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Elastic behavior of composites containing multi-layer coated particles with imperfect interface bonding conditions and application to size effects and mismatch in these composites

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ABSTRACT

This paper proposes a procedure to deal with n-layered inclusion based composites with imperfect interfaces (which conditions consist of displacement or stress vector jumps) respecting spherical symmetry. For that purpose, "discontinuity matrices" have been introduced. These matrices have been derived for several classical interface-models and an asymptotic method has been used to determine some of them. A self-consistent condition based on a strain-energy equivalence in the case of inclusion-matrix type composite materials is restated for n-layered inclusions with imperfect interfaces and applied to get estimates of such composites materials. The remarkable feature of the presently self consistent approach is that it does not need any tedious algebra providing the attached interface models respect the spherical symmetry. The present Generalized Self Consistent Model (GSCM) is then used to study size effects and mismatch in composites reinforced by coated inclusions.

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1. Introduction

The present is motivated by the need for a better understanding of the role of interfaces on the elastic behavior of multi-layered inclusions-matrix type composites with imperfect interfaces. At a perfect interface the displacement vector and the stress vector is classically considered as continuous. When these vectors are no longer continuous across the interface, these interfaces are called *imperfect*. In literature, the "behavior" of these imperfect interfaces have been described by either interface models which directly give the discontinuity of the displacement vectors or of the stress vector present at the interface or, by asymptotic methods which transform thin interphases in an *interface-model*. It is the purpose of Benveniste and Miloh (2001) who have studied the influence of a constant-thicknessed layer between two elastic isotropic media and have used an asymptotic expansion for the elastic field in this interphase to exhibit seven distinct regimes of interface conditions. All theses seven different regimes have not been yet introduced in a Generalized Self Consistent framework. However the Spring-type interface has been studied by Hashin (1991) who established the link between the parameters of a spring-type interface model and the properties of an isotropic soft and thin interphase between two media. In addition, Hashin (2002) has considered thin interphases but with no more restriction on the magnitude of the properties of this interphase regarding the properties of the two abutting phases and only plane and cylindrical interfaces have been analyzed as special cases (no spherical interfaces have been studied). Wang et al. (2005) and Benveniste (2006) have shown that a thin and stiff interphase is equivalent to an interface which displacement/stress discontinuities are described by the so-called generalized Young–Laplace equations (Povstenko, 1993; Le Quang and He, 2008). Gurtin and Murdoch's model (Gurtin and Murdoch, 1975; Kushch et al., 2011), compared to this generalized Young–Laplace model, considers an extra term in the interface stress which depends on the surface gradient of displacement. This model has not been considered in this paper. In Le Quang and He (2008), Le Quang and He provide first-order upper and lower bounds for the effective elastic moduli of such composite materials and, Brisard et al. (2010a) and Brisard et al. (2010b) Hashin–Shtrikman bounds.

Wang et al. (2005) and Duan et al. (2005a) have determined the effective elastic behavior of solids containing inclusions with discontinuity in the tractions across the interface between their inclusions and the matrix around but no extension have been carried out for n-layered inclusion-based composites. For that purpose they have used three micromechanicals models (Duan et al.,

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2005c), Hashin's Composite Spheres Assemblage (CSA), the Mori–Tanaka method (MTM) and the Generalized Self-Consistent Method (GSCM) and have predicted size-dependent effective behavior of nano-composites.

Marcadon et al. (2007) and Zaoui et al. (2006) have in the meantime improved micromechanicals models derived from the so-called "Morphological Representative Pattern (MRP) approach (Bornert et al., 1996) to predict size effects in nanoparticle-reinforced. For that purpose, they have considered that either the distance between the nearest inclusion decreases leading to the stiffening of a part of the matrix (confined matrix) or, for a fixed volume fraction of the inclusion the particle diameter can change while the thickness of a disturbed matrix around the inclusion is almost unchanged.

The aim of this paper is to present a method that can predict the effective elastic constants of composites containing multi-layer coated particles with imperfect bonding conditions. For this purpose, the (n + 1)-phase model of Hervé and Zaoui (1993) is extended to the case of imperfect interfaces where the imperfect bonding conditions are expressed thanks to "discontinuity matrices" and where transfert matrices are still used.

This paper is organized as follows. The main result is the derivation of the elastic stress and strain fields in an infinite medium constituted of an n-layered isotropic spherical inclusion embedded in a matrix subjected to uniform stress or strain conditions at infinity and where the behavior of the imperfect interfaces are described by two "discontinuity matrices". This derivation is presented in Section 2. These "discontinuity matrices" characterizing the bonding conditions fulfilled at a given interface are expressed in the case of the linear spring model, the dislocation-like model and in the case of the presence of a thin interphase replaced by an interface. It is shown that in the case of a very thin interphase the explicit forms of these "discontinuity matrices" can emerge equivalently from two asymptotic methods: one using transfert matrices and another one using Hashin's procedure. In Section 3 a link is made between the problem of predicting the effective behavior of composites containing n-layered spherical inclusions with imperfect interfaces and a n-layered spherical problem as the one presented in Section 2. This is an extension of the work of Marcadon et al. (2007) to n-layered spherical inclusion-reinforced composites with imperfect interfaces. The two "discontinuity matrices" attached to very thin soft interphases or rigid ones are given and compared for the latter case to the ones attached to generalized Young Laplace Conditions. Section 4 shows that the classical self-consistent energy condition given first by Christensen (1979) is still valid in this particular context of imperfect interfaces. Finally, in Section 5 some illustrative examples are given.

2. n-layered spherical inclusion with imperfect interfaces, embedded in an infinite matrix

This section is concerned with the derivation of the elastic strain and stress fields in an infinite medium constituted of a n-layered spherical inclusion, embedded in a matrix subjected to uniform stress conditions ($\vec{T}^0 = \boldsymbol{\sigma}^0\vec{n}$) or strain conditions ($\vec{u}^0 = \boldsymbol{\varepsilon}^0\vec{x}$) at infinity, where $\boldsymbol{\sigma}^0$ and $\boldsymbol{\varepsilon}^0$ denote constant tensors, $\vec{x}$ is the position vector and $\vec{n}$ denotes the unit normal of the considered surface oriented from the inside of the inhomogeneous inclusion towards its outside.

Each phase is assumed to be homogeneous and linearly elastic. The interfaces between the different phases can be perfect or not. In this last case, abutting phases are imperfectly bonded and the interface energy is no more negligible compared to the bulk energy. Different cases of imperfect interfaces are dealt with in this paper: the traction vector across the interfaces can be assumed to be continuous while the displacement vector at the same place can suffer a jump, or the displacement vector across the interfaces can be assumed to be continuous while the traction vector is discontinuous. In order to solve all these problems, a general procedure is proposed. This procedure uses "discontinuity matrices" to represent the interface models attached to imperfect interfaces.

Throughout the following a second-order tensor will be denoted by bold letters and a fourth-order one by calligraphic letters. Moreover, Einstein's convention of summation over repeated indices will not be adopted. Let $\boldsymbol{\sigma}$ and $\boldsymbol{\varepsilon}$ be respectively the Cauchy stress tensor and infinitesimal strain tensors. The letter (i) will be use to denote a generic phase of the n-layered spherical inclusion.

Let phase (1) constitute the central core and phase (i) lie within the shell limited by the spheres with the radii R_{i-1} and R_i $(i \in [1, n+1],\ R_{n+1} \to \infty)$ (Fig. 1(a)). The phase, referred to as phase $(n+1)$ denotes here the matrix and will represent in the next section the unknown equivalent homogeneous medium (EHM).

The linear elastic behavior of each phase (i) is characterized by Hooke's law:

$$\boldsymbol{\sigma}^{(i)} = \mathcal{L}^{(i)} : \boldsymbol{\epsilon}^{(i)} \quad \text{or} \quad \boldsymbol{\epsilon}^{(i)} = \mathcal{M}^{(i)} : \boldsymbol{\sigma}^{(i)}, \tag{1}$$

where $\boldsymbol{\sigma}^{(i)}, \boldsymbol{\varepsilon}^{(i)}$ are respectively the Cauchy stress tensor and infinitesimal strain tensors of phase (i), $\mathcal{L}^{(i)}$ and $\mathcal{M}^{(i)}$ stand for the elastic stiffness and compliance tensors of phase (i) and the summation over two indices is denoted by two points $(:)$.

Note that all the considered phases are assumed to be isotropic and thus all these elastic stiffness and compliance can be written in the following form:

$$\mathcal{L}^{(i)} = 3k_i\mathcal{J} + 2\mu_i\mathcal{K} \quad \text{and} \quad \mathcal{M}^{(i)} = \frac{1}{3k_i}\mathcal{J} + \frac{1}{2\mu_i}\mathcal{K} \tag{2}$$

where $(\mu_i,\ k_i)$ are respectively the shear modulus and bulk modulus of phase (i).

Let v_i be the Poisson's ratio of phase (i), $i \in [1, n+1],\ R_{n+1} \to \infty$ (Fig. 1(a)). $\mathcal{J} = \frac{1}{3}\boldsymbol{I} \otimes \boldsymbol{I}$, $\mathcal{K} = \mathcal{I} - \mathcal{J}$ with $\boldsymbol{I}$ and $\mathcal{I}$ being respectively the second-order and fourth-order identity tensor.

It is worth noting that the imposed conditions at infinity ($\vec{T}^0 = \boldsymbol{\sigma}^0\vec{n}$) or ($\vec{u}^0 = \boldsymbol{\varepsilon}^0\vec{x}$) can be written as:

$$\left.\begin{array}{l} \vec{T}^0 = (\mathcal{J} + \mathcal{K}) : \boldsymbol{\sigma}^0\vec{n} = \left(\frac{1}{3}Tr\boldsymbol{\sigma}^0 + \mathbf{s}_0\right)\vec{n} \\ \text{or} \\ \vec{u}^0 = (\mathcal{J} + \mathcal{K}) : \boldsymbol{\varepsilon}^0\vec{x} = \left(\frac{1}{3}Tr\boldsymbol{\varepsilon}^0 + \mathbf{e}_0\right)\vec{x} \end{array}\right\} \tag{3}$$

where $\boldsymbol{s}_0$ and $\boldsymbol{e}_0$ denote respectively the deviatoric parts of $\boldsymbol{\sigma}^0$ and $\boldsymbol{\varepsilon}^0$. Consequently, the general solution of such an n-layered spherical inclusion embedded in an infinite matrix and subjected to uniform conditions may be obtained from the solution of two elementary problems: hydrostatic pressure and simple shear applied at infinity (Hervé and Zaoui, 1993).

The interface bonding conditions between phase (i) and phase $(i+1)$ (interface denoted as $\Gamma^{(i)}$ for $r = R_i$) corresponding to a chosen interface model are assumed to be described by a "discontinuity matrix" defined from the solutions of the equilibrium equations.

We shall use a spherical (r, θ, ϕ) coordinate system with the origin at the center of the above-mentioned n-layered sphere. Let $[\cdot]$ denote the interfacial jump in the quantity under consideration.

The interface energy $E^{int}(R_i)$ at the Γ_i interface is defined as:

$$\left.\begin{array}{ll} \frac{1}{2}\int\int_{\Gamma_i} \vec{u}\,[\boldsymbol{\sigma}]\,\vec{n}\,dS, & \text{(jump of the traction vector)} \\ \frac{1}{2}\int\int_{\Gamma_i} [\vec{u}]\,\boldsymbol{\sigma}\,\vec{n}\,dS, & \text{(jump of the displacement vector)} \end{array}\right\} \tag{4}$$

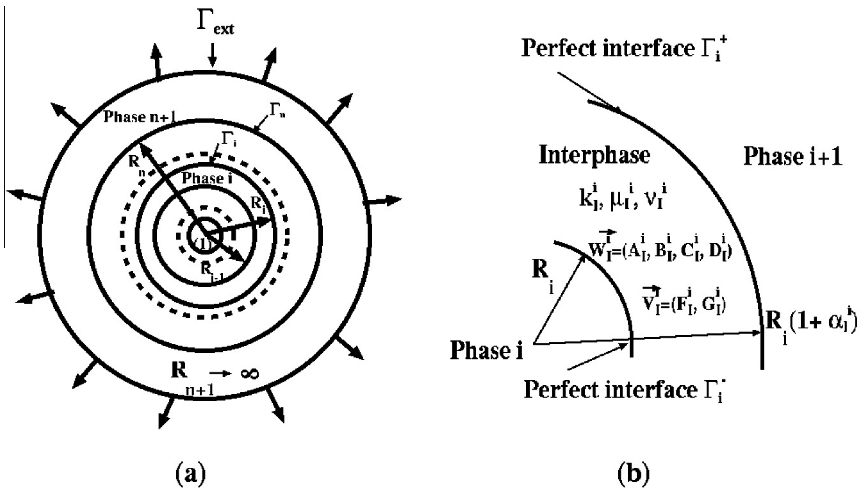

Fig. 1. (a) The n-layered spherical inclusion and its interfaces. (b) Thin interphase between phase (i) and phase $(i+1)$.

2.1. Hydrostatic pressure

One of the two following uniform conditions is applied at infinity:

$$\left.\begin{array}{l}\vec{T}^0 = \frac{\sigma_0}{3}\vec{e}_r \\ \text{or} \\ \vec{u}^0 = r\frac{\theta_0}{3}\vec{e}_r\end{array}\right\} \tag{5}$$

where σ_0 and θ_0 are constant and $\vec{e}_r$ is the first vector of the spherical coordinate system.

The solution for the non-zero displacement component u_r, of the single equilibrium equation is given in phase (i) Love (1944) by,

$$u_r^{(i)} = F_i r + \frac{G_i}{r^2} \tag{6}$$

where F_i and G_i are constants.

