_t \right)}}{\sqrt{1 - \frac{GJ_t}{EI_{zz}} - \frac{EI_{\omega}}{EI_{zz}} \left( \frac{n\pi}{L} \right)^2}} \quad \begin{aligned} &EI_{yy} < EI_{zz} \quad \& \\ &\left( GJ_t + EI_{\omega} \left( \frac{n\pi}{L} \right)^2 \right) < EI_{zz} \end{aligned} \quad (121)$$

For the case  $EI_{yy} = EI_{zz}$ , the relevant equations are again Eqs. (109) and (108) which reduce to:

$$EI_{yy} \frac{d\delta\kappa'}{dx} + \delta\tilde{M}_1 \frac{M_3^0}{EI_{zz}} - M_3^0 \delta\tau = 0 \quad (122)$$

$$\frac{d\delta\tilde{M}_1}{dx} = (GJ_t - M_3^0 E^* \beta_1^*) \delta\tau_x - EI_{\omega} \delta\tau_{xxx} = 0 \quad (123)$$

Applying the boundary conditions and looking at the determinate of the system of equations, non-trivial buckling moment solutions exist only for monosymmetric sections and are:

$$M_{cr} = \left\{ \frac{GJ_t}{E^* \beta_1^*}, \frac{GJ_t - EI_{zz}}{E^* \beta_1^*}, \frac{GJ_t + EI_{\omega} \left( \frac{n\pi}{L} \right)^2}{E^* \beta_1^*} \right\} \quad EI_{yy} = EI_{zz} \quad (124)$$

The first two solutions in the above equation are independent of the beam length and hence would indicate localised buckling at the section where the moment reaches these critical values. Further discussion about localised torsional buckling can be found in Attard and Lawther (1989). In the lateral buckling analysis of beams, secondary warping or thickness warping is usually ignored. For angle and tee beams where the contour warping is zero, the secondary warping is, however, small but finite. Secondary warping was shown in Attard and Lawther (1989), to have a significant effect on the lateral buckling load of simply supported beams under moment gradient.

## 6. Flexural–torsional buckling of a simply supported prismatic beam under axial compression

Consider a simply supported beam of length  $L$ , whose ends are prevented from twisting but are free to warp and bend. The beam is loaded with an axial force  $N^0$  applied at each end of the beam at the position of the centroid causing only an axial initial stress  $S_o^{nm} = \frac{N^0}{A}$ . There is therefore an initial axial deformation given by:

$$N^0 = EA(\hat{\gamma}_1^0 - 1) \quad \therefore \hat{\gamma}_1^0 = 1 + \frac{N^0}{EA} \quad (125)$$

The boundary conditions are:

$$\delta\kappa(0) = \delta\kappa(L) = 0 \quad \delta\kappa'(0) = \delta\kappa'(L) = 0 \quad \delta\tau_x(0) = \delta\tau_x(L) = 0 \quad (126)$$

Applying variations to the force equilibrium equations (101)–(103) and incorporating the constitutive relationships (63) and (64) we conclude the following:

$$\frac{d\delta N}{dx} = 0 \quad \left( \delta\hat{\gamma}_{1,x} + y_s \delta\kappa_x - z_s \delta\kappa'_x \right) = 0 \quad (127)$$

$$\frac{d\delta Q_2}{dx} + N^0 \delta\kappa = 0 \quad \therefore \delta\hat{\gamma}_{2,x} = \frac{-N^0}{GA} \delta\kappa \quad (128)$$

$$\frac{d\delta Q_3}{dx} - N^0 \delta\kappa' = 0 \quad \therefore \delta\hat{\gamma}_{3,x} = \frac{N^0}{GA} \delta\kappa' \quad (129)$$

Now turning to the variations of the equilibrium equations (104)–(106), substituting the constitutive relations (65)–(68), incorporating Eqs. (86) and (94) and the above results, Eqs. (127)–(129), we derive:

$$\begin{aligned} \frac{d\delta\tilde{M}_1}{dx} + N^0 z_s \delta\kappa + N^0 y_s \delta\kappa' &= 0 \\ \therefore GJ_t \delta\tau_x - EI_{\omega} \delta\tau_{xxx} + N^0 \frac{(I_{p0} - J_t)}{A} \delta\tau_x + N^0 z_s \left( 1 - \frac{N^0}{GA} \right) \delta\kappa \\ &+ N^0 y_s \left( 1 - \frac{N^0}{GA} \right) \delta\kappa' = 0 \end{aligned} \quad (130)$$

$$\frac{d(\delta M_3 + \delta N y_s)}{dx} - N^0 z_s \delta \tau = N^0 \delta \hat{\gamma}_2 - \delta Q_2 \hat{\gamma}_1^0 \quad (131)$$

$$\therefore El_{zz} \delta \kappa_{,xx} + \left( (N^0)^2 \left( \frac{E}{G} - 1 \right) - N^0 \right) \delta \kappa = N^0 z_s \delta \tau_{,x}$$

$$\frac{d(\delta M_2 - \delta N z_s)}{dx} - N^0 y_s \delta \tau = -N^0 \delta \hat{\gamma}_3 + \delta Q_3 \hat{\gamma}_1^0 \quad (132)$$

$$\therefore El_{yy} \delta \kappa'_{,xx} + \left( (N^0)^2 \left( \frac{E}{G} - 1 \right) - N^0 \right) \delta \kappa' = N^0 y_s \delta \tau_{,x}$$

Eqs. (130)–(132) are differential equations in the three variables  $\delta \kappa$ ,  $\delta \kappa'$  &  $\delta \tau$ . These equations are uncoupled if the shear centre coincides with the centroid. Adopting here a similar technique used by Vlasov (1959), we first select the following eigenfunctions:

$$\delta \kappa = A \sin \left( \frac{n\pi x}{L} \right) \quad \delta \kappa' = B \sin \left( \frac{n\pi x}{L} \right) \quad \delta \tau_{,x} = C \sin \left( \frac{n\pi x}{L} \right) \quad (133)$$

which satisfy the boundary conditions (126) and where  $A$ ,  $B$  and  $C$  are scalar coefficients, and  $n$  is an arbitrary positive integer. Assuming axial compression with  $N^0 = -P$ , substituting Eq. (133) into the three Eqs. (130)–(132), the determinate of the resulting system of equations is

$$\begin{vmatrix} El_{zz} + \left( P^2 \left( \frac{E}{G} - 1 \right) + P \right) & 0 & -P z_s \\ 0 & El_{yy} + \left( P^2 \left( \frac{E}{G} - 1 \right) + P \right) & -P y_s \\ -P z_s \left( 1 + \frac{P}{GA} \right) & -P y_s \left( 1 + \frac{P}{GA} \right) & GJ_t + El_{\omega} \left( \frac{n\pi}{L} \right)^2 - P \frac{(I_{po} - J_t)}{A} \end{vmatrix} = 0 \quad (134)$$

