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Effective elastic moduli of a particulate composite in terms of the dipole moments and property contribution tensors

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ABSTRACT

The paper focuses on the comparison of two approaches used for calculation of the effective elastic properties of particulate composites: the dipole moments representation and the technique based on property contribution tensors. Its specific goal is to bridge the gap between the two methods and to identify the key microstructural parameters affecting overall elastic stiffness of heterogeneous materials. The basic concepts of the homogenization theory including a consistent way of introducing the macroscopic field parameters are discussed and clarified. We provide a detailed comparison of the analytical expressions for the dipole moment tensors obtained by the multipole expansion method and for the stiffness contribution tensors and show that they coincide.

1. Introduction

In the present paper we discuss the connection between two approaches that can be applied to calculate effective elastic properties of heterogeneous materials: (1) the multipole expansion and (2) the property contribution tensors. It can be considered as extension of the work of the present authors (Kushch and Sevostianov, 2014), where conductive properties were discussed, to the case of effective elastic properties.

Connection between the compliance contribution tensors and far-field asymptotes received some attention in literature. Jasiuk et al. (1994) and Jasiuk (1995) considering 2-D polygonal holes, made an observation that the far-field asymptotic of the hole-generated fields fully determines the compliance contribution of the hole.

Actually, sufficiency of the far-fields for proper description of the contributions of the inhomogeneities to effective properties extends to the general 3-D case as shown by Sevostianov and Kachanov (2011). The extra overall strain due to the presence of an inhomogeneity in reference volume V is given by the well-known expression in terms of an integral over the boundary ∂V (Hill, 1963):

$$\Delta\boldsymbol{\varepsilon} = \frac{1}{2V}\int_{\partial V}(\Delta\mathbf{u}\mathbf{n} + \mathbf{n}\Delta\mathbf{u})dS, \tag{1.1}$$

where $\Delta\mathbf{u}$ are extra displacements due to the inhomogeneity and n_i is the outward unit normal to ∂V. Volume V can be arbitrarily large, hence the far-field asymptotics of $\Delta\mathbf{u}$ is sufficient for determination of the compliance contribution of an inhomogeneity. Formula (1.1) gives the compliance contribution of an inhomogeneity in terms of experimentally measurable quantities – displacements of the specimen boundaries; in this context, volume V must be large to neglect the inhomogeneity-boundary interaction thus making the far-field asymptotic necessary.

The far-field asymptotics of elastic field is *shape-dependent*, even in cases when the inhomogeneity compliance contribution is isotropic (for example, when the inhomogeneity shape has the symmetry of any equilateral polygon, except square).This is in contrast with shape *independence* of the inhomogeneity contributions to the physical properties characterized by *second-rank* tensors (see Kushch and Sevostianov, 2014), such as the conductive or dielectric ones: for them, the isotropic case is characterized by only one constant, hence any isotropic – in regard to these properties – shape (such as any equilateral polygon including square) can be replaced by a circle of appropriate radius.

The structure of the far-field and its shape dependence can be clarified using the multipole expansions (Kushch, 2013). Batchelor (1974) suggested to calculate average stress – and thus the effective stiffness of composite – in terms of the induced dipole moments of particles populating the representative volume element (RVE). The elastic dipole moment is formally defined (see, for example, Vakulenko and Kosheleva, 1980; Kosheleva, 1983) as the coefficient in the multipole series expansion of displacement

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disturbance field associated with the dipole term. Multipole expansions can be illustrated on a system of forces distributed in volume V. At distance r that is much larger than linear dimensions of V, elastic fields can be represented as a sum of terms: the first one is generated by the principal vector of forces (it decreases as r^{-2} for stresses and r^{-1} for displacements); the second one – by dipoles, i.e. pairs of equal and opposite point forces applied at closely spaced points (it decreases as r^{-3} and r^{-2}); the third one – by quadrupoles – closely spaced dipoles of opposite signs (it decreases as r^{-4} and r^{-3}), etc. The first term (generated by the principal vector) is a dominant one. Such expansions can be extended from a discrete system of forces to a distribution of stresses (or strains) in V: the role of the principal vector is played then by the integral $\int_V \sigma_{ij} dV$ and higher order moments take the form $\int_V x_k \sigma_{ij} dV$, $\int_V x_k x_l \sigma_{ij} dV$, etc. We refer to the book of Lur'e (1964) for the case of discrete system of forces and the book of Kanaun and Levin (2008) for a more general form of distributions.

The connection between the property contribution tensors and multipole expansion method is not yet well recognized and understood. This work aims at establishing the connection between two different approaches to the problem of homogenization and to identify and discuss the key microstructural parameters affecting overall elastic properties of heterogeneous materials. It follows the idea proposed by the present authors (Kushch and Sevostianov, 2014) for overall conductivity (thermal or electric) of heterogeneous materials.

2. Background material

For readers convenience, in this section we briefly outline the concepts of (1) property contribution tensors and (2) the multipole expansion. These topics, being known for several decades, are not widely used in the problems of homogenization.

2.1. Compliance and stiffness contribution tensors

Compliance contribution tensors have been first introduced by Horii and Nemat-Nasser (1983) for pores of ellipsoidal shape (explicit formulas connecting compliance contribution tensor and Eshelby tensor for an ellipsoidal pore are given in the appendix of the mentioned paper). Components of this tensor for two-dimensional pores of arbitrary shape were given by Kachanov et al. (1994) and for ellipsoidal inhomogeneities – by Sevostianov and Kachanov (1999). Connection between compliance and stiffness contribution tensors has been discussed by Sevostianov and Kachanov (2007b). The significance of these tensors for the homogenization theory is that their sum is the proper microstructural parameter in whose terms the considered effective property has to be expressed. In other words, it is *these* tensors that have to be summed up, or averaged over a RVE to calculate overall elastic properties.

In the context of linear elastic properties, the average, over representative volume V strain can be represented as a sum

$$\langle \boldsymbol{\varepsilon} \rangle = \mathbf{S}_0 : \boldsymbol{\sigma}^\infty + \Delta\boldsymbol{\varepsilon}, \tag{2.1}$$

where $\mathbf{S}_0$ is the compliance tensor of the matrix and $\boldsymbol{\sigma}^\infty$ represents the homogeneous boundary conditions (tractions on ∂V have the form $\mathbf{t}|_{\partial V} = \boldsymbol{\sigma}^\infty \cdot \mathbf{n}$ where $\boldsymbol{\sigma}^\infty$ is a constant tensor); $\boldsymbol{\sigma}^\infty$ can be viewed as a far-field, or remotely applied, stress. The material is assumed to be linear elastic, hence the extra strain $\Delta\boldsymbol{\varepsilon}$ due to inhomogeneity of volume V_1 is proportional to applied stress and compliance contribution tensor is the proportionality factor in this relation:

$$\Delta\boldsymbol{\varepsilon} = (V_1/V)\mathbf{H} : \boldsymbol{\sigma}^\infty. \tag{2.2}$$

In the case of multiple inhomogeneities, $\Delta\boldsymbol{\varepsilon} = (1/V)\sum_i V_i \mathbf{H}^{(i)} : \boldsymbol{\sigma}^\infty$ so that the extra compliance due to inhomogeneities is given by

$$\Delta\mathbf{S} = (1/V)\sum_i V_i \mathbf{H}^{(i)}. \tag{2.3}$$

Alternatively, one can consider the extra average stress $\Delta\boldsymbol{\sigma}$ due to an inhomogeneity under given applied displacement homogeneous boundary conditions (displacements on ∂V have the form $\mathbf{u}|_{\partial V} = \boldsymbol{\varepsilon}^\infty \cdot \mathbf{x}$ where $\boldsymbol{\varepsilon}^\infty$ is a constant tensor). This defines the *stiffness contribution tensor* of an inhomogeneity:

$$\Delta\boldsymbol{\sigma} = (V_1/V)\mathbf{N} : \boldsymbol{\varepsilon}^\infty, \tag{2.4}$$

In the case of multiple inhomogeneities, the extra stiffness due to inhomogeneities is given by

$$\Delta\mathbf{C} = (1/V)\sum_i V_i \mathbf{N}^{(i)}. \tag{2.5}$$

The property contribution tensors, obviously, have the same rank and symmetry as the tensors characterizing the property: $\mathbf{H}$ and $\mathbf{N}$ are fourth-rank tensors with $ijkl$ components symmetric with respect to $i \leftrightarrow j$, $k \leftrightarrow l$ and $ij \leftrightarrow kl$.

The $\mathbf{H}$- and $\mathbf{N}$-tensors are determined by the shape of the inhomogeneity, as well as properties of the matrix and of the inhomogeneity material.

Remark. The property contribution tensors defined via Eqs. (2.2) and (2.5) do not depend on the size of inhomogeneity. This definition is different from those used, for example, by Sevostianov and Kachanov, 2002, where multiplier (V_1/V) was absorbed by the tensors. The present definition has a number of advantages. For example, the problem of distinction between infinite cylinder and a needle is irrelevant. The difference between these two shapes is in the multiplier (V_1/V) only.

The compliance and stiffness contribution tensors are also affected by elastic interactions. In the non-interaction approximation, they are taken by treating the inhomogeneities as isolated ones. These tensors for a given inhomogeneity are interrelated, as follows. The overall compliance of certain volume containing one inhomogeneity $\mathbf{S}_0 + \mathbf{H}$ is an inverse of its stiffness tensor $\mathbf{C}_0 + \mathbf{N}$, i.e. their product equals the fourth-rank unit tensor implying that $\mathbf{N} = -\mathbf{C}_0 : \mathbf{H} : \mathbf{C}_0 - \mathbf{N} : \mathbf{H} : \mathbf{C}_0$. The $\mathbf{H}$- and $\mathbf{N}$-tensors scale as the ratio l^3/V that can be made arbitrarily small by enlarging V. Hence the second term can be neglected so that

$$\mathbf{N} = -\mathbf{C}_0 : \mathbf{H} : \mathbf{C}_0 \tag{2.6}$$

or, in the case of the isotropic matrix,

$$-N_{ijkl} = \lambda_0^2 H_{mmnn}\delta_{ij}\delta_{kl} + \mu_0^2 H_{ijkl} + \lambda_0\mu_0(\delta_{ij}H_{mmkl} + \delta_{kl}H_{mmij}), \tag{2.7}$$

where λ_0 and μ_0 are Lame constants of the matrix.

For an *ellipsoidal inhomogeneity*, compliance and stiffness contribution tensors can be explicitly expressed in terms of Hill's tensors $\mathbf{Q}$ and $\mathbf{P}$ (Walpole, 1966) or in terms of Eshelby's tensor $\mathbf{s}$ (given for example in book of Mura, 1987) and, therefore, in terms of ellipsoid geometry. For compliance contribution tensor, one can write (Sevostianov and Kachanov, 1999):

$$\mathbf{H} = \left[(\mathbf{S}_1 - \mathbf{S}_0)^{-1} + \mathbf{Q}\right]^{-1}. \tag{2.8}$$

where $\mathbf{S}_1$ is compliance of the inhomogeneity material. In the case of a pore, $\mathbf{H} = \mathbf{Q}^{-1}$. Similarly, the stiffness contribution tensor is obtained as

$$\mathbf{N} = \left[(\mathbf{C}_1 - \mathbf{C}_0)^{-1} + \mathbf{P}\right]^{-1}. \tag{2.9}$$

with $\mathbf{C}_1$ being inhomogeneity stiffness. For a perfectly rigid inhomogeneity, $\mathbf{N} = \mathbf{P}^{-1}$.

