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A rigid-plastic micromechanical modeling of a random packing of frictional particles

Francesco Trentadue *

Politecnico di Bari, Dipartimento di Ingegneria per l'Ambiente e lo Sviluppo Sostenibile (DIASS), Viale del Turismo, 8, Taranto 74123, Italy

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ABSTRACT

The mechanical behavior of a random packing of rigid particles, in which interparticle contact forces follow the Coulomb friction law, is analyzed with the aim of establishing a link between the microscopic frictional behavior of contacts, the equilibrium of particles and the macroscopic plasticity of this material. A Reference Volume Element (RVE) containing a very large number of particles is examined and, under a rather general assumption on the shapes of particles, it is shown that in the macro stress space the yield surface of this material is a cone. Further, linear displacement boundary conditions are prescribed on the RVE and the plastic macro strain of this material is examined. It is shown that in case of frictional particles the plastic macro strain cannot be associated, while it is associated in case of frictionless particles. Further, in the particular case of frictional identical spheres, it is shown that only the deviatoric component of plastic macro strain is associated. Finally, a micromechanical derivation of the constitutive inequality relating the friction and dilatancy coefficients is given.

1. Introduction

Granular materials are packings of discrete particles in which relative movements are due both to deformations of particles and to sliding or opening of contacts. Despite several decades of extensive studies, a link between the microscopic properties of these materials and their general macroscopic mechanical behavior is not definitively established.

At the macroscopic scale granular materials generally exhibit vanishing elasticity and predominantly irreversible deformations. It is then customary to represent their macroscopic mechanical behavior by plasticity models. Well known plasticity models for soils are the Mohr–Coulomb model and the Drucker-Prager model (1952). Other plasticity models for granular materials which provide a better matching with the experimental results were more recently proposed by Lade (1977), Matsuoka and Nakai (1977), Krenk (2000). All these models predict convex conic yield surfaces.

The plastic macro deformation of granular materials, with either negative or positive dilatancy, is frequently described by non-associated flow rules. In particular, it is usually assumed that only the deviatoric part of the plastic strain follows the normality rule, while the volumetric plastic strain is not associated. This assumption, usually referred as deviatoric associativity, is widely adopted in soil mechanics: Gudehus (1972), Lade and Duncan (1973), Baker and Desai (1982).

Several micromechanical models have been adopted to study the macroscopic mechanical behavior of granular materials. Among these, the models developed to study the elasticity exhibited by granular materials when small arbitrary stresses are superimposed to a confining pressure, neglect the opening or the sliding of contacts and assume that elastic deformations are localized in small neighborhoods of particle contacts. Duffy and Mindlin (1957), Emeriault and Cambou (1996), Chang and Liao (1990), Chang and Misra (1990), Jenkins (1987), developed elastic models based on the uniform strain (Voigt) hypothesis. Other models, Misra and Chang (1993), Trentadue (2001, 2004), Jenkins et al. (2005), considered the effect of local equilibrium conditions on the macroscopic elasticity of granular materials.

The more general elasto-plastic behavior of granular materials is more complex and different micromechanical approaches can be distinguished. Plasticity models with a fabric tensor were proposed by Oda (1993), Wan and Guo (2001), Nemat-Nasser and Zhang (2002), Zhu et al. (2006). In these models the material parameters are defined at the macro level and are functions of the fabric tensor, in order to consider the effects of packing structure. Other micromechanical plasticity models represent the material as a random packing of particles, where some material parameters are defined at the particle level, and other parameter are defined at the macro level. This approach has been followed by Suiker and Chang (2004), Chang and Hicher (2005), Nicot and Darve (2006, 2007).

The nature of the distribution of contact forces in granular media was also investigated in numerical simulations on systems of particles under quasi-static loading (Radjai et al., 1996, 1997, 1998, 1999; Radjai and Wolf, 1998; Antony and Kuhn, 2004). These studies shows that the normal components of contact force provide the major contribution to the deviatoric stress and that load

* Tel.: +39 0994733219; fax: +39 0805963719.
E-mail address: f.trentadue@poliba.it

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is mainly transmitted by relatively rigid, heavily stressed chains of particles which form a sparse network of contacts. The remaining groups of particles, which separate the strong force chains, are only lightly loaded.

Further, simpler plasticity models rely on the consideration that when low interparticle contact forces are exerted, the macroscopic plasticity of granular materials can be related only to sliding and opening of contacts, while deformations or breakings of particles can be neglected. Micromechanical models based on the above assumption were proposed by Rowe (1962), Vardoulakis and Sulem (1992) who considered ordered arrays of identical rigid spherical particles, by Vardulakis (1981), Vardoulakis and Sulem (1992) who considered a packing of disks under particular kinematic assumptions and by Cambou et al. (2009), who adopted a mean field approach and considered particular rigid plastic collapse mechanisms.

In this work, granular materials are modeled as statistically homogeneous random packings of rigid particles with frictional contacts. The modeling is relatively simple and based on quite clear theoretical motivations. All material parameters are defined at the microscopic level and no assumptions regarding the number and the orientations of contacts are required. Further, it is able to capture many essential features of the plastic behavior of general granular materials when low macro-stresses are applied, so that no contact damages or crushing of particles are produced and no significant changes in the material microstructure occur until yielding is reached.

A Reference Volume Element (RVE) containing a very large number of particles is examined: under a rather general condition which relates the shapes of particles to the contact friction coefficients, it is shown that statically admissible equilibrium states can exist if and only if the mean macro stress $p\boldsymbol{I} = tr(\boldsymbol{T})/3 = (\sigma_{11} + \sigma_{22} + \sigma_{33})/3$ is a pressure ($p \leqslant 0$). Further, it is shown that, given a bounded pressure $p \leqslant 0$, every statically admissible macro stress $\boldsymbol{T}$ must be bounded. Finally, it is shown that the yield surfaces are cones.

Next, linear displacement conditions are prescribed on the boundary of RVE and the initial plastic flow is examined in two particular cases. In the first case a material made of frictionless convex particles is examined, showing that the initial plastic flow is associated and that the yield surface is a convex cone. Next, a more particular material made of identical spherical particles is considered, showing that also in this case the yield surfaces is a convex cone, but only the deviatoric part of the plastic flow is associated. Finally, it is shown that in a general granular material the plastic macro strain cannot be associated and a micromechanical derivation of the well known constitutive inequality relating the friction and the dilatancy coefficients, first proposed by Taylor (1948), is given.

It must be highlighted that this modeling is not able to describe any evolution of the material microstructure either before or after the macroscopic plastic flow occurs. The rigid-plastic behavior predicted at the macro scale is therefore influenced by this limitation and physical phenomena such as hardening or softening or the evolution of the material dilatancy are not captured.