The corresponding stresses in phase (i) are found to be (Hervé and Zaoui, 1993):

$$\left.\begin{array}{l}\sigma_{rr}^{(i)}(r) = 3k_iF_i - \frac{4\mu_i}{r^3}G_i \\ \sigma_{\theta\theta}^{(i)}(r) = \sigma_{\phi\phi}^{(i)}(r) = 3k_iF_i + \frac{2\mu_i}{r^3}G_i \\ \sigma_{r\theta}^{(i)} = \sigma_{r\phi}^{(i)} = \sigma_{\theta\phi}^{(i)} = 0\end{array}\right\} \tag{7}$$

Let us introduce the (2×1) following matrices:

$$\left.\begin{array}{l}\vec{t}(r) = [u_r(r),\ \sigma_{rr}(r)]^T \\ \vec{t}^{(i)} = \left[u_r^{(i)}(r),\ \sigma_{rr}^{(i)}(r)\right]^T\end{array}\right\} \tag{8}$$

expressed in a canonical base $(\vec{e}_1, \vec{e}_2)$. $\vec{t}^{(i)}(r)$ is then given in phase (i) by:

$$\vec{t}^{(i)}(r) = \left[\left(F_i r + \frac{G_i}{r^2}\right),\ \left(3k_iF_i - \frac{4\mu_i}{r^3}G_i\right)\right]^T = \mathbf{J}^{(i)}(r)\,\vec{V}_i \tag{9}$$

where $\vec{V}_i = (F_i,\ G_i)^T$ and where $\mathbf{J}^i(r)$ is the following matrix:

$$\mathbf{J}^{(i)}(r) = \begin{bmatrix} r & \frac{1}{r^2} \\ 3k_i & -4\frac{\mu_i}{r^3}\end{bmatrix} \tag{10}$$

We now consider all the interface bonding conditions which can be written in the following manner:

$$[\vec{t}]_{R_i} = \vec{t}^{(i+1)}(R_i) - \vec{t}^{(i)}(R_i) = \Delta\mathbf{J}(R_i)\,\vec{V}_i = \left(\delta\mathbf{J}(R_i)\,\mathbf{J}^{(i)}(R_i)\right)\vec{V}_i \tag{11}$$

where $[\vec{t}]$ denotes the interfacial jump of $\vec{t}$ at $r = R_i$ (surface Γ_i).

$\delta\mathbf{J}(R_i)$ is called the "pressure discontinuity matrix" on Γ_i.

The interface conditions, taking into account the interface bonding conditions (9) and then (11) can be written as:

$$\left.\begin{array}{l}\vec{V}_{i+1} = \left(\mathbf{J}^{(i+1)}\right)^{-1}(R_i)\left[\mathbf{J}^{(i)}(r)\,\vec{V}_i + [\vec{t}\,]_{R_i}\right] \\ \vec{V}_{i+1} = \left(\mathbf{J}^{(i+1)}\right)^{-1}(R_i)\left[\mathbf{J}^{(i)}(R_i) + \Delta\mathbf{J}^{(i)}(R_i)\right]\vec{V}_i \\ \vec{V}_{i+1} = \left(\mathbf{N}^{(i)} + \Delta\mathbf{N}(R_i)\right)\vec{V}_i \\ \vec{V}_{i+1} = \mathbf{N}^{S(i)}\vec{V}_i = \left(\prod_{j=i}^{1}\mathbf{N}^{S(j)}\right)\vec{V}_1 = \mathbf{Q}^{S(i)}\vec{V}_1\end{array}\right\} \tag{12}$$

with:

$$\left.\begin{array}{l}\mathbf{N}^{(i)} = \left(\mathbf{J}^{(i+1)}\right)^{-1}(R_i)\,\mathbf{J}^{(i)}(R_i) \\ \Delta\mathbf{N}(R_i) = \left(\mathbf{J}^{(i+1)}\right)^{-1}(R_i)\,\delta\mathbf{J}^{(i)}(R_i)\,\mathbf{J}^{(i)}(R_i) \\ \mathbf{N}^{S(i)} = \left(\mathbf{N}^{(i)} + \Delta\mathbf{N}(R_i)\right) \\ \mathbf{Q}^{S(i)} = \left(\prod_{j=i}^{1}\mathbf{N}^{S(j)}\right)\end{array}\right\} \tag{13}$$

$\delta\mathbf{J}^{(i)}(R_i)$ and consequently $\Delta\mathbf{J}^{(i)}(R_i)$ depend on the chosen interface model (see different expressions of $\delta\mathbf{J}^{(i)}(R_i)$ for several interface models in Section 2.3).

All the coefficients $(F_i,\ G_i)$ can be determined by using (12) and by taking into account, on the one hand that the coefficient G_1 must vanish in order to avoid a singularity at the origin and, on the other hand, that the constant F_{n+1} is determined by the applied state of hydrostatic pressure at infinity (as in Hervé and Zaoui (1993)) and expressed as:

$$\left.\begin{array}{l}F_{n+1} = \frac{\sigma_0}{9k_{n+1}}\ \text{ in the stress approach} \\ F_{n+1} = \frac{\theta_0}{3}\ \text{ in the displacement approach}\end{array}\right\} \tag{14}$$

The coefficients $(F_i,\ G_i)$ are then given by:

$$\left.\begin{array}{l}F_i = \frac{Q_{11}^{S(i-1)}}{Q_{11}^{S(n)}}F_{n+1} \\ \qquad\qquad i\in[1,n] \\ G_i = \frac{Q_{21}^{S(i-1)}}{Q_{11}^{S(n)}}F_{n+1}\end{array}\right\} \tag{15}$$

The interface energy $E^{int}(R_i)$ at the Γ_i interface can be expressed as:

$$E^{int}(R_i) = 2\pi R_i^2\, \vec{V}_i\, \Delta\mathbf{E}\, \vec{V}_i \tag{16}$$

with

$$\Delta\mathbf{E} = \left[\overrightarrow{\Delta\mathbf{J}_2}(R_i) \otimes \overrightarrow{\mathbf{J}_1^{(i)}}(R_i) + \overrightarrow{\Delta\mathbf{J}_1}(R_i) \otimes \overrightarrow{\mathbf{J}_2^{(i)}}(R_i)\right] \tag{17}$$

and where $\overrightarrow{\Delta\mathbf{J}_p}(R_i)$ and $\overrightarrow{\mathbf{J}_p^{(i)}}(R_i)$ denote respectively the (2×1) matrices made of the pth row of $\Delta\mathbf{J}(R_i)$ and of $\mathbf{J}^{(i)}(R_i)$. It is worth noting that either $\overrightarrow{\Delta\mathbf{J}_1}(R_i) = 0$ ($[\vec{u}] = \vec{o}$) or $\overrightarrow{\Delta\mathbf{J}_2}(R_i) = 0$ ($[\boldsymbol{\sigma}]\,\vec{n} = \vec{o}$).

2.2. Simple shear

Following Hervé and Zaoui (1993), the boundary conditions imposed at infinity and written in the spherical coordinate system (r, θ, ϕ) are in the displacement approach expressed as:

$$\left.\begin{aligned} u_r^0 &= \gamma r \sin^2\theta \cos 2\phi \\ u_\theta^0 &= \gamma r \sin\theta\cos\theta\cos 2\phi \\ u_\phi^0 &= -\gamma r \sin\theta \sin 2\phi \end{aligned}\right\} \tag{18}$$

and in the stress approach as:

$$\left.\begin{aligned} T_r^0 &= \tau \sin^2\theta\cos 2\phi \\ T_\theta^0 &= \tau\sin\theta\cos\theta\cos 2\phi \\ T_\phi^0 &= \tau\sin\theta\sin 2\phi \end{aligned}\right\} \tag{19}$$

In phase (i), the components of displacement field $\vec{u}$ in the spherical coordinate system has the following form Love (1944) after the resolution of the equilibrium equations:

$$\left.\begin{aligned} u_r^{(i)}(r,\theta,\phi) &= \left(A_i r - \tfrac{-6\nu_i}{1-2\nu_i} B_i r^3 + \tfrac{3C_i}{r^4} + \tfrac{5-4\nu_i}{1-2\nu_i}\tfrac{D_i}{r^2}\right)\sin^2\theta\cos 2\Phi \\ u_\theta^{(i)}(r,\theta,\phi) &= \left(A_i r - \tfrac{7-4\nu_i}{1-2\nu_i} B_i r^3 - 2\tfrac{C_i}{r^4} + 2\tfrac{D_i}{r^2}\right)\sin\theta\cos\theta\cos 2\Phi \\ u_\Phi^{(i)}(r,\theta,\phi) &= -\left(A_i r - \tfrac{7-4\nu_i}{1-2\nu_i} B_i r^3 - 2\tfrac{C_i}{r^4} + 2\tfrac{D_i}{r^2}\right)\sin\theta\sin 2\Phi \end{aligned}\right\} \tag{20}$$

The corresponding stresses in phase (i) are found to be:

$$\left.\begin{aligned} \sigma_{rr}^{(i)}(r,\theta,\phi) &= \mu_i\left(A_i + \tfrac{3\nu_i}{1-2\nu_i} B_i r^2 - \tfrac{12C_i}{r^5} + \tfrac{2(\nu_i-5)}{1-2\nu_i}\tfrac{D_i}{r^3}\right)2\sin^2\theta\cos 2\Phi \\ \sigma_{r\theta}^{(i)}(r,\theta,\phi) &= \mu_i\left(A_i - \tfrac{7+2\nu_i}{1-2\nu_i} B_i r^2 + \tfrac{8C_i}{r^5} + \tfrac{2(1+\nu_i)}{1-2\nu_i}\tfrac{D_i}{r^3}\right)2\sin\theta\cos\theta\cos 2\Phi \\ \sigma_{r\phi}^{(i)}(r,\theta,\phi) &= -\mu_i\left(A_i - \tfrac{7+2\nu_i}{1-2\nu_i} B_i r^2 + \tfrac{8C_i}{r^5} + \tfrac{2(1+\nu_i)}{1-2\nu_i}\tfrac{D_i}{r^3}\right)2\sin\theta\sin 2\Phi \end{aligned}\right\} \tag{21}$$

where $(A_i,\ B_i,\ C_i,\ C_i)$ are constants.

The key tool to solve easily the problem of simple shear is to introduce these quantities into the two following (4×1) matrices:

$$\left.\begin{aligned} \vec{l}(r,\theta,\phi) &= [u_r(r,\theta,\phi),\ u_\theta(r,\theta,\phi),\ \sigma_{rr}(r,\theta,\phi),\ \sigma_{r\theta}(r,\theta,\phi)]^T \\ \vec{l}^{(i)}(r,\theta,\phi) &= \left[u_r^{(i)}(r,\theta,\phi),\ u_\theta^{(i)}(r,\theta,\phi),\ \sigma_{rr}^{(i)}(r,\theta,\phi),\ \sigma_{r\theta}^{(i)}(r,\theta,\phi)\right]^T \end{aligned}\right\} \tag{22}$$

expressed in a canonical base $(\vec{e}_1, \vec{e}_2, \vec{e}_3, \vec{e}_4)$. Let $\vec{l}^{(i)}$ denote the value of $\vec{l}(r)$ in phase (i). In order to deal separately with the dependence over θ and ϕ, let us denote by $\vec{g}(\theta,\phi)$ the following vector:

$$\vec{g}(\theta,\phi) = \sin\theta\cos 2\phi[\sin\theta\vec{e}_1 + \cos\theta\vec{e}_2 + 2\sin\theta\vec{e}_3 + 2\cos\theta\vec{e}_4] \tag{23}$$

For the sake of simplicity of the expression of the bonding conditions over the interface Γ_i, and using Eqs. (20) and (21), $\vec{l}^{(i)}(r,\theta,\phi)$ is written in the following manner:

$$\vec{l}^{(i)}(r,\theta,\phi) = \mathbf{L}^{(i)}(r)\vec{W}_i \boxplus \vec{g}(\theta,\phi) \tag{24}$$

where $\vec{W}_i = [A_i,\ B_i,\ C_i,\ D_i]^T$ and $\vec{H} \boxplus \vec{g}$ denotes for any (4×1) matrix $\vec{H} = [H_1,\ H_2,\ H_3,\ H_3]^T$:

$$\vec{H} \boxplus \vec{g} = \sin\theta\cos 2\phi[H_1\sin\theta,\ H_2\cos\theta,\ 2H_3\sin\theta,\ 2H_4\cos\theta]^T \tag{25}$$

where the matrix $\mathbf{L}^{(i)}(r)$ (already defined in Hervé and Zaoui (1993)) is given by:

$$\mathbf{L}^{(i)}(r) = \begin{bmatrix} r & \frac{-6\nu_i}{1-2\nu_i} r^3 & \frac{3}{r^4} & \frac{5-4\nu_i}{1-2\nu_i}\frac{1}{r^2} \\ r & -\frac{7-4\nu_i}{1-2\nu_i} r^3 & -\frac{2}{r^4} & \frac{2}{r^2} \\ \mu_i & \frac{3\nu_i}{1-2\nu_i}\mu_i r^2 & -\frac{12}{r^5}\mu_i & \frac{2(\nu_i-5)}{1-2\nu_i}\frac{\mu_i}{r^3} \\ \mu_i & -\frac{7+2\nu_i}{1-2\nu_i}\mu_i r^2 & \frac{8}{r^5}\mu_i & \frac{2(1+\nu_i)}{1-2\nu_i}\frac{\mu_i}{r^3} \end{bmatrix} \tag{26}$$

It is worth noting that:

$$\left.\begin{aligned} u_\Phi^{(i)}(r,\theta,\phi) &= -\vec{\mathbf{L}}_2^{(i)}(r).\vec{W}_i\sin\theta\sin 2\Phi \\ \sigma_{r\Phi}^{(i)}(r,\theta,\phi) &= -2\vec{\mathbf{L}}_4^{(i)}(r).\vec{W}_i\sin\theta\sin 2\Phi \end{aligned}\right\} \tag{27}$$

where $\vec{\mathbf{L}}_p^{(i)}(r)$ denotes the (4×1) matrix made of the pth row of $\mathbf{L}^{(i)}(r)$.