This leads to a fifth order polynomial in  $P$ , which can be solved for the critical buckling load. When the shear centre coincides with the centroid, the buckling load is uncoupled and consists of either the modified Haringx column buckling load  $P_{Har}$  about either of the principal axis  $y$  or  $z$  (see Attard and Hunt, 2008a), or the torsional buckling load  $P_{\omega}$  defined below:

$$\delta^2 W = \int_0^L \left[ \frac{1}{2} \delta N (\hat{\gamma}_1 + y_s \kappa - z_s \kappa') + \frac{1}{2} \delta Q_2 \delta \hat{\gamma}_2 + \frac{1}{2} \delta Q_3 \delta \hat{\gamma}_3 + \frac{1}{2} \delta M_2 \delta \kappa' + \frac{1}{2} \delta M_3 \delta \kappa \right. \\ \left. + \frac{1}{2} \delta M_t \delta \tau + \frac{1}{2} \delta M_{\omega} \delta \tau_{,x} + N^0 \delta^2 (\hat{\gamma}_1 + y_s \kappa - z_s \kappa') + Q_2^0 \delta^2 (\hat{\gamma}_2) + M_3^0 \delta^2 \kappa \right] dx - \int \int_S \mathbf{q} \cdot \delta^2 \mathbf{u} dS \quad (139)$$

$$\frac{P_{Har}}{EA} = \frac{1}{2 \left( 1 - \frac{E}{G} \right)} \left( 1 \pm \sqrt{1 - \frac{4n^2 P_{euler}}{EA} \left( 1 - \frac{E}{G} \right)} \right) \quad (135)$$

$$P_{euler} = \frac{n^2 \pi^2 El_{zz}}{L^2} \quad \text{or} \quad \frac{n^2 \pi^2 El_{yy}}{L^2} \quad (136)$$

$$P_{\omega} = \frac{GJ_t + \frac{n^2 \pi^2 El_{\omega}}{L^2}}{r_{po}^2} \quad r_{po}^2 = \frac{(I_{po} - J_t)}{A} \quad (137)$$

$$\delta^2 W = \int_0^L \left[ \frac{1}{2} EA (\delta (\hat{\gamma}_1 + y_s \kappa - z_s \kappa'))^2 + \frac{1}{2} GA (\delta \hat{\gamma}_2)^2 + \frac{1}{2} GA (\delta \hat{\gamma}_3)^2 + \frac{1}{2} El_{yy} (\delta \kappa')^2 \right. \\ \left. + \frac{1}{2} El_{zz} (\delta \kappa)^2 + \frac{1}{2} GJ_t (\delta \tau)^2 + \frac{1}{2} El_{\omega} (\delta \tau_{,x})^2 + N^0 \delta^2 (\hat{\gamma}_1) \right. \\ \left. + (M_3^0 + N^0 y_s) \{ \delta^2 \kappa - \frac{1}{2} \delta (\hat{\gamma}_3) \delta \tau \} + (M_2^0 - N^0 z_s) \{ \delta^2 \kappa' - \frac{1}{2} \delta (\hat{\gamma}_2) \delta \tau \} \right. \\ \left. + Q_2^0 \delta^2 (\hat{\gamma}_2) + Q_3^0 \delta^2 (\hat{\gamma}_3) + \frac{1}{2} \left( N^0 \frac{I_{po} - J_t}{A} - M_3^0 \beta_1^* E^* - M_2^0 \beta_2^* E^* \right) (\delta \tau)^2 \right] dx - \int \int_S \mathbf{q} \cdot \delta^2 \mathbf{u} dS \quad (140)$$

The expression for the torsional buckling load  $P_{\omega}$  is different to conventional equations with the polar second moment of area replaced by  $I_{po} - J_t$ . As shown in Attard (2003a) torsional buckling is impossible for a cross-section which does not warp such as circular sections with  $I_{po} = J_t$ .

## 7. Second variation of total potential/work

Closed form solutions for beams with moment gradient can only be derived using approximate methods. Approximate solutions to the buckling problem can be obtained by exploiting numerical methods such as the Rayleigh–Ritz method or the finite element method. These methods can be used to determine the buckling load by either manipulating the variation of the equilibrium differential equations or by using the second variation of the total potential. Appendix F contains the linearized lateral buckling equations for a prismatic monosymmetric beam under transverse loading. A static equilibrium configuration is considered stable if the second variation of the total potential with respect to all kinematically admissible variations is positive definite. An analogous approach based on the second variation of work was proposed by Bažant and Cedolin (1991). The second variation of the total potential/work is analogous to taking a variation of the equilibrium equations and then applying the principle of virtual displacements transforming the equilibrium equations into variational form, see Vlasov (1959).

The virtual work expression derived in Eq. (85) is rewritten here:

$$\delta W = \int_0^L \left[ N \delta (\hat{\gamma}_1 + y_s \kappa - z_s \kappa') + Q_2 \delta \hat{\gamma}_2 + Q_3 \delta \hat{\gamma}_3 \right. \\ \left. + M_2 \delta \kappa' + M_3 \delta \kappa + (M_t + M_{wagner}) \delta \tau + M_{\omega} \delta \tau_{,x} \right] dx + \delta W_E \quad (138)$$

Consider a prismatic beam with conservative loads, which in the unbuckled state has initial internal actions  $N^0$ ,  $Q_2^0$ ,  $Q_3^0$ ,  $M_2^0$  &  $M_3^0$  and an initial longitudinal stress  $S_o^{nn} = \frac{N^0}{A} - \frac{M_2^0 y}{I_{zz}} + \frac{M_3^0 z}{I_{yy}}$  prior to instability. The loading causes no torsion prior to buckling. The initial axial deformation, bending curvature and shear deformations prior to instability are not taken into account in this paper but are left for future examination. Taking a variation of Eq. (138) and incorporating the initial conditions gives for the second variation of work. That is:

In the above,  $\mathbf{q}$  is the vector of conservative surface tractions acting on the boundary surface defined by the domain  $S$ . Incorporating the constitutive law from Eqs. (63)–(68), and using Eqs. 73, 74, 75 and (80), we have:

Using Eqs. (160), (161) and (158), we can write for the first variation of the deformation terms to first order:

$$\delta \kappa = \delta \omega_{3,x} \quad \delta \kappa' = \delta \omega_{2,x} \quad \delta \tau = \delta \omega_{1,x} \quad \delta \tau_{,x} = \delta \omega_{1,xx} \\ \delta \hat{\gamma}_1 = \delta u_{s,x} \quad \delta \hat{\gamma}_2 = \delta \gamma_2 \quad \delta \hat{\gamma}_3 = \delta \gamma_3 \quad (141)$$

The second variation deformation terms are required to second order for consistency, that is:

$$\begin{aligned}\delta^2 \kappa &= -\delta\omega_1 \delta\omega_{2,x} - \delta\omega_3 \delta u_{s,xx} - \delta\omega_{3,x} \delta u_{s,x} \\ \delta^2 \kappa' &= \delta\omega_1 \delta\omega_{3,x} - \delta\omega_2 \delta u_{s,xx} - \delta\omega_{2,x} \delta u_{s,x} \\ \delta^2 \hat{\gamma}_2 &= \delta\omega_1 \delta\gamma_3 \quad \delta^2 \hat{\gamma}_3 = -\delta\omega_1 \delta\gamma_2 \\ \delta^2 \hat{\gamma}_1 &= \frac{1}{2}(\delta\omega_2)^2 + \delta\omega_2 \delta\gamma_3 + \delta\omega_3 \delta\gamma_2 + \frac{1}{2}(\delta\omega_3)^2\end{aligned}\quad (142)$$

$$\begin{aligned}\delta^2 u^1 &\cong (y - y_s) \delta\omega_1 \delta\omega_2 + (z - z_s) \delta\omega_1 \delta\omega_3 \\ &\quad - \frac{1}{2} \omega (\delta\omega_{2,x} \delta\omega_3 - \delta\omega_2 \delta\omega_{3,x}) \\ \delta^2 u^2 &\cong \frac{1}{2} (z - z_s) \delta\omega_2 \delta\omega_3 - \frac{1}{2} (y - y_s) (\delta\omega_1^2 + \delta\omega_3^2) \\ &\quad + \omega \delta\omega_{1,x} \delta\omega_2 \\ \delta^2 u^3 &\cong \frac{1}{2} (y - y_s) \delta\omega_2 \delta\omega_3 - \frac{1}{2} (z - z_s) (\delta\omega_1^2 + \delta\omega_2^2) \\ &\quad - \omega \delta\omega_{1,x} \delta\omega_3\end{aligned}\quad (143)$$