2. Microstructural continuum model

Let us consider a statistically homogeneous granular material formed by a random packing of rigid particles. No interparticle cohesion is considered and contact forces are assumed to follow the Coulomb friction law. A reference volume element (RVE) of volume V containing a very large number of particles is examined, as shown in Fig. 1. The RVE is bounded by a fictitious surface S_v enveloping the contact points $\mathbf{x}^{i,e}$ between the external particles e and the internal particles i.

2.1. Kinematically admissible systems of particle displacements

We call kinematically admissible any system of infinitesimal rigid displacements $\mathbf{u}(\mathbf{x})$ that does not produce interpenetrations between particles. So that, for every pair of particles i and j, the following condition holds:

$$\begin{array}{l}\forall \mathbf{x}_{(j)} \in V_j^0, \\ \forall \mathbf{x}_{(k)} \in V_k^0\end{array} : \quad \mathbf{x}_{(j)} + \mathbf{u}(\mathbf{x}_{(j)}) \neq \mathbf{x}_{(k)} + \mathbf{u}(\mathbf{x}_{(k)}), \tag{1}$$

where V_j^0 and V_j^0 are the interiors of the space domains occupied by the two particles. In particular, for every pair of particles a and b in contact, (1) implies that:

$$\begin{cases} (\mathbf{u}^b + \mathbf{W}^b \mathbf{r}^{a_b}) - (\mathbf{u}^a + \mathbf{W}^a \mathbf{r}^{b_a}) = \xi^{b,a} \mathbf{n}^{b_a} + \gamma^{b,a} \boldsymbol{\nu}^{b_a}, \\ \xi^{b,a} \geqslant 0, \\ \gamma^{b,a} \geqslant 0, \end{cases} \tag{2}$$

where: $\mathbf{u}^a = \mathbf{u}(\mathbf{x}^a)$ *and* $\mathbf{u}^b = \mathbf{u}(\mathbf{x}^b)$ are the displacements of two internal points $\boldsymbol{x}^a \in \boldsymbol{V}_a^0$ *and* $\boldsymbol{x}^b \in \boldsymbol{V}_b^0$; $\mathbf{W}^a$ and $\mathbf{W}^b$ are the rotation tensors of the rigid particles; $\mathbf{r}^{b_a} = \mathbf{x}^{b,a} - \mathbf{x}^a$ is the radius vector joining the contact point $\mathbf{x}^{b,a}$ with the internal point $\mathbf{x}^a$; $\mathbf{n}^{b_a}$ is the external unit normal to the surface S_a of the particle a at the contact point $\mathbf{x}^{b,a}$; $\boldsymbol{\nu}^{b_a}$ is a unit tangent vector to this surface at the same point (Fig. 1).

A positive value of $\xi^{b,a}$ implies a contact opening, while a null value of $\xi^{b,a}$ together with a positive value of $\gamma^{b,a}$ implies sliding. When both parameters are null the two particles remain attached.

2.2. Contact force-displacement laws

Contact forces can be only compressive and must follow the Coulomb friction law, so that:

$$\begin{cases} N^{j_a} = \mathbf{f}^{j_a} \cdot \mathbf{n}^{j_a} \leqslant 0, \\ V^{j_a} = |\mathbf{f}^{j_a} - \mathbf{n}^{j_a}(\mathbf{f}^{j_a} \cdot \mathbf{n}^{j_a})| \leqslant \mu^{j,a}\left(-N_s^{j_a}\right), \end{cases} \tag{3}$$

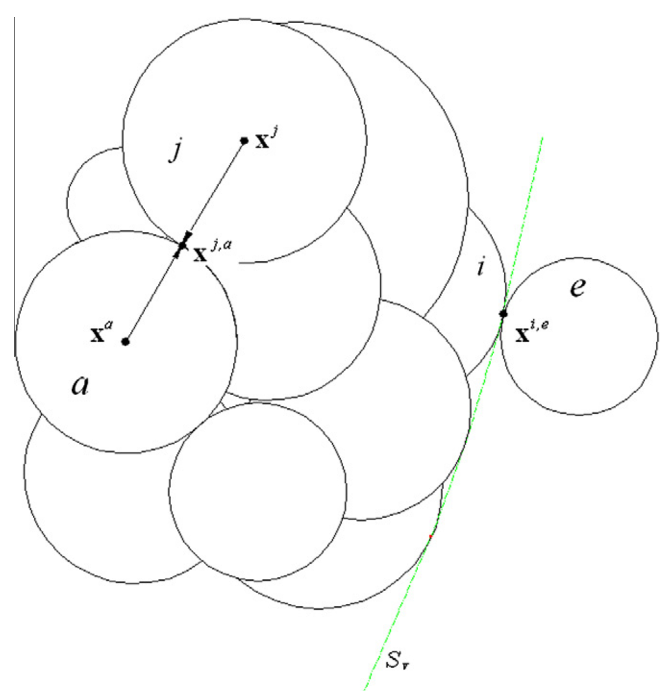

Fig. 1. A random packing of rigid particles.

where: $\mathbf{f}^{j_a}$ denotes the contact force exerted by the particle j to the particle a at the contact point $\mathbf{x}^{j,a}$; $N^{j_a}\mathbf{n}^{j_a}$ is the normal component of the contact force; $\mathbf{V}^{j_a} = \mathbf{f}^{j_a} - \mathbf{n}^{j_a}(\mathbf{f}^{j_a} \cdot \mathbf{n}^{j_a})$ is the shear component; $\mu^{j,a} = \tan(\varphi^{j,a}) \geqslant 0$ is the contact friction coefficient, where $\varphi^{j,a}$ is the contact friction angle.

A contact opening ($\xi^{j,a} > 0$) can occur only when the contact force $\mathbf{f}^{j_a}$ is null, while a sliding ($\gamma^{j,a} > 0$) can occur only when the Coulomb inequality is strictly satisfied. Then the following contact force-displacement laws hold:

$$\begin{cases} N^{j_a} \cdot \xi^{j,a} = 0, \\ (V^{j_a} + \mu^{j,a} N^{j_a}) \cdot \gamma^{j,a} = 0. \end{cases} \tag{4}$$

Finally, contacts are assumed to be isotropic. Therefore, when a contact sliding occurs, the shear contact force $\mathbf{V}^{j_a}$ must have the same direction of the sliding displacement $\boldsymbol{\gamma}^{j_a}$:

$$\gamma^{j,a} > 0 \;\Rightarrow\; \begin{cases} \mathbf{V}^{j_a} = \mu^{j,a}(-N^{j_a})\boldsymbol{v}^{j_a}, \\ \boldsymbol{\gamma}^{j_a} = \gamma^{j,a}\boldsymbol{v}^{j_a}. \end{cases} \tag{5}$$