We consider now all the interface bonding conditions (over the interface Γ_i) which can be written in the following manner:

$$\left.\begin{aligned} \left[\vec{l}\right]_{R_i}(R_i,\theta,\phi) &= \vec{l}^{(i+1)}(R_i,\theta,\phi) - \vec{l}^{(i)}(R_i,\theta,\phi) = \Delta\mathbf{L}(R_i)\vec{W}_i \boxplus \vec{g}(\theta,\phi) \\ \left[\vec{l}\right]_{R_i}(R_i,\theta,\phi) &= \delta\mathbf{L}(R_i)\mathbf{L}^{(i)}(R_i)\vec{W}_i \boxplus \vec{g}(\theta,\phi) \end{aligned}\right\} \tag{28}$$

where $\left[\vec{l}\right]_{R_i}(R_i,\theta,\phi)$ denotes the interfacial jump of $\vec{l}$ at $r = R_i$ (surface Γ_i).

The different displacement and stress discontinuities at the interface Γ_i are taken into account thanks to the "shear discontinuity matrix" $\delta\mathbf{L}(R_i)$ which depends on the chosen interface models. To respect the spherical symmetry, we also assume that:

$$\left.\begin{aligned} [u_\Phi^{(i)}]_{R_i} &= -\overrightarrow{\Delta\mathbf{L}}_2(R_i).\vec{W}_i\sin\theta\sin 2\Phi \\ \left[\sigma_{r\Phi}^{(i)}\right]_{R_i} &= -2\,\overrightarrow{\Delta\mathbf{L}}_4(R_i).\vec{W}_i\sin\theta\sin 2\Phi \end{aligned}\right\} \tag{29}$$

where $\overrightarrow{\Delta\mathbf{L}}_p(R_i)$ denotes the (4×1) matrix made of the pth row of $\Delta\mathbf{L}(R_i)$.

The interface conditions, taking into account the interface bonding conditions (28) and using the definition (24) yield:

$$\left.\begin{aligned} \vec{l}^{(i+1)}(R_i,\theta,\phi) &= \vec{l}^{(i)}(R_i,\theta,\phi) + \left[\vec{l}\right]_{R_i}(R_i,\theta,\phi) \\ \vec{l}^{(i+1)}(R_i,\theta,\phi) &= \left(\mathbf{L}^{(i)}(R_i)) + \Delta\mathbf{L}(R_i)\right)\vec{W}_i \boxplus \vec{g}(\theta,\phi) \\ \mathbf{L}^{(i+1)}(R_i)\vec{W}_{i+1} \boxplus \vec{g}(\theta,\phi) &= \left(\mathbf{L}^{(i)}(R_i)) + \Delta\mathbf{L}(R_i)\right)\vec{W}_i \boxplus \vec{g}(\theta,\phi) \end{aligned}\right\} \tag{30}$$

It follows that:

$$\left.\begin{aligned} \vec{W}_{i+1} &= \left(\mathbf{L}^{(i+1)}\right)^{-1}(R_i)\left[\mathbf{L}^{(i)}(R_i) + \Delta\mathbf{L}(R_i)\right]\vec{W}_i \\ \vec{W}_{i+1} &= \left(\mathbf{M}^{(i)} + \Delta\mathbf{M}(R_i)\right)\vec{W}_i \\ \vec{W}_{i+1} &= \mathbf{M}^{S(i)}\vec{W}_i = \left(\prod_{j=i}^{1}\mathbf{M}^{S(j)}\right)\vec{W}_1 = \mathbf{P}^{S(i)}\vec{W}_1 \end{aligned}\right\} \tag{31}$$

with

$$\left.\begin{aligned} \mathbf{M}^{(i)} &= \left(\mathbf{L}^{(i+1)}\right)^{-1}(R_i)\ \mathbf{L}^{(i)}(R_i) \\ \Delta\mathbf{M}(R_i) &= \left(\mathbf{L}^{(i+1)}\right)^{-1}(R_i)\ \delta\mathbf{L}^{(i)}(R_i)\ \mathbf{L}^{(i)}(R_i) \\ \mathbf{M}^{S(i)} &= \left(\mathbf{M}^{(i)} + \Delta\mathbf{M}(R_i)\right) \\ \mathbf{P}^{S(i)} &= \left(\prod_{j=i}^{1}\mathbf{M}^{S(j)}\right) \end{aligned}\right\} \tag{32}$$

$\delta\mathbf{L}^{(i)}(R_i)$ and consequently $\Delta\mathbf{L}^{(i)}(R_i)$ depend on the chosen interface model (see different expressions of $\delta\mathbf{L}^{(i)}(R_i)$ for several interface models in Section 2.3).

In order to avoid singularity, C_1, D_1 and B_{n+1} vanish leading by using (31) (See Hervé and Zaoui, 1993) to:

$$\vec{W}_i = \frac{A_{n+1}}{P_{22}^{S(n)}P_{11}^{S(n)} - P_{12}^{S(n)}P_{21}^{S(n)}}\mathbf{P}^{S(i-1)}\vec{V}_P^{(n)} \tag{33}$$

with $\vec{V}_P^{(n)} = P_{22}^{S(n)}\vec{e}_1 - P_{21}^{S(n)}\vec{e}_2$ and:

$$\begin{cases} A_{n+1} = \frac{\tau}{2\mu_{n+1}} & \text{in the stress approach} \\ A_{n+1} = \gamma & \text{in the displacement approach} \end{cases} \tag{34}$$

Using Eqs. (4), (22), (24), and (28), it can been shown that the interface energy $E^{int}(R_i)$ at the Γ_i interface, due to this simple shear sollicitation, can be expressed as:

$$E^{int}(R_i) = \frac{8\pi}{15}R_i^2\,\vec{W}_i\,\Delta\mathbf{e}\,\vec{W}_i \tag{35}$$

with $\Delta\mathbf{e}$ given by:

$$\left.\begin{array}{l} 2\,\overrightarrow{\mathbf{L}}_3(R_i)\otimes\overrightarrow{\Delta\mathbf{L}}_1(R_i) + 3\,\overrightarrow{\mathbf{L}}_4(R_i)\otimes\overrightarrow{\Delta\mathbf{L}}_2(R_i) \text{ interface displacement jumps} \\ 2\,\overrightarrow{\Delta\mathbf{L}}_3(R_i)\otimes\overrightarrow{\mathbf{L}}_1(R_i) + 3\,\overrightarrow{\Delta\mathbf{L}}_4(R_i)\otimes\overrightarrow{\mathbf{L}}_2(R_i) \text{ interface traction jumps} \end{array}\right\} \tag{36}$$

The replacement of $\vec{W}_i$ by Eq. (33) in E^{int} provides:

$$E^{int}(R_i) = \frac{8\pi}{15}\frac{R_i^2 A_{n+1}^2}{\left(P_{22}^{S(n)}P_{11}^{S(n)} - P_{12}^{S(n)}P_{21}^{S(n)}\right)^2}\vec{V}_P^{(n)t}\mathbf{P}^{S(i-1)}\Delta\mathbf{e}\mathbf{P}^{S(i-1)}\vec{V}_P^{(n)} \tag{37}$$

It is worth noting that applications will be carried out only to interface models which consider that either $\overrightarrow{\Delta\mathbf{L}}_1(R_i) = \overrightarrow{\Delta\mathbf{L}}_2(R_i) = 0$ (jump of the normal and tangential components of traction across Γ_i with a continuous displacement vector) or $\overrightarrow{\Delta\mathbf{L}}_3(R_i) = \overrightarrow{\Delta\mathbf{L}}_4(R_i) = 0$ (jump of the normal and tangential components of displacement across Γ_i with a continuous traction vector).

2.3. Expression of the discontinuity matrices for several interface models

The aim of this section is to give for several classical interface models the two discontinuity matrices $\delta\mathbf{J}^{(i)}(R_i)$ and $\delta\mathbf{L}^{(i)}(R_i)$ (defined in Sections 2.1 and 2.2) which will characterize the displacement/stress discontinuities over the interfaces Γ_i $(i \in [1, n-1])$. These results will be used in Section 4 to get the effective behavior of multi-layered inclusion reinforced composites having such imperfect interfaces.

2.3.1. Linear spring models

The interface conditions over Γ_i can then be written as:

$$\left.\begin{array}{l} [\boldsymbol{\sigma}]_{\Gamma_i}\vec{n} = \vec{0} \\ \boldsymbol{P}\,\boldsymbol{\sigma}\,\vec{n} = \zeta_s^i\,\mathbf{P}\,[\vec{u}]_{\Gamma_i} \\ \vec{n}\,\boldsymbol{\sigma}\,\vec{n} = \iota_s^i[\vec{u}]_{\Gamma_i}\,\vec{n} \end{array}\right\} \tag{38}$$

where ζ_s^i and ι_s^i denote the interface parameters of the interface Γ_i in the tangential and normal directions and where $\boldsymbol{P} = \boldsymbol{I} - \vec{n}\otimes\vec{n}$.

For an applied hydrostatic pressure, the "pressure discontinuity matrix" $\delta\mathbf{J}^{(LS)}(R_i)$ attached to this "linear spring" over Γ_i may be found by substituting Eqs. (8) and (9) in Eq. (38) and taking into account Eqs. (10) and (11):

$$[\vec{t}]_{R_i} = \begin{bmatrix} [u_r]_{R_i} \\ [\sigma_{rr}]_{R_i} \end{bmatrix} = \begin{bmatrix} \frac{\sigma_{rr}(R_i)}{\iota_s^i} \\ 0 \end{bmatrix} = \frac{1}{\iota_s^i}\begin{bmatrix} 3k_i & -\frac{4\mu_i}{R_i^3} \\ 0 & 0 \end{bmatrix}\begin{bmatrix} F_i \\ G_i \end{bmatrix} \tag{39}$$

which takes the form:

$$[\vec{t}]_{R_i} = \frac{1}{\iota_s^i}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\mathbf{J}^{(i)}(R_i)\vec{V}_i = \delta\mathbf{J}^{(LS)}(R_i)\mathbf{J}^{(i)}(R_i)\vec{V}_i \tag{40}$$

For a simple shear sollicitation, the shear discontinuity matrix $\delta\mathbf{L}^{(LS)}(R_i)$) attached to this "linear spring" may be found by substituting Eqs. (22) and (24) in Eq. (38) and taking into account Eqs. (26) and (28).

For that purpose, let us write the expression of $[\vec{l}]_{R_i}$ which is given by $[\vec{l}]_{R_i} = \left[\frac{1}{\iota_s^i}\sigma_{rr}(R_i),\ \frac{1}{\zeta_s^i}\sigma_{r\theta}(R_i),\ 0,\ 0\right]^T$ or:

$$\begin{bmatrix} \frac{2\mu_i}{\iota_s^i} & \frac{6\nu_i}{\iota_s^i(1-2\nu_i)}\mu_i R_i^2 & -\frac{24}{\iota_s^i R_i^5}\mu_i & \frac{2(\nu_i-5)}{\iota_s^i(1-2\nu_i)}\frac{\mu_i}{R_i^3} \\ \frac{2\mu_i}{\zeta_s^i} & -\frac{7+2\nu_i}{\zeta_s^i(1-2\nu_i)}2\mu_i R_i^2 & \frac{16}{\zeta_s^i R_i^5}\mu_i & \frac{4(1+\nu_i)}{\zeta_s^i(1-2\nu_i)}\frac{\mu_i}{R_i^3} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} A_i \\ B_i \\ C_i \\ D_i \end{bmatrix} \boxplus \vec{g}(\theta,\phi) \tag{41}$$

We can also write $[\vec{l}]_{R_i}$ as:

$$[\vec{l}]_{R_i} = \begin{pmatrix} 0 & 0 & \frac{2}{\iota_s^i} & 0 \\ 0 & 0 & 0 & \frac{2}{\zeta_s^i} \\ 0 & 0 & 0 & 0 \end{pmatrix}\mathbf{L}^{(i)}(R_i)\vec{W}_i \boxplus \vec{g}(\theta,\phi) = \delta\mathbf{L}^{(LS)}(R_i)\mathbf{L}^{(i)}(R_i)\vec{W}_i \boxplus \vec{g}(\theta,\phi) \tag{42}$$

The two discontinuity matrices ($\delta\mathbf{J}^{(LS)}(R_i)$ and $\delta\mathbf{L}^{(LS)}(R_i)$) attached to this "linear spring" interface model can then be written as follows:

$$\left.\begin{array}{l} \delta\mathbf{J}^{(LS)}(R_i) = \frac{1}{\iota_s^i}\vec{e}_1\otimes\vec{e}_2 \\ \delta\mathbf{L}^{(LS)}(R_i) = 2\left(\frac{1}{\iota_s^i}\vec{e}_1\otimes\vec{e}_3 + \frac{1}{\zeta_s^i}\vec{e}_2\otimes\vec{e}_4\right) \end{array}\right\} \tag{43}$$

2.3.2. Dislocation-like model

In the Dislocation-like interface model (Yu, 1998; Duan et al., 2005a) the following conditions prevail across the interface $\Gamma_i$which separates the two phases (i) and $(i+1)$:

$$\vec{u}^{(i+1)}(R_i) = (\eta_T^i\mathbf{P} + \eta_N^i\vec{n}\otimes\vec{n})\vec{u}^{(i)}(R_i)\;\} \tag{44}$$

leading to:

$$[\vec{u}]_{R_i} = \vec{u}^{(i+1)}(R_i) - \vec{u}^{(i)}(R_i) = \left((\eta_T^i - 1)\mathbf{P} + (\eta_N^i - 1)\vec{n}\otimes\vec{n}\right)\vec{u}^{(i)}(R_i)\;\} \tag{45}$$

The two discontinuity matrices attached to this interface "dislocation-like model" can easily be written using the tools presented in the present paper and can be immediately derived as follows:

$$\left.\begin{array}{l} \delta\mathbf{J}^{(DL)}(R_i) = (\eta_N^i - 1)\vec{e}_1\otimes\vec{e}_1 \\ \delta\mathbf{L}^{(DL)}(R_i) = (\eta_N^i - 1)\vec{e}_1\otimes\vec{e}_1 + (\eta_T^i - 1)\vec{e}_2\otimes\vec{e}_2 \end{array}\right\} \tag{46}$$

2.3.3. Asymptotic method to deal with thin elastic interphase

An asymptotic approach can be used to exhibit the (imperfect or not) interfacial bonding as the effect of a thin elastic interphase. Hashin (2002) has already derived such an interface model by a Taylor expansion method in terms of interface displacement and traction jumps.