We now consider a *spheroidal* inhomogeneity with the rotation axis $\mathbf{i}_3$, semi-axes $a_1 = a_2 = a$ and a_3 and aspect ratio $\gamma = a_3/a$. The Eshelby tensor and all the property contribution tensors are elementary functions of γ. To express the $\mathbf{H}$ and $\mathbf{N}$-tensors in terms of γ, we represent them as linear combinations of six tensors $\mathbf{T}^{(1)}, \ldots, \mathbf{T}^{(6)}$ that form a tensor basis for transversely-isotropic fourth-rank tensors (see Appendix C):

$$\mathbf{H} = \sum_{m=1}^{6} h_m \mathbf{T}^{(m)}; \quad \mathbf{N} = \sum_{m=1}^{6} n_m \mathbf{T}^{(m)}. \tag{2.10}$$

This reduces the problem to calculation of scalar coefficients h_m and n_m as functions of γ and material constants.

Although various applications require quantitative characterization of inhomogeneities of irregular shape, most of the existing results are based on Eshelby (1957, 1961) solution for the ellipsoidal inhomogeneity. While for 2-D non-elliptical inhomogeneities many analytical and numerical results have been obtained, only a limited number of numerical results and approximate estimates are available for more complex 3-D shapes (see literature review of Sevostianov and Giraud, 2012).

2.2. Multipole expansion for displacement disturbance field

The local stress equilibrium equations in the constituents of composite are

$$\nabla \cdot \boldsymbol{\sigma}^{(i)} = 0, \quad \boldsymbol{\sigma}^{(i)} = \mathbf{C}_i : \boldsymbol{\varepsilon}^{(i)}, \quad 2\boldsymbol{\varepsilon}^{(i)} = \nabla \mathbf{u}^{(i)} + \left(\nabla \mathbf{u}^{(i)}\right)^T. \tag{2.11}$$

Here, $\mathbf{u}^{(i)}$ is the displacement vector, $\boldsymbol{\varepsilon}^{(i)}$ is the strain tensor, $\boldsymbol{\sigma}^{(i)}$ is the stress tensor and $\mathbf{C}_i$ is the elastic stiffness tensor of ith phase: $i = 0$ for matrix and $i = 1$ for inhomogeneities. Both the matrix and inhomogeneities are assumed to be anisotropic. Due to linearity of the considered problem, the displacement field in the matrix domain $\mathbf{u}^{(0)}$ in a vicinity of inhomogeneity can be splitted as

$$\mathbf{u}^{(0)} = \mathbf{u}_{far} + \mathbf{u}_{dis}, \tag{2.12}$$

where $\mathbf{u}_{far}$ is an incident field and $\mathbf{u}_{dis}$ is a disturbance field due to this inhomogeneity. From the physical reasonings, $\mathbf{u}_{dis} \to 0$ for $\|\mathbf{x}\| \to \infty$ that enables its series expansion over the multipoles. For our study, the dipole moment $\mathbf{t}$ of $\mathbf{u}_{far}$ multipole expansion is of particular interest. In order to avoid possible ambiguities in the terminology and interpretation of results, we start with the definitions.

In the conductivity theory, the induced dipole moment of inclusion is commonly defined as a vector governing asymptotic behavior of the disturbance field (Landau and Lifshitz, 1951). This quantity was *de-facto* employed by Maxwell to derive his famous formula for the effective conductivity of a composite containing spherical particles. Its counterpart, an elastic dipole moment tensor draws more attention of mathematicians (Ammari et al., 2007; Nazarov, 2009) and physicists (e.g., Puls, 1985; Pfeiffer and Mahan, 1993; Balluffi, 2012). In the theory of multipoles, the dipole moment is understood as a series coefficient, associated with the dipole term. The formal multipole expansion of the displacement field around the inclusion (Vakulenko and Kosheleva, 1980; Kosheleva, 1983) is written in our notations as

$$\mathbf{u}_{dis}(\mathbf{x}) = \sum_{k=1}^{\infty} \underbrace{\nabla\nabla\ldots\nabla}_{k} \mathbf{G}(\mathbf{x}) \underbrace{\cdots}_{(k+1)} \mathbf{T}^k, \tag{2.13}$$

where $\mathbf{x} = x_j \mathbf{i}_j$ is the position vector and $\mathbf{G}(\mathbf{x})$ is Green's tensor (Kelvin, 1848). The $(k+1)$-rank tensors $\mathbf{T}^k$ are defined as

$$\mathbf{T}^k = \frac{(-1)^{k-1}}{(k-1)!} \int_{V_1} \underbrace{\mathbf{x}'\mathbf{x}'\ldots\mathbf{x}'}_{k-1} (\mathbf{C}_1 - \mathbf{C}_0) : \boldsymbol{\varepsilon}(\mathbf{x}') \mathrm{d}\mathbf{x}'. \tag{2.14}$$

So the dipole approximation is given by the first term of series in Eq. (2.14), namely,

$$\nabla \mathbf{G}(\mathbf{x}) : \mathbf{T}^1, \tag{2.15}$$

where the elastic dipole moment

$$\mathbf{T}^1 = (\mathbf{C}_1 - \mathbf{C}_0) : \int_{V_1} \boldsymbol{\varepsilon}(\mathbf{x}') \mathrm{d}\mathbf{x}'. \tag{2.16}$$

An alternative less formal definition has been suggested by Batchelor (1974) who introduced the induced *dipole moment* $\mathbf{t}$ of a single inhomogeneity as "*a measure of the net additional dipole strength* ... *resulting from the replacement of matrix material there by particle material*". In the elasticity theory framework, $\mathbf{t}$ is the second rank tensor written as

$$\mathbf{t}^{(i)} = \int_{V_i} (\boldsymbol{\sigma} - \mathbf{C}_0 : \boldsymbol{\varepsilon}) \mathrm{d}\mathbf{x}, \tag{2.17}$$

where V_i is the volume of ith inhomogeneity with stiffness $\mathbf{C}_1$. As easy to see, $\mathbf{t}^{(i)}$ is consistent with $\mathbf{T}^1$ defined by Eq. (2.14). Sometimes, this tensor is also referred as the "average elastic polarization" (Lipton, 1993). In what follows, we call $\mathbf{t}$ the *induced elastic dipole moment tensor* or, briefly, the dipole moment.

It has been shown (Kushch, 2013) that for any finite domain V with surface S,

$$\mathbf{C}_0 : \int_S (\mathbf{n} \cdot \nabla \mathbf{u}^{(0)} \mathbf{x} - \mathbf{u}^{(0)} \mathbf{n}) \mathrm{d}S = \begin{cases} \mathbf{t} = \text{const}, & V_1 \subset V, \\ 0, & \text{otherwise}. \end{cases} \tag{2.18}$$

Alternatively, Eq. (2.18) can be viewed as a *definition* of the dipole moment. The advantage of such definition is that, in contrast to Eqs. (2.16) and (2.16), it does not require the knowledge of the field inside the inhomogeneity. As a consequence, Eq. (2.18) is valid for any (not necessarily linear) far field, general type anisotropy of matrix material and *arbitrary* (in terms of shape, structure, properties, interface bonding conditions, etc.) inhomogeneities including the cavities and cracks. It appears that integral in Eq. (2.18) equals zero for any (not necessarily linear) regular field $\mathbf{u}_{far}$ and only the disturbance field $\mathbf{u}_{dis}$ contributes to the dipole moment: $\mathbf{t}(\mathbf{u}^{(0)}) = \mathbf{t}(\mathbf{u}_{dis})$.

Due to linearity of the problem, the induced dipole moment is proportional to the far-field strain, so $\mathbf{t} = (\mathbf{C}_1 - \mathbf{C}_0)\mathbf{W} : \boldsymbol{\varepsilon}^{\infty}$, where $(\mathbf{C}_1 - \mathbf{C}_0)\mathbf{W}$ is referred as the elastic polarization tensor (Lipton, 1993) and $\mathbf{W}$ is known as Wu strain tensor (see Wu, 1966). What important in the context of our study, is that the integral in Eq. (2.14) over RVE containing a number of inhomogeneities equals to the sum of induced dipole moments of each separate inhomogeneity. It will be clear from the analysis to follow that the elastic polarization tensor quite closely relate the stiffness contribution tensor, so this summation rule is consistent with the above mentioned observation that these tensors contribute to the overall elastic properties additively.

3. Isotropic matrix

3.1. Compliance and stiffness contribution tensors for a spheroidal inhomogeneity

We assume here that both the matrix and the inhomogeneity are isotropic. Results can be extended to cases when the inhomogeneities – but not the matrix – possess an arbitrary anisotropy since the latter would only affect the field of the fictitious body force. Components of tensors $\mathbf{P}$ and $\mathbf{Q}$ for a spheroidal inhomogeneity have been first given by Walpole (1966). They can be written in terms of the tensor basis $\mathbf{T}^{(1)}, \ldots, \mathbf{T}^{(6)}$ as follows (Sevostianov and Kachanov, 1999):

$$\mathbf{P} = \sum_{m=1}^{6} p_m \mathbf{T}^{(m)}; \quad \mathbf{Q} = \sum_{m=1}^{6} q_m \mathbf{T}^{(m)}; \tag{3.1}$$

where factors p_m and q_m are given by:

$$\begin{aligned} p_1 &= \frac{1}{2\mu}[(1-\kappa)f_0 + \kappa f_1], \quad p_2 = \frac{1}{2\mu}[(2-\kappa)f_0 + \kappa f_1], \\ p_3 &= p_4 = -\frac{\kappa}{\mu} f_1, \quad p_5 = \frac{1}{\mu}(1 - f_0 - 4\kappa f_1), \\ p_6 &= \frac{1}{\mu}[(1-\kappa)(1-2f_0) + 2\kappa f_1] \end{aligned} \tag{3.2}$$

and

$$\begin{aligned} q_1 &= \mu[4\kappa - 1 - 2(3\kappa - 1)f_0 - 2\kappa f_1], \\ q_2 &= 2\mu[1 - (2-\kappa)f_0 - \kappa f_1], \quad q_3 = q_4 = 2\mu[(2\kappa - 1)f_0 + 2\kappa f_1], \\ q_5 &= 4\mu(f_0 + 4\kappa f_1), \quad q_6 = 8\mu\kappa(f_0 - f_1). \end{aligned} \tag{3.3}$$

In Eqs. (3.2) and (3.3), $\kappa = 1/[2(1-\nu_0)]$ and the following shape functions are introduced:

$$f_0 = \frac{1-g}{2(1-\gamma^2)}; \quad f_1 = \frac{1}{4(1-\gamma^2)^2}[(2+\gamma^2)g - 3\gamma^2]; \tag{3.4}$$

$$g(\gamma) = \begin{cases} \frac{\gamma^2}{\sqrt{\gamma^2-1}} \arctan\sqrt{\gamma^2-1}, & \text{oblate spheroid, } \gamma \geqslant 1; \\ \frac{\gamma^2}{2\sqrt{1-\gamma^2}} \ln\frac{1+\sqrt{1-\gamma^2}}{1-\sqrt{1-\gamma^2}}, & \text{prolate spheroid, } \gamma \leqslant 1. \end{cases} \tag{3.5}$$

Representing tensors of the isotropic elastic constants in the same basis and doing inversions in formulas (2.8) and (2.9) using (C.4), we arrive at the following results (see Sevostianov and Kachanov, 1999).