2.3. Statics

For every particle a of the RVE the following equilibrium conditions must be fulfilled:

$$\begin{aligned} &\sum_{\mathbf{x}^{j,a} \in S_a} \mathbf{f}^{j_a} = \mathbf{0}, \\ &\sum_{\mathbf{x}^{j,a} \in S_a} \mathbf{f}^{j_a} \otimes \mathbf{r}^{j_a} - \mathbf{r}^{j_a} \otimes \mathbf{f}^{j_a} = \mathbf{0}, \end{aligned} \tag{6}$$

where: $(\mathbf{a} \otimes \mathbf{b})_{ij} = a_i b_j$ is the dyadic product of two vector $\mathbf{a}$ and $\mathbf{b}$; the summations are extended to all contact points $\mathbf{x}^{j,a}$ of the surfaces S_a of the particles a. Further, for every pair of contact forces, the action-reaction principle holds:

$$\mathbf{f}^{j_a} = -\mathbf{f}^{a_j}. \tag{7}$$

Since the system of contact forces is equilibrated, according to the mean stress theorem (Nemat-Nasser and Hori, 1993, Cambou et al., 2009), the macroscopic stress tensor $\boldsymbol{T}$ can be written as:

$$V\mathbf{T} = \sum_{a \in V} V_a \int_{V_a} \mathbf{T}_{(\mathbf{x})} dV = \sum_{a \in V} \sum_{\mathbf{x}^{j,a} \in S_a} \mathbf{r}^{j_a} \otimes \mathbf{f}^{j_a}, \tag{8}$$

where: V_a here denotes both the volume and the space domain occupied by a generic particle a; the summation is extended to all particles a of the RVE and to all contacts $\mathbf{x}^{j,a}$ on the particles surfaces S_a. In view of the rotational equilibrium conditions (6), the macro stress $\mathbf{T}$ is symmetric. Further, according to (6) and (7), the macro stress $\mathbf{T}$ can also be derived only from the contact forces on the RVE boundary surface S_v:

$$\begin{aligned} V\mathbf{T} &= \sum_{a \in V} \sum_{\mathbf{x}^{j,a} \in S_a} \mathbf{r}^{j_a} \otimes \mathbf{f}^{j_a} = \sum_{a \in V} \sum_{\mathbf{x}^{j,a} \in S_a} (\mathbf{x}^{j,a} - \mathbf{x}^a) \otimes \mathbf{f}^{j_a} \\ &= \sum_{a \in V} \left(\sum_{\mathbf{x}^{j,a} \in S_a} (\mathbf{x}^{j,a} - \mathbf{x}^a) \otimes \mathbf{f}^{j_a} + \mathbf{x}^a \otimes \sum_{\mathbf{x}^{j,a} \in S_a} \mathbf{f}^{j_a} \right) \\ &= \sum_{a \in V} \sum_{\mathbf{x}^{j,a} \in S_a} \mathbf{x}^{j,a} \otimes \mathbf{f}^{j_a} = \sum_{\mathbf{x}^{i,e} \in S_v} \mathbf{x}^{i,e} \otimes \mathbf{f}^{e_i}, \end{aligned} \tag{9}$$

where, from action-reaction principle, all terms due to internal contact forces vanish and then the last summation is extended only to the contacts $\mathbf{x}^{i,e}$ on the RVE boundary surface S_v.

The macro stress tensor $\mathbf{T}$ is then expressed as the sum of its deviatoric part $\mathbf{S}$ and its hydrostatic part $p\mathbf{I}$, so that: $\mathbf{T} = \mathbf{S} + p\mathbf{I}$, where $tr(\mathbf{S}) = 0$ and $p\mathbf{I} = (tr(\mathbf{T})/3)\mathbf{I}$. By applying the trace operator to (8), the following relations is established:

$$3pV = \sum_{a \in V} \sum_{\mathbf{x}^{j,a} \in S_a} \mathbf{f}^{j_a} \cdot \mathbf{r}^{j_a}. \tag{10}$$

In view of (7), (10) can also be written as:

$$3pV = \sum_{\mathbf{x}^{j,a} \in V} \mathbf{l}^{j_a} \cdot \mathbf{f}^{j_a}, \tag{11}$$

where $\mathbf{l}^{j_a} = \mathbf{r}^{j_a} - \mathbf{r}^{a_j} = \mathbf{x}^j - \mathbf{x}^a$ is the branching vector and the summation is extended to all contacts $\mathbf{x}^{j,a}$ in the RVE.

2.4. Existence and boundedness of statically admissible systems of contact forces

We call statically admissible any equilibrated system of contact forces for which the contact laws (3) are satisfied. A macro stress is called statically admissible if at least a statically admissible system of contact forces can be determined for it.

As already stated, no assumptions regarding the number and orientations of the contacts on particles are requested by the present model. However, it must be underlined that a random packing of rigid particles can be conceived as a solid (or fluid) material only if its microstructure is such that at least a statically admissible macro stress can be exerted on it. Therefore, in the following we will assume that material microstructure is such that at least a macro stress $\mathbf{T}_s = p_0\mathbf{I}$, ($p_0 < 0$) is statically admissible [1]. Under this assumption, it is easy to show that every hydrostatic macrostress $\mathbf{T}_s = p\mathbf{I}$ (with $p \leqslant 0$) is also statically admissible.

Next, in the following analysis we will consider only shapes of particles (Fig. 2) for which the condition:

$$\alpha^{j_a} + \varphi^{j,a} \leqslant \pi/2, \tag{12}$$

is satisfied for every contact point, where: α^{j_a} is the angle between the radius vector $\mathbf{r}^{j_a}$ and the external unit normal $\mathbf{n}^{j_a}$; $\varphi^{j,a}$ is the contact friction angle. It can be noted that if the contact friction angles $\varphi^{j,a}$ are null, the assumption (12) is satisfied for any convex shape and that, in the particular case of spherical particles, it is satisfied for any value of the contact friction angle.