The purpose of this section is to use the previous developed approach using transfert matrices to deal with n-layered isotropic spherical inclusions with thin elastic interphases and to exhibit the two discontinuity matrices attached to a thin elastic interphase.

Let us consider that in the n-layered isotropic spherical inclusion presented in Fig. 1(a), flexible bond layer of a thickness $\alpha_I^i R_i$ is introduced between phase (i) and phase $(i+1)$ (Fig. 1(b)). The two interfaces (Γ_i^- and Γ_i^+) delimiting these three phases are assumed perfect ($\vec{T} = \boldsymbol{\sigma}\vec{n}$ and $\vec{u}$ are continuous on Γ_i^- and Γ_i^+). Let

$\left(\mu_I^i,\ k_I^i,\ \nu_I^i\right)$ be respectively the shear modulus, bulk modulus and Poisson's ration of this interphase.

2.3.3.1. Method using the above-described procedure. When an hydrostatic pressure is applied at infinity (boundary conditions (5)), the solution $\vec{t}^{(int)}(r)$ inside this thin interphase is given by:

$$\vec{t}^{(int)}(r) = \mathbf{J}^{(int)}(r)\vec{V}_I^i \tag{47}$$

with $\vec{V}_I^i = \left[F_I^i,\ G_I^i\right]^T$ and where $\mathbf{J}^{int}(r)$ has the same definition as (10) except that $(\mu_i,\ k_i)$ are respectively replaced by $\left(\mu_I^i,\ k_I^i\right)$. Consequently, the interface bonding conditions can be written in the following manner:

$$\left[\vec{t}\right]_{R_i} = \left[\mathbf{J}^{(int)}\left(R_i\left(1+\alpha_I^i\right)\right) - \mathbf{J}^{(int)}(R_i)\right]\vec{V}_I^i \tag{48}$$

Since Γ_i^- is a perfect interface it follows that:

$$\mathbf{J}^{(i)}(R_i)\vec{V}_i = \mathbf{J}^{(int)}(R_i)\vec{V}_I^i \text{ or } \vec{V}_I^i = \mathbf{J}^{(int)^{-1}}(R_i)\mathbf{J}^{(i)}(R_i)\vec{V}_i \tag{49}$$

$\left[\vec{t}\right]_{R_i}$ becomes then:

$$\left[\vec{t}\right]_{R_i} = \left[\mathbf{J}^{(int)}(R_i)\left(1+\alpha_I^i\right)\right)\mathbf{J}^{(int)^{-1}}(R_i) - \mathbf{I}\right]\mathbf{J}^{(i)}(R_i)\vec{V}_i \tag{50}$$

It is worth noting that using a taylor expansion in terms of α_I^i :

$$\mathbf{J}^{(int)}\left(R_i\left(1+\alpha_I^i\right)\right) = \mathbf{J}^{(int)}(R_i)) + \alpha_I^i R_i \left.\frac{\partial \mathbf{J}^{(int)}}{\partial r}\right|_{R_i} \tag{51}$$

Finally

$$\left[\vec{t}\right]_{R_i} = \alpha_I^i R_i \left.\frac{\partial \mathbf{J}^{(int)}}{\partial r}\right|_{R_i} \mathbf{J}^{(int)^{-1}}(R_i)\mathbf{J}^{(i)}(R_i)\vec{V}_i \tag{52}$$

and

$$\delta\mathbf{J}^{As}(R_i) = \alpha_I^i R_i \left.\frac{\partial \mathbf{J}^{(int)}}{\partial r}\right|_{R_i} \mathbf{J}^{(int)^{-1}}(R_i) \tag{53}$$

Using the expressions of $\mathbf{J}^{(int)^{-1}}(R_i)$ (Eq. (A.1)) and of $\left.\frac{\partial \mathbf{J}^{(int)}}{\partial r}\right|_{R_i}$ (Eq. (A.2)), it follows that:

$$\delta\mathbf{J}^{As}(R_i) = \frac{\alpha_I^i}{3k_I^i + 4\mu_I^i}\begin{pmatrix} 4\mu_I^i - 6k_I^i & 3R_i \\ \frac{36k_I^i\mu_I^i}{R_i} & -12\mu_I^i \end{pmatrix} \tag{54}$$

or

$$\delta\mathbf{J}^{As}(R_i) = \frac{\alpha_I^i}{2\mu_I^i\left(1-\nu_I^i\right)}\begin{pmatrix} -4\mu_I^i\nu_I^i & R_i\left(1-2\nu_I^i\right) \\ \frac{8\left(\mu_I^i\right)^2\left(1+\nu_I^i\right)}{R_i} & -4\mu_I^i\left(1-2\nu_I^i\right) \end{pmatrix} \tag{55}$$

In the asymptotic limit ($\alpha_I^i \to 0$) several interface models can be derived depending on the functions $\mu_I^i(\alpha_I^i)$ and $k_I^i(\alpha_I^i)$.

The same procedure can be used when a simple shear is applied at infinity (boundary conditions (19) or (18)), the solution $\vec{l}^{(int)}(r)$ inside this thin interphase is given by:

$$\vec{l}^{(int)}(r) = \mathbf{L}^{(int)}(r)\vec{W}_I^i \boxplus \vec{g} \tag{56}$$

with $\vec{W}_I^i = \left[A_I^i,\ B_I^i,\ C_I^i,\ D_I^i\right]^T$ and where $\mathbf{L}^{int}(r)$ has the same definition as (26) except that $(\mu_i,\ \nu_i)$ are respectively replaced by $\left(\mu_I^i,\ \nu_I^i\right)$.

$$\left[\vec{l}\right]_{R_i} = \left[\mathbf{L}^{(int)}\left(R_i\left(1+\alpha_I^i\right)\right) - \mathbf{L}^{(int)}(R_i)\right]\vec{W}_I^i \boxplus \vec{g} \tag{57}$$

Since Γ_i^- is a perfect interface it follows that:

$$\mathbf{L}^{(i)}(R_i)\vec{W}_i = \mathbf{L}^{(int)}(R_i)\vec{W}_I^i \quad \text{or} \quad \vec{W}_I^i = \mathbf{L}^{(int)^{-1}}(R_i)\mathbf{L}^{(i)}(R_i)\vec{W}_i \tag{58}$$

Consequently, by using Eq. (58) in Eq. (57), the interface bonding conditions can be written in the following manner:

$$\left[\vec{l}\right]_{R_i} = \left[\mathbf{L}^{(int)}(R_i)\left(1+\alpha_I^i\right)\right)\mathbf{L}^{(int)^{-1}}(R_i) - \mathbf{I}\right]\mathbf{L}^{(i)}(R_i)\vec{W}_i \boxplus \vec{g} \tag{59}$$

It is worth noting that using a taylor expansion in terms of α_I^i :

$$\mathbf{L}^{(int)}\left(R_i\left(1+\alpha_I^i\right)\right) = \mathbf{L}^{(int)}(R_i)) + \alpha_I^i R_i \left.\frac{\partial \mathbf{L}^{(int)}}{\partial r}\right|_{R_i} \tag{60}$$

Finally

$$\left[\vec{l}\right]_{R_i} = \alpha_I^i R_i \left.\frac{\partial \mathbf{L}^{(int)}}{\partial r}\right|_{R_i} \mathbf{L}^{(int)^{-1}}(R_i)\mathbf{L}^{(i)}(R_i)\vec{W}_i \boxplus \vec{g} \tag{61}$$

and

$$\delta\mathbf{L}^{As}(R_i) = \alpha_I^i R_i \left.\frac{\partial \mathbf{L}^{(int)}}{\partial r}\right|_{R_i} \mathbf{L}^{(int)^{-1}}(R_i) \tag{62}$$

Using the expressions of $\mathbf{L}^{(int)^{-1}}(R_i)$ (Eq. (A.3)) and of $\left.\frac{\partial \mathbf{L}^{(int)}}{\partial r}\right|_{R_i}$ (Eq. (A.4)), $\delta\mathbf{L}^{As}(R_i)$ is then given by the following matrix:

$$\frac{\alpha_I^i}{\left(\nu_I^i - 1\right)}\begin{pmatrix} 2\nu_I^i & -3\nu_I^i & \frac{R_i\left(2\nu_I^i-1\right)}{\mu_I^i} & 0 \\ -2\left(\nu_I^i - 1\right) & \left(\nu_I^i - 1\right) & 0 & \frac{2R_i\left(\nu_I^i-1\right)}{\mu_I^i} \\ -\frac{2\mu_I^i\left(\nu_I^i+1\right)}{R_i} & \frac{3\mu_I^i\left(\nu_I^i+1\right)}{R_i} & -2\left(2\nu_I^i - 1\right) & 3\left(\nu_I^i - 1\right) \\ \frac{2\mu_I^i\left(\nu_I^i+1\right)}{R_i} & -\frac{\mu_I^i\left(\nu_I^i+5\right)}{R_i} & 2\nu_I^i & -3\left(\nu_I^i - 1\right) \end{pmatrix} \tag{63}$$

2.3.3.2. Method using Hashin's procedure. Hashin's method Hashin (2002) can also be used and we check here that it leads to the same discontinuity matrices.

It should be pointed out here that Hashin's model is equivalent to take into account the solution of the field equations in the thin interphase which lays between phase (i) and phase ($i+1$) and to consider the asymptotic limit when the thickness of the interphase (here $\alpha_I^i R_i) \to 0$.

Following Hashin, it is necessary to express all the jumps of the displacement and the traction vectors by C.I quantities (continuous across the interfaces Γ_i^- and Γ_i^+). Because of the form of the displacement and tractions vector in each phase (Eqs. (8), (9), (18), (22), (25) and (24)), and considering that $u_r, u_\theta, u_\phi, \sigma_{rr}, \sigma_{r\theta}$ and $\sigma_{r\phi}$ are C.I, it follows that their tangential derivatives with respect to θ and ϕ are also C.I and also $\varepsilon_{rr}, \varepsilon_{r\theta}$ and $\varepsilon_{r\phi}$. Using Hooke's law (1) and (2), it is easy to show that $\sigma_{\phi\phi}$ and $\sigma_{\theta\theta}$ are also C.I.

Moreover, following Hashin's procedure, it is a long way to get the two discontinuity matrices attached to this method.

We can express the jump of the displacement vector as:

$$\left.\begin{aligned} [u_r]_{R_i} &= \alpha_I^i R_i u_{r,r}(R_i) = \alpha_I^i R_i \varepsilon_{rr}(R_i) \\ [u_\theta]_{R_i} &= \alpha_I^i R_i u_{\theta,r}(R_i) = \alpha_I^i R_i \left[\frac{\sigma_{r\theta(R_i)}}{\mu} + \frac{u_\theta(R_i) - u_{r,\theta}(R_i)}{R_i}\right] \\ \left[u_\phi\right]_{R_i} &= \alpha_I^i R_i u_{\phi,r}(R_i) = \alpha_I^i R_i \left[\frac{\sigma_{r\phi(R_i)}}{\mu} + \frac{\sin\theta\ u_\phi(R_i) - u_{r,\phi}(R_i)}{\sin\ \theta R_i}\right] \end{aligned}\right\} \tag{64}$$

where commas denote partial differentiation.