(A) Coefficients h_m of the compliance contribution tensor:

$$\begin{aligned} h_1 &= \frac{1}{2\Delta}\left(K_* + \frac{4}{3}\mu_* + q_6\right); \quad h_2 = \frac{1}{2\mu_* + q_2}; \\ h_5 &= \frac{4}{4\mu_* + q_5}; \quad h_3 = h_4 = -\frac{1}{\Delta}\left(K_* - \frac{2}{3}\mu_* + q_3\right); \\ h_6 &= \frac{2}{\Delta}\left(K_* + \frac{1}{3}\mu_* + q_1\right); \end{aligned} \tag{3.6}$$

where K is a bulk modulus, μ is a shear modulus, factors q_i are given by (3.3) and the following notations for elastic constants are used:

$$\begin{aligned} K_* &= K_1 K_0/(K_0 - K_1), \quad \mu_* = \mu_1\mu_0/(\mu_0 - \mu_1), \\ \Delta &= 2[3\mu_* K_* + K_*(q_1 + q_6 - 2q_3) + (\mu_*/3)(4q_1 + q_6 + 4q_3) \\ &\quad + (q_1 q_6 - q_3^2)]. \end{aligned} \tag{3.7}$$

(B) Coefficients n_m of the stiffness contribution tensor are:

$$\begin{aligned} n_1 &= \frac{1}{2\Delta_1}\left[\frac{\delta\lambda + \delta\mu}{\delta\mu(3\delta\lambda + 2\delta\mu)} + p_6\right]; \quad n_2 = \frac{2\delta\mu}{1 + 2p_2\delta\mu}; \\ n_3 &= n_4 = -\frac{1}{\Delta_1}\left[-\frac{\delta\lambda}{2\delta\mu(3\delta\lambda + 2\delta\mu)} + p_3\right]; \quad n_5 = \frac{4\delta\mu}{1 + \delta\mu p_5}; \\ n_6 &= \frac{1}{\Delta_1}\left[\frac{\delta\lambda + 2\delta\mu}{2\delta\mu(3\delta\lambda + 2\delta\mu)} + 2p_1\right]; \end{aligned} \tag{3.8}$$

where the p_i factors are given by (3.2) and the following notations are used:

$$\begin{aligned} \delta\mu &= \mu_1 - \mu_0, \quad \delta\lambda = \lambda_1 - \lambda_0, \\ \Delta_1 &= \frac{1 + (\delta\lambda + 2\delta\mu)p_6 + 4(\delta\lambda + \delta\mu)p_1 + 4\delta\lambda p_3}{2\delta\mu(3\delta\lambda + 2\delta\mu)} + 2p_1p_6 - 2p_3^2. \end{aligned} \tag{3.9}$$

In the case of a *pore* ($K_1 = \mu_1 = 0$), formulas for h_i simplified as:

$$\begin{aligned} h_1 &= \frac{q_6}{4(q_1q_6 - q_3^2)}; \quad h_2 = \frac{1}{q_2}; \quad h_3 = h_4 = -\frac{q_3}{2(q_1q_6 - q_3^2)}; \\ h_5 &= \frac{4}{q_5}; \quad h_6 = \frac{q_1}{q_1q_6 - q_3^2}. \end{aligned} \tag{3.10}$$

For a *perfectly rigid* inhomogeneity ($\delta\lambda \to \infty, \delta\mu \to \infty$) formulas for coefficients n_i simplified as

$$\begin{aligned} n_1 &= \frac{p_6}{4(p_1p_6 - p_3^2)}; \quad n_2 = \frac{1}{p_2}; \quad n_3 = n_4 = -\frac{p_3}{2(p_1p_6 - p_3^2)}; \\ n_5 &= \frac{4}{p_5}; \quad n_6 = \frac{p_1}{p_1p_6 - p_3^2}. \end{aligned} \tag{3.11}$$

For the *spheroidal* geometry, the $\mathbf{H}$ – tensor reduces, after some algebra, to three groups of terms: isotropic terms expressed in unit tensors of the second and fourth ranks, $\mathbf{I}$ and $\mathbf{J}$, terms containing the dyad $\mathbf{nn}$ and a term containing $\mathbf{nnnn}$:

$$\mathbf{H} = \left[\underbrace{W_1\mathbf{II} + W_2\mathbf{J}}_{\text{isotropic terms}} + W_3(\mathbf{Inn} + \mathbf{nnI}) + W_4(\mathbf{J}\cdot\mathbf{nn} + \mathbf{nn}\cdot\mathbf{J}) + W_5\mathbf{nnnn}\right], \tag{3.12}$$

where scalar factors W_i are expressed in terms of coefficients h_i:

$$\begin{aligned} W_1 &= h_1 - h_2/2; \quad W_2 = h_2; \quad W_3 = 2h_3 + h_2 - 2h_1; \\ W_4 &= h_5 - 2h_2; \quad W_5 = h_6 + h_1 + h_2/2 - 2h_3 - h_5. \end{aligned} \tag{3.13}$$

Similarly, for the $\mathbf{N}$ -tensor we have

$$\mathbf{N} = \left[\underbrace{U_1\mathbf{II} + U_2\mathbf{J}}_{\text{isotropic terms}} + U_3(\mathbf{Inn} + \mathbf{nnI}) + U_4(\mathbf{J}\cdot\mathbf{nn} + \mathbf{nn}\cdot\mathbf{J}) + U_5\mathbf{nnnn}\right], \tag{3.14}$$

where factors U_i are expressed in terms of coefficients n_i by formulas similar to (3.13):

$$\begin{aligned} U_1 &= n_1 - n_2/2; \quad U_2 = n_2; \quad U_3 = 2n_3 + n_2 - 2n_1; \\ U_4 &= n_5 - 2n_2; \quad U_5 = n_6 + n_1 + n_2/2 - 2n_3 - n_5. \end{aligned} \tag{3.15}$$

3.2. Induced dipole moments

3.2.1. General form for isotropic matrix

We denote R_c the radius of minimal sphere fully encompassing the inhomogeneity. For $r = \|\mathbf{x}\| > R_c$, the multipole expansion of the disturbance displacement field $\mathbf{u}_{dis}$ caused by this inhomogeneity is given by the series

$$\mathbf{u}_{dis}(\mathbf{x}) = \sum_{i=1}^{3}\sum_{t=0}^{\infty}\sum_{|s|\leqslant t} a_{ts}^{(i)}\mathbf{U}_{ts}^{(i)}(\mathbf{x}), \tag{3.16}$$

where $\mathbf{U}_{ts}^{(i)}$ are the irregular vector solutions of Lame equation defined in Appendix B and $a_{ts}^{(i)}$ are the expansion coefficients, or multipole moments. The Cartesian components of the displacement vector are real numbers, hence $\mathbf{u} = \overline{\mathbf{u}}$. Since $\mathbf{U}_{t,-s}^{(i)} = (-1)^s\overline{\mathbf{U}_{ts}^{(i)}}$, the series expansion coefficients $a_{ts}^{(i)}$ obey the similar relations, i.e., $a_{t,-s}^{(i)} = (-1)^s\overline{a_{ts}^{(i)}}$. For $r \to \infty$, the leading, $O(r^{-2})$ asymptotic term is the dipole sum

$$\sum_{i=1}^{3}\sum_{|s|\leqslant i-1} a_{i-1,s}^{(i)}\mathbf{U}_{i-1,s}^{(i)}(\mathbf{x}). \tag{3.17}$$

Although integration in Eq. (2.18) can be done over any surface S encompassing the inclusion, the spherical surface of radius $r > R_c$ is the most convenient choice. It has been shown by Kushch (2013) that only the terms entering Eq. (3.17) contribute to $\mathbf{t}$ in Eq. (2.18).

Omitting the details, we write the final formulas for the components of the dipole moment tensor $\mathbf{t}$ of inhomogeneity in terms of the multipole expansion coefficients:

$$\begin{aligned} t_{11}+t_{22}+t_{33} &= 24\pi\frac{\mu_0(1-\nu_0)}{(1-2\nu_0)}a_{00}^{(1)}; \\ 2t_{33}-t_{11}-t_{22} &= -32\pi\mu_0(1-\nu_0)a_{20}^{(3)}; \\ t_{11}-t_{22}-2\mathrm{i}t_{12} &= -64\pi\mu_0(1-\nu_0)a_{22}^{(3)}; \\ t_{13}-\mathrm{i}t_{23} &= -16\pi\mu_0(1-\nu_0)a_{21}^{(3)}; \end{aligned} \tag{3.18}$$

where $\mathrm{i}^2=-1$. As expected, two representations of the asymptotic (dipole) term, by Eqs. (3.17) and (2.15), are equivalent:

$$\nabla\mathbf{G}(\mathbf{x}):\mathbf{t}\equiv\sum_{i=1}^{3}\sum_{|s|\leqslant i-1}a_{i-1,s}^{(i)}\mathbf{U}_{i-1,s}^{(i)}(\mathbf{x}). \tag{3.19}$$

Eqs. (3.17) and (2.15) are general and valid for an arbitrary inhomogeneity, in terms of shape, structure, properties and interface bonding. On the contrary, the constants $a_{ts}^{(i)}$ and t_{ij} are case-dependent. In a few particular cases, the explicit expressions for them can be obtained in closed form. Below, we consider two of them.

3.2.2. Spherical inhomogeneity

Consider an unbounded domain containing a single spherical inhomogeneity of radius R. At the interface S, the perfect mechanical contact between the constituents is assumed:

$$[\![\mathbf{u}]\!]_S=0;\quad [\![\mathbf{T}_n]\!]_S=0; \tag{3.20}$$

where $[\![f]\!]_S=(f^{(0)}-f^{(1)})|_S$ means a jump of quantity f across the interface S and $\mathbf{T}_n=\boldsymbol{\sigma}\cdot\mathbf{n}$ is the normal traction vector. Due to regularity of $\mathbf{u}_{far}$ in a vicinity of inhomogeneity, its local expansion is

$$\mathbf{u}_{far}(\mathbf{x})=\sum_{i=1}^{3}\sum_{t=0}^{\infty}\sum_{|s|\leqslant t}c_{ts}^{(i)}\mathbf{u}_{ts}^{(i)}(\mathbf{x}), \tag{3.21}$$

where $\mathbf{u}_{ts}^{(i)}$ are the regular vector solutions of Lame equation defined in Appendix B. We consider the particular case of linear displacement $\mathbf{u}_{far}=\boldsymbol{\varepsilon}^\infty\cdot\mathbf{x}$, where $\boldsymbol{\varepsilon}^\infty=\{\varepsilon_{ij}^\infty\}$ is the uniform far-field strain tensor. In this case,

$$\mathbf{u}_{far}=c_{00}^{(3)}\mathbf{u}_{00}^{(3)}+\sum_{s=-2}^{2}c_{2s}^{(1)}\mathbf{u}_{2s}^{(1)}, \tag{3.22}$$

where

$$\begin{aligned} c_{00}^{(3)} &= \frac{(\varepsilon_{11}^\infty+\varepsilon_{22}^\infty+\varepsilon_{33}^\infty)}{2(2\nu_0-1)};\quad c_{20}^{(1)}=\frac{(2\varepsilon_{33}^\infty-\varepsilon_{11}^\infty-\varepsilon_{22}^\infty)}{3}; \\ c_{21}^{(1)} &= \varepsilon_{13}^\infty-\mathrm{i}\varepsilon_{23}^\infty;\quad c_{22}^{(1)}=\varepsilon_{11}^\infty-\varepsilon_{22}^\infty-2\mathrm{i}\varepsilon_{12}^\infty;\quad c_{2,-s}^{(i)}=(-1)^s\overline{c_{2s}^{(i)}} \end{aligned} \tag{3.23}$$

are the only non-zero series expansion coefficients in Eq. (3.21). Moreover, in the case of linear $\mathbf{u}_{far}$ the disturbance $\mathbf{u}_{dis}$ in Eq. (3.16) reduces to Eq. (3.17). I.e., an asymptotic field coincides with the actual field in a whole matrix domain.