Now, noting that only compressive contact normal forces ($N_s^{j_a} \leqslant 0$) can be exerted, under the assumption (12) it can be shown that in a statically admissible state all terms $\mathbf{f}_s^{j_a} \cdot \mathbf{r}^{j_a}$ in (10) must be negative or null:

$$\begin{aligned} \mathbf{f}_s^{j_a} \cdot \mathbf{r}^{j_a} &= \left[N_s^{j_a}\mathbf{n}^{j_a} + V_s^{j_a}\boldsymbol{v}_s^{j_a} \right] \cdot r^{j_a}\bar{\mathbf{r}}^{j_a} \\ &= r^{j_a}\left[N_s^{j_a}(\mathbf{n}^{j_a} \cdot \bar{\mathbf{r}}^{j_a}) + V_s^{j_a}(\boldsymbol{v}_s^{j_a} \cdot \bar{\mathbf{r}}^{j_a}) \right] \\ &= r^{j_a}\left[N_s^{j_a}\cos(\alpha^{j_a}) + V_s^{j_a}(\boldsymbol{v}_s^{j_a} \cdot \bar{\mathbf{r}}^{j_a}) \right] \\ &\leqslant r^{j_a}\left[N_s^{j_a}\cos(\alpha^{j_a}) + \left(-\mu^{j,a}N_s^{j_a} \right) Max_{\boldsymbol{v}_s^{j_a} \in \pi^{h,a}}\left(\boldsymbol{v}_s^{j_a} \cdot \bar{\mathbf{r}}^{j_a} \right) \right] \\ &= N_s^{j_a}r^{j_a}\left[\cos(\alpha^{j_a}) - \mu^{j,a}\sin(\alpha^{j_a}) \right] \\ &= N_s^{j_a}r^{j_a}\left[\cos(\alpha^{j_a}) - \tan(\varphi^{j,a})\sin(\alpha^{j_a}) \right] \\ &= N_s^{j_a}r^{j_a}\frac{\cos(\alpha^{j_a} + \varphi^{j,a})}{\cos(\varphi^{j,a})} \leqslant 0, \end{aligned} \tag{13}$$

where $\bar{\mathbf{r}}^{j_a} = \mathbf{r}^{j_a}/r^{j_a}$ is the unit radius vector and the unit tangent vector $\boldsymbol{v}^{j_a}$ lies on the plane $\pi^{j,a}$ tangent to the particle surface at the contact points $\mathbf{x}^{j,a}$. Then, in view of (10) and (13), we find:

$$3pV = \sum_{a \in V} \sum_{\mathbf{x}^{j,a} \in S_a} \mathbf{f}_s^{j_a} \cdot \mathbf{r}^{j_a} \leqslant 0, \tag{14}$$

which implies that no statically admissible systems of contact forces can exist if the mean macro stress p is positive.

[1] The subscript s refers to any quantity related to a statically admissible state.

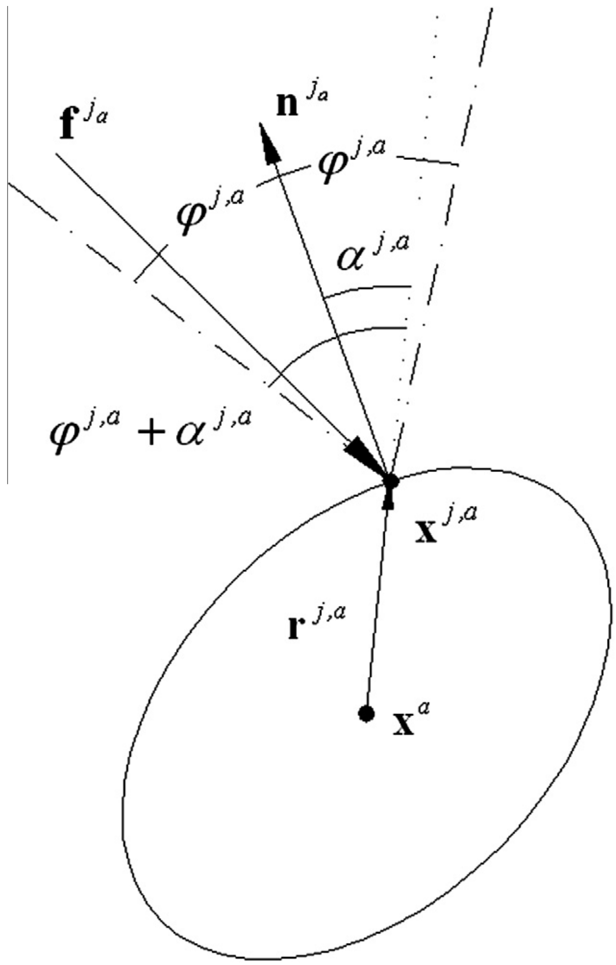

Fig. 2. A statically admissible contact force.

Under the same assumption (12) it can also be established that, given a bounded non positive macro mean stress p, every statically admissible macro stress must be bounded. In fact, from (14) and (13):

$$3|p|V=\left|\sum_{a\in V}\sum_{\mathbf{x}^{j,a}\in S_a}\mathbf{f}_s^{j_a}\cdot\mathbf{r}^{j_a}\right|\geqslant Max_{\mathbf{x}^{j,a}\in V}\left\{\left|\mathbf{f}_s^{j_a}\cdot\mathbf{r}^{j_a}\right|\right\}$$
$$\geqslant Max_{\mathbf{x}^{j,a}\in V}\left\{\left|N_s^{j_a}\right|\frac{\cos(\alpha^{j_a}+\varphi^{j,a})}{\cos(\varphi^{j,a})}r^{j_a}\right\}$$
$$\geqslant Max_{\mathbf{x}^{j,a}\in V}\left\{\left|N_s^{j_a}\right|\right\}\cdot\min_{\mathbf{x}^{j,a}\in V}\left\{r^{j_a}\frac{\cos(\alpha^{j_a}+\varphi^{j,a})}{\cos(\phi^{j,a})}\right\}\geqslant 0 \qquad (15)$$

and, by multiplying the above inequalities by $\sqrt{1+\mu_{Max}^2}$, where $\mu_{Max}=Max_{\mathbf{x}^{j,a}\in V}\{\mu^{j,a}\}$, we get:

$$3|p|V\sqrt{1+\mu_{Max}^2}\geqslant\left(\sqrt{1+\mu_{Max}^2}Max_{\mathbf{x}^{j,a}\in V}\left\{\left|N_s^{j_a}\right|\right\}\right)$$
$$\cdot\min_{\mathbf{x}^{j,a}\in V}\left\{r^{j_a}\frac{\cos(\alpha^{j_a}+\varphi^{j,a})}{\cos(\varphi^{j,a})}\right\}$$
$$\geqslant\left|\mathbf{f}_s^{Max}\right|\cdot\min_{\mathbf{x}^{j,a}\in V}\left\{r^{j_a}\frac{\cos(\alpha^{j_a}+\varphi^{j,a})}{\cos(\varphi^{j,a})}\right\}, \qquad (16)$$

where $\left|\mathbf{f}_s^{Max}\right|=Max_{\mathbf{x}^{j,a}\in V}\left|\mathbf{f}_s^{i_a}\right|$. So we find the inequality:

$$\left|\mathbf{f}_s^{Max}\right|\leqslant\frac{3|p|V\sqrt{1+\mu_{Max}^2}}{\min_{\mathbf{x}^{j,a}\in V}\left\{\frac{\cos(\alpha^{j_a}+\varphi^{j,a})}{\cos(\varphi^{j,a})}r^{j_a}\right\}}, \qquad (17)$$

which proves that under a bounded pressure every statically admissible system of contact forces is bounded. The above inequality allows us to state that also the Euclidean norm[2] $(\mathbf{T}_s\cdot\mathbf{T}_s)^{1/2}$ of the macrostress $\mathbf{T}_s$ is bounded:

