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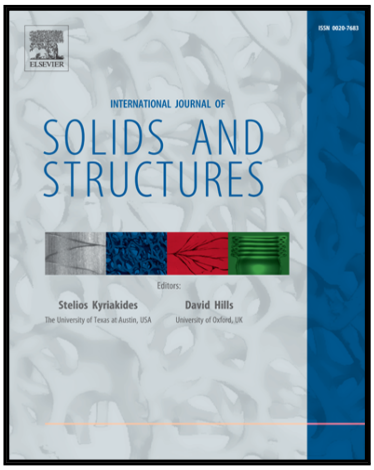

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Highlights

- Dislocation-based fracture is successfully applied to nonlocal and gradient elasticity of bi-Helmholtz type.
- Gradient elasticity of bi-Helmholtz type results in a full nonsingular fracture theory.
- Nonlocal elasticity of bi-Helmholtz type results in a nonsingular nonlocal stress field for plane weakened by a crack.
- Crack tip plasticity is captured without any ad hoc assumption.

Dislocation-based fracture mechanics within nonlocal and gradient elasticity of bi-Helmholtz type - Part I: Antiplane analysis

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Abstract

In the present paper, the dislocation-based antiplane fracture mechanics is employed for the analysis of Mode III crack within nonlocal and (strain) gradient elasticity of bi-Helmholtz type. These frameworks are appropriate candidates of generalized continua for regularization of classical singularities of defects such as dislocations. Within nonlocal elasticity of bi-Helmholtz type, nonlocal stress is regularized, while the strain field remain singular. Interestingly, gradient elasticity of bi-Helmholtz type (second gradient elasticity) eliminates all physical singularities of discrete dislocation including stress and strain fields and dislocation density while the so-called total stress tensor still contains singularity at the dislocation core. Based on the distribution of dislocations, a fracture theory with nonsingular stress field is formulated in these nonlocal and gradient theories. Strain and displacement fields within nonlocal fracture theory are identical to the classical ones. In contrast, gradient elasticity of bi-Helmholtz type leads to a full nonsingular fracture theory in which stress, strain and dislocation density are regularized. However, the singular total stress of a discrete dislocation results in singular total stress of the plane weakened by a crack. Within classical fracture mechanics, Barenblatt's cohesive fracture theory assumes that cohesive forces is distributed ahead of the crack tip to model crack tip plasticity and remove the stress singularity. Here, considering the dislocations as the carriers of plasticity, the crack tip plasticity is captured without any assumption. Once the crack is modelled by distributing the dislocations along its surface, due to the gradient theory, the distribution function gives rise to a non-zero plastic distortion ahead of the crack. Consequently, regularized solutions of crack are developed incorporating crack tip plasticity.

Keywords: crack, antiplane, dislocation, nonlocal elasticity of bi-Helmholtz type, gradient elasticity of bi-Helmholtz type, nonsingular.

1. Introduction

Dislocations play a key role in fracture and plasticity of crystalline materials. In line with this observation, within continuum mechanics of crystalline materials, the dislocations are interpreted as the building blocks of cracks in fracture mechanics and also are known as the carriers of plasticity. In this regard, a unified dislocation-based theory for plasticity and fracture is an ambitious long-standing goal.

The equivalence of a crack and a continuous distribution of translational dislocations was already recognized in the 1970s and earlier (e.g., Bilby and Eshelby [1]). The stress field of cracks may be calculated as a convolution of the stress of dislocations with a so-called dislocation density function which was proposed by Weertman [2] as dislocation-based fracture mechanics and by Hills et al. [3] as distributed dislocation technique (DDT) for crack problems in classical elasticity.

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In the framework of classical elasticity, stress at the tip of a crack is singular and behaves as $O(r^{-1/2})$, where r is the distance from the crack tip. In reality and as confirmed by atomistic simulations, there is no stress singularity at the crack tip. A remedy for this shortcoming of classical elasticity can be achieved through the assumption of a plastic zone at the crack tip. A model incorporating crack tip plasticity was initially introduced by Dugdale [4] for cracks in metal sheet specimens pulled in tension (Mode I) under plane stress conditions. Later, in order to resolve the paradox of stress singularity, Barenblatt [5] introduced a cohesive fracture theory. He considered a nonlinear region in which cohesive stresses are active. This theory is based on the idea that cohesive forces must be distributed in such a way to close the crack faces smoothly and remove the stress singularity at the crack tip. Later, the theory for the crack tip plastic zone was developed by Bilby, Cottrell and Swinden [6]. Weertman [2] gave a dislocation-based approach for the analysis of this plastic zone. Therein, a friction stress is defined in the plastic zone, and accordingly, the dislocation density is determined beyond the crack tip. A sound physical fracture theory should lead to regularized solution for crack as well as crack tip plasticity without any assumption of the cohesive zone.

Generalized continua extend the classical continuum mechanics towards more realistic frameworks via building bridges between continuum and atomic physics. The founders of generalized continuum mechanics (Cosserat, Krner, Toupin, Rivlin, Mindlin, Eringen and others) introduced various extensions of classical elasticity to capture the corresponding physical phenomena [7]. Among those, nonlocal elasticity and gradient elasticity are able to give regularized fields of defects such as dislocation. Nonlocal theory considers the inner structure of materials and takes into account long-range (nonlocal) interactions [8]. Within this framework, various nonlocal kernels have been proposed. Employing a Green function of bi-Helmholtz operator as the nonlocal kernel in Eringen's model of nonlocal elasticity, nonsingular stress fields are derived for dislocations [9]. This continuum model of nonlocal elasticity involves two material length scales. However, within Eringen's theory of nonlocal elasticity, no nonlocal strain appears. Thus, the strain and the displacement fields are identical to the classical ones in Eringen's theory of nonlocal elasticity. It is to be noted that a refined nonlocal theory can be developed based on nonlocal strain [10]. It is depicted that a nonlocal elastic material is featured by a strain energy depending on both local and nonlocal strain [11], which is in contrast to gradient elasticity where the strain energy depends on strain and its gradient.