The jump of the traction vector is expressed by using the equilibrium equations:

$$\left.\begin{aligned} [\sigma_{rr}]_{R_i} &= \alpha_I^i\left[-\sigma_{r\theta,\theta} - \frac{\sigma_{r\phi,\phi}}{\sin\theta} - \text{cotan}\ \theta\ \sigma_{r\theta} - 2\sigma_{rr} + \left(\sigma_{\theta\theta} + \sigma_{\phi\phi}\right)\right] \\ [\sigma_{r\theta}]_{R_i} &= \alpha_I^i\left[-\sigma_{\theta\theta,\theta} - \frac{\sigma_{\theta\phi,\phi}}{\sin\theta} - 3\sigma_{r\theta} - \text{cotan}\ \theta\ \left(\sigma_{\theta\theta} - \sigma_{\phi\phi}\right)\right] \\ \left[\sigma_{r\phi}\right]_{R_i} &= \alpha_I^i\left[-\sigma_{\theta\phi,\theta} - \frac{\sigma_{\phi\phi,\phi}}{\sin\theta} - 3\sigma_{r\phi} - 2\text{cotan}\ \theta\ \sigma_{\theta\phi}\right] \end{aligned}\right\} \tag{65}$$

After tedious calculations and using the definitions presented in Section 2, we obtain the following results:

- Hydrostatic pressure:

$$\left.\begin{aligned} [u_r]_{R_i} &= \tfrac{\alpha_I^i}{3k_I^i+4\mu_I^i}\left[2\left(2\mu_I^i-3k_I^i\right)t_1^{(i)}(R_i)-3R_it_2^{(i)}(R_i)\right] \\ [\sigma_{rr}]_{R_i} &= \tfrac{\alpha_I^i}{3k_I^i+4\mu_I^i}\left[36k_I^i\mu_I^it_1^{(i)}(R_i)-12\mu_I^it_2^{(i)}(R_i)\right] \end{aligned}\right\} \tag{66}$$

- Simple shear:

$$[u_r]_{R_i} = \frac{\sin^2\theta\cos 2\phi\ \alpha_I^i}{\nu_I^i-1}\left[2\nu_I^i\vec{L}_1^{(i)}(R_i)-3\nu_I^i\vec{L}_2^{(i)}(R_i)+\cdots\frac{(2\nu_I^i-1)R_i}{\mu_I^i}\vec{L}_3^{(i)}(R_i)\right].\vec{W}_i$$

$$[u_\theta]_{R_i} = \cos\theta\sin\theta\cos 2\phi\ \alpha_I^i\left[-2\vec{L}_1^{(i)}(R_i)+\vec{L}_2^{(i)}(R_i)+\frac{2R_i\vec{L}_4^{(i)}(R_i)}{\mu_I^i}\right].\vec{W}_i$$

$$[\sigma_{rr}]_{R_i} = \frac{2\sin^2\theta\cos 2\phi\ \alpha_I^i}{\nu_I^i-1}\left[-\frac{2\mu_I^i(1+\nu_I^i)}{R_i}\vec{L}_1^{(i)}(R_i)+\frac{3\mu_I^i(1+\nu_I^i)}{R_i}\vec{L}_2^{(i)}(R_i)\right.$$
$$\left.+\cdots-2(2\nu_I^i-1)\vec{L}_3^{(i)}(R_i)+3(\nu_I^i-1)\vec{L}_4^{(i)}(R_i)\right].\vec{W}_i$$

$$[\sigma_{r\theta}]_{R_i} = \frac{2\sin\theta\cos\theta\cos 2\phi\ \alpha_I^i}{\nu_I^i-1}\left[\frac{2\mu_I^i(1+\nu_I^i)}{R_i}\vec{L}_1^{(i)}(R_i)\right.$$
$$\left.-\frac{\mu_I^i(\nu_I^i+5)}{R_i}\vec{L}_2^{(i)}(R_i)+\cdots-2\nu_I^i\vec{L}_3^{(i)}(R_i)-3(\nu_I^i-1)\vec{L}_4^{(i)}(R_i)\right].\vec{W}_i \tag{67}$$

This comparison shows obviously that Hashin's procedure leads (with tedious calculations) to the same discontinuity matrices $\delta\mathbf{J}^{As}(R_i)$ and $\delta\mathbf{L}^{As}(R_i)$ (Eqs. (55) and (63)) already easily obtained thanks to the method presented in Section 2.1 and in Section 2.2.

3. n-layered spherical inclusion-reinforced composites

In order to predict the behavior of n-layered spherical inclusion-reinforced composites with imperfect interfaces a link must be made between this composite and a n-layered spherical problem as the one presented in Section 2 (cf Fig. 1(a)). It is worth noticing that this n-layered spherical problem is an auxilliary problem that will be used to determine in Section 4 the effective properties of the studied composites by the generalized self-consistent scheme (GSCS). For this purpose, we consider that the volume fraction of each phase is the same in the composite as in the attached auxilliary n-layered spherical problem. This is a generalization of Eq. (20), Marcadon et al. (2007) to n-layered spherical inclusion-reinforced composites and this equivalence is also justified by Benveniste (2008). This equivalence of volume fractions is applied in the interphase between phase i and phase $i+1$ and leads to:

$$\frac{R_{i+1}^3-R_i^3}{R_i^3} = \frac{\left(R_{0i}+t_I^i\right)^3-R_{0i}^3}{R_{0i}^3} \tag{68}$$

and, in the case of thin interphase with $R_{i+1}=R_i(1+\alpha_I^i)$ ($\alpha_I^i\to 0$) to:

$$\alpha_I^i = \frac{t_I^i}{R_{0i}} \tag{69}$$

where R_{0i} denotes the external radius of the inhomogeneous inclusions, part of the n-phase inclusions where such an interface model is used, and t_I^i the thickness of the interphase laying between phase i and phase $i+1$.

Eq. (69) can be used to clarify the discontinuity matrices attached to soft or rigid interphases:

1. In the case of rigid thin interphase the shear modulus of the thin interphase between phase i and phase $i+1$, μ_s^i is defined by $\mu_s^i=\mu_I^it_I^i$ leading, thanks to Eq. (69), to $\mu_I^i=\frac{\mu_s^i}{t_I^i}=\frac{\mu_s^i}{\alpha_I^iR_{0i}}$. $\delta\mathbf{J}^{As}(R_i)$ (Eq. (55)) becomes then $\delta\mathbf{J}^{AsRigid}(R_i)$:

$$\delta\mathbf{J}^{AsRigid}(R_i) = \frac{4\mu_I^i(1+\nu_I^i)t_I^i}{R_{0i}R_i(1-\nu_I^i)}\begin{bmatrix}0 & 0\\ 1 & 0\end{bmatrix} \tag{70}$$

and $\delta\mathbf{L}^{As}(R_i)$ (Eq. (63)) becomes $\delta\mathbf{L}^{AsRigid}(R_i)$:

$$\delta\mathbf{L}^{AsRigid}(R_i) = \frac{\mu_I^it_I^i}{R_{0i}R_i(\nu_I^i-1)}\begin{bmatrix}0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ -2(1+\nu_I^i) & 3(1+\nu_I^i) & 0 & 0\\ 2(1+\nu_I^i) & -(5+\nu_I^i) & 0 & 0\end{bmatrix} \tag{71}$$

2. In the case of soft thin interphase ($\mu_I^i=\mu_s^it_I^i$, with μ_s^i denoting the shear modulus of the interface in this case), it is well known that this thin interphase is equivalent to a linear spring interface model and, $\delta\mathbf{J}^{As}(R_i)$ and $\delta\mathbf{L}^{As}(R_i)$ write like the discontinuity matrices attached to the linear spring interface model (see Eq. (43)) with:

$$\left.\begin{aligned}\zeta_s^i &= \tfrac{\mu_I^i}{\alpha_I^iR_i}\\ \iota_s^i &= \tfrac{2\mu_I^i(1-\nu_I^i)}{\alpha_I^iR_i(1-2\nu_I^i)}\end{aligned}\right\} \tag{72}$$

It is worth mentioning here that interface effects have been intensively used to study size effects in composites reinforced by nano-sized particles. In this case, a stress discontinuity is assumed on an interface Γ_i ($i\in[1,n-1]$) and the displacement vector is continuous across Γ_i. The interface conditions across each Γ_i, as given by Povstenko (1993), taking into account the jump of the traction vector on Γ_i and resulting from the analysis of the mechanical equilibrium of the interface between two different media are also called generalized Young–Laplace equations (Duan et al., 2005c). These interface conditions have been used by several authors like for instance (Brisard et al., 2010a; Brisard et al., 2010b; Kushch et al., 2011; Le Quang and He, 2008; Wang et al., 2005; Duan et al., 2005b; Duan et al., 2005a; Duan et al., 2007; Duan et al., 2008) and are written as:

$$\left.\begin{aligned}[\vec{u}] &= \vec{0}\\ \vec{n}.[\boldsymbol{\sigma}].\vec{n} &= -\boldsymbol{\sigma}^s:\boldsymbol{\kappa},\\ \boldsymbol{P}.[\boldsymbol{\sigma}].\vec{n} &= -\nabla_S.\boldsymbol{\sigma}^s\end{aligned}\right\} \tag{73}$$

where $\boldsymbol{\kappa}$ is the curvature tensor, $\nabla_S.\boldsymbol{\tau}$ denotes the surface divergence of the so-called surface stress tensor $\boldsymbol{\sigma}^s$. This surface stress tensor is linked to the surface strain tensor $\boldsymbol{\varepsilon}^s$ by the shuttleworth equation and will be written here as in Brisard et al. (2010b):

$$\boldsymbol{\sigma}^s = \mathcal{C}^S:\boldsymbol{\varepsilon}^s \tag{74}$$

where $\mathcal{C}^S$ denotes the stiffness surface tensor of the interface which only operates on the tangential components of the bulk strain tensor $\boldsymbol{\varepsilon}$.

In the case of the generalized Young–Laplace conditions, we will used as in Wang et al. (2005), Brisard et al. (2010a) and Brisard et al. (2010b) the following constitutive equations at the interfaces Γ_i:

$$\left.\begin{aligned}\sigma_{\theta\theta}^s &= \left(\beta_s^i-\mu_s^i\right)\left(\varepsilon_{\theta\theta}(R_i)+\varepsilon_{\phi\phi}(R_i)\right)+2\mu_s^i\varepsilon_{\theta\theta}(R_i),\\ \sigma_{\phi\phi}^s &= \left(\beta_s^i-\mu_s^i\right)\left(\varepsilon_{\theta\theta}(R_i)+\varepsilon_{\phi\phi}(R_i)\right)+2\mu_s^i\varepsilon_{\phi\phi}(R_i),\\ \sigma_{\theta\phi}^s &= 2\mu_s^i\varepsilon_{\theta\phi}(R_i)\end{aligned}\right\} \tag{75}$$

where the components of the tangential strain in the abutting bulk materials are taking into account.

β_s^i and μ_s^i denote the elastic coefficients of the interface Γ_i. They have been determined for a three-dimensional problem by Wang et al. (2005) thanks to a connection between a thin stiff interphase and Young–Laplace interface model and are here given by:

$$\left.\begin{aligned}\beta_s^i &= \tfrac{\mu_I^i(1+\nu_I^i)t_I^i}{(1-\nu_I^i)}\\ \mu_s^i &= \mu_I^i\ t_I^i\end{aligned}\right\} \tag{76}$$

The same convention has been chosen as Brisard et al. (2010a) and Brisard et al. (2010b) ($\beta_s^i = k^s$ and $\mu_s^i = \mu^s$) and Le Quang and He (2008) ($\beta_s^i = k_{si}$ and $\mu_s^i = \mu_{si}$) but a different convention as Duan et al. (2005c) ($\beta_s^i = k_s/2$ and $\mu_s^i = \mu_s$). It is worth noting that $1 \leqslant \frac{\beta_s^i}{\mu_s^i} \leqslant 3$ when $\nu_I^i \in [0, 0.5]$.

The bonding conditions over Γ_i are expressed as:

$$\left.\begin{array}{l}[\sigma_{rr}]_{\Gamma_i} = \frac{1}{R_{0i}}\left(\sigma^s_{\theta\theta} + \sigma^s_{\phi\phi}\right)\\ [\sigma_{r\theta}]_{\Gamma_i} = -\frac{1}{R_{0i}}\left(\frac{\partial\sigma^s_{\theta\theta}}{\partial\theta} + \frac{1}{\sin\theta}\frac{\partial\sigma^s_{\theta\phi}}{\partial\phi} + \left(\sigma^s_{\theta\theta} - \sigma^s_{\phi\phi}\right)\cot\theta\right)\\ [\sigma_{r\theta}]_{\Gamma_i} = -\frac{1}{R_{0i}}\left(\frac{\partial\sigma^s_{\theta\phi}}{\partial\theta} + \frac{1}{\sin\theta}\frac{\partial\sigma^s_{\phi\phi}}{\partial\phi} + 2\sigma^s_{\theta\phi}\cot\theta\right)\end{array}\right\} \tag{77}$$

In the case of an hydrostatic pressure applied to the boundary of the configuration presented in Fig. 1(a) the bonding conditions over Γ_i may be found by performing the calculation of $\boldsymbol{\varepsilon}$ from Eq. (6), substituting $\boldsymbol{\varepsilon}$ in Eq. (75) and then $\boldsymbol{\sigma}$ (Eq. (7)) and $\boldsymbol{\sigma}^s$ in (Eq. (77)).

In this simple case: $\varepsilon_{\theta\theta}(R_i) = \varepsilon_{\Phi\Phi}(R_i) = \frac{u_r(R_i)}{R_i} = \frac{\overrightarrow{\mathbf{J}_1^{(i)}}(R_i)}{R_i}.\vec{V}_i$

$$\left.\begin{array}{l}\left[u_r^{(i)}\right]_{\Gamma_i} = 0\\ [\sigma_{rr}]_{\Gamma_i} = \frac{4\beta_s^i}{R_{0i}R_i}\vec{J}_1^i(R_i).\vec{V}_i\end{array}\right\} \tag{78}$$

leading to the same discontinuity matrix as Eq. (70)

$$\delta\mathbf{J}^{(YL)}(R_i) = \delta\mathbf{J}^{(AsRigid)}(R_i) \tag{79}$$

In the case of a simple shear applied to the boundary of the configuration presented in Fig. 1(a), the bonding conditions over Γ_i may be found in that case by performing the calculation of $\boldsymbol{\varepsilon}$ from Eq. (20), substituting $\boldsymbol{\varepsilon}$ in Eq. (75) and then $\boldsymbol{\sigma}$ (Eq. (21)) and $\boldsymbol{\sigma}^s$ in (Eq. (77)).