$$(\mathbf{T}_s\cdot\mathbf{T}_s)^{\frac{1}{2}}=\frac{1}{V}\left[\left(\sum_{a\in V}\sum_{\mathbf{x}^{j,a}\in S_a}\mathbf{r}^{j_a}\otimes\mathbf{f}_s^{j_a}\right)\cdot\left(\sum_{b\in V}\sum_{\mathbf{x}^{i,b}\in S_b}\mathbf{r}^{i_b}\otimes\mathbf{f}_s^{i_b}\right)\right]^{\frac{1}{2}}$$
$$=\frac{1}{V}\left[\left(\sum_{a\in V}\sum_{\mathbf{x}^{j,a}\in S_a}\sum_{b\in V}\sum_{\mathbf{x}^{i,b}\in S_b}(\mathbf{r}^{j_a}\cdot\mathbf{r}^{i_b})\left(\mathbf{f}_s^{i_b}\cdot\mathbf{f}_s^{j_a}\right)\right)\right]^{\frac{1}{2}}$$
$$\leqslant\frac{2n}{V}\left|\mathbf{f}_s^{Max}\right||\mathbf{r}^{Max}|, \qquad (18)$$

where n is the number of contact in the RVE and $|\mathbf{r}^{Max}|=Max_|\mathbf{r}^{i_a}|$. Then, we conclude that, given a bounded pressure $p\leqslant 0$, every statically admissible macro stress must be bounded.

2.5. Virtual work

Since particles are rigid, according to the virtual work theorem, for every kinematically admissible system of particle displacements and for every equilibrated system of contact forces, the virtual work of all contact forces exerted on the particles of the RVE is null. Then it must be:

$$\sum_{\mathbf{x}^{e,i}\in S_v}\mathbf{f}^{e_i}\cdot\mathbf{u}(\mathbf{x}^{i,e})-\sum_{x^{j,a}\in V^0}(\mathbf{V}^{j_a}\cdot\gamma^{j_a}+\xi^{j,a}N^{j_a})=0, \qquad (19)$$

where the first summation is extended to all contacts $\mathbf{x}^{e,i}$ on the RVE boundary surface S_v and the second summation is extended to all inner contacts $\mathbf{x}^{j,a}\in V_0=V\backslash S_v$ (when in a contact no opening or sliding occurs, $\xi^{j,a}$ and $\gamma^{j,a}$ have null values).

Further, we assume that linear displacement conditions are imposed on the boundary of RVE, so that the displacement $\mathbf{u}(\mathbf{x}^{i,e})$ of a material point $\mathbf{x}^{i,e}$ on the RVE boundary surface S_v is given by:

$$\mathbf{u}(\mathbf{x}^{i,e})=\mathbf{u}_0+\mathbf{H}\mathbf{x}^{i,e}, \qquad (20)$$

where $\mathbf{H}$ is a given macro displacement gradient tensor and $\mathbf{u}_0$ is a translation vector. The macro strain $\mathbf{E}$ is then defined as the symmetric part of $\mathbf{H}$. In view of (9), the virtual work of the boundary contact forces (Nemat-Nasser and Hori, 1993) can be expressed as:

$$\sum_{\mathbf{x}^{i,e}\in S_v}\mathbf{f}^{e_i}\cdot\mathbf{u}(\mathbf{x}^{i,e})=\sum_{\mathbf{x}^{i,e}\in S_v}\mathbf{f}^{e_i}\cdot(\mathbf{u}_0+\mathbf{H}\mathbf{x}^{i,e})$$
$$=\left(\sum_{\mathbf{x}^{i,e}\in S_v}\mathbf{f}^{e_i}\otimes\mathbf{x}^{i,e}\right)\cdot\mathbf{H}=V\mathbf{T}^T\cdot\mathbf{H}$$
$$=V\mathbf{T}\cdot\left(\frac{\mathbf{H}+\mathbf{H}^T}{2}\right)=V\mathbf{T}\cdot\mathbf{E}, \qquad (21)$$

where it has been considered that the system of boundary contact forces is equilibrated and that the macro stress tensor $\mathbf{T}$ is symmetric. Finally, from (19) and (21) we find:

$$V\mathbf{T}\cdot\mathbf{E}=\sum_{x^{j,a}\in V_0}(\mathbf{V}^{j_a}\cdot\gamma^{j_a}+\xi^{j,a}N^{j_a}). \qquad (22)$$

[2] $\mathbf{A}\cdot\mathbf{B}=\sum_{i,j=1}^{3}a_{ij}b_{ij}$ denotes the inner product of two tensor and the Euclidean norm of $\mathbf{T}$ is equal to $(\mathbf{T}\cdot\mathbf{T})^{\frac{1}{2}}=\left(\sum_{i,j=1}^{3}\sigma_{ij}^2\right)^{\frac{1}{2}}$.

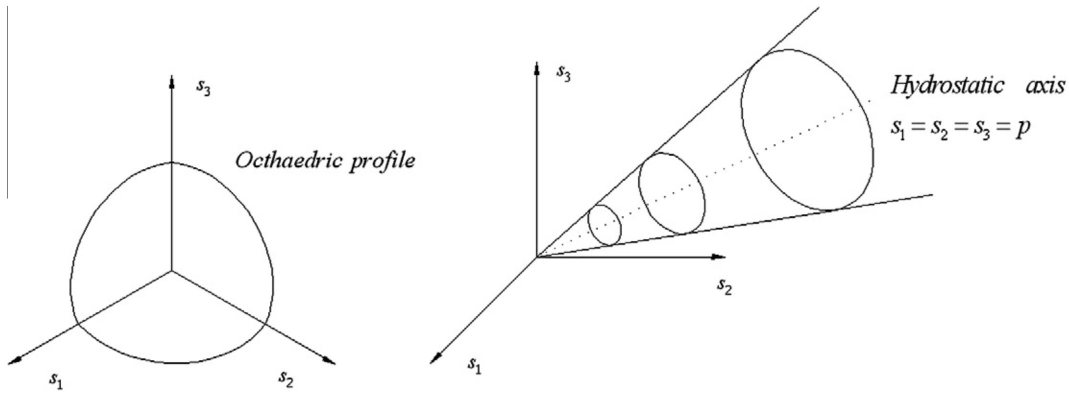

Fig. 3. A conic yield surface.

2.6. Yield surface. Flow rule

We denote as *rigid-plastic collapse state* a state in which under a constant macro stress $\mathbf{T}_c$ a macro strain $\mathbf{E}_c$ occurs.[3] More exactly, a collapse state is a statically admissible state $(\mathbf{T}_c, \mathbf{f}_c^{j_a})$ in which the material develops a kinematically admissible system of particle displacements $\left(\mathbf{u}_c^a, \mathbf{W}_c^a, \gamma_c^{j_a}, \xi_c^{j,a}\right)$ satisfying the contact force-displacement laws (4) and (5) and consistent with a macro strain $\mathbf{E}_c$.