The load potential expression is then:

$$\int \int_S \mathbf{q} \cdot \delta^2 \mathbf{u} dS = \int \int_S \left[ q_1 (y - y_s) \delta\omega_1 \delta\omega_2 + q_1 (z - z_s) \delta\omega_1 \delta\omega_3 - \frac{1}{2} q_1 \omega (\delta\omega_{2,x} \delta\omega_3 - \delta\omega_2 \delta\omega_{3,x}) \right. \\ \left. + \frac{1}{2} [q_2 (z - z_s) + q_3 (y - y_s)] \delta\omega_2 \delta\omega_3 - \frac{1}{2} q_2 (y - y_s) (\delta\omega_1^2 + \delta\omega_3^2) \right. \\ \left. - \frac{1}{2} q_3 (z - z_s) (\delta\omega_1^2 + \delta\omega_2^2) + (q_2 \omega \delta\omega_2 - q_3 \omega \delta\omega_3) \delta\omega_{1,x} \right] dS \quad (144)$$

Here, we consider conservative tractions, which produce only bending and shear (no bimoments or torsion prior to buckling), and are applied either at the member ends or are uniformly distributed along the length of the beam. Incorporating Eqs. (141) and (142) into Eq. (140), and evaluating Eq. (144) at the boundaries gives:

$$\delta^2 W = \frac{1}{2} \int_0^L \left[ EA(\delta u_{s,x} + y_s \delta\omega_{3,x} - z_s \delta\omega_{2,x})^2 + GA(\delta\gamma_2)^2 + GA(\delta\gamma_3)^2 + EI_{yy}(\delta\omega_{2,x})^2 \right. \\ \left. + EI_{zz}(\delta\omega_{3,x})^2 + GJ_t(\delta\omega_{1,x})^2 + EI_\omega(\delta\omega_{1,xx})^2 + 2Q_2^0 \delta\gamma_3 \delta\omega_1 - 2Q_3^0 \delta\gamma_2 \delta\omega_1 \right. \\ \left. + N^0 \{ (\delta\omega_2)^2 + 2\delta\gamma_3 \delta\omega_2 + 2\delta\gamma_2 \delta\omega_3 + (\delta\omega_3)^2 \} \right. \\ \left. - (M_3^0 + N^0 y_s) \{ 2\delta\omega_1 \delta\omega_{2,x} + 2\delta\omega_3 \delta u_{s,xx} + 2\delta\omega_{3,x} \delta u_{s,x} + \delta\gamma_3 \delta\omega_{1,x} \} \right. \\ \left. + (M_2^0 - N^0 z_s) \{ 2\delta\omega_1 \delta\omega_{3,x} - 2\delta\omega_2 \delta u_{s,xx} - 2\delta\omega_{2,x} \delta u_{s,x} - \delta\gamma_2 \delta\omega_{1,x} \} \right. \\ \left. + \left( N^0 \frac{I_{\omega\omega} - J_t}{A} - M_3^0 \beta_1^* E^* - M_2^0 \beta_2^* E^* \right) (\delta\omega_{1,x})^2 \right. \\ \left. + q_y^0 e_y (\delta\omega_1^2 + \delta\omega_3^2) + q_z^0 e_z (\delta\omega_1^2 + \delta\omega_2^2) \right] dx + \left[ (M_3^0 + N^0 y_s) \delta\omega_1 \delta\omega_2 - (M_2^0 - N^0 z_s) \delta\omega_1 \delta\omega_3 \right. \\ \left. + \frac{1}{2} Q_2^0 a_y (\delta\omega_1^2 + \delta\omega_3^2) + \frac{1}{2} Q_3^0 a_z (\delta\omega_1^2 + \delta\omega_2^2) \right]_0^L \quad (145)$$

where  $q_y^0$  &  $q_z^0$  are transverse uniformly distributed loads acting through the shear centre in the  $y$ - and  $z$ -directions, respectively, and act at a distance,  $e_y$  &  $e_z$ , respectively, from the shear centre. The terms  $a_y$  &  $a_z$  refer to the distance in the  $y$ - or  $z$ -direction, respectively, of the applied boundary forces whose line of action passes through the shear centre. We can reduce Eq. (145) further by restricting ourselves to the case of no initial axial force and ignoring the axial displacement terms. Hence, Eq. (145) simplifies to:

$$\delta^2 W = \frac{1}{2} \int_0^L \left[ GA(\delta\gamma_2)^2 + GA(\delta\gamma_3)^2 + EI_{yy}(\delta\omega_{2,x})^2 + EI_{zz}(\delta\omega_{3,x})^2 + GJ_t(\delta\omega_{1,x})^2 \right. \\ \left. + EI_\omega(\delta\omega_{1,xx})^2 + 2Q_2^0 (\delta\gamma_3 - \delta\omega_2) \delta\omega_1 + M_3^0 (2\delta\omega_{1,x} \delta\omega_2 - \delta\gamma_3 \delta\omega_{1,x}) \right. \\ \left. - 2Q_3^0 (\delta\gamma_2 + \delta\omega_3) \delta\omega_1 - M_2^0 (2\delta\omega_1 \delta\omega_{3,x} + \delta\gamma_2 \delta\omega_{1,x}) \right. \\ \left. - (M_3^0 \beta_1^* E^* + M_2^0 \beta_2^* E^*) (\delta\omega_{1,x})^2 + q_y^0 e_y (\delta\omega_1^2 + \delta\omega_3^2) + q_z^0 e_z (\delta\omega_1^2 + \delta\omega_2^2) \right] dx + \left[ \frac{1}{2} Q_2^0 a_y (\delta\omega_1^2 + \delta\omega_3^2) + \frac{1}{2} Q_3^0 a_z (\delta\omega_1^2 + \delta\omega_2^2) \right]_0^L \quad (146)$$

## 8. Generalised non-dimensional lateral buckling equation

We now derive a generalised non-dimensional equation for the lateral buckling of a prismatic simply supported monosymmetric beam ( $\beta_2^*, z_s = 0$ ) under either uniform bending or transverse loading. The method adopted here follows that in Attard (1990). The second variation of total potential for this problem is derived from Eq. (146) and is:

$$\delta^2 W = \frac{1}{2} \int_0^L \left[ \frac{GA(\delta\gamma_3)^2 + EI_{yy}(\delta\omega_{2,x})^2 + GJ_t(\delta\omega_{1,x})^2 + EI_\omega(\delta\omega_{1,xx})^2}{2Q_2^0 (\delta\gamma_3 - \delta\omega_2) \delta\omega_1 + M_3^0 (2\delta\omega_{1,x} \delta\omega_2 - \delta\gamma_3 \delta\omega_{1,x})} \right. \\ \left. - M_3^0 \beta_1^* E^* (\delta\omega_{1,x})^2 + q_y^0 e_y \delta\omega_1^2 \right. \\ \left. + \frac{1}{2} P_y^0 a_y \delta\bar{\omega}_1^2 \right] dx \quad (147)$$