It is worth noting that, with the above-introduced notation we have:

$$\left\{\begin{array}{l}\varepsilon_{\theta\theta}(R_i) = \frac{\cos 2\phi}{R_i}\left(\overrightarrow{\mathbf{L}_1^{(i)}}(R_i)\sin^2\theta + \overrightarrow{\mathbf{L}_2^{(i)}}(R_i)\cos 2\theta\right).\vec{W}_i\\ \varepsilon_{\phi\phi}(R_i) = \frac{\cos 2\phi}{R_i}\left(\overrightarrow{\mathbf{L}_1^{(i)}}(R_i)\sin^2\theta + \overrightarrow{\mathbf{L}_2^{(i)}}(R_i)(\cos^2\theta - 2)\right).\vec{W}_i\\ \varepsilon_{\theta\phi}(R_i) = -\frac{\sin 2\phi\cos\theta}{R_i}\overrightarrow{\mathbf{L}_2^{(i)}}(R_i).\vec{W}_i\end{array}\right. \tag{80}$$

leading to:

$$\left[\vec{l}\right]_{\Gamma_i} = \frac{1}{R_{0i}R_i}\left\{\beta_s^i\left(2\vec{\mathbf{L}}_1^{(i)}(R_i) - 3\vec{\mathbf{L}}_2^{(i)}(R_i)\right).\vec{W}_i(\vec{e}_3 - \vec{e}_4) + \cdots 2\mu_s^i\vec{\mathbf{L}}_2^{(i)}(R_i).\vec{W}_i\vec{e}_4\right\} \boxplus \vec{g} \tag{81}$$

Consequently, using the definition of $\delta\mathbf{L}(R_i)$ given in Eq. (28), we can get the value of $\delta\mathbf{L}^{(YL)}(R_i)$ from Eq. (81) and show that the discontinuity matrix $\delta\mathbf{L}^{(YL)}(R_i) = \delta\mathbf{L}^{(AsRigid)}(R_i)$.

The displacement and stress fields in configuration Fig. 1(a) with imperfect interfaces can be found easily thanks to the present method based on discontinuity matrices when this configuration is submitted at infinity to an hydrostatic pressure or to a simple shear sollicitation. More precisely, this problem needs no more to use a replacement procedure as the one presented in Duan et al. (2007) in the case of generalized Young Laplace condition.

It is also worth noting that, each time it will be possible to characterize the imperfect interface models by discontinuity matrices such as $\delta\mathbf{J}$ and $\delta\mathbf{L}$, it will be easy to solve such auxilliary problems.

These solutions will be used in the following section to determine the effective elastic behavior of composites containing n-layered spherical inclusions with imperfect interfaces.

4. $(n+1)$-phase model with imperfect interfaces

The expression of the energy balance has to be revised in the context of imperfect interfaces. The well-known work of Christensen (1979) derives the calculation of strain energy in systems containing inhomogeneities with perfect interfaces.

Let us consider now the previous defined n-layered spherical inclusion (delimited by the Γ_n interface) embedded in an homogeneous elastic matrix whose elastic stiffness tensor is $\mathcal{L}^{(n+1)} = 3k_{n+1}\mathcal{J} + 2\mu_{n+1}\mathcal{K}$. where (μ_{n+1}, k_{n+1}) are respectively the shear modulus and bulk modulus of phase $(n+1)$ (Cf Fig. 1(a)).

All the Γ_i interface $(i \in [1, n-1])$ can be imperfect or not except the Γ_n one which is supposed to be perfect.

Let us denote by Γ_{ext} the external surface. This homogeneous matrix is subjected to uniform stress ($\vec{T}^0 = \boldsymbol{\sigma}^0\vec{n}$) or strain conditions ($\vec{u}^0 = \boldsymbol{\varepsilon}^0\vec{x}$) on its external surface.

The elastic strain energy in the heterogeneous body presented in Fig. 1(a) is defined by:

$$U = \frac{1}{2}\int_V \boldsymbol{\sigma} : \boldsymbol{\varepsilon} dv \tag{82}$$

where $\boldsymbol{\sigma}, \boldsymbol{\varepsilon}$ are respectively the stress and strain tensor in configuration Fig. 1(a) and the elastic strain energy in the homogeneous configuration made only of phase $(n+1)$ is defined by:

$$U_0 = \frac{1}{2}\int_V \boldsymbol{\sigma}^0 : \boldsymbol{\varepsilon}^0 dv \tag{83}$$

where $\boldsymbol{\sigma}^0, \boldsymbol{\varepsilon}^0$ are respectively the stress and strain tensor in this configuration. V denotes the volume of the region inside the surface Γ_{ext}. Eq. (82) can also be written as:

$$\left.\begin{array}{l}U = U_0 + \frac{1}{2}\int_V(\boldsymbol{\sigma} : \boldsymbol{\varepsilon} - \boldsymbol{\sigma}^0 : \boldsymbol{\varepsilon}^0)dv\\ U = U_0 + \frac{1}{2}\int_V(\boldsymbol{\sigma} : \mathbf{grad}\,\vec{u} - \boldsymbol{\sigma}^0 : \mathbf{grad}\,\vec{u}^0)dv\\ U = U_0 + W_p - \frac{1}{2}\int_V \boldsymbol{\sigma}^0 : \mathbf{grad}\,\vec{u}^0 dv\end{array}\right\} \tag{84}$$

with $W_p = \frac{1}{2}\int_V(\boldsymbol{\sigma} : \mathbf{grad}\,\vec{u})dv$

4.1. Interface with jump of the stress vector

It is worth noticing that $\boldsymbol{\sigma}$ and $\boldsymbol{\sigma}^0$ in Eq. (84) satisfy both the equilibrium equation which can be respectively expressed as $\overrightarrow{\mathrm{div}}\,\boldsymbol{\sigma} = 0$ and $\overrightarrow{\mathrm{div}}\,\boldsymbol{\sigma}^0 = 0$ but where $\overrightarrow{\mathrm{div}}\,\boldsymbol{\sigma}$ denotes the divergence with distribution derivatives meaning respecting:

$$\overrightarrow{\mathrm{div}}\,\boldsymbol{\sigma} = \left\{\overrightarrow{\mathrm{div}}\,\boldsymbol{\sigma}\right\} + [\boldsymbol{\sigma}]\,\vec{n}\,\delta_{\Gamma_i} \tag{85}$$

where $\left\{\overrightarrow{\mathrm{div}}\,\boldsymbol{\sigma}\right\}$ denotes the divergence with function derivatives meaning and δ_{Γ_i} the Dirac distribution over the interface Γ_i. $\left\{\overrightarrow{\mathrm{div}}\,\boldsymbol{\sigma}\right\}$ can then be replaced by $-[\boldsymbol{\sigma}]\,\vec{n}\,\delta_{\Gamma_i}$.

$$W_p = \frac{1}{2}\int_V(\boldsymbol{\sigma} : \mathbf{grad}\,\vec{u})dv = \frac{1}{2}\int_V\{\mathrm{div}\,(\boldsymbol{\sigma}\vec{u})\}dv - \frac{1}{2}\int_V\left\{\overrightarrow{\mathrm{div}}\,\boldsymbol{\sigma}\right\}\vec{u}dv \tag{86}$$

Taking into account the jump of the stress vector over all the imperfect interfaces Γ_i (sum over i with $i \in S_{imp}$):

$$W_p = \frac{1}{2}\int_{\Gamma_{ext}}(\vec{u}\,\boldsymbol{\sigma}\,\vec{n})\,ds - \sum_{i\in S_{imp}}\int_{\Gamma_i}\frac{\vec{u}\,[\boldsymbol{\sigma}]\,\vec{n}}{2}ds - \sum_{i\in S_{imp}}\int_V -\frac{(\vec{u}\,[\boldsymbol{\sigma}]\,\vec{n}\,\delta_{\Gamma_i})}{2}dv \tag{87}$$

With $\int_V(\vec{u}\,[\boldsymbol{\sigma}]\,\vec{n}\,\delta_{\Gamma_i})dv = \int_{\Gamma_i}\vec{u}\,[\boldsymbol{\sigma}]\,\vec{n}\,ds, W_p$ becomes:

$$W_p = \frac{1}{2}\int_{\Gamma_{ext}}(\vec{n}\,\boldsymbol{\sigma}\,\vec{u})\,ds \tag{88}$$

Consequently by taking into account that $\overrightarrow{\text{div}}\,\boldsymbol{\sigma}^0 = 0$ and by substituting Eq. (88) in Eq. (84), the energy U can be written in the following form:

$$U = U_0 + \frac{1}{2}\int_{\Gamma_{ext}} \vec{n}(\boldsymbol{\sigma}\vec{u} - \boldsymbol{\sigma}^0\vec{u}^0)dS \tag{89}$$

Using Eshelby's procedure which consists in replacing the effect of the inclusion by a particular distribution of body forces, Christensen (1979), it can be easily shown that the following relation is still valid when there may be a jump of the stress vector on the Γ_i interfaces:

$$U = U_0 + \frac{1}{2}\int_{\Gamma_n} \left(\vec{T}^0.\vec{u} - \vec{T}.\vec{u}^0\right)dS \tag{90}$$

where uniform stress has been applied to the external surface. It is worth noting that Γ_n can be replaced, in Eq. (90), by any Γ_i interface providing there is no jump of stress vector outside this interface (as already pointed out by Benveniste (1985)). The corresponding result when uniform displacement has been applied to the external surface can be written as:

$$U = U_0 - \frac{1}{2}\int_{\Gamma_n} \left(\vec{T}^0.\vec{u} - \vec{T}.\vec{u}^0\right)dS \tag{91}$$

4.2. Interface with jump of the displacement vector

The previous energy approach have already been generalized by Benveniste (1985) to the case in which there may be a jump in the displacements at the interfaces. It is still valid when we consider a composite material with n-layered inclusions.

$$U = U_0 \pm \frac{1}{2}\int_{\Gamma_n} \left(\vec{T}^0.\vec{u} - \vec{T}.\vec{u}^0\right)dS \tag{92}$$

The sign in front of the integral in Eq. (92) depends on the fact that uniform stress or uniform strain conditions are applied to the external surface. It is worth noting that in the case of jump of the displacement vector the possible interpenetration has to be prevented.

The hereabove presented generalization of the Generalized Self Consistent Scheme (GSCS) to n-layered inclusions with imperfect conditions can account for several type of interfaces with for some of them undergoing a jump of the traction vector, for some others undergoing a jump of the displacement vector and for some others being perfect, all this in the same n-layered inclusion.

4.3. (n + 1)-phase model with imperfect interfaces

We can use the solution, presented in Section 2, of the n-phases inclusion with imperfect interfaces embedded in an infinite matrix when the bulk and shear modulus of this matrix named respectively $K^{Seff}_{(n)}$ and $\mu^{Seff}_{(n)}$ denote the effective bulk and shear modulus of the n-layered inclusion based composite ($\mathcal{L}^{(n+1)} = 3K^{Seff}_{(n)}\mathcal{J} + 2\mu^{Seff}_{(n)}\mathcal{K}$).

Our criterion for determining the effective properties is to use the Christensen-Lo's energy condition $U = U_0$. As already shown by Hervé and Zaoui (1993), this energy condition, which can be here, thanks to Eqs. ()()()(90)–(92), still replaced by:

$$\int_{\Gamma_n} \left(\vec{T}^0 \cdot \vec{u} - \vec{T} \cdot \vec{u}^0\right)dS = 0 \tag{93}$$

reduces to G_{n+1} =0 in the case of an applied hydrostatic pressure and to $D_{n+1} = 0$ in the case of an applied simple shear sollicitation when the proper stress and displacement expressions are used from Section 2.

The two conditions G_{n+1} =0 and $D_{n+1} = 0$ lead to the expression of the effective bulk modulus $K^{Seff}_{(n)}$ and the effective shear modulus $\mu^{Seff}_{(n)}$ as shown in Hervé and Zaoui (1993).

These effective moduli are here determined from Eqs. (45) and (51) of Hervé and Zaoui (1993) where the matrix $\mathbf{N}^{(n)}$ (respectively the matrix $\mathbf{M}^{(n)}$) has to be replaced by the matrix $\mathbf{N}^{S(n)}$ (respectively the matrix $\mathbf{M}^{S(n)}$) defined in the present paper (see Appendix B, where Eq. (45) and (51) of Hervé and Zaoui (1993) have become respectively Eqs. (B.3) and (B.4) in the case of imperfect interfaces).

5. Applications to interface models leading to size effects in nano-composites or mismatch in composites

Before considering some new applications with $n > 3$, we have first verified that, in the particular case where $n = 2$ our extended theory provides the same results as the numerical results available in the literature.

- In the case of the considered interface model between phase 1 and phase 2 is a linear spring model, we have compared our predictions of the effective stiffness of composite with such interface, with Sangani and Mo's results (Sangani and Mo, 1997). For that purpose, the discontinuity matrices given by Eq. (43) have been used with $\iota^1_s = \frac{\mu(2)D_n}{R_1}$ and $\zeta^i_s = \frac{\mu(2)D_t}{R_1}$, where D_n and D_t are non-dimensional coefficients introduced by the authors to characterize the interface behavior. This application provides exactly the same results as for example, their Figs. 3 ($D_n \to \infty$) and 4 ($D_t \to \infty$).
- In the case of the considered interface model between phase 1 and phase 2 is a Young–Laplace model, we have compared our predictions of effective stiffness of porous solids with Duan et al.'s results Duan et al. (2005c) who have considered two sets of surface moduli (k_S, μ_S) (referred as cases A and B) taken from the paper of Miller and Shenoy (2000). For this purpose the two discontinuity matrices given in Eqs. (70) and (71) have been used using Eq. (76) where $(\mu^1_I,\ \nu^1_I)$are determined from the two sets of moduli by the link: $\beta^1_I = \frac{k_S}{2}$, $\mu^1_I = \mu_S$. This application provides exactly the same results as for example, their Figs. 1 (effective bulk modulus function of void radius) and 4 (effective shear modulus function of void radius). it is worth noting that Kushch et al. (2013) have made comparison between these results and the predictions of the effective behavior of periodic particulate nanocomposites with the same interface model in the case of the two sets A and B. They have found a very good agreement for porosity lower than 0.4.