In order to model a granular material as a rigid plastic material it must to be assumed that a collapse state is independent on the loading history and then a *yield function* $f(\mathbf{T})$ exists, such that the condition $f(\mathbf{T}_c) = 0$ holds for all collapse macro stress $\mathbf{T}_c$.[4] We denote as *yield surface* the locus of points of the macro stress space such that $f(\mathbf{T}_c) = 0$.

It is easy to verify that if $\left(-\mathbf{I} - \bar{\mathbf{S}}_c, -\bar{\mathbf{f}}_\mathbf{c}^{\mathbf{j_a}}, \mathbf{E_c}, \mathbf{u_c^a}, \mathbf{W_c^a}, \gamma_\mathbf{c}^{\mathbf{j_a}}, \xi_\mathbf{c}^{\mathbf{j,a}}\right)$ is a plastic collapse state in which the macro pressure is equal to $p = -1$, then also $\left(p_c\mathbf{I} + p_c\bar{\mathbf{S}}_c, p_c\bar{\mathbf{f}}_c^{j_a}, \mathbf{E}_c, \mathbf{u}^a, \mathbf{W}_c^a, \gamma_c^{j_a}, \xi_c^{j,a}\right)$, where $p_c \leqslant 0$, is a plastic collapse state. Therefore the yield surface must be a cone, with vertex at the origin of stress space.

Now, the initial plastic flow of this material is examined in two particular cases in which (12) holds. First, an ideal random packing of rigid convex particles, with null contact friction, is taken into account. A generic collapse state $\left(\mathbf{T}_c, N_c^{j_a}\mathbf{n}^{j_a}, \mathbf{E}_c, \mathbf{u}_c^a, \mathbf{W}_c^a, \gamma_c^{j_a}, \xi_c^{j,a}\right)$ and a generic statically admissible state $\left(\boldsymbol{T}_s, N_s^{j_a}\mathbf{n}^{j_a}\right)$ are considered. Since $\left(\mathbf{T}_c - \mathbf{T}_s, \left(N_c^{j_a} - N_s^{j_a}\right)\mathbf{n}^{j_a}\right)$ is an equilibrium state and the particle displacements $\left(\mathbf{u}_c^a, \mathbf{W}_c^a, \gamma_c^{j_a}, \xi_c^{j,a}\right)$ are consistent with the macro strain $\mathbf{E}_c$, from (22) it must be:

$$V(\mathbf{T}_c - \mathbf{T}_s) \cdot \mathbf{E}_c = \sum_{\mathbf{x}^{j,a} \in V_0} \left(N_c^{j_a} - N_s^{j_a}\right)\xi_c^{j,a}, \tag{23}$$

where, from (2) and (4): $N_c^{j_a}\xi_c^{j,a} = 0; N_s^{j_a} \leqslant 0$ and $\xi_c^{j,a} \geqslant 0$. Then we get the inequality:

$$V(\mathbf{T}_c - \mathbf{T}_s) \cdot \mathbf{E}_c = \sum_{\mathbf{x}^{j,a} \in V_0} \xi_c^{j,a}\left(-N_s^{j_a}\right) \geqslant 0, \tag{24}$$

which proves that the yield surface is a convex cone and that the plastic macro strain $\mathbf{E}_c$ is associated.

In Fig. 3, in the particular case of an isotropic granular material, a convex conic yield surface and an intersection of this surface with a deviatoric plane of equation $s_1 + s_2 + s_3 = p$, here denoted as octhaedric profile, are shown in the principal stress space.

It can be shown that an ideal frictionless granular material can react to a not null macro stress only if its microstructure is such that to produce a collapse dilatant behavior ($\varepsilon_c^V \geqslant 0$).[5] In order to prove this property, let us consider a generic collapse state $\left(p_c\mathbf{I} + \mathbf{S}_c, \mathbf{f}_c^{j_a}, \varepsilon_c^V\mathbf{I}/3 + \mathbf{E}_c^D, \mathbf{u}_c^a, \mathbf{W}_c^a, \gamma_c^{j_a}, \xi_c^{j,a}\right)$ and a statically admissible macro stress $\left(p_c\mathbf{I}, N_s^{j_a}\mathbf{n}_s^{j_a}\right)$. From 24 we find that the in the collapse state the work done by the deviatoric part of stress is always non negative: $\mathbf{S}_c \cdot \mathbf{E}_c^D \geqslant 0$. Further, by setting ($\mathbf{T}_s = \mathbf{0}$, $N_s^{j_a}\mathbf{n}_s^{j_a} = \mathbf{0}$), we find that the total work in the collapse state is always null: $p_c\varepsilon_c^V + \mathbf{S}_c \cdot \mathbf{E}_c^D = 0$. Therefore in a collapse state the internal work done by the hydrostatic part of the stress is always non positive: $p_c\varepsilon_c^V \leqslant 0$. Since $p_c \leqslant 0$, the volumetric collapse strain ε_c^V must be non negative.

Next, the particular case of a random packing of frictional rigid spherical particles of equal diameters and with equal friction coefficients is examined. For every rigid-plastic collapse state and for every statically admissible state, the following identity holds:

$$V(\mathbf{T}_c - \mathbf{T}_s) \cdot \mathbf{E}_c = \sum_{x^{j,a} \in V_0} \left[\left(\mathbf{V}_c^{j_a} - \mathbf{V}_s^{j_a}\right) \cdot \gamma_c^{j_a} - \xi_c^{j,a}N_s^{j_a}\right]. \tag{25}$$

In order to discuss the consequences of Eq. (25) we first note that, from the local contact force displacement laws (4) and (5), we have:

$$\begin{aligned} &\mathbf{V}_s^{j_a} \cdot \gamma_c^{j_a} = V_s^{j_a}\gamma_c^{j,a}\cos(\vartheta_s^{j_a}) = \rho^{j_a}\mu\left(-N_s^{j_a}\right)\gamma_c^{j,a}, \\ &\rho^{j_a} = \cos\left(\vartheta_s^{j_a}\right)\frac{V_s^{j_a}}{\mu\left(-N_s^{j_a}\right)}; \quad -1 \leqslant \rho^{j_a} \leqslant 1, \end{aligned} \tag{26}$$

where $\vartheta_s^{j_a}$ is the angle between the statically admissible shear contact force $\mathbf{V}_s^{j_a}$ and the sliding contact displacement $\gamma_c^{j_a}$. Further:

$$\mathbf{V}_c^{j_a} \cdot \gamma_c^{j_a} = V_c^{j_a}\gamma_c^{j,a} = \mu\left(-N_c^{j_a}\right)\gamma_c^{j,a}. \tag{27}$$

Eqs. (26) and (27) show that, if equal values of the normal contact force $\left(N_c^{j_a} = N_s^{j_a}\right)$ are applied, the virtual work done by a statically admissible contact shear force $\mathbf{V}_s^{j_a}$ is always less or equal than the work done by the collapse shear force $\mathbf{V}_c^{j_a}$ that produces the sliding displacement $\gamma_c^{j_a}$.