with  $P_y^0$  being a transverse force applied in the  $y$ -direction at mid-span and  $\bar{\omega}_1$  being the twist at the mid-span. The following substitutions are firstly made:

$$X = \frac{x}{L} \quad M_3^0 = M_{cr} m(X) \quad Q_2^0 = -\frac{M_{cr} m_X}{L} \quad (148)$$

$$\delta\gamma_3 = \bar{\gamma}_3 h(X) \quad \delta\omega_1 = \bar{\omega}_1 g(X) \quad \delta\omega_2 = \bar{\omega}_2 f(X)$$

where  $m(X)$  is a function describing the bending moment distribution;  $\bar{\omega}_2$  is the variation of the lateral bending rotation of a characteristic point along the beam;  $\bar{\gamma}_3$  is the variation of the shear rotation evaluated at a characteristic point;  $f(X)$ ,  $g(X)$  &  $h(X)$  are functions describing the distribution of the deformations which

satisfy the kinematic boundary conditions. The term  $M_{cr}$  refers to the critical buckling moment. Substituting Eq. (148) into (147), we have for the second variation of the total potential

$$\frac{1}{2} \begin{bmatrix} \bar{\omega}_2 & \bar{\gamma}_3 & \bar{\omega}_1 \end{bmatrix} \begin{bmatrix} \frac{EI_{yy}}{L} C_5 & 0 & M_{cr} C_6 \\ 0 & GJ_t C_7 & -M_{cr} C_4 \\ M_{cr} C_6 & -M_{cr} C_4 & C \end{bmatrix} \begin{bmatrix} \bar{\omega}_2 \\ \bar{\gamma}_3 \\ \bar{\omega}_1 \end{bmatrix} \quad (149)$$

in which

$$\begin{aligned}
 C &= \frac{GJ}{L} C_2 + \frac{EI_{\omega}}{L^3} C_1 - \frac{M_{cr}}{L} \beta_1^* E^* C_3 + P_y^0 a_y^0 + q_y^0 L e_y^0 C_7 \\
 C_1 &= \int_0^1 (g_{xx})^2 dXC_2 = \int_0^1 (g_x)^2 dXC_3 \\
 &= \int_0^1 m(g_x)^2 dXC_4 = \int_0^1 m_x g h dX + \int_0^1 \frac{1}{2} m h g_x dX \\
 C_5 &= \int_0^1 (f_x)^2 dXC_6 = \int_0^1 -(mg)_x f dXC_7 \\
 &= \int_0^1 g^2 dXC_8 = \int_0^1 h^2 dX
 \end{aligned} \quad (150)$$

Setting the determinant of (149) to zero gives for the critical load:

$$\frac{M_{cr} L}{\sqrt{EI_{yy} GJ}} = \frac{A_1 \pi \left\{ -\Psi \pm \sqrt{\Psi^2 + 4(1 + K^2 A_4)(1 + S A_6)} \right\}}{2(1 + S A_6)} \quad (151)$$

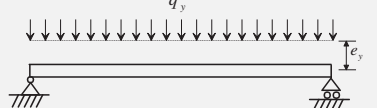
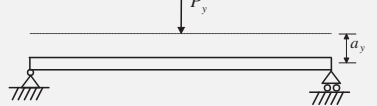
with the following defined parameters:

$$\begin{aligned}
 K^2 &= \frac{\pi^2 EI_{\omega}}{GJ L^2} \quad S = \frac{\pi^2 EI_{yy}}{GAL^2} \\
 \Psi &= (\beta_1^* E^* A_3 - a_y A_7 - e_y A_8) \sqrt{\frac{\pi^2 EI_{yy}}{GJ L^2}}
 \end{aligned} \quad (152)$$

$$\begin{aligned}
 A_1 &= \frac{\sqrt{C_2 C_5}}{\pi C_6} \quad A_3 = \frac{C_3}{\pi C_6} \sqrt{\frac{C_5}{C_2}} \quad A_4 = \frac{C_1}{\pi^2 C_2} \\
 A_6 &= \frac{C_4 C_5}{\pi^2 C_6^2 C_8} \quad A_7 = \frac{4}{\pi C_6} \sqrt{\frac{C_5}{C_2}} \quad A_8 = \frac{8 C_7}{\pi C_6} \sqrt{\frac{C_5}{C_2}}
 \end{aligned} \quad (153)$$

The non-dimensional shear parameter  $S = \frac{\pi^2 EI_{yy}}{GAL^2}$  was introduced by Lee et al. (submitted for publication) to study parametrically the influence of the shear stiffness on the lateral buckling load. For most beam sections made of an isotropic homogenous material, the shear parameter will be very small, varying between about 0.0001–0.1. For example, the  $S$  parameter for a rectangular section with a Poisson's ratio of 0.3 would be  $2.14(b/L)^2$  where “ $b$ ” is the width of the section. The buckling coefficients  $A_1, \dots, A_8$  can be approximated by solving the buckling problem numerically and integrating using the derived eigenmode shape. Table 1 lists the buckling coefficients estimated using trigonometric functions, which satisfy the boundary conditions and the Rayleigh–Ritz approach for three loading conditions.

**Table 1**  
Buckling coefficients.

| $M_{cr}$              | $A_1$ | $A_3$ | $A_4$ | $A_6$ | $A_7$ | $A_8$ | Loading                                                                               |
|-----------------------|-------|-------|-------|-------|-------|-------|---------------------------------------------------------------------------------------|
| $M_{cr}$              | 1     | 1     | 1     | –     | –     | –     |  |
| $\frac{q_y^0 L^2}{8}$ | 1.13  | 0.53  | 1     | 0.66  | –     | 0.92  |  |
| $\frac{P_y^0 L}{4}$   | 1.36  | 0.65  | 1     | 1.06  | 1.11  | –     |  |

To examine the importance of shear deformations during lateral buckling, consider Eq. (151) but with the parameter  $\Psi$  set to zero, hence

$$\frac{M_{cr} L}{\sqrt{EI_{yy} GJ}} = A_1 \pi \sqrt{\frac{(1 + K^2 A_4)}{(1 + S A_6)}} \quad (154)$$

We see that the shear parameter decreases the buckling capacity. For a stiff shear rigidity  $GA$  or a long span the  $S$  parameter would be much less than unity and shear deformations would have little impact on the capacity. Only for very short spans with flexible shear rigidity would shear deformations have an impact on the buckling capacity.

## 9. Summary

A consistent finite strain hyperelastic constitutive model for a linearly elastic isotropic material was used to develop the equilibrium and buckling equations for the lateral buckling of a prismatic straight beam which included shear deformations. Several assumptions were made and are summarised here.

- The cross-sectional shape was assumed to be maintained during deformation.
- The stresses associated with maintaining the cross-sectional shape  $\Pi^{22}$ ,  $\Pi^{23}$  &  $\Pi^{33}$  were ignored.
- The assumed cross-sectional displacements were based on three finite Euler angles and two shear finite rotations.
- Warping was taken normal to the displaced cross-section plane.
- The constitutive relations were derived assuming the volume invariant  $J$  was close to unity.
- The constitutive relations were expanded to second order in terms of the curvatures, torsion, shear rotations and stretch.
- The material parameter  $2G + \lambda$  was replaced by the elastic modulus  $E$ .
- The deformation measure  $\frac{(\lambda_n - \omega \tau_x)}{J}$  was taken as close to unity.