Analytical solution for this problem is straightforward and yields

$$\begin{aligned} \frac{a_{00}^{(1)}}{R^3} &= \frac{(3k_1-3k_0)}{(3k_1+4\mu_0)}\frac{2(2\nu_0-1)}{3}c_{00}^{(3)}; \\ \frac{a_{2s}^{(3)}}{R^3} &= -\frac{Mc_{2s}^{(1)}}{(2+s)!(2-s)!};\quad M=\frac{15(\mu_1-\mu_0)}{[\mu_1(8-10\nu_0)+\mu_0(7-5\nu_0)]}. \end{aligned} \tag{3.24}$$

Comparison with Eq. (3.18) gives us an explicit formula for the elastic dipole moment of spherical inhomogeneity of volume $V_1=\frac{4}{3}\pi R^3$ in terms of remote strain field $\boldsymbol{\varepsilon}^\infty$:

$$\begin{aligned} \frac{(t_{11}+t_{22}+t_{33})}{2\mu_0(1-\nu_0)V_1} &= \frac{3}{(1-2\nu_0)}\frac{(3k_1-3k_0)}{(3k_1+4\mu_0)}(\varepsilon_{11}^\infty+\varepsilon_{22}^\infty+\varepsilon_{33}^\infty); \\ \frac{2t_{33}-t_{11}-t_{22}}{2\mu_0(1-\nu_0)V_1} &= M(2\varepsilon_{33}^\infty-\varepsilon_{11}^\infty-\varepsilon_{22}^\infty); \\ \frac{t_{13}-\mathrm{i}t_{23}}{2\mu_0(1-\nu_0)V_1} &= M(\varepsilon_{13}^\infty-\mathrm{i}\varepsilon_{23}^\infty); \\ \frac{t_{11}-t_{22}-2\mathrm{i}t_{12}}{2\mu_0(1-\nu_0)V_1} &= M(\varepsilon_{11}^\infty-\varepsilon_{22}^\infty-2\mathrm{i}\varepsilon_{12}^\infty). \end{aligned} \tag{3.25}$$

Or, in compact form,

$$\begin{aligned} \mathrm{Tr}(\mathbf{t}) &= 3(k_1-k_0)\frac{(3k_0+4\mu_0)}{(3k_1+4\mu_0)}\mathrm{Tr}(\boldsymbol{\varepsilon}^\infty)V_1; \\ \mathrm{Dev}(\mathbf{t}) &= 2\mu_0(1-\nu_0)M\mathrm{Dev}(\boldsymbol{\varepsilon}^\infty)V_1. \end{aligned} \tag{3.26}$$

3.2.3. Spheroidal inhomogeneity

Consider now an unbounded domain containing a single spheroidal inhomogeneity with the inter-foci distance $2d$ and boundary defined by $\xi=\xi_0$, where ξ_0 relates the aspect ratio $\gamma=a_3/a_1$ by $\gamma=\xi_0/\overline{\xi_0}>1$ for prolate spheroid and $\gamma=\overline{\xi_0}/\xi_0<1$ for oblate one. Here, $\overline{\xi_0}=\sqrt{\xi_0^2-1}$; for the spheroidal coordinates and other notations, see Appendix A.

The displacement field outside the inhomogeneity is taken in the form of Eq. (2.12), with

$$\mathbf{u}_{dis}(\mathbf{r})=\sum_{i=1}^{3}\sum_{t=0}^{\infty}\sum_{|s|\leqslant t}b_{ts}^{(i)}\mathbf{V}_{ts}^{(i)}(\mathbf{r},d), \tag{3.27}$$

$\mathbf{V}_{ts}^{(i)}$ are the spheroidal vector solutions of Lame equation and $b_{ts}^{(i)}$ are the unknown multipole strengths: $b_{t,-s}^{(i)}=(-1)^{s+i-1}\overline{b_{ts}^{(i)}}$. For the explicit form of $\mathbf{V}_{ts}^{(i)}$, see Kushch (1996). In the case of uniform remotely applied field $\mathbf{u}_{far}=\boldsymbol{\varepsilon}^\infty\cdot\mathbf{x}$, only a few first terms of the series in Eq. (3.26) contribute to the solution:

$$\mathbf{u}_{dis}(\mathbf{r})=\sum_{i=1}^{3}\sum_{|s|\leqslant i-1}b_{i-1,s}^{(i)}\mathbf{V}_{i-1,s}^{(i)}(\mathbf{x}). \tag{3.28}$$

To determine an asymptotic behavior of the disturbance filed of the spheroid, we observe that, at some distance from inhomogeneity, $\mathbf{u}_{dis}$ can be also expanded over the spherical solutions $\mathbf{U}_{ts}^{(i)}$. This is readily done by applying the re-expansion formula

$$(-1)^s\mathbf{V}_{ts}^{(i)}(\mathbf{r},d)=\sum_{j=1}^{i}\sum_{k=t}^{\infty}K_{tks}^{(i)(j)}(d)\mathbf{U}_{k+j-i,s}^{(j)}(\mathbf{r}),\quad (\|\mathbf{r}\|>d); \tag{3.29}$$

(explicit expressions for constants $K_{tks}^{(i)(j)}$ are given in the book of Kushch (2013)). It follows from Eq. (3.28) that, for $r=\|\mathbf{x}\|\to\infty$,

$$\mathbf{V}_{i-1,s}^{(i)}(\mathbf{r},d)\approx(-1)^{s+1}\frac{d^2}{3}\mathbf{U}_{i-1,s}^{(i)}(\mathbf{r}), \tag{3.30}$$

so we get

$$a_{i-1,s}^{(i)}=\frac{d^2}{3}(-1)^{s+1}b_{i-1,s}^{(i)}. \tag{3.31}$$

By substitution of Eq. (3.30) into Eq. (3.24), we obtain the desired expression of elastic dipole moment in terms of $b_{ts}^{(i)}$:

$$\begin{aligned} t_{11}+t_{22}+t_{33} &= -\frac{3k}{2(1-2\nu_0)}b_{00}^{(1)};\quad 2t_{33}-t_{11}-t_{22}=2kb_{20}^{(3)}; \\ t_{11}-t_{22}-2\mathrm{i}t_{12} &= 4kb_{22}^{(3)};\quad t_{13}-\mathrm{i}t_{23}=-kb_{21}^{(3)}; \end{aligned} \tag{3.32}$$

where $k=\frac{16}{3}\pi d^2\mu_0(1-\nu_0)$.

Again, these formulas are valid for the disturbance caused by the spheroid due to any (not necessarily uniform) far field. To evaluate the elastic dipole moment $\mathbf{t}$, one has to know $b_{i-1,s}^{(i)}$. For a single inhomogeneity under uniform remote load, this task is ready. In particular, the explicit expressions for the multipole strengths of rigid prolate spheroidal inclusion are

$$\begin{aligned}
&b_{00}^{(1)}Q_1^0 - 2b_{20}^{(3)}\left[\xi_0 Q_2^0 + \frac{2}{3}(1-2\nu)Q_1^0\right] = d\xi_0\varepsilon_{33}^{\infty};\\
&b_{00}^{(1)}Q_1^1 + b_{20}^{(3)}\left(-\xi_0 Q_2^1 + \frac{5-4\nu}{3}Q_1^1\right) = d\overline{\xi_0}(\varepsilon_{11}^{\infty}+\varepsilon_{22}^{\infty});\\
&b_{21}^{(3)}\left[(1-\nu_0)\left(1/\overline{\xi_0}^2 + Q_1^0\right) - 2\xi_0 Q_2^0\right] = -d\xi_0(\varepsilon_{13}^{\infty} - \mathrm{i}\varepsilon_{23}^{\infty});\\
&b_{22}^{(3)}\left[(7-8\nu)Q_1^1 - \xi_0 Q_2^1\right] = -d\overline{\xi_0}(\varepsilon_{11}^{\infty} - \varepsilon_{22}^{\infty} - 2\mathrm{i}\varepsilon_{12}^{\infty});
\end{aligned} \tag{3.33}$$

where $Q_t^s = Q_t^s(\xi_0)$ are the associated Legendre functions of second kind (Hobson, 1931). The multipole strengths for the oblate shape is also given by Eq. (3.32) where ξ_0 is replaced with $\mathrm{i}\overline{\xi_0}$ and $Q_t^s(\mathrm{i}\overline{\xi})$ are the real-valued functions of imaginary argument (Kushch, 2013).

4. Transversely-isotropic matrix

In the Cartesian coordinate system $Ox_1x_2x_3$ with Ox_3 axis aligned with the anisotropy axis of transversely isotropic material, the generalized Hooke's law is written in explicit form as

$$\begin{aligned}
&\sigma_{11} = C_{11}\varepsilon_{11} + C_{12}\varepsilon_{22} + C_{13}\varepsilon_{33}; \quad \sigma_{13} = 2C_{44}\varepsilon_{13};\\
&\sigma_{22} = C_{12}\varepsilon_{11} + C_{11}\varepsilon_{22} + C_{13}\varepsilon_{33}; \quad \sigma_{23} = 2C_{44}\varepsilon_{23};\\
&\sigma_{33} = C_{13}\varepsilon_{11} + C_{13}\varepsilon_{22} + C_{33}\varepsilon_{33}; \quad \sigma_{12} = 2C_{66}\varepsilon_{12}.
\end{aligned} \tag{4.1}$$

Hereafter two-indices notation C_{ij} is adopted for components of the fourth rank stiffness tensor $\mathbf{C}$ and $2C_{66} = (C_{11} - C_{12})$.