Now, let us consider a generic collapse state $(p_c\mathbf{I} + \mathbf{S}_c, \mathbf{f}_c^{j_a}, \varepsilon_c^V\mathbf{I}/3 + \mathbf{E}_c^D, \mathbf{u}_c^a, \mathbf{W}_c^a, \gamma_c^{j_a}, \xi_c^{j,a})$ and a generic statically admissible macro stress $(p_c\mathbf{I} + \mathbf{S}_s, \mathbf{f}_s^{j_a})$ with the same hydrostatic stress $p_c\mathbf{I}$. From (25)–(27) we get:

$$V(\mathbf{S}_c - \mathbf{S}_s) \cdot \mathbf{E}_c^D = \mu\sum_{\mathbf{x}^{j,a} \in V_0} \gamma_c^{j,a}\left(\rho_s^{j_a}N_s^{j_a} - N_c^{j_a}\right) + \sum_{\mathbf{x}^{j,a} \in V_0} \xi_c^{j,a}\left(-N_s^{j_a}\right). \tag{28}$$

In this case the signs of the virtual works $\left(\mathbf{V}_c^{j_a} - \mathbf{V}_s^{j_a}\right) \cdot \gamma_c^{j_a} = \mu\left(\rho_s^{j_a}N_s^{j_a} - N_c^{j_a}\right)\gamma_c^{j,a}$ are not known. However, we

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[3] The subscript c refers to any quantity related to a collapse state.

[4] For every statically admissible macrostress $\mathbf{T}_s$ it must be $f(\mathbf{T}_s) \leqslant 0$, while for every inadmissible macrostress $\mathbf{T}$ it must be $f(\mathbf{T}) > 0$.

[5] $\varepsilon_c^V = tr(\mathbf{E}_c)/3$ is the volumetric collapse strain and $\mathbf{E}_c^D = \mathbf{E}_c - \varepsilon_c^V\mathbf{I}$ is the deviatoric component of the collapse strain.

can consider that there is not a direct relation between the sliding contact displacements $\gamma_c^{j,a}$ and the quantities $\mu\left(\rho_s^{j_a}N_s^{j_a} - N_c^{j_a}\right)$.

To this regard, it can be noted that there is not a direct relation between the statically admissible contact shear forces $\mathbf{V}_s^{j_a}$ and the sliding collapse displacements $\boldsymbol{\gamma}_c^{j_a}$, because these quantities are related to different states of the material. Further, when a not null sliding displacement $\boldsymbol{\gamma}_c^{j_a}$ occurs, its magnitude $\gamma_c^{j,a}$ depends on the interaction of the examined particle with the neighbouring particles and on the kinematic conditions imposed on the RVE, but it is not directly depending on the magnitude $\mu\left(-N_c^{j_a}\right)$ of the shear contact force $\mathbf{V}_c^{j_a}$. Then the assumption that the amplitudes $\gamma_c^{j,a}$ of the sliding contact displacements are statistically independent or poorly correlated to the quantities $\mu\left(\rho_s^{j_a}N_s^{j_a} - N_c^{j_a}\right)$ can be accepted as an approximation of the real behavior of this material. Under this assumption, as it will be shown, it is possible to assert that also in this case the virtual work $(\mathbf{S}_c - \mathbf{S}_s)\cdot\mathbf{E}_c^D$ is positive.

First, according to (11), we can note that the expected value of the normal contact forces is proportional to the mean stress p:

$$3pV = \sum_{\mathbf{x}^{j,a}\in V} \mathbf{l}^{j_a}\cdot\mathbf{f}^{j_a} = \sum_{\mathbf{x}^{j,a}\in V} \mathbf{l}^{j_a}\cdot N^{j_a}\mathbf{n}^{j_a} = l\sum_{\mathbf{x}^{j,a}\in V} N^{j_a}, \tag{29}$$

from which:

$$E[N] = \frac{1}{n}\sum_{\mathbf{x}^{j,a}\in V} N^{j_a} = \frac{3pV}{nl}$$

where it has been considered that the number n of contact in the RVE is very large and then the expected value $E[N]$ of the contact normal forces is equal to their arithmetic average. Further, we have chosen a collapse state and a statically admissible state with the same pressure p, then it must be:

$$\begin{aligned} E[\rho_s N_s - N_c] &= \frac{1}{n}\sum_{\mathbf{x}^{j,a}\in V}\left(\rho_s^{j_a}N_s^{j_a} - N_c^{j_a}\right) = \frac{3pV}{nl}(\bar{\rho}_s - 1) \\ &\geqslant 0 \quad -1 \leqslant \bar{\rho}_s \leqslant 1. \end{aligned} \tag{30}$$

Next, under the assumption that the sliding contact displacements $\gamma_c^{j,a}$ are statistically independent or poorly correlated to the quantities $\left(\rho_s^{j_a}N_s^{j_a} - N_c^{j_a}\right)$, the expected value of their product can be confused with the product of their expected values and we can write:

$$\begin{aligned} \mu\sum_{\mathbf{x}^{j,a}\in V_0}\gamma_c^{j,a}\left(\rho_s^{j_a}N_s^{j_a} - N_c^{j_a}\right) &= n_0\mu E[\gamma_c(\rho_s N_s - N_c)] \\ &\simeq n_0\mu E[\gamma_c]E[(\rho_s N_s - N_c)] \geqslant 0, \end{aligned} \tag{31}$$

where n_0 is the number of inner contact in the RVE and where it has been considered that:

$$E[\gamma_c] = \frac{1}{n}\sum_{\mathbf{x}^{j,a}\in V}\gamma_c^{j,a} > 0. \tag{32}$$

Finally, (28) can be written as:

$$V(\mathbf{S}_s - \mathbf{S}_c)\cdot\mathbf{E}_c^D \simeq n_0\mu E[\gamma_c]E[(\rho_s N_s - N_c)] + \sum_{\mathbf{x}^{j,a}\in V_0}\xi_c^{j,a}\left(-N_s^{j_a}\right) \tag{33}$$

and, from (2), (4) and (31) we get the inequality:

$$(\mathbf{S}_c - \mathbf{S}_s)\cdot\mathbf{E}_c^D \geqslant 0, \tag{34}$$

that allows us to establish that the conic yield surface is convex and that the deviatoric plastic macro strain $\mathbf{E}_c^D$ is associated. Then the following relations hold:

$$\begin{cases} \forall\Delta\mathbf{S}\neq\mathbf{0}: \quad \Delta\mathbf{S}\cdot\frac{\partial^2}{\partial\mathbf{S}\partial\mathbf{S}}f_{(\mathbf{T}_c)}[\Delta\mathbf{S}] > 0, \\ \mathbf{E}_c^D = \kappa\frac{\partial}{\partial\mathbf{S}}f(\mathbf{T}_c); \quad \kappa > 0. \end{cases} \tag{35}$$