Second order approximations to the displacements, curvatures and twist were also derived. Lagrangian physical stresses were defined with respect to the unwarped displaced cross-sectional plane. Because warping was taken normal to the displaced cross-sectional plane, the constitutive relationships for the bending moments, torsion and bimoment revealed new coupling terms which are functions of the cross-sectional warping and shear deformations. Terms normally associated with the Wagner effect were derived from a consideration of warping and were coupled to the twisting and shear deformations of the cross-section. A closed form

solution for the lateral buckling load for a prismatic monosymmetric beam under pure bending was presented. An important conclusion was that the lateral buckling load under pure bending for simply supported boundary conditions, involves no shear. The flexural–torsional buckling problem under axial compression was also solved. The Euler type column buckling formulas were consistent with Haringx's column buckling formula which included shear. The torsional buckling formula was different to conventional equations with the polar second moment of area replaced by  $I_{po} - J_t$ . The torsional buckling formula agrees with Attard (2003a) where it was shown that torsional buckling is impossible for a cross-section which does not warp such as for a circular section. The second variation of the total potential was also derived. The effects of shear deformations were explored by examining the non-dimensional lateral buckling equation for a simply supported beam. The lateral buckling capacity of most isotropic prismatic straight beams are unaffected by shear deformations. Shear deformations do become important for short spans and soft flexible shear rigidity. The effect of pre-buckling deformations, warping shear and buckling under torsional loading have not been examined in this paper but are subjects for future work.

#### Appendix A. Second order approximation for displacement components and internal actions

Here the expressions for the displacements and internal actions will be truncated to second order so that comparisons can be made to other lateral buckling formulations which are based on second order approximations. Truncated expressions for the displacements are also useful for finite element buckling formulations. Consider Eqs. (20)–(22). The angles  $\theta$  &  $\psi$  can be written as:

$$\theta = \tan^{-1} \left( \frac{\sqrt{\bar{w}_{s,x}^2 + \bar{v}_{s,x}^2}}{1 + \bar{u}_{s,x}} \right) \quad \psi = \tan^{-1} \left( \frac{\bar{w}_{s,x}}{\bar{v}_{s,x}} \right) \quad (155)$$

For small rotations ( $\bar{v}_{s,x}, \bar{w}_{s,x}$ ) we also assert that the twist about the 1 axis is denoted by  $\omega_1$  such that approximately:

$$\omega_1 = \phi + \psi \quad (156)$$

Also, we set:

$$\begin{aligned} u_{s,x} &= \bar{u}_{s,x} & \omega_3 &= \bar{v}_{s,x} & \omega_2 &= -\bar{w}_{s,x} & \therefore \gamma_2 &= v_{s,x} - \omega_3 \\ \gamma_3 &= w_{s,x} + \omega_2 \end{aligned} \quad (157)$$

Which for small rotations ( $\bar{v}_{s,x}, \bar{w}_{s,x}$ ) represent rotations about the z- and y-axes, respectively (see Fig. 3). The displacement compo-

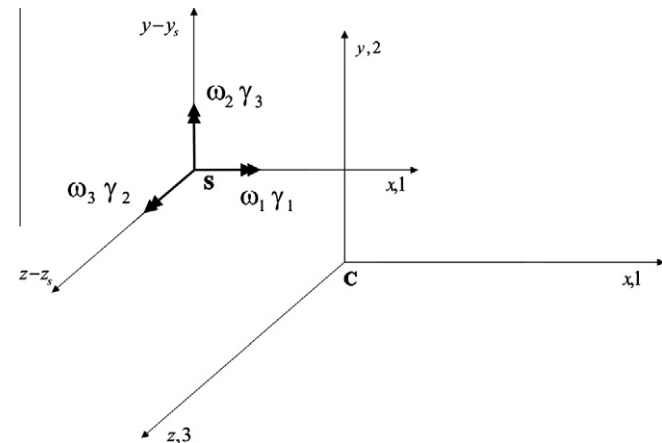


Fig. 3. Rotations at the shear centre.

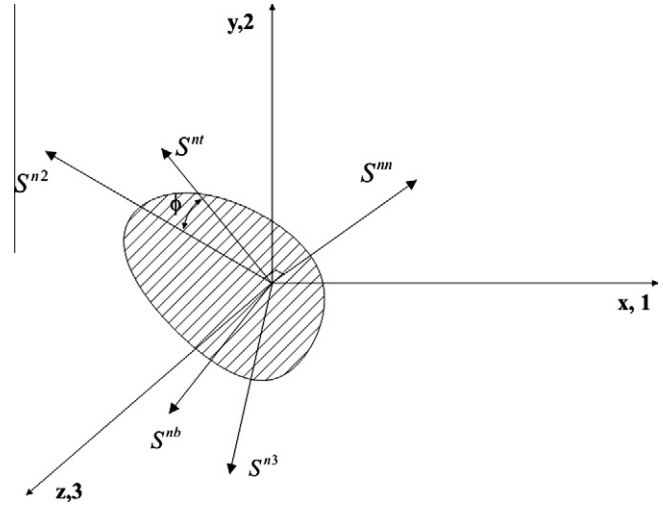


Fig. 4. Direction of physical Lagrangian stresses.

nents to second order in the 1, 2 & 3 directions are then obtained from Eq. (43), that is:

$$\begin{aligned} u^1 &\cong u_s - (y - y_s)(\omega_3(1 - u_{s,x}) - \omega_1\omega_2) \\ &\quad + (z - z_s)(\omega_2(1 - u_{s,x}) + \omega_1\omega_3) \\ &\quad - \omega(y, z) \left[ \omega_{1,x} + \frac{1}{2}(\omega_{2,x}\omega_3 - \omega_2\omega_{3,x}) \right] \\ u^2 &\cong v_s - (z - z_s) \left( \omega_1 - \frac{1}{2}\omega_2\omega_3 \right) \\ &\quad - \frac{1}{2}(y - y_s)(\omega_1^2 + \omega_3^2) + \omega(y, z)\omega_{1,x}\omega_2 \\ u^3 &\cong w_s + (y - y_s) \left( \omega_1 + \frac{1}{2}\omega_2\omega_3 \right) \\ &\quad - \frac{1}{2}(z - z_s)(\omega_1^2 + \omega_2^2) - \omega(y, z)\omega_{1,x}\omega_3 \end{aligned} \quad (158)$$

The shear angles  $\phi_0$  &  $\alpha_0$  and shear deformations  $\hat{\gamma}_2$  &  $\hat{\gamma}_3$  can also be approximated using Eqs. (18) and (19), by:

$$\phi_0 \cong \frac{\gamma_2\omega_3 - \gamma_3\omega_2}{\sqrt{(\omega_2)^2 + (\omega_3)^2}} \quad \alpha_0 \cong \frac{\gamma_2\omega_2 + \gamma_3\omega_3}{\sqrt{(\omega_2)^2 + (\omega_3)^2}} \quad (159)$$

$$\begin{aligned} \hat{\gamma}_1 &= \lambda_{1s} \cos \phi_0 \cos \alpha_0 \cong 1 + u_{s,x} + \frac{1}{2}(\omega_2)^2 + \omega_2\gamma_3 + \omega_3\gamma_2 + \frac{1}{2}(\omega_3)^2 \\ \hat{\gamma}_2 &= \lambda_{1s} \sin \phi_0 \cos \phi + \lambda_{1s} \sin \alpha_0 \cos \phi_0 \sin \phi \cong \gamma_2 + \gamma_3\omega_1 \\ \hat{\gamma}_3 &= -\lambda_{1s} \sin \phi_0 \sin \phi + \lambda_{1s} \sin \alpha_0 \cos \phi_0 \cos \phi \cong \gamma_3 - \gamma_2\omega_1 \end{aligned} \quad (160)$$