4.1. Compliance- and stiffness contribution tensors

The explicit results in elementary functions can be obtained only in the case of a transversely isotropic matrix with symmetry axis x_3, containing a spheroidal inhomogeneity provided the spheroid axis is parallel to x_3. In this case, Hill's tensor $\mathbf{P}$ has the following coefficients in representation (3.1) (Sevostianov et al., 2005):

$$\begin{aligned}
&p_1 = \frac{\pi}{2}\sum_{q=1}^{3}(b_q - A_q a_q)J_1^{(q)}; \quad p_2 = \frac{\pi}{2}\sum_{q=1}^{3}(2b_q - A_q a_q)J_1^{(q)};\\
&p_3 = p_4 = -\frac{\pi}{2}\sum_{q=1}^{3}c_q\left(J_1^{(q)} - \gamma^2 A_q J_2^{(q)}\right);\\
&p_5 = \pi\sum_{q=1}^{3}\left[\gamma^2(2b_q - A_q a_q)J_2^{(q)} - c_q\left(J_1^{(q)} - \gamma^2 A_q J_2^{(q)}\right) + d_q J_1^{(q)}\right];\\
&p_6 = 2\pi\sum_{q=1}^{3}d_q\gamma^2 J_2^{(q)};
\end{aligned} \tag{4.2}$$

where coefficients a_l, b_l, c_l, and d_l depend on elastic stuffiness as follows:

$$\begin{aligned}
&a_l = \frac{1}{\varepsilon_l}\left[(C_{66} - C_{11})(C_{33} - A_l C_{44}) + (C_{13} + C_{44})^2\right];\\
&b_l = \frac{1}{\varepsilon_l}\left[(C_{44} - A_l C_{11})(C_{33} - A_l C_{44}) + A_l(C_{13} + C_{44})^2\right];\\
&c_l = \frac{1}{\varepsilon_l}(C_{13} + C_{44})(C_{44} - A_l C_{66});\\
&d_l = \frac{1}{\varepsilon_l}(C_{44} - A_l C_{11})(C_{44} - A_l C_{66});\\
&\varepsilon_l = 4\pi C_{11}C_{44}C_{66}\prod_{j=1(j\neq l)}^{3}(A_j - A_l);
\end{aligned} \tag{4.3}$$

$A_1 = C_{44}/C_{66}$, and A_2 and A_3 are roots of

$$C_{11}C_{44}\phi^2 + \left[(C_{13})^2 + 2C_{13}C_{44} - C_{11}C_{33}\right]\phi + C_{33}C_{44} = 0. \tag{4.4}$$

The shape factors $J_i^{(q)}$ (functions of the aspect ratio γ) are given by

$$\begin{aligned}
J_1^{(q)} &= A_q\int_{-1}^{1}\frac{(1-u^2)du}{[1+(\gamma^2-1)u^2][A_q+(1-A_q)u^2]^{3/2}}\\
&= \lambda_q^2\left[2 - \gamma^2 A_q\lambda_q ln\left(\frac{\lambda_q+1}{\lambda_q-1}\right)\right];\\
J_2^{(q)} &= A_q\int_{-1}^{1}\frac{u^2du}{[1+(\gamma^2-1)u^2][A_q+(1-A_q)u^2]^{3/2}} = \lambda_q^2\left[\lambda_q ln\left(\frac{\lambda_q+1}{\lambda_q-1}\right) - 2\right],
\end{aligned} \tag{4.5}$$

where $\lambda_q = 1/\sqrt{1 - A_q\gamma^2}$.

Calculations of property contribution tensors are simplified in the cases of strongly oblate and strongly prolate inhomogeneities, especially in the cases of perfectly rigid inhomogeneities and pores. We consider these cases below. Note that unlike the case of the isotropic matrix, *spherical shape* ($\gamma = 1$) does not offer drastic simplifications.

For a *perfectly rigid disk*, $\mathbf{N}$-tensor is given by

$$\mathbf{N} = (n_1\mathbf{T}^{(1)} + n_2\mathbf{T}^{(2)}), \tag{4.6}$$

where

$$\begin{aligned}
n_1 &= \frac{4}{3}\frac{\sqrt{A_2A_3}(\sqrt{A_2}+\sqrt{A_3})\left(C_{11}^0 + C_{12}^0 + 2C_{66}^0\right)}{\sqrt{A_2A_3}C_{44}^0 + C_{33}^0};\\
n_2 &= \frac{32}{3}\left[\sqrt{\frac{C_{44}^0}{C_{66}^0}} + 2\frac{\sqrt{A_2A_3}C_{44}^0 + C_{33}^0}{\sqrt{A_2A_3}(\sqrt{A_2}+\sqrt{A_3})\left(C_{11}^0 + C_{12}^0 + 2C_{66}^0\right)}\right].
\end{aligned} \tag{4.7}$$

In the case of *a crack-like pore*, $\mathbf{H}$-tensor is given by

$$\mathbf{H} = (h_5\mathbf{T}^{(5)} + h_6\mathbf{T}^{(6)}), \tag{4.8}$$

where

$$\begin{aligned}
h_5 &= \frac{8}{3C_{44}^0}\left[\sqrt{\frac{C_{66}^0}{C_{44}^0}} + \frac{-2\left(C_{13}^0\right)^2 + C_{33}^0\left(C_{11}^0 + C_{12}^0 + 2C_{66}^0\right)}{\sqrt{2}C_{44}^0(\sqrt{A_2}+\sqrt{A_3})\sqrt{C_{33}^0\left(C_{11}^0 + C_{12}^0 + 2C_{66}^0\right)}}\right]^{-1};\\
h_6 &= \frac{8}{3}\frac{(\sqrt{A_2}+\sqrt{A_3})\left(C_{11}^0 + C_{12}^0 + 2C_{66}^0\right)}{C_{33}^0\left(C_{11}^0 + C_{12}^0 + 2C_{66}^0\right) - 2\left(C_{13}^0\right)^2}.
\end{aligned} \tag{4.9}$$

Perfectly rigid cylindrical fiber is described by the following n_i coefficients of representation (2.10):

$$\begin{aligned}
&n_1 = \frac{1}{2}(C_{11}^0 + C_{12}^0) + C_{66}^0; \quad n_2 = \frac{4C_{66}^0(C_{11}^0 + C_{12}^0 + 2C_{66}^0)}{C_{11}^0 + C_{12}^0 + 4C_{66}^0};\\
&n_3 = n_4 = C_{13}^0 + C_{44}^0; \quad n_5 = 8C_{44}^0, \quad n_6 \to \infty.
\end{aligned} \tag{4.10}$$

Note that $n_6 \to \infty$ implies that the presence of the fiber makes the RVE perfectly rigid in the axial direction. The h_i factors of tensor $\mathbf{H}$ in Eq. (2.10) for a *cylindrical pore* are:

$$\begin{aligned}
&h_1 = \frac{2(C_{13}^0)^2 - C_{33}^0(C_{11}^0 + C_{12}^0 + 2C_{66}^0)}{4C_{66}^0(2(C_{13}^0)^2 - C_{33}^0(C_{11}^0 + C_{12}^0))}; \quad h_2 = \frac{2}{C_{11}^0 + C_{12}^0} + \frac{1}{C_{66}^0};\\
&h_3 = h_4 = \frac{C_{13}^0}{2(C_{13}^0)^2 - C_{33}^0(C_{11}^0 + C_{12}^0)};\\
&h_5 = \frac{2}{C_{44}^0}; \quad h_6 = \frac{C_{11}^0 + C_{12}^0}{(C_{11}^0 + C_{12}^0)C_{33}^0 - 2(C_{13}^0)^2}.
\end{aligned} \tag{4.11}$$

4.2. Induced dipole moments

Eq. (2.15) holds for a solid of any elastic anisotropy. I.e., an asymptotic of the disturbance field of *arbitrary* inclusion is given by the formula

$$\mathbf{u}_{dis} \underset{\|\mathbf{x}\|\to\infty}{\to} \nabla\mathbf{G}(\mathbf{x}) : \mathbf{t}, \tag{4.12}$$

where $\mathbf{t}$ is the dipole moment defined by Eq. (2.18). An explicit form of the elastostatic Green's function can be obtained for a transversely-isotropic material only (Lifshitz and Rosentsweig, 1947).

In this case, we introduce the vector functions

$$\begin{aligned}
&\mathbf{W}_{ts}^{(j)}(\mathbf{x}) = Y_t^{s-1}(\mathbf{x}_j)\mathbf{e}_1 - Y_t^{s+1}(\mathbf{x}_j)\mathbf{e}_2 + \frac{k_j}{\sqrt{A_j}} Y_t^s(\mathbf{x}_j)\mathbf{e}_3 \quad (j = 1, 2);\\
&\mathbf{W}_{ts}^{(3)}(\mathbf{x}) = Y_t^{s-1}(\mathbf{x}_3)\mathbf{e}_1 + Y_t^{s+1}(\mathbf{x}_3)\mathbf{e}_2 \quad (t = 0, 1, 2, \ldots; |s| \leqslant t+1).
\end{aligned} \tag{4.13}$$

Here, Y_t^s are the scalar solid spherical harmonics (Appendix A), $\mathbf{e}_1 = (\mathbf{i}_1 + i\mathbf{i}_2)/2$, $\mathbf{e}_2 = \overline{\mathbf{e}_1}$ and $\mathbf{e}_3 = \mathbf{i}_3$ are the complex Cartesian basis vectors, A_j are roots of Eq. (4.4), and

$$k_j = \frac{\nu_j (C_{13} + C_{44})}{C_{33} - \nu_j C_{44}} \quad (j = 1, 2). \tag{4.14}$$

Also, $\mathbf{x}_i = x_{ij}\mathbf{i}_j$ and x_{ij} are the scaled Cartesian variables

$$x_{1j} = x_1, \quad x_{2j} = x_2, \quad x_{3j} = x_3/\sqrt{A_j}; \quad r_j^2 = (x_{ij})^2. \tag{4.15}$$

It is straightforward to show that the functions $\mathbf{W}_{ts}^{(j)}(\mathbf{x})$ obey the equilibrium equation (2.11). Moreover, it appears that the identity

$$\nabla\mathbf{G}(\mathbf{x}) : \mathbf{t} \equiv \sum_{i=1}^{3} \sum_{|s|\leqslant 2} A_{1s}^{(i)} \mathbf{W}_{1s}^{(i)}(\mathbf{x}) \tag{4.16}$$

analogous to Eq. (3.19) takes place provided the dipole moments t_{ij} and the expansion coefficients $A_{1s}^{(i)}$ are related by

$$\begin{aligned}
&t_{11} + t_{22} = 2C_{11}\sum_{j=1}^{2}\sqrt{A_j}A_{10}^{(j)}; \quad t_{33} = C_{33}\sum_{j=1}^{2}\frac{k_j}{\sqrt{A_j}}A_{10}^{(j)};\\
&t_{11} - t_{22} - 2it_{12} = -2C_{44}\sum_{j=1}^{3}\frac{(1+k_j)}{\sqrt{A_j}}A_{12}^{(j)},\\
&t_{13} - it_{23} = -C_{44}\left[\sum_{j=1}^{2}(1+2k_j)A_{11}^{(j)} - A_{11}^{(3)}\right].
\end{aligned} \tag{4.17}$$

Noteworthy, Eqs. (4.16) and (4.17) enable an efficient way of $\nabla\mathbf{G}(\mathbf{x})$ evaluation. To illustrate this, we take $t_{kj} = \delta_{j3}\delta_{k3}$. The left side of Eq. (4.16) simplifies to $u_i = \partial G_{i3}/\partial x_3$. Also, we find form Eq. (4.17) that

$$A_{10}^{(1)} = -\frac{1}{C_{33}\sqrt{A_1}(\tilde{k}_2 - \tilde{k}_1)}; \quad A_{10}^{(2)} = -\frac{\sqrt{A_1}}{\sqrt{A_2}}A_{10}^{(1)}; \tag{4.18}$$

where $\tilde{k}_j = k_j/A_j$; all other $A_{1s}^{(i)} = 0$. Then, Eq. (4.16) yields

$$\begin{aligned}
\frac{\partial G_{33}}{\partial x_3} &= \left[A_{10}^{(1)}\mathbf{W}_{10}^{(1)}(\mathbf{x}) + A_{10}^{(2)}\mathbf{W}_{10}^{(2)}(\mathbf{x})\right] \cdot \mathbf{e}_3\\
&= \frac{1}{C_{33}\left(\tilde{k}_2 - \tilde{k}_1\right)}\left(\tilde{k}_1\frac{x_{31}}{r_1^3} - \tilde{k}_2\frac{x_{32}}{r_2^3}\right)
\end{aligned} \tag{4.19}$$

and two analogous expressions for $\partial G_{13}/\partial x_3$ and $\partial G_{23}/\partial x_3$. By taking appropriate values of t_{kj}, all the derivatives $\partial G_{ij}/\partial x_k$ can be expressed in terms of $\mathbf{W}_{1s}^{(i)}$. Obtained here for the first time, these formulas are remarkably simple and transparent as compared to those available in literature (e.g., Lee, 2009).