For $n > 3$ the following Illustrative examples correspond to different n-layered spherical inhomogeneities represented in Fig. 2. The considered imperfect interfaces correspond either to Asymptotic Rigid interface (named here Γ_{ARi} and characterized by the two discontinuity matrices (70) and (71)) or to Asymptotic Soft interface (named here Γ_{ASo} and characterized by the discontinuity matrices (43) with (72)).

The first application is devoted to the study of size effects in a particle-reinforced composite where the inclusions are surrounded by an interphase which thickness is named t_{int} (Cf Fig. 2(a)). Let f be the volume fraction of inclusions, $(k_{inc}, \mu_{inc}, \nu_{inc})$, $(k_{int}, \mu_{int}, \nu_{int})$, $(k_{mat}, \mu_{mat}, \nu_{mat})$ be respectively the bulk modulus, the shear modulus and the Poisson's ratio of the inclusions, the interphase and the matrix and R_{inc} be the radius of the inclusions.

Let us define the normalized effective shear modulus by μ^{eff}/μ^{2p} where μ^{2p} is the effective shear modulus given by the GSCM considering only two phases in the composite (inclusions + matrix) and let us report in Fig. 3 some normalized effective shear moduli versus the ratio R_{inc}/t_{int}.

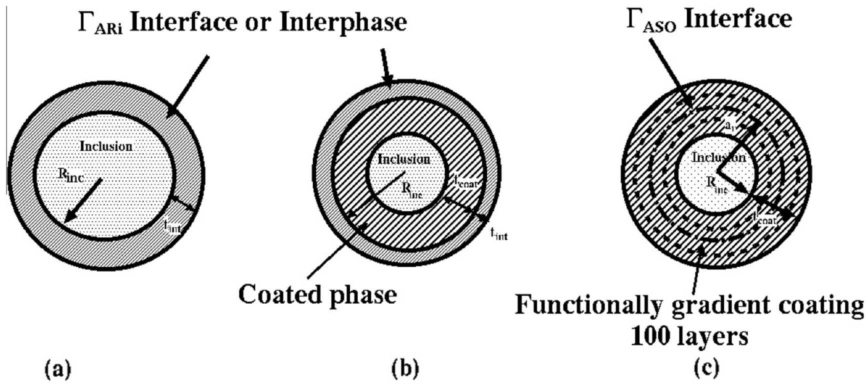

Fig. 2. Applications to different n-layered inhomogeneities.

The normalized effective shear moduli obtained by an asymptotic approach (corresponding to a thin and rigid interface), μ^{ARi}/μ^{2p} are compared in Fig. 3 on the one hand with the normalized ones given by the GSCM considering the three phases of the composite (inclusions + interphase + matrix) without an asymptotic approximation (μ^{3p}/μ^{2p}), and, on the other hand, with the normalized ones given by the GSCM considering that the inclusions and the interphase are rigid μ^{Rigid}/μ^{2p} . Similar results, valid for the bulk moduli are not presented here.

Fig. 4 shows an application to composite with coated inclusions and an interphase is present around these coated inclusions (Fig. 2(b)). Let f be now the volume fraction of coated inclusions and $k_{coat}, \mu_{coat}, \nu_{coat}, t_{coat}$ be respectively the bulk modulus, the shear modulus, the Poisson's ratio and the thickness of the coated phase. Moreover we have considered that the thicknesses of the coating phase and of the interphase are linked by $t_{coat}/t_{int} = 5$. The approach consisting in performing first the calculation of the effective behavior of the coated inclusion and then to replace these coated inclusions by their effective behavior for determining the effective behavior of the whole composite (k^{2steps} and μ^{2steps}) has been compared in Fig. 4 to the correct homogeneization approach (k^{4p} and μ^{4p}) consisting in considering all the phases together in the n-layered spherical inhomogeneity with $n = 4$. R_{inc} denotes in Fig. 2(b) the radius of the coated inclusions. These results show as already known, that this replacement works only for the determination of the effective bulk modulus ($k^{4p}/k_{2steps} = 1$, cf the recurrence relation Eq. (46) (Hervé and Zaoui, 1993)). It is worth noticing that the coated inclusions are homogeneous when $R_{inc} \to t_{coat}$ (the coated inclusions are made only of the coating phase) and for $R_{inc} \to \infty$ (the coated inclusions are only made of the inclusion phase) leading, for these two particular values of R_{inc} , to the same results for the two approaches also regarding the effective shear modulus, $\mu^{4p}/\mu^{2steps} = 1$. The approach consisting of replacing the interphase by a thin rigid interface (effective behavior: k^{ARi} and μ^{ARi}) has also been compared in Fig. 4 to the correct homogeneization approach (k^{4p} and μ^{4p}). We can see that this approach is a good approximation for $R_{inc}/t_{int} \geqslant 100$.

In Fig. 6 we have considered a composite containing inclusions embedded in an accommodating coating phase with linearly variable behavior discretized into various steps (here 100 steps) (cf Fig. 5 for the normalized value k/k_{mat} of the bulk modulus of the different phases) and we account for a mismatch occuring at the Γ_{ASo} interface located in the coating phase, at the distance a_Γ from the center of the inclusions. (cf FigS. 2(c) and 5). (k^{ASo}, μ^{ASo}) and (k^{ud}, μ^{ud}) represent respectively the effective bulk and shear modulus of the composite with the presence of the Γ_{ASo} interface or not (undamaged material in this last case). The fully damaged material is represented by a composite where all the domain inside the Γ_{ASo} interface has become porous.

For the composite studied in Fig. 6, we have represented in Fig. 7 the jump of displacement at the Γ_{ASo} interface $(3[u_r]_{HP}/(\theta_0 R_{inc}))$ obtained from (Eq. (11)) when the composite material is subjected to an hydrostatic pressure (displacement condition Eq. (5)) and the jump of displacement at the Γ_{ASo} interface $[U_r]_S/(\gamma R_{inc})$ obtained from (28) where a simple shear is applied (Eq. (18)), where $u_r^{(i)}(r, \theta, \phi) = U_r(r)\sin^2\theta\cos 2\Phi$, with still fixed 30% volume fraction of inclusions and 21.8 % of coated phase and $\nu_{inc} = \nu_{coat} = \nu_{int} = \nu_{mat} = 0.3, k_{inc}/k_{mat} = 100$ and for different values of k_{int}.

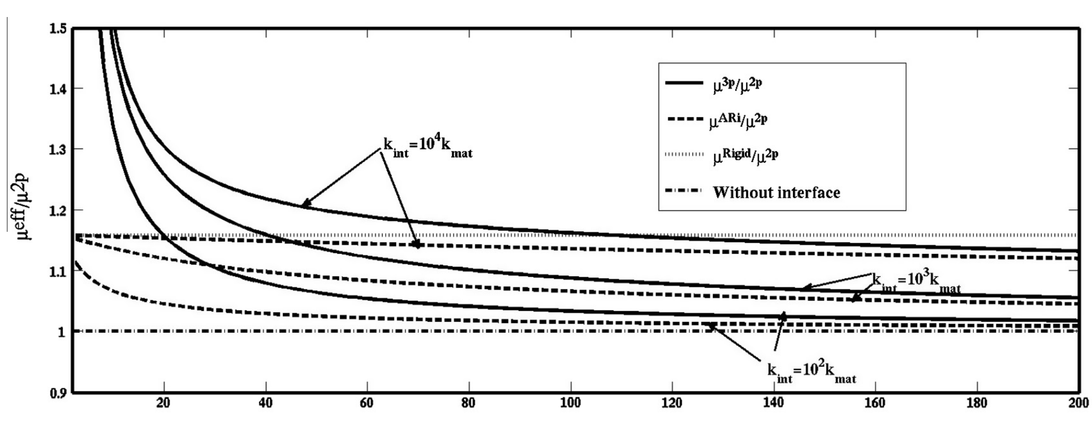

Fig. 3. Comparison between the normalized shear moduli for different values of k_{int}. ($f = 0.3, k_{inc} = 10\ k_{mat}, \nu_{inc} = \nu_{int} = \nu_{mat} = 0.3$).

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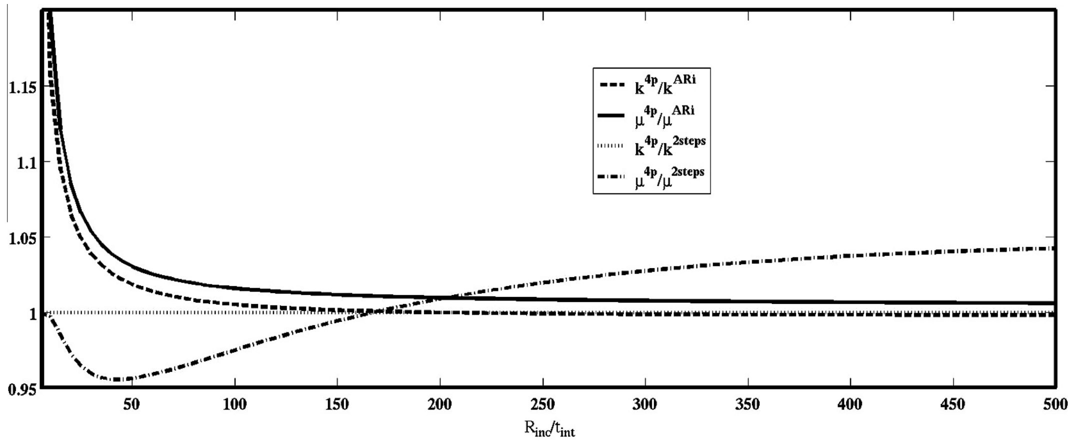

Fig. 4. Comparison between the one, the two steps homogeneization approaches and the asymptotic approach, $f = 30\%$, $\nu_{inc} = \nu_{coat} = \nu_{int} = \nu_{mat} = 0.3$, $k_{inc} = 0$, $k_{coat}/k_{mat} = 30$, $k_{int}/k_{mat} = 5$.

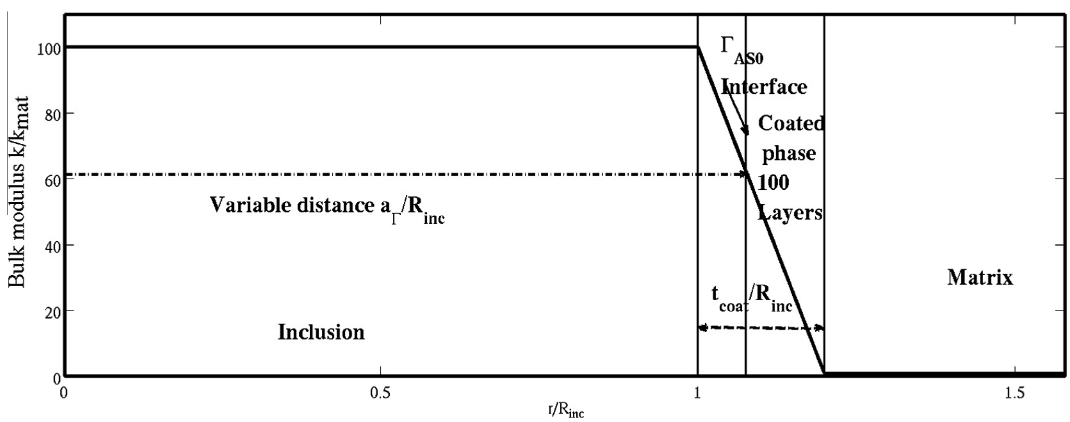

Fig. 5. Bulk modulus inside the basic morphological pattern where a Γ_{ASo} interface is located in the coating phase, at the distance a_Γ from the center of the inclusions.

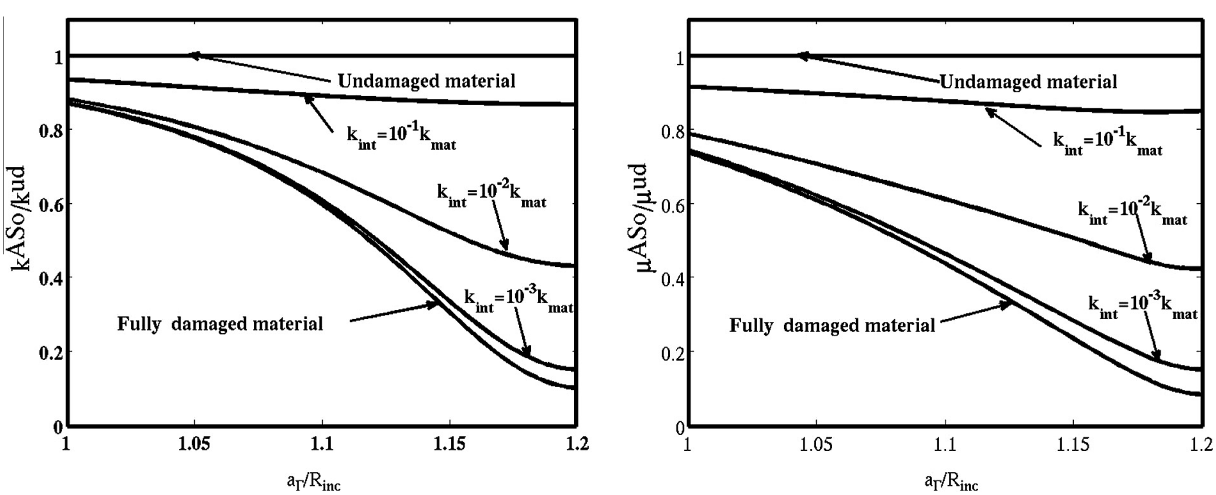

Fig. 6. Effect, on the effective bulk modulus of the composite (left) and on the effective shear modulus (right), of the normalized distance (a_Γ/R_{inc}) of a partially damaged interface, for different values of the k_{int} bulk modulus of the thin soft interphase/interface Γ_{ASo} with fixed 30% volume fraction of inclusions and 21.8 % of coated phase, $\nu_{inc} = \nu_{coat} = \nu_{int} = \nu_{mat} = 0.3$, $k_{inc}/k_{mat} = 100$.