Likewise, the curvature and torsion defined above in Eqs. (29)–(31), can also be approximated.

$$\begin{aligned} \kappa &\cong \omega_{3,x}(1 - u_{s,x}) - \omega_1\omega_{2,x} - \omega_3u_{s,xx} \\ \kappa' &\cong \omega_{2,x}(1 - u_{s,x}) + \omega_{3,x}\omega_1 - \omega_2u_{s,xx} \\ \tau &\cong \omega_{1,x} + \frac{1}{2}(\omega_3\omega_{2,x} - \omega_2\omega_{3,x}) \end{aligned} \quad (161)$$

The constitutive relationships (63)–(68) can be simplified to second order terms as:

$$\begin{aligned} N &\cong EA \left( (1 - y_s\omega_{3,x} + z_s\omega_{2,x})u_{s,x} - \omega_{2,x}(y_s\omega_1 + z_s) \right. \\ &\quad \left. - \omega_{3,x}(z_s\omega_1 - y_s) + \frac{1}{2}(\omega_2^2 + \omega_3^2) \right. \\ &\quad \left. + \omega_2(\gamma_3 + z_su_{s,xx}) + \omega_3(\gamma_2 - y_su_{s,xx}) \right) \\ &\quad + (G + A)A(Z_s\gamma_2 - y_s\gamma_3)\omega_{1,x} + (G + A)(I_{po} - J_t)\omega_{1,x}^2 \end{aligned} \quad (162)$$

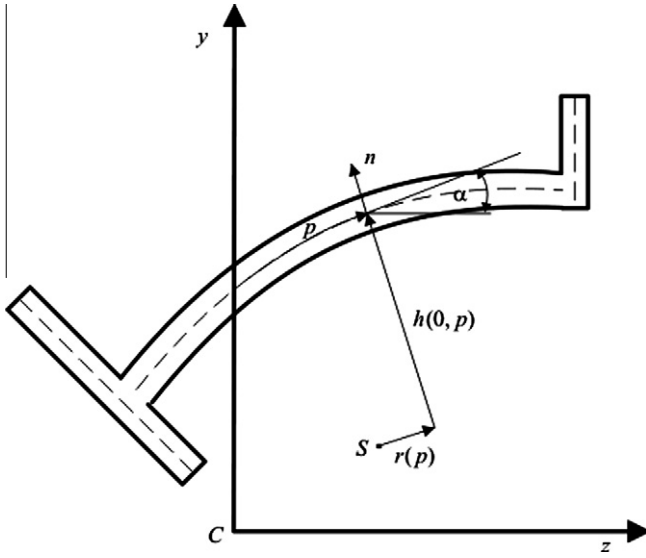


Fig. 5. Cross-section of a thin-walled beam showing curvilinear profile coordinate system.

$$Q_2 \cong GA(\gamma_2 + \gamma_3 \omega_1) \quad Q_3 \cong GA(\gamma_3 - \gamma_2 \omega_1) \quad (163)$$

$$M_3 \cong EI_{zz} \left( (1 - u_{s,x}) \omega_{3,x} - \omega_{2,x} \omega_1 - \omega_3 u_{s,xx} - \beta_1^* \omega_{1,x}^2 \right) - (G + A) \omega_{1,x} (C_{yy} \gamma_2 + C_{yz} \gamma_3) \quad (164)$$

$$M_2 \cong EI_{yy} \left( (1 - u_{s,x}) \omega_{2,x} + \omega_{3,x} \omega_1 - \omega_2 u_{s,xx} - \beta_2^* \omega_{1,x}^2 \right) + (G + A) \omega_{1,x} (C_{zy} \gamma_2 + C_{zz} \gamma_3) \quad (165)$$

$$M_t \cong GJ_t \left( \omega_{1,x} + \frac{1}{2} (\omega_3 \omega_{2,x} - \omega_2 \omega_{3,x}) \right) + 2G(C_{\omega z} \omega_{1,x} \omega_{2,x} + C_{\omega y} \omega_{1,x} \omega_{3,x}) \quad (166)$$

$$M_{\omega} \cong EI_{\omega} \left( \omega_{1,xx} + \frac{1}{2} (\omega_3 \omega_{2,xx} - \omega_2 \omega_{3,xx}) - U_{\omega}^* \omega_{1,x}^2 \right) - (G + A) \omega_{1,x} (C_{\omega y} \gamma_2 + C_{\omega z} \gamma_3) \quad (167)$$

The expression for the Wagner component of the twisting moment  $M_{\text{wagner}}$ , Eq. (86), can be reduced to second order using Eqs. (160), (161), (72), (75) and (80), as well as the expression for the normal stress in Eq. (188). Hence:

$$M_{\text{wagner}} \cong \left( E^* \{ -M_3 \beta_1^* - M_2 \beta_2^* + M_{\omega} U_{\omega}^* \} + N \frac{I_{p0} - J_t}{A} \right) \times \left( \omega_{1,x} + \frac{1}{2} \{ \omega_3 \omega_{2,x} - \omega_2 \omega_{3,x} \} \right) - \left( M_2 - Nz_s + \frac{M_{\omega} C_{\omega y}}{I_{\omega}} \right) (\gamma_2 + \omega_1 \gamma_3) - \left( M_3 + Ny_s + \frac{M_{\omega} C_{\omega z}}{I_{\omega}} \right) (\gamma_3 - \omega_1 \gamma_2) \quad (167)$$

## Appendix B. Monosymmetric parameters for a thin-walled open beam

Consider a thin-walled open profile as shown in Fig. 5. The middle surface of the cross-section denotes the contour of the cross-section. A curvilinear coordinate system  $p, n$  is shown, where  $p$  is the profile coordinate measured from a pole  $D$  on the middle surface and  $n$  is the normal thickness coordinate. The wall thickness is  $t(p)$ . The angle between the tangent at the contour and the positive

$z$ -axis is denoted by  $\alpha$ . The magnitude of the perpendicular distance from the tangent at the contour, to the shear centre (see Fig. 5) is represented by:

$$h(n, p) = n + h(0, p) = (y - y_s) \cos \alpha - (z - z_s) \sin \alpha \quad (168)$$

The magnitude of the perpendicular distance from the normal at the contour to shear centre is given by:

$$r(p) = (y - y_s) \sin \alpha + (z - z_s) \cos \alpha \quad (169)$$

From Eqs. (168) and (169) the rectangular coordinates can be expressed in terms of the curvilinear coordinates as:

$$\begin{aligned} (y - y_s) &= h(n, p) \cos \alpha + r(p) \sin \alpha \\ (z - z_s) &= -h(n, p) \sin \alpha + r(p) \cos \alpha \end{aligned} \quad (170)$$

The radial distance  $R$  from the shear centre to the any point within the cross-section can be written in either the curvilinear coordinate system or the rectangular system, that is:

$$R^2 = r(p)^2 + h(n, p)^2 = (y - y_s)^2 + (z - z_s)^2 \quad (171)$$

The warping function for a thin-walled open member (Attard, 1984) is given by

$$\omega(n, p) = \int_D^p h(0, p) dp - nr(p) \quad (172)$$

Consisting of a contour warping term  $\int_D^p h(0, p) dp$  and a thickness warping term  $nr(p)$ . The function  $\bar{\omega}$  defined in Eq. (54) can be simplified using Eqs. (170) and (171), and assuming that the thin-walled cross-section is composed of straight elements so that  $\frac{d\alpha}{dp} = 0$ , we can write:

$$\begin{aligned} \bar{\omega} &= \omega_z(y - y_s) - \omega_y(z - z_s) \\ &= h(n, p)(\omega_z \cos \alpha + \omega_y \sin \alpha) + r(p)(\omega_z \sin \alpha - \omega_y \cos \alpha) \\ &= h(n, p)\omega_p - r(p)\omega_n \\ &= h(n, p) \left( h(0, p) \left( 1 - n \frac{d\alpha}{dp} \right) - n \right) + r(p)^2 \\ &= h(n, p)^2 + r(p)^2 - 2nh(n, p) \\ &= R^2 - 2nh(n, p) \end{aligned} \quad (173)$$

Integrating over the area gives:

$$\int_A \bar{\omega} dA = \int_A [R^2 - 2nh(n, p)] dA = I_{p0} - J_t \quad (174)$$

Note for a thin-walled open section the torsion constant is defined by  $J_t = \int_A 2nh dA$ .