The problem about a single inhomogeneity embedded into an infinite transversely isotropic solid is a particular case of more general, multiple inclusion model derived by Kushch (2003). The multipole series expansion of the disturbance field $\mathbf{u}_{dis}$ of a single inhomogeneity is

$$\mathbf{u}_{dis} = \sum_{i=1}^{3}\sum_{t=0}^{\infty}\sum_{|s|\leqslant t+1} B_{ts}^{(j)}\mathbf{W}_{ts}^{(j)}(\mathbf{x}). \tag{4.20}$$

The irregular vector solutions $\mathbf{W}_{ts}^{(j)}$ entering Eq. (4.20) are defined by Eq. (4.13), with replace $Y_t^s(\mathbf{x}_j)$ to $F_t^s(\mathbf{x}_j, d_j)$, Eq. (A.5) of Appendix A. The corresponding spheroidal coordinates (ξ_j, η_j, ϕ_j) are given by Eqs. (A.1) and (A.2). Eq. (4.20) is valid for arbitrary spheroidal inclusion (including the strongly oblate and prolate shapes) whose symmetry axis coincides with the transverse isotropy axis of matrix solid. In the case of spherical inclusion of radius R the coordinate systems are chosen to get $\xi_j = \xi_{j0} = \text{const}$ at the surface $r = R$ (Kushch, 2003):

$$d_j = R/\xi_{j0}, \quad \xi_{j0} = \sqrt{\nu_j/|\nu_j - 1|}. \tag{4.21}$$

The leading, dipole term of the sum Eq. (4.20) is

$$\sum_{i=1}^{3}\sum_{|s|\leqslant 2} B_{1s}^{(j)}\mathbf{W}_{1s}^{(j)(j)}(\mathbf{x}), \tag{4.22}$$

where $B_{1s}^{(j)}$ are the problem-related multipole strengths. The magnitude of $B_{1s}^{(j)}$ is affected by the inclusion size, properties, orientation and interface bonding type – but their relation to the dipole moments t_{ij} is always given by Eq. (4.17) where now

$$A_{1s}^{(j)} = \frac{4\pi}{3}R^3 d_j^2 B_{1s}^{(j)}. \tag{4.23}$$

This is a direct consequence of the asymptotic formula (Kushch, 2013)

$$F_t^s(\mathbf{x}_j, d_j) \underset{\|\mathbf{x}_j\|\to\infty}{\to} \frac{d_j^2}{3} Y_t^s(\mathbf{x}_j). \tag{4.24}$$

Provided $B_{1s}^{(j)}$ are found by solving the corresponding boundary value problem, an explicit expression of the dipole moment $\mathbf{t}$ can be written. For the details of solution, see Kushch (2003) and Kushch and Sevostianov (2004).

5. Connection between the dipole moments and stiffness contribution tensors

The connection between the induced dipole moment and stiffness contribution tensor of inhomogeneity is readily found. Direct comparison of Eqs. (2.4) and (2.5) with the Batchelor's (1974) formula

$$\langle\boldsymbol{\sigma}\rangle = \mathbf{C}_0 : \langle\boldsymbol{\varepsilon}\rangle + n\langle\mathbf{t}\rangle, \tag{5.1}$$

where n is a number density of inhomogeneities, yields an expression of the dipole moment $\mathbf{t}$ for a single inhomogeneity of volume V_1 in terms of the stiffness contribution tensor $\mathbf{N}$ and arbitrary far-field strain $\boldsymbol{\varepsilon}^\infty$:

$$\mathbf{t} = V_1\mathbf{N} : \boldsymbol{\varepsilon}^\infty. \tag{5.2}$$

In particular, Eq. (5.2) must be valid for the explicit expressions of $\mathbf{t}$ and $\mathbf{N}$ derived in the previous sections. Noteworthy, such a comparison is not straightforward due to different notations. We illustrate their equivalence on the particular case of rigid spheroidal inclusion in isotropic elastic solid, subject to uniform far load. By combining Eqs. (3.31) and (3.32), we get the following set of equalities:

$$\begin{aligned}&\left[2\xi_0 Q_2^0+2(1-2\nu_0)Q_1^0\right]t_{33}-\xi_0 Q_2^0(t_{11}+t_{22})=d\xi_0 K\varepsilon_{33}^\infty;\\&2\left(Q_1^1-\xi_0 Q_2^1\right)t_{33}-\left[(3-4\nu_0)Q_1^1-\xi_0 Q_2^1\right](t_{11}+t_{22})=2Kd\overline{\xi_0}(\varepsilon_{11}^\infty+\varepsilon_{22}^\infty);\\&\left[(7-8\nu_0)Q_1^1-\xi_0 Q_2^1\right](t_{11}-t_{22}-2\mathrm{i}t_{12})=-4Kd\overline{\xi_0}(\varepsilon_{11}^\infty-\varepsilon_{22}^\infty-2\mathrm{i}\varepsilon_{12}^\infty);\\&\left[(1-\nu_0)\left(1/\overline{\xi_0}^2+Q_1^0\right)-2\xi_0 Q_2^0\right](t_{13}-\mathrm{i}t_{23})=Kd\xi_0(\varepsilon_{13}^\infty-\mathrm{i}\varepsilon_{23}^\infty).\end{aligned}\tag{5.3}$$

For a perfectly rigid inhomogeneity, $\mathbf{N}=\mathbf{P}^{-1}$ and Eq. (5.2) is reduced to $\mathbf{t}=V_1\mathbf{P}^{-1}:\boldsymbol{\varepsilon}^\infty$. Its inversion yields

$$V_1\boldsymbol{\varepsilon}^\infty=\mathbf{P}:\mathbf{t}=\sum_{m=1}^{6}p_m(\mathbf{T}^{(m)}:\mathbf{t}).\tag{5.4}$$

By taking the explicit form of basic tensors $\mathbf{T}^{(m)}$ from Eq. (C.1) into account, one finds that

$$\begin{aligned}V_1\boldsymbol{\varepsilon}^\infty=&[p_1(t_{11}+t_{22})+p_3t_{33}](\mathbf{i}_1\mathbf{i}_1+\mathbf{i}_2\mathbf{i}_2)+\frac{p_2}{2}(t_{11}-t_{22})(\mathbf{i}_1\mathbf{i}_1-\mathbf{i}_2\mathbf{i}_2)\\&+[p_4(t_{11}+t_{22})+p_6t_{33}]\mathbf{i}_3\mathbf{i}_3+p_2t_{12}(\mathbf{i}_1\mathbf{i}_2+\mathbf{i}_2\mathbf{i}_1)\\&+\frac{p_5}{2}[t_{13}(\mathbf{i}_1\mathbf{i}_3+\mathbf{i}_3\mathbf{i}_1)+t_{23}(\mathbf{i}_2\mathbf{i}_3+\mathbf{i}_3\mathbf{i}_2)].\end{aligned}\tag{5.5}$$

Next, we compare this formula with the identity

$$\begin{aligned}\boldsymbol{\varepsilon}^\infty=&\frac{1}{2}(\varepsilon_{11}^\infty+\varepsilon_{22}^\infty)(\mathbf{i}_1\mathbf{i}_1+\mathbf{i}_2\mathbf{i}_2)+\frac{1}{2}(\varepsilon_{11}^\infty-\varepsilon_{22}^\infty)(\mathbf{i}_1\mathbf{i}_1-\mathbf{i}_2\mathbf{i}_2)\\&+\varepsilon_{33}^\infty\mathbf{i}_3\mathbf{i}_3+\varepsilon_{12}^\infty(\mathbf{i}_1\mathbf{i}_2+\mathbf{i}_2\mathbf{i}_1)+\varepsilon_{13}^\infty(\mathbf{i}_1\mathbf{i}_3+\mathbf{i}_3\mathbf{i}_1)+\varepsilon_{23}^\infty(\mathbf{i}_2\mathbf{i}_3+\mathbf{i}_3\mathbf{i}_2).\end{aligned}\tag{5.6}$$

to get the following, quite similar to Eq. (5.3) set of equalities:

$$\begin{aligned}&p_1(t_{11}+t_{22})+p_3t_{33}&&=(V_1/2)(\varepsilon_{11}^\infty+\varepsilon_{22}^\infty);\\&p_4(t_{11}+t_{22})+p_6t_{33}&&=V_1\varepsilon_{33}^\infty;\\&p_2(t_{11}-t_{22}-2\mathrm{i}t_{12})&&=V_1(\varepsilon_{11}^\infty-\varepsilon_{22}^\infty-2\mathrm{i}\varepsilon_{12}^\infty);\\&p_5(t_{13}-\mathrm{i}t_{23})&&=2V_1(\varepsilon_{13}^\infty-\mathrm{i}\varepsilon_{23}^\infty).\end{aligned}\tag{5.7}$$

Recall now that p_m are expressed in terms of f_0 and f_1, see Eq. (3.2). The functions f_0 and f_1 are related to the associate Legendre functions Q_t^s as follows:

$$f_0=\frac{1}{2}\left(1-\overline{\xi_0}^2Q_1^0\right)=-\frac{1}{2}\xi_0\overline{\xi_0}Q_1^1;\quad f_1=\frac{\overline{\xi_0}^2\xi_0Q_2^0}{4(1-\nu_0)}.\tag{5.8}$$

For example,

$$p_5=\frac{1}{\mu_0}[1-f_0-4f_1]=\frac{1}{\mu_0}\left[\frac{1}{2}\left(1+\overline{\xi_0}^2Q_1^0\right)-\frac{\overline{\xi_0}^2\xi_0Q_2^0}{(1-\nu_0)}\right].\tag{5.9}$$

In view of $V_1=\frac{4}{3}\pi d^3\xi_0\overline{\xi_0}^2$, the last lines of Eqs. (5.3) and (5.7) are identical. It is straightforward to check that the lines 1 to 3 of Eqs. (5.3) and (5.7) coincide as well, that confirms Eq. (5.2).

6. Effective stiffness of a composite

Now, we proceed to the composites and show how the developed theory applies to evaluation of their effective elastic stiffness. Homogenization is old and well-known – but still not always correctly formulated – problem of micromechanics. In what follows, we discuss some basic concepts of this theory including a consistent way of introducing the macroscopic field parameters.

6.1. Definition of macro parameters: volume vs. surface averaging

The effective elastic stiffness tensor $\mathbf{C}^*=\{C_{ijkl}^*\}$ relates the macroscopic strain $\langle\boldsymbol{\varepsilon}\rangle$ and stress $\langle\boldsymbol{\sigma}\rangle$ fields by

$$\langle\boldsymbol{\sigma}\rangle=\mathbf{C}^*:\langle\boldsymbol{\varepsilon}\rangle,\tag{6.1}$$

The macroscopic strain $\langle\boldsymbol{\varepsilon}\rangle$ and stress $\langle\boldsymbol{\sigma}\rangle$ tensors in Eq. (6.1) are conventionally defined as the representative volume-averaged quantities:

$$\langle\boldsymbol{\varepsilon}\rangle\overset{def}{=}\frac{1}{V}\int_V\boldsymbol{\varepsilon}d\mathbf{x};\quad\langle\boldsymbol{\sigma}\rangle\overset{def}{=}\frac{1}{V}\int_V\boldsymbol{\sigma}d\mathbf{x}.\tag{6.2}$$

For the matrix type composite, $V=\sum_{i=0}^N V_i$, V_i being the volume of ith inhomogeneity and V_0 being the matrix volume inside RVE. Hence, the total integral is a sum

$$\langle\boldsymbol{\varepsilon}\rangle=\frac{1}{V}\sum_{i=0}^{N}\int_{V_i}\boldsymbol{\varepsilon}^{(i)}d\mathbf{x},\tag{6.3}$$

where $\boldsymbol{\varepsilon}^{(i)}$ is the strain in ith inhomogeneity.