Now, we want to show that in a general frictional granular material the volumetric plastic macrostrain cannot be associated. To this end, let us consider (Fig. 4) the intersection of the yield surface with a plane of equation $\mathbf{T} = p\mathbf{I} + \lambda\widetilde{\mathbf{S}}$, where $\widetilde{\mathbf{S}}$ is a fixed unit deviatoric tensor ($\widetilde{\mathbf{S}}\cdot\widetilde{\mathbf{S}} = 1$). This intersection is here denoted as meridian profile. Since the yield surface is a cone, the meridian profile is composed by two rays, whose equations are $\lambda_c = (-p_c)\tan\varphi_{\widetilde{\mathbf{S}}} > 0$ and $\lambda_c = p_c\tan\varphi_{(-\widetilde{\mathbf{S}})} < 0$.

Now, let us consider a plastic collapse state $\left(p_c\mathbf{I} - p_c\tan\varphi_{\widetilde{\mathbf{S}}}\widetilde{\mathbf{S}}, p_c\bar{\mathbf{f}}_c^{j_a}, \varepsilon_c^V\mathbf{I}/3 + \mathbf{E}_c^D, \mathbf{u}_c^a, \mathbf{W}_c^a, \gamma_c^{j_a}, \xi_c^{j,a}\right)$, where $p_c < 0$. According to the Virtual Work Theorem, we find:

$$\begin{aligned} &V(p_c\mathbf{I} - p_c\tan\varphi_{\widetilde{\mathbf{S}}}\widetilde{\mathbf{S}})\cdot\left(\frac{\varepsilon_c^V}{3}\mathbf{I} + \mathbf{E}_c^D\right) \\ &\quad = Vp_c\left(\varepsilon_c^V - \tan\varphi_{\widetilde{\mathbf{S}}}\left(\widetilde{\mathbf{S}}\cdot\mathbf{E}_c^D\right)\right) = \sum_{x^{j,a}\in V_0} p_c\overline{\mathbf{V}}_c^{j_a}\cdot\boldsymbol{\gamma}_c^{j,a} \geqslant 0. \end{aligned} \tag{36}$$

It is worth noting that if we assume that besides the above micro state a different micro state, corresponding to the same macro state, $\left(\mathbf{f}_c^{\prime j_a}, \mathbf{u}_c^{\prime a}, \mathbf{W}_c^{\prime a}, \gamma_c^{\prime j_a}, \xi_c^{\prime j,a}\right)$ exists, according to the Virtual Work Theorem, we find:

$$\sum_{x^{j,a}\in V_0} p_c\overline{\mathbf{V}}_c^{j_a}\cdot\boldsymbol{\gamma}_c^{j,a} = \sum_{x^{j,a}\in V_0}\mathbf{V}_c^{\prime j_a}\cdot\boldsymbol{\gamma}_c^{\prime j,a}. \tag{37}$$

Further, dividing (36) by $|p_c|$ we get:

$$-\varepsilon_c^V + \tan\Phi_{\widetilde{\mathbf{S}}}\varepsilon_c^{\widetilde{\mathbf{S}}} = \frac{1}{V}\sum_{x^{j,a}\in V_0}\left(-\overline{\mathbf{V}}_c^{j_a}\right)\cdot\boldsymbol{\gamma}_c^{j,a} \geqslant 0, \tag{38}$$

where $\varepsilon_c^{\widetilde{\mathbf{S}}} = \widetilde{\mathbf{S}}\cdot\mathbf{E}_c^D$ is the component of the deviatoric plastic macro strain in the given meridian plane. It easy to note that the inequality (38) is strictly satisfied only when friction is null and, further, Fig. 4 shows us that the component of the plastic macro strain in the given meridian plane is normal to the meridian profile only in this last case. Then, inequality (38) shows us that in a frictional material the plastic macro strain cannot be associated.

Now, we consider a collapse state for which $\varepsilon_c^{\widetilde{\mathbf{S}}} > 0$. If we define the dilatancy coefficient $d = \varepsilon_c^V/\varepsilon_{\widetilde{\mathbf{S}}}$ as the ratio of the volumetric strain over the component $\varepsilon_c^{\widetilde{\mathbf{S}}}$ of the deviatoric plastic macro strain in the actual meridian plane, from (38) we get the inequality:

$$\tan\Phi_{\widetilde{\mathbf{S}}} - d \geqslant 0, \tag{39}$$

that extends to a general triaxial stress state the well known constitutive inequality first proposed by Taylor (1948).

3. Conclusions

A micromechanical modeling of a granular material consisting of a random package of rigid particles has been developed in order to establish a conceptual link between the general properties of macroscopic plasticity of granular materials, the frictional behavior

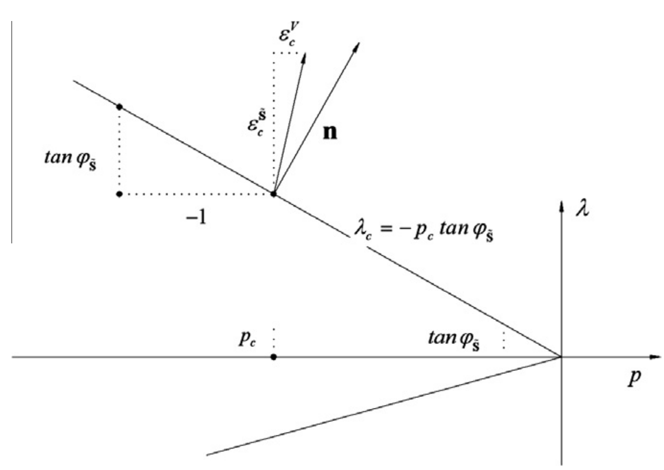

Fig. 4. A mendian profile of a conic yield surface.

of contacts and the local equilibrium of particles. A RVE containing a very large number of particles with prescribed linear displacement boundary conditions has been examined and the general properties of the effective rigid plastic constitutive behavior of this material has been determined. The model has allowed us to deduce many essential features of the plastic behavior of general granular materials when no contact damages or particles crushing are produced and no relevant changes in the material microstructure occur until the yielding. It has been shown that the yield surface of a random packing of frictional rigid particles is a cone and that the plastic macro strain cannot be associated, while it is associated in case of frictionless particles. The model is relatively simple and based on clear theoretical motivations.

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