The parameters defined in Eq. (74) can be determined using Eq. (173), therefore:

$$\begin{aligned} E^* I_{zz} \beta_1^* &= \int_A y \bar{\omega} dA = \int_A y (R^2 - 2nh) dA \\ &= \left( \int_A y^3 + yz^2 dA - 2I_{zz} y_s \right) \\ &\quad - \int_A 2nh(h \cos \alpha + r \sin \alpha + y_s) dA \\ &= \left( \int_A y^3 + yz^2 dA - 2I_{zz} y_s \right) \\ &\quad - y_s J_t - \int_A 2nh(h \cos \alpha + r \sin \alpha) dA \end{aligned} \quad (175)$$

The term in brackets relates to the conventional definition for the monosymmetric parameter  $\beta_1$ .

$$\beta_1 = \frac{1}{2I_{zz}} \int_A y^3 + yz^2 dA - y_s \quad (176)$$

The other parameters in Eq. (74) are:

$$\begin{aligned} E^* I_{yy} \beta_2^* &= \left( - \int_A z^3 + y^2 z dA + 2 I_{yy} z_s \right) + z_s J_t \\ &\quad + \int_A 2nh(-h \sin \alpha + r \cos \alpha) dA \\ E^* I_{\omega} U_{\omega}^* &= \int_A -\omega R^2 dA + \int_A 2nh\omega dA \end{aligned} \quad (177)$$

### Appendix C. Wagner twisting moment

Eq. (86) for the twisting moment produced by the normal stresses is rewritten here:

$$M_{\text{wagner}} = \int_A [S^{nn}(\hat{\gamma}_2 \omega_y + \hat{\gamma}_3 \omega_z + \tau \bar{\omega})] dA \quad (178)$$

We can derive certain equations to be satisfied by the warping function by examining this expression. Consider the case of normal stresses only involving an axial force  $N$  at the centroid and variable bending about the  $z$ -axis at the shear centre,  $M_3$  which only causes a transverse and axial displacement of the shear centre. There will also be a shear  $Q_2$  associated with the variable bending moment  $M_3$  and thus,  $(\hat{\gamma}_2 \neq 0, \hat{\gamma}_3 = 0)$ . Hence, there must be bending about the centroidal  $y$  axis of  $M_2 = Nz_s$ . The normal stress is then given by

$$S^{nn} = \frac{N}{A} - \frac{M_3 y}{I_{zz}} + \frac{N z_s z}{I_{yy}} \quad (179)$$

Substituting this into the second order twisting moment defined in Eq. (178) and assuming that this should be zero, we have:

$$M_{\text{wagner}} = \hat{\gamma}_2 \left( \frac{N z_s}{I_{yy}} \left( I_{yy} + \int_A z \omega_y dA \right) \right) - \hat{\gamma}_2 \frac{M_3}{I_{zz}} \int_A y \omega_y dA = 0 \quad (180)$$

Using Eq. (75) we can conclude that

$$\int_A y \omega_y dA = 0 \quad \int_A -z \omega_y dA = I_{yy} \quad (181)$$

Similarly, by considering bending about the  $y$ -axis we can conclude:

$$\int_A z \omega_z dA = 0 \quad \int_A y \omega_z dA = I_{zz} \quad (182)$$

### Appendix D. Nonlinear axial shortening and bending under torsion

Consider the case of a beam under pure torque, where there is no resultant axial force, bending moments or shear forces, but only a twisting moment. From Eqs. (63)–(68), we have:

$$\begin{aligned} N &= EA(\lambda_{1s} + y_s \kappa - z_s \kappa' - 1) + (G + A)(I_{p0} - J_t) \tau^2 = 0 \\ M_3 &= EI_{zz}(\kappa - \beta_1^* \tau^2) = 0 \\ M_2 &= EI_{yy}(\kappa' - \beta_2^* \tau^2) = 0 \\ M_{\omega} &= EI_{\omega}(\tau_x - U_{\omega}^* \tau^2) \\ M_t &= GJ_t \tau \quad Q_2 = 0 \quad Q_3 = 0 \end{aligned} \quad (183)$$

Therefore, there must be nonlinear second order axial shortening and curvatures induced in the beam given by:

$$\begin{aligned} \kappa &= \beta_1^* \tau^2 \quad \kappa' = \beta_2^* \tau^2 \\ \lambda_{1s} - 1 &= \left[ - \frac{(I_{p0} - J_t)}{EA} - y_s \beta_1^* + z_s \beta_2^* \right] \tau^2 \end{aligned} \quad (184)$$

Using the above and Eqs. (32) and (33) we can relate the torsion to the deflections in the axial, as well as transverse  $y$ - and  $z$ -directions, hence:

$$\begin{aligned} \kappa \sin \phi - \kappa' \cos \phi &= \sin \theta \frac{d\psi}{dx} \\ \therefore (\beta_1^* \sin \phi - \beta_2^* \cos \phi) \tau^2 &= \frac{\bar{v}_{s,x} \bar{w}_{s,xx} - \bar{w}_{s,x} \bar{v}_{s,xx}}{\lambda_{1s} \sqrt{\bar{w}_{s,x}^2 + \bar{v}_{s,x}^2}} \end{aligned} \quad (185)$$

$$\begin{aligned} \kappa \cos \phi + \kappa' \sin \phi &= \frac{d\theta}{dx} \\ \therefore (\beta_1^* \cos \phi + \beta_2^* \sin \phi) \tau^2 &= \frac{(\bar{v}_{s,x} \bar{v}_{s,xx} + \bar{w}_{s,x} \bar{w}_{s,xx})(1 + \bar{u}_{s,x}) - \bar{u}_{s,xx}(\bar{w}_{s,x}^2 + \bar{v}_{s,x}^2)}{\lambda_{1s} \sqrt{\bar{w}_{s,x}^2 + \bar{v}_{s,x}^2}} \end{aligned} \quad (186)$$

### Appendix E. Expression for normal stress

Using Eqs. (62)–(68), we can write the expression for the normal stress as

$$\begin{aligned} S^{nn} &= \frac{N}{A} - \frac{M_3 y}{I_{zz}} + \frac{M_2 z}{I_{yy}} - \frac{M_{\omega} \omega}{I_{\omega}} + E(-y \beta_1^* + z \beta_2^* - \omega U_{\omega}^*) \tau^2 \\ &\quad + (G + A) \tau (z_s \hat{\gamma}_2 - y_s \hat{\gamma}_3) - (G + A) \left[ \frac{I_{p0} - J_t}{A} \right] \tau^2 \\ &\quad - \frac{(G + A) y \tau}{I_{zz}} (C_{yy} \hat{\gamma}_2 + C_{yz} \hat{\gamma}_3) - \frac{(G + A) z \tau}{I_{yy}} (C_{zy} \hat{\gamma}_2 + C_{zz} \hat{\gamma}_3) \\ &\quad - \frac{(G + A) \omega \tau}{I_{\omega}} (C_{\omega y} \hat{\gamma}_2 + C_{\omega z} \hat{\gamma}_3) \\ &\quad + (G + A) (\tau \hat{\gamma}_2 \omega_y + \tau \hat{\gamma}_3 \omega_z + \tau^2 \bar{\omega}) \end{aligned} \quad (187)$$