The definition Eq. (6.2) looks self-obvious – but, in fact, is not always correct. The simple counter-example is a porous material where $\boldsymbol{\varepsilon}^{(i)}$ is *not defined* – and hence Eqs. (6.2) and (6.3) do not apply. The more substantial counter-examples concern the composites with imperfect interfaces. Citing form Böhm (2013), "*If the displacements show discontinuities, … correction terms involving the displacement jumps across imperfect interfaces or cracks must be introduced*". The opposite case is a nanocomposite with coherent interface (e.g., Duan et al., 2005) where the normal traction jump across the interface is non-zero due to the surface stress. It appears that Eq. (6.2) is valid only for composites with perfectly bonded constituents.

Alternate, surface averaging-based definition of the macroscopic strain and stress parameters (Hill, 1963)

$$\langle\boldsymbol{\varepsilon}\rangle\overset{def}{=}\frac{1}{2V}\int_{S_0}(\mathbf{nu}+\mathbf{un})dS;\quad\langle\boldsymbol{\sigma}\rangle\overset{def}{=}\frac{1}{V}\int_{S_0}\mathbf{x}(\boldsymbol{\sigma}\cdot\mathbf{n})dS;\tag{6.4}$$

eliminates the problem. In the case of perfect interfaces, this definition is consistent with the conventional one, Eq. (6.2) but holds true for the composites with imperfect interfaces.

The definition Eq. (6.4) is advantageous for it involves only the observable quantities, namely, displacement and stress, at the surface of composite specimen. In essence, we consider RVE as a "black box" whose interior structure may affect numerical value of the macro parameters – but not the way they are defined. This makes the definition general, valid for composites with arbitrary interior microstructure and arbitrary interface bonding as well as for porous and cracked materials. What is also important, numerical simulation becomes directly linked to experimental study.

6.2. RVE-averaged stress tensor

The Betti reciprocity theorem written for the matrix domain V_0 of RVE states that the equality

$$\sum_{q=0}^{N}\int_{S_q}\left[\mathbf{T}_n(\mathbf{u}^{(0)})\cdot\mathbf{u}'-\mathbf{T}_n(\mathbf{u}')\cdot\mathbf{u}^{(0)}\right]dS=0\tag{6.5}$$

is valid for any displacement vector $\mathbf{u}'$ obeying the equilibrium equation $\nabla\cdot(\mathbf{C}:\nabla\mathbf{u})=0$. Following Kushch and Sevostianov (2004), we take it in the form $\mathbf{u}'_{ij}=\mathbf{i}_ix_j$. After somewhat tedious algebra (for the details, see Kushch, 2013), we come out with the formula

$$\langle\sigma_{ij}\rangle=C_{ijkl}^0\langle\varepsilon_{kl}\rangle+\frac{1}{V}\sum_{q=1}^{N}\int_{S_q}\left[\mathbf{T}_n(\mathbf{u}^{(0)})\cdot\mathbf{u}'_{ij}-\mathbf{T}_n(\mathbf{u}'_{ij})\cdot\mathbf{u}^{(0)}\right]dS,\tag{6.6}$$

consistent with the result of Russel and Acrivos (1972). In tensor form,

$$\begin{aligned}&\langle\boldsymbol{\sigma}\rangle=\mathbf{C}_0:\langle\boldsymbol{\varepsilon}\rangle+\frac{1}{V}\sum_{q=1}^{N}\mathbf{t}^{(q)};\\&\mathbf{t}^{(q)}=\mathbf{C}_0:\int_{S_q}(\mathbf{n}\cdot\nabla\mathbf{u}^{(0)}\mathbf{x}-\mathbf{u}^{(0)}\mathbf{n})dS.\end{aligned}\tag{6.7}$$

where the second term in Eq. (6.7) is entirely due to qth inhomogeneity. As seen from Eq. (2.18), $\mathbf{t}^{(q)}$ is exactly the dipole moment of the disturbance field of this inhomogeneity.

Eq. (6.7), together with Eq. (6.1), provides accurate evaluation of the effective stiffness tensor of composite, with the interactions between the inclusions taken into account. No restrictions are imposed on the shape of inclusions, elastic properties of constituents and interface bonding type. Noteworthy, the integrals in Eq. (6.7) involve only the matrix phase displacement field, $\mathbf{u}^{(0)}$. These integrals are identically zero for all but dipole term in the $\mathbf{u}^{(0)}$ multipole expansion in a vicinity of each inhomogeneity and represent contribution of these inhomogeneities to the overall stiffness tensor. In the multipole expansion method, where the disturbance field is initially written as a series over the multipoles, analytical integration in Eq. (6.7) is ready and yields the exact, finite form expressions for the macroscopic strain, stress and effective elastic moduli.

6.3. Relation to dipole moments and stiffness contribution tensors

In Eq. (6.7) and below, the surface averaging-based definition of the macro quantities Eq. (6.4) is used, where domain V is a representative volume element (RVE) of composite and S is an outer boundary of V. To determine effective stiffness tensor of composite, we employ Eqs. (5.2) and (6.7). By combining them, we get

$$\langle \boldsymbol{\sigma} \rangle = \mathbf{C}_0 : \langle \boldsymbol{\varepsilon} \rangle + \frac{1}{V}\sum_i \mathbf{t}^{(i)} = \mathbf{C}_0 : \langle \boldsymbol{\varepsilon} \rangle + \sum_i \frac{V_i}{V}\mathbf{N}^{(i)} : \langle \boldsymbol{\varepsilon} \rangle. \tag{6.8}$$

Then, comparison with Eq. (6.1) yields

$$\mathbf{C}^* = \mathbf{C}_0 + \sum_i \frac{V_i}{V}\mathbf{N}^{(i)}. \tag{6.9}$$

In the general case, $\mathbf{t}^{(i)}$ and $\mathbf{N}^{(i)}$ are affected by interaction between the inhomogeneities (more specifically, by the superposition of disturbance fields caused by other inhomogeneities): $\mathbf{N} = \mathbf{N}(c, \ldots)$ where c is the volume content of particles. In the dilute limit $c \to 0$, $\mathbf{N} \to \mathbf{N}(0)$ and can be evaluated from the single inhomogeneity problem, see the previous Section. Expectedly, Eq. (6.9) for low c reduces to NIA theory for a dilute composite (e.g., Sevostianov and Kachanov, 2007). For a finite c, the interaction effects are important and must be taken into account in order to provide an adequate estimate of the effective conductivity tensor. This, in turn, necessitates consideration of the multiple inhomogeneity model with aid of the appropriate analytical (e.g., Kushch, 2013) or numerical (e.g., Sevostianov et al., 2008) method.

Eqs. (6.8) and (6.9) are the most general formulas for the average stress and effective elastic moduli, respectively. All we need in each specific case is to substitute there the appropriate formulas for the dipole moments. So, for the spherical particle composite with isotropic matrix they are given by Eq. (3.18). The substitution yields

$$\begin{aligned}
&\langle \sigma_{11} \rangle + \langle \sigma_{22} \rangle + \langle \sigma_{33} \rangle = 2\mu_0 \frac{(1+\nu_0)}{(1-2\nu_0)}\langle \varepsilon_{kk} \rangle + 3\tilde{a}_{00}^{(1)};\\
&2\langle \sigma_{33} \rangle - \langle \sigma_{11} \rangle - \langle \sigma_{22} \rangle = 2\mu_0[2\langle \varepsilon_{33} \rangle - \langle \varepsilon_{11} \rangle - \langle \varepsilon_{22} \rangle] - 4\tilde{a}_{20}^{(3)};\\
&\langle \sigma_{11} \rangle - \langle \sigma_{22} \rangle - 2\mathrm{i}\langle \sigma_{12} \rangle = 2\mu_0[\langle \varepsilon_{11} \rangle - \langle \varepsilon_{22} \rangle - 2\mathrm{i}\langle \varepsilon_{12} \rangle] - 8\tilde{a}_{22}^{(3)};\\
&\langle \sigma_{13} \rangle - \mathrm{i}\langle \sigma_{23} \rangle = 2\mu_0[\langle \varepsilon_{13} \rangle - \mathrm{i}\langle \varepsilon_{23} \rangle] - 2\tilde{a}_{21}^{(3)};
\end{aligned} \tag{6.10}$$

where

$$\tilde{a}_{ts}^{(j)} = \frac{8\pi}{a^3}\mu_0(1-\nu_0)\sum_i a_{ts}^{(j)(i)} \tag{6.11}$$

and summation is made over the particles inside RVE. Noteworthy, Eq. (6.10) is consistent with the results of Kushch (1987) and Sangani (1987).

Quite analogously, the formulas of Eq. (4.17) are appropriate for the composite with transversely isotropic matrix containing transversely isotropic spherical inhomogeneities. By combining them with Eq. (6.8) we get

$$\begin{aligned}
&\langle \sigma_{11} \rangle + \langle \sigma_{22} \rangle = \left(C_{1k}^0 + C_{2k}^0\right)\langle \varepsilon_{kk} \rangle + 2C_{11}^0 \sum_{j=1}^{2}\sqrt{A_j}\tilde{A}_{10}^{(j)};\\
&\langle \sigma_{33} \rangle = C_{3k}^0\langle \varepsilon_{kk} \rangle + C_{33}^0 \sum_{j=1}^{2}\frac{k_j}{\sqrt{A_j}}\tilde{A}_{10}^{(j)};\\
&\langle \sigma_{11} \rangle - \langle \sigma_{22} \rangle - 2\mathrm{i}\langle \sigma_{12} \rangle = -2\mathrm{i}C_{66}^0\langle \varepsilon_{12} \rangle - 2C_{44}^0 \sum_{j=1}^{3}\frac{(1+k_j)}{\sqrt{A_j}}\tilde{A}_{12}^{(j)};\\
&\langle \sigma_{13} \rangle - \mathrm{i}\langle \sigma_{23} \rangle = C_{44}^0(\langle \varepsilon_{13} \rangle - \mathrm{i}\langle \varepsilon_{23} \rangle) - C_{44}^0\left[\sum_{j=1}^{2}(1+2k_j)\tilde{A}_{11}^{(j)} - \tilde{A}_{11}^{(3)}\right];
\end{aligned} \tag{6.12}$$

where

$$\tilde{A}_{1s}^{(j)} = \frac{4\pi}{3a^3}\sum_i (d_i^-)^2 R_i^3 A_{1s}^{(j)(i)}. \tag{6.13}$$

Again, summation is made over all the inclusions populating the RVE. These formulas are consistent with those derived earlier (Kushch, 1997; Kushch and Sevostianov, 2004).

It should be clearly understood that the dipole-related expansion coefficients in Eqs. (6.10), (6.11) and other analogous formulas must be found from the appropriate model of composite. For dilute composites, the explicit expressions of $\tilde{a}_{ts}^{(j)}$ Eq. (3.23) can be substituted into Eq. (6.10) to get the formulas of NIA theory (e.g., Sevostianov and Kachanov, 2007). For composites with moderate and high volume content of disperse phase, where the interactions contribute to the effective properties quite significantly, determination of the dipole moments from the multiple particle finite cluster or representative unit cell models is the preferable option. For the comparative numerical study of particulate composite with transversely isotropic constituents by means of the NIA, effective field method (Kanaun and Levin, 2008) and RUC-based multipole expansion method, see Sevostianov et al. (2005).