6. Conclusion

The approach developed in this paper gives a general procedure to study the elastic behavior of composite materials reinforced by n-layered inclusions with imperfect bonding conditions between their different phases. The main point of this procedure is to use transfert matrices to express the bonding conditions between the different phases. In the case of imperfect interfaces "two discontinuity matrices" are attached to the interface model characterizing the imperfect interfaces. The "discontinuity matrices" attached to classical imperfect interfaces (linear spring, dislocation-like models, generalized Young Laplace conditions) have been derived.

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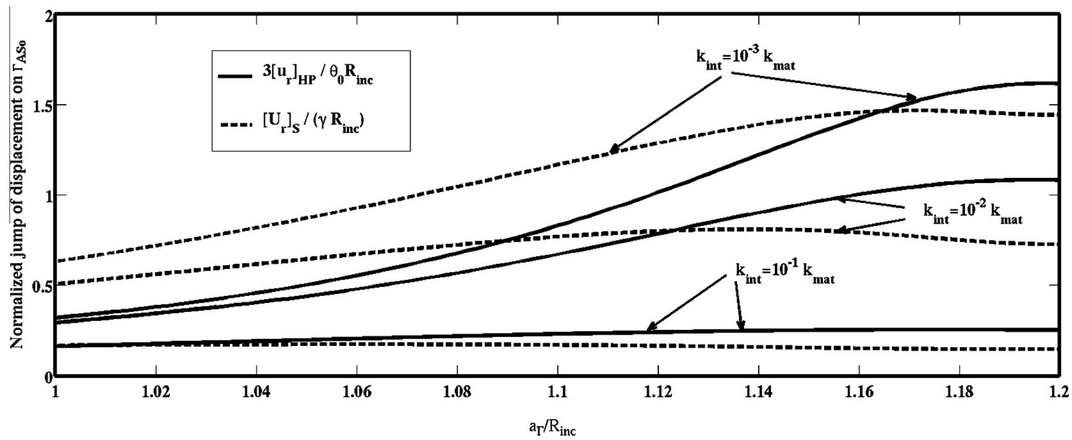

Fig. 7. Normalized jump of displacement in the basic morphological pattern at the Γ_{ASo} interface, $(3[u_r]_{HP}/(\theta_0 R_{inc})$ when an hydrostatic pressure is applied or $[U_r]_S/(\gamma R_{inc})$ where a simple shear is applied, with still fixed 30% volume fraction of inclusions and 21.8 % of coated phase and $\nu_{inc} = \nu_{coat} = \nu_{int} = \nu_{mat} = 0.3$, $k_{inc}/k_{mat} = 100$, for different values of k_{int}.

Section 3 shows in an easy way that we obtain the same "discontinuity matrices" for the interface model based on the generalized Young–Laplace interface conditions for solids as for the interface model obtained by an asymptotic method considering a thin stiff interphase present in a spherical inhomogeneity inside which the volume fraction of each phase is the same as the one in the composite. The homogeneization method developed by Marcadon et al. (2007) when a single Morphological Representative Pattern is used appears to be a generalization of the one developed by Wang et al. (2005) and Duan et al. (2005a). It is also possible, thanks to the present approach to introduce not yet studied interface models in micromechanicals models and to use the resolution of the n-layered spherical inclusion with imperfect interfaces problem in an approach using several Morphological Representative Patterns (Bornert et al., 1996; Marcadon et al., 2007).

Moreover the problem of size effect due to the presence of a constant-thicknessed interphase around inclusions embedded in a matrix has been solved and some results have been presented. The influence of the distance between the center of the inclusions and the place where a mismatch inside a functionally gradient coating occurs has also been presented.

The same procedure, using "discontinuity matrices" can also be used in the case of multiply coated fiber reinforced composites (work in progress).

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Appendix A. Useful matrices

$$\mathbf{J}^{int^{-1}}(R_i) = \frac{1}{3k_I^i + 4\mu_{I,}^i}\begin{pmatrix} \frac{4\mu_{I,}^i}{R_i} & 1 \\ 3k_I^i R_i^2 & -R_i^3 \end{pmatrix} \tag{A.1}$$

$$\left.\frac{\partial \mathbf{J}^{int}}{\partial r}\right|_{R_i} = \begin{pmatrix} 1 & -\frac{2}{R_i^3} \\ 0 & \frac{12\mu_{I,}^i}{R_i^4} \end{pmatrix} \tag{A.2}$$

$\mathbf{L}^{(int)^{-1}}(R_i)$ is given by the following matrix:

$$\frac{1}{5(\nu_I^i - 1)}\begin{bmatrix} \frac{2(\nu_I^i-5)}{3R_i} & \frac{(\nu_I^i+1)}{R_i} & \frac{(4\nu_I^i-5)}{3\mu_I^i} & \frac{(2\nu_I^i-1)}{\mu_I^i} \\ \frac{4(2\nu_I^i-1)}{7R_i^3} & -\frac{4(2\nu_I^i-1)}{7R_i^3} & \frac{(2\nu_I^i-1)}{7\mu_I^i R_i^2} & -\frac{(2\nu_I^i-1)}{7\mu_I^i R_i^2} \\ -\frac{\nu_I^i R_i^4}{7} & \frac{R_i^4(2\nu_I^i+7)}{14} & -\frac{2\nu_I^i R_i^5}{7\mu_I^i} & \frac{(4\nu_I^i-7)R_i^5}{14\mu_I^i} \\ \frac{R_i^2(2\nu_I^i-1)}{3} & \frac{R_i^2(2\nu_I^i-1)}{2} & -\frac{R_i^3(2\nu_I^i-1)}{3\mu_I^i} & -\frac{R_i^3(2\nu_I^i-1)}{2\mu_I^i} \end{bmatrix} \tag{A.3}$$

$$\left.\frac{\partial \mathbf{L}^{(int)}}{\partial r}\right|_{R_i} = \begin{bmatrix} 1 & \frac{-18\nu_I^i}{1-2\nu_I^i}R_i^2 & -\frac{12}{R_i^5} & -\frac{2(5-4\nu_I^i)}{(1-2\nu_I^i)R_i^3} \\ 1 & -\frac{3(7-4\nu_I^i)}{1-2\nu_I^i}R_i^2 & \frac{8}{R_i^5} & -\frac{4}{R_i^3} \\ 0 & \frac{6\mu_I^i\nu_I^i}{1-2\nu_I^i}R_i & \frac{60\mu_I^i}{R_i^6} & -\frac{6\mu_I^i(\nu_I^i-5)}{(1-2\nu_I^i)R_i^4} \\ 0 & -\frac{2\mu_I^i(7+2\nu_I^i)}{1-2\nu_I^i}R_i & -\frac{40\mu_I^i}{R_i^6} & -\frac{6\mu_I^i(1+\nu_I^i)}{(1-2\nu_I^i)R_i^4} \end{bmatrix} \tag{A.4}$$

Appendix B. (*n* + 1)-phase model

From Hervé and Zaoui (1993), the matrix $\mathbf{N}^{(i)}$ defined as $\left(\mathbf{J}^{(i+1)}\right)^{-1}(R_i)\mathbf{J}^{(i)}(R_i)$ is given by:

$$\mathbf{N}^{(i)} = \frac{1}{3k_{i+1} + 4\mu_{i+1}}\begin{pmatrix} 3k_i + 4\mu_{i+1} & \frac{4}{R_i^3}(\mu_{i+1} - \mu_i) \\ 3(k_{i+1} - k_i)R_i^3 & 3k_{i+1} + 4\mu_{i)} \end{pmatrix} \tag{B.1}$$

and the matrix $\mathbf{M}^{(i)} = \left(\mathbf{L}^{(i+1)}\right)^{-1}(R_i)\mathbf{L}^{(i)}(R_i)$ can be expressed as:

$$\mathbf{M}^{(i)} = \frac{1}{5(1-\nu_{i+1})}\left(\begin{matrix} \frac{c_i}{3} & \frac{R_i^2(3b_i-7c_i)}{5(1-2\nu_i)} \\ 0 & \frac{(1-2\nu_{i+1})b_i}{7((1-2\nu_i)} \cdots \\ \frac{R_i^5\alpha_i}{2} & -\frac{R_i^7(2a_i+147\alpha_i)}{70(1-2\nu_i)} \\ -\frac{5}{6}(1-2\nu_{i+1})\alpha_i R_i^3 & \frac{7(1-2\nu_{i+1})\alpha_i R_i^5}{2(1-2\nu_i)} \end{matrix}\right.$$

$$\left.\begin{matrix} -\frac{12\alpha_i}{R_i^5} & \frac{4(f_i-27\alpha_i)}{15(1-2\nu_i)R_i^3} \\ \cdots -\frac{20((1-2\nu_{i+1})\alpha_i}{7R_i^7} & -\frac{12\alpha_i(1-2\nu_{i+1})}{7(1-2\nu_i)R_i^5} \\ \frac{d_i}{7} & \frac{R_i^2\left[105(1-\nu_{i+1})+12\alpha_i(7-10\nu_{i+1})-7e_i\right]}{35(1-2\nu_i)} \\ 0 & \frac{e_i(1-2\nu_{i+1})}{3(1-2\nu_i)} \end{matrix}\right) \tag{B.2}$$

with

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$$\begin{cases} a_i &= (7+5\nu_i)(7-10\nu_{i+1})\frac{\mu_i}{\mu_{i+1}} - (7-10\nu_i)(7+5\nu_{i+1}) \\ b_i &= 4(7-10\nu_i) + (7+5\nu_i)\frac{\mu_i}{\mu_{i+1}} \\ c_i &= (7-5\nu_{i+1}) + 2(4-5\nu_{i+1})\frac{\mu_i}{\mu_{i+1}} \\ d_i &= (7+5\nu_{i+1}) + 4(7-10\nu_{i+1})\frac{\mu_i}{\mu_{i+1}} \\ e_i &= 2(4-5\nu_i) + (7-5\nu_i)\frac{\mu_i}{\mu_{i+1}} \\ f_i &= (4-5\nu_i)(7-5\nu_{i+1}) - (4-5\nu_{i+1})(7-5\nu_i)\frac{\mu_i}{\mu_{i+1}} \\ \alpha_i &= \frac{\mu_i}{\mu_{i+1}} - 1 \end{cases}$$

Let us define $Z^S_{\alpha\beta}$ by $Z^S_{\alpha\beta} = P^{S(n-1)}_{\alpha 1}P^{S(n-1)}_{\beta 2} - P^{S(n-1)}_{\beta 1}P^{S(n-1)}_{\alpha 2}$ with $\alpha \in [1,4]$ and $\beta \in [1,4]$.

The effective behavior $(k^{Seff}_{(n)}, \mu^{Seff}_{(n)})$, respectively the bulk modulus and the shear modulus, of such materials are given by:

$$k^{Seff}_{(n)} = \frac{3k_n R_n^3 Q^{S(n-1)}_{11} - 4\mu_n Q^{S(n-1)}_{21}}{3(R_n^3 Q^{S(n-1)}_{11} + Q^{S(n-1)}_{21})} \tag{B.3}$$

and $\mu^{Seff}_{(n)}$ is the positive root of the following second order equation:

$$A\left(\frac{\mu}{\mu_m})\right)^2 + B\left(\frac{\mu}{\mu_m})\right) + C = 0 \tag{B.4}$$

with:

$$\begin{aligned} A = {} & 4R_n^{10}(1-2\nu_n)(7-10\nu_n)Z^S_{12} + 20R_n^7(7-12\nu_n+8\nu_n^2)(7-10\nu_n)Z^S_{42} \\ & + \cdots + \cdots 12R_n^5(1-2\nu_n)(Z^S_{14}-7Z^S_{23}) + 20R_n^3(1-2\nu_n)^2 Z^S_{13} \\ & + 16(4-5\nu_n)(1-2\nu_n)Z^S_{43} \\ B = {} & 3R_n^{10}(1-2\nu_n)(15\nu_n-7)Z^S_{12} + 60R_n^7(\nu_n-3)\nu_n Z^S_{42} + \cdots \\ & + \cdots - 24R_n^5(1-2\nu_n)(Z^S_{14}-7Z^S_{23}) - 40R_n^3(1-2\nu_n)^2 Z^S_{13} \\ & -8(1-5\nu_n)(1-2\nu_n)Z^S_{43} \\ C = {} & -R_n^{10}(1-2\nu_n)(7+5\nu_n)Z^S_{12} + 10R_n^7(7-\nu_n^2)Z^S_{42} + \cdots \\ & + \cdots 12R_n^5(1-2\nu_n)(Z^S_{14}-7Z^S_{23}) + 20R_n^3(1-2\nu_n)^2 Z^S_{13} \\ & -8(7-5\nu_n)(1-2\nu_n)Z^S_{43} \end{aligned}$$

Appendix C. Supplementary data

Supplementary data associated with this article can be found, in the online version, at http://dx.doi.org/10.1016/j.ijsolstr.2014.04.008.

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