Using Eqs. (160), (161), (75), (72) and (80), the expression for the normal stress can be reduced to second order as:

$$\begin{aligned} S^{nn} &\cong \frac{N}{A} - \frac{M_3 y}{I_{zz}} + \frac{M_2 z}{I_{yy}} - \frac{M_{\omega} \omega}{I_{\omega}} + E(-y \beta_1^* + z \beta_2^* - \omega U_{\omega}^*) (\omega_{1,x})^2 \\ &\quad - (G + A) \left[ \frac{I_{p0} - J_t}{A} \right] (\omega_{1,x})^2 + (G + A) \left[ \bar{\omega} (\omega_{1,x})^2 \right. \\ &\quad \left. + \omega_{1,x} \hat{\gamma}_2 \left( z_s + z + \omega_y - \frac{\omega C_{\omega y}}{I_{\omega}} \right) + \omega_{1,x} \hat{\gamma}_3 \left( -y_s - y + \omega_z - \frac{\omega C_{\omega z}}{I_{\omega}} \right) \right] \end{aligned} \quad (188)$$

### Appendix F. Linearized lateral buckling equations for a prismatic monosymmetric beam under transverse loading

The buckling analysis of statically determinate prismatic monosymmetric beams subjected to transverse loads such as concentrated forces, or uniformly distributed load, is analysed here by developing the linearized governing equations. Firstly, the relationship between the axial force and shears with the internal force resultants in the  $x$ -,  $y$ - and  $z$ -direction is determined and linearized. Thus we have:

$$\begin{aligned} N &= P_x \cos \theta + P_y \sin \theta \cos \psi + P_z \sin \theta \sin \psi \\ &\cong P_x + P_y \omega_3 - P_z \omega_2 \end{aligned} \quad (189)$$

$$\begin{aligned} Q_2 &= -P_x \sin \theta \cos \phi + P_y (\cos \theta \cos \psi \cos \phi - \sin \psi \sin \phi) \\ &\quad + P_z (\cos \theta \sin \psi \cos \phi + \cos \psi \sin \phi) \\ &\cong -P_x \omega_3 + P_y - P_z \omega_1 \end{aligned} \quad (190)$$

$$\begin{aligned} Q_3 &= P_x \sin \theta \sin \phi + P_y (-\cos \theta \cos \psi \sin \phi - \sin \psi \cos \phi) \\ &\quad + P_z (-\cos \theta \sin \psi \sin \phi + \cos \psi \cos \phi) \\ &\cong -P_x \omega_2 - P_y \omega_1 + P_z \end{aligned} \quad (191)$$

In some instances, it is more convenient for problems which are statically determinate to deal with the transverse force resultants  $P_x, P_y, P_z$  rather than the internal actions. The equilibrium equations (101)–(106) are now linearized to first order in displacements, hence:

$$\frac{dN}{dx} - Q_2\omega_{3,x} + Q_3\omega_{2,x} + p_n = 0 \quad (192)$$

$$\frac{dQ_2}{dx} + N\omega_{3,x} - Q_3\omega_{1,x} + p_2 = 0 \quad (193)$$

$$\frac{dQ_3}{dx} - N\omega_{2,x} + Q_2\omega_{1,x} + p_3 = 0 \quad (194)$$

$$\frac{d\tilde{M}_1}{dx} - (M_2 - Nz_s)\omega_{3,x} + (M_3 + Ny_s)\omega_{2,x} + m_n = 0 \quad (195)$$

$$\begin{aligned} \frac{d(M_3 + Ny_s)}{dx} - \tilde{M}_1\omega_{2,x} + (M_2 - Nz_s)\omega_{1,x} + m_3 \\ = N\gamma_2 - (1 + u_{s,x})Q_2 = P_x(\gamma_2 + \omega_3) - P_y(1 + u_{s,x}) - P_z\omega_1 \end{aligned} \quad (196)$$

$$\begin{aligned} \frac{d(M_2 - Nz_s)}{dx} + \tilde{M}_1\omega_{3,x} - (M_3 + Ny_s)\omega_{1,x} + m_2 \\ = -N\gamma_3 + (1 + u_{s,x})Q_3 \\ = -P_x(\gamma_3 + \omega_2) + P_z(1 + u_{s,x}) - P_y\omega_1 \end{aligned} \quad (197)$$

Let us now consider a monosymmetric beam with transverse loading  $P_y^0$  such that  $P_x, P_z = 0$  with bending about the  $z$ -axis producing  $M_3^0$ . Eqs. (189)–(191) reduce to:

$$N \cong P_y\omega_3 \quad Q_2 \cong P_y \quad Q_3 \cong -P_y\omega_1 \quad (198)$$

Perturbations of the linearized equilibrium equations are considered. Eq. (196) becomes

$$\frac{dM_3^0}{dx} = -P_y^0 \quad (199)$$

Substituting Eq. (198) and the constitutive relationship (163) into the shear equilibrium equation in the lateral direction, Eq. (194), becomes

$$Q_3 = -P_y\omega_1 \Rightarrow GA\delta\gamma_3 = -P_y^0\delta\omega_1 \quad (200)$$

Incorporating the constitutive relationships (165) and (166) into the twisting moment rate equilibrium equation (195) we have:

$$\begin{aligned} GJ_t\delta\omega_{1,xx} - EI_\omega\delta\omega_{1,xxxx} + M_3^0\delta\omega_{2,x} + m_n \\ = \frac{d}{dx} [E^*\beta_1^*M_3^0\delta\omega_{1,x} + M_3^0\delta\gamma_3] \end{aligned} \quad (201)$$

Substituting the constitutive law, Eq. (164) and making use of Eq. (199), the lateral bending rate equilibrium equation (197) is then:

$$EI_{yy}\delta\omega_{2,xx} - (M_3^0\delta\omega_1)_{,x} = 0 \quad \therefore \delta\omega_{2,x} = \frac{M_3^0\delta\omega_1}{EI_{yy}} + C_1 \quad (202)$$

where  $C_1$  is a constant of integration. Eq. (201) is now transformed using the above equation and (200), resulting in an differential equation in one displacement unknown:

$$\begin{aligned} GJ_t\delta\omega_{1,xx} - EI_\omega\delta\omega_{1,xxxx} + M_3^0\left(\frac{M_3^0\delta\omega_1}{EI_{yy}} + C_1\right) + m_n \\ = \frac{d}{dx} \left[ E^*\beta_1^*M_3^0\delta\omega_{1,x} - M_3^0\frac{P_y^0\delta\omega_1}{GA} \right] \end{aligned} \quad (203)$$

This is similar to the classical lateral buckling equation with the added term  $-M_3^0\frac{P_y^0\delta\omega_1}{GA}$  due to a consideration of shear. We see that if there is no transverse load  $P_y^0 = 0$  as in the case of pure bending,

this additional term would vanish in agreement with our earlier derivation. Numerical solutions can be derived to Eq. (203).

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