7. Conclusions

We considered the homogenization problem for an elastic particulate composite with a specific goal to bridge the gap between the two different approaches to the problem that use (a) multipole expansion and (b) property contribution tensors. We also identified the key microstructural parameters affecting overall stiffness of heterogeneous materials. The basic concepts of the homogenization theory including a consistent way of introducing the macroscopic field parameters are discussed and clarified.

The multipole expansions of elastic disturbance fields of inclusion in both isotropic and anisotropic matrix have been obtained. The dipole moment conservation law has been suggested as an alternative definition of the dipole moment. The induced dipole moment tensor of inhomogeneity is written in terms of the multipole expansion coefficients. The relations between the far field asymptotic, derivatives of Green function for anisotropic solid and dipole moment have been established. As a bi-product, the derivatives of anisotropic Green function are expressed in terms of the vector solutions for the elastic equilibrium equations. Obtained for the first time, these formulas are remarkably simple and transparent as compared to those available in literature. The explicit expression of the multipole strengths and thus the dipole moment have been obtained for the inhomogeneities of spheroidal shape.

The dipole moments are shown to be closely related to the compliance and stiffness contribution tensors. The significance of these tensors is that their sum is the proper structural parameter in whose terms the considered effective property has to be expressed. In other words, it is these tensors that have to be summed up, or averaged over a RVE for calculation of effective elastic properties. It is shown that the mathematical expressions for dipole moments and the stiffness contribution tensors coincide. In particular, it allows one to use available results obtained by multipole expansion method to write formulas for stiffness and compliance contribution tensors and vice versa. Also, the multipole expansion method can be used to validate applicability of various approximate schemes (effective media, differential, effective field etc.) that use compliance and stiffness contribution tensors as the basic building block.

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Appendix A. Spherical and spheroidal solid harmonics

The Cartesian (x_1, x_2, x_3), spherical (r, θ, φ) and spheroidal (ξ, η, φ) coordinates relate each others by Hobson (1931)

$$x_1 + \mathrm{i}x_2 = r\sin\theta \exp(\mathrm{i}\varphi) = d\bar{\xi}\bar{\eta}\exp(\mathrm{i}\varphi), \quad x_3 = r\cos\theta = d\xi\eta, \tag{A.1}$$

where

$$\bar{\xi}^2 = \xi^2 - 1, \quad \bar{\eta}^2 = 1 - \eta^2 \quad (1 \leqslant \xi < \infty, -1 \leqslant \eta \leqslant 1, 0 \leqslant \varphi < 2\pi). \tag{A.2}$$

In the case $\mathrm{Re}\, d > 0$, the formulas of Eqs. (A.1) and (A.2) define a family of confocal prolate spheroids with rotation axis $\mathbf{i}_3$ and inter-foci distance $2d$. In the case of oblate spheroid, one must replace ξ with $\mathrm{i}\bar{\xi}$ and d with $(-\mathrm{i}d)$ in all relevant formulas.

The regular y_t^s and irregular Y_t^s solid spherical harmonics are defined as

$$y_t^s(\mathbf{x}) = \frac{r^t}{(t+s)!}\chi_t^s(\theta,\varphi); \quad Y_t^s(\mathbf{x}) = \frac{(t-s)!}{r^{t+1}}\chi_t^s(\theta,\varphi); \tag{A.3}$$

where

$$\chi_t^s(\theta,\varphi) = P_t^s(\cos\theta)\exp(\mathrm{i}s\varphi) \tag{A.4}$$

are the scalar surface harmonics and P_t^s are the associated Legendre functions of first kind. The regular f_t^s and irregular F_t^s spheroidal solid harmonics are given by

$$f_t^s(\mathbf{x}, d) = P_t^{-s}(\xi)\chi_t^s(\eta,\varphi); \quad F_t^s(\mathbf{x}, d) = Q_t^{-s}(\xi)\chi_t^s(\eta,\varphi); \tag{A.5}$$

Q_t^s being the associated Legendre functions of second kind (Hobson, 1931).

Appendix B. Vector solutions of Lame's equation in spherical coordinates

The regular (finite at $r = 0$) functions $\mathbf{u}_{ts}^{(i)} = \mathbf{u}_{ts}^{(i)}(\mathbf{x})$ $(\mathbf{x} = \{x_1, x_2, x_3\}^{\mathrm{T}})$ satisfying the Lame equation are defined as

$$\begin{aligned} &\mathbf{u}_{ts}^{(1)} = \frac{r^{t-1}}{(t+s)!}\left(\mathbf{S}_{ts}^{(1)} + t\mathbf{S}_{ts}^{(3)}\right); \quad \mathbf{u}_{ts}^{(2)} = -\frac{1}{(t+1)}\frac{r^t}{(t+s)!}\mathbf{S}_{ts}^{(2)}; \\ &\mathbf{u}_{ts}^{(3)} = \frac{r^{t+1}}{(t+s)!}\left(\beta_t\mathbf{S}_{ts}^{(1)} + \gamma_t\mathbf{S}_{ts}^{(3)}\right); \end{aligned} \tag{B.1}$$

where ν is Poisson ratio, $\beta_t = \frac{t+5-4\nu}{(t+1)(2t+3)}$ and $\gamma_t = \frac{t-2+4\nu}{2t+3}$ (Kushch, 1987). In Eq. (B.1), $\mathbf{S}_{ts}^{(i)} = \mathbf{S}_{ts}^{(i)}(\theta,\varphi)$ are the vector spherical harmonics (Morse and Feshbach, 1953) taken here as in the following form:

$$\begin{aligned} &\mathbf{S}_{ts}^{(1)} = \mathbf{e}_\theta\frac{\partial}{\partial\theta}\chi_t^s + \frac{\mathbf{e}_\varphi}{\sin\theta}\frac{\partial}{\partial\varphi}\chi_t^s, \quad \mathbf{S}_{ts}^{(2)} = \frac{\mathbf{e}_\theta}{\sin\theta}\frac{\partial}{\partial\varphi}\chi_t^s - \mathbf{e}_\varphi\frac{\partial}{\partial\theta}\chi_t^s, \\ &\mathbf{S}_{ts}^{(3)} = \mathbf{e}_r\chi_t^s; \quad t = 0, 1, \ldots |s| \leqslant t. \end{aligned} \tag{B.2}$$

The singular (infinite at $r = 0$ and vanishing at infinity) functions $\mathbf{U}_{ts}^{(i)}$ are given by $\mathbf{U}_{ts}^{(i)} = \mathbf{u}_{-(t+1),s}^{(i)}$, where $\mathbf{u}_{-(t+1),s}^{(i)}$ is defined by Eq. (B.1) with use $\mathbf{S}_{-(t+1),s}^{(i)} = \mathbf{S}_{ts}^{(i)}$ and replace $1/(-t-1+s)!$ to $(t-s)!$. The functions $\mathbf{u}_{ts}^{(i)}(\mathbf{x})$ and $\mathbf{U}_{ts}^{(i)}(\mathbf{x})$ are vectorial counterparts of the solid spherical harmonics $y_t^s(\mathbf{x})$ and $Y_t^s(\mathbf{x})$, respectively.

Appendix C. Analytic inversion and multiplication of 4th rank tensors with transversely-isotropic symmetry

In this appendix we outline a convenient technique of analytic inversion and multiplication of 4th rank tensors with transversely-isotropic symmetry. It is based on expressing tensors in "standard" tensor bases (Kunin, 1983; Walpole, 1984; Kanaun and Levin, 2008). In the case of the transversely isotropic elastic symmetry, the following basis is most convenient:

$$\mathbf{T}^{(m)} = \left\{T_{ijkl}^{(m)}\right\}, \quad m = 1, 2, \ldots, 6;$$

where

$$\begin{aligned} &T_{ijkl}^{(1)} = \theta_{ij}\theta_{kl}, \quad T_{ijkl}^{(2)} = (\theta_{ik}\theta_{lj} + \theta_{il}\theta_{kj} - \theta_{ij}\theta_{kl})/2, \\ &T_{ijkl}^{(3)} = \theta_{ij}m_km_l, \quad T_{ijkl}^{(4)} = \theta_{kl}m_im_j, \quad T_{ijkl}^{(6)} = m_im_jm_km_l, \\ &T_{ijkl}^{(5)} = (\theta_{ik}m_lm_j + \theta_{il}m_km_j + \theta_{jk}m_lm_i + \theta_{jl}m_km_i)/4, \end{aligned} \tag{C.1}$$

$\theta_{ij} = \delta_{ij} - m_im_j$ and $\mathbf{m} = m_1\mathbf{i}_1 + m_2\mathbf{i}_2 + m_3\mathbf{i}_3$ is a unit vector along the axis of transverse symmetry.

These tensors form a closed algebra with respect to the operation of (non-commutative) multiplication (contraction over two indices):

$$(\mathbf{T}^{(\alpha)} : \mathbf{T}^{(\beta)})_{ijkl} \equiv T_{ijpq}^{(\alpha)}T_{pqkl}^{(\beta)}. \tag{C.2}$$

Then, the inverse of any fourth rank tensor $\mathbf{X}$, as well as the product $\mathbf{X} : \mathbf{Y}$ of two such tensors are readily found in the closed form, as soon as the representations in the basis

$$\mathbf{X} = \sum_{m=1}^{6}X_m\mathbf{T}^{(m)}, \quad \mathbf{Y} = \sum_{m=1}^{6}Y_m\mathbf{T}^{(m)} \tag{C.3}$$

are established. Indeed:

(a) inverse tensor $\mathbf{X}^{-1}$ defined by $X_{ijpq}^{-1}X_{pqkl} = X_{ijpq}X_{pqkl}^{-1} = J_{ijkl}$ is given by

$$\mathbf{X}^{-1} = \frac{X_6}{2\Delta}\mathbf{T}^{(1)} + \frac{1}{X_2}\mathbf{T}^{(2)} - \frac{X_3}{\Delta}\mathbf{T}^{(3)} - \frac{X_4}{\Delta}\mathbf{T}^{(4)} + \frac{4}{X_5}\mathbf{T}^{(5)} + \frac{2X_1}{\Delta}\mathbf{T}^{(6)} \tag{C.4}$$

where $\Delta = 2(X_1X_6 - X_3X_4)$.

(b) product of two tensors $\mathbf{X} : \mathbf{Y}$ (tensor with $ijkl$ components equal to $X_{ijpq}Y_{pqkl}$) is

$$\begin{aligned} \mathbf{X} : \mathbf{Y} = &(2X_1Y_1 + X_3Y_4)\mathbf{T}^{(1)} + X_2Y_2\mathbf{T}^{(2)} + (2X_1Y_3 + X_3Y_6)\mathbf{T}^{(3)} \\ &+ (2X_4Y_1 + X_6Y_4)\mathbf{T}^{(4)} + \frac{1}{2}X_5Y_5\mathbf{T}^{(5)} \\ &+ (X_6Y_6 + 2X_4Y_3)\mathbf{T}^{(6)}. \end{aligned} \tag{C.5}$$

General transversely isotropic fourth-rank tensor, being represented in this basis

$$\Psi_{ijkl} = \sum_{m=1}^{6} \psi_m T_{ijkl}^m \tag{C.6}$$

has the following components:

$$\begin{aligned} &\psi_1 = (\Psi_{1111} + \Psi_{1122})/2; \quad \psi_2 = 2\Psi_{1212}; \quad \psi_3 = \Psi_{1133}; \\ &\psi_4 = \Psi_{3311}; \quad \psi_5 = 4\Psi_{1313}; \quad \psi_6 = \Psi_{3333}. \end{aligned} \tag{C.7}$$

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