Strain gradient elasticity is another extension of the classical theory of elasticity [12]. In first strain gradient elasticity, the potential energy function is assumed to be a quadratic function in terms of strain and gradients of strain. Mindlin [13] formulated a second strain gradient elasticity theory in which the potential energy function is assumed to be a quadratic function in terms of strain, second-order gradient displacement (first-order gradient strain) and third-order gradient displacement (second-order gradient strain). Using a variational approach, Mindlin also formulated the higher order boundary conditions. This general form of second strain gradient elasticity was later simplified to a so-called gradient elasticity of bi-Helmholtz type by Lazar et al. [14]. In that paper, the nonsingular stress and strain fields as well as nonsingular dislocation density were derived for dislocations. The gradient elasticity of bi-Helmholtz type can be simplified to a so-called gradient elasticity of Helmholtz type [15] which is a special case of first strain gradient elasticity. Recently, Polizzotto [16] formulated a second strain gradient elasticity theory with second velocity gradient inertia for which the non-standard boundary conditions were also determined.

Generalized continuum theories are candidates for improving the classical elasticity towards more realistic frameworks. Considering the interesting aspects of the generalized continua, it is motivated to investigate the dislocation-based fracture mechanics within these frameworks. However, a proper higher-order or higher-grade theory should be carefully validated with the results from experimental observations of real materials behaviour. An extension of the dislocation-based fracture or distributed dislocation technique (DDT) towards couple stress elasticity was given by Gourgiotis and Georgiadis [18]. The calculated stress fields for cracks of Mode I, Mode II and Mode III in couple-stress elasticity were more singular than in classical elasticity while the rotation field was bounded.

Other candidates for the generalization and extension of DDT towards generalized elasticity are gradient elasticity and nonlocal elasticity. Within nonlocal elasticity, Eringen [8, 19] found a nonsingular stress of a Mode III crack. Recently, Mousavi and Lazar [20] formulated distributed dislocation technique for cracks based on nonsingular dislocations in nonlocal elasticity of Helmholtz type. Later, Mousavi and Korsunsky [21] presented the dislocation-based nonsingular fracture theory within nonlocal anisotropic elasticity.

Without using the distributed dislocation technique, gradient elasticity theory has been applied to study the antiplane [22] and inplane [23, 24, 25] crack problems. Mousavi et al. [26] generalized the classical DDT to the framework of (first and second) gradient elasticity. They reported nonsingular stress fields of single crack as well as multiple cracks of Mode III. Later Mousavi and Aifantis [27] generalized this work considering the non-standard boundary conditions of gradient elasticity of Helmholtz type. A limitation of the dislocation-based approach is the fact that it requires an analytical solution to the problem of dislocation. Within gradient and nonlocal elasticity of bi-Helmholtz type, such analytical solutions are available, and the crack problem can be solved based on the analytical expressions of dislocation field.

The purpose of this paper is to formulate a dislocation-based fracture mechanics for which all the physical singularities are regularized. It is also intended to capture the crack tip plasticity without any assumption of cohesive zone. To achieve these goals, among generalized continua, two frameworks including nonlocal and gradient elasticity of bi-Helmholtz type will be employed. In order to implement the dislocation-based fracture within the generalized continua, the generalized dislocation solutions together with the generalized boundary conditions will be used. Within the framework of nonlocal elasticity of bi-Helmholtz type, the nonsingular stress fields of cracks based on nonsingular stress fields of straight dislocations are derived. Since the strain and the displacement fields are identical to the classical ones in nonlocal elasticity, this framework does not improve the classical singularity of dislocation density and strain field. In contrast, the gradient elasticity of bi-Helmholtz type eliminates both stress and strain singularities of classical fracture mechanics. Moreover, nonsingular expressions are derived for double and triple stresses as well as dislocation density. Within gradient elasticity of bi-Helmholtz type, the crack tip plasticity is also captured without any assumption of classical Barenblatt's cohesive fracture theory.

The paper is organized as follows. Fundamentals of nonlocal and gradient elasticity of bi-Helmholtz type are reviewed in section 2. Section 3 summarizes the solution to the screw dislocation within nonlocal and gradient elasticity of bi-Helmholtz type. Sections 4 and 5 present the generalization of the dislocation-based fracture to nonlocal and gradient elasticity of bi-Helmholtz type, respectively. The effective dislocation density of the crack, crack opening displacement, plastic distortion, strain field and stress field of the cracked plane are derived. Numerical examples are provided in section 6 in which the results are compared to those from classical elasticity and nonlocal and gradient elasticity of Helmholtz type. Finally, the conclusion is given in section 7.

2. Generalized Continua

In this section, the fundamentals of generalized continua including nonlocal elasticity of bi-Helmholtz type and gradient elasticity of bi-Helmholtz type are reviewed from the literature. The boundary conditions for nonlocal as well as gradient elasticity will also be summarized.

2.1. Nonlocal elasticity of bi-Helmholtz type

For an isotropic medium and in the absence of body forces, the fundamental static field equations of nonlocal elasticity reads (e.g., [8, 28])

$$\partial_j t_{ij} = 0, \tag{1}$$

$$t_{ij}(\boldsymbol{x}) = \int_V \alpha(|\boldsymbol{x}-\boldsymbol{y}|)\sigma_{ij}(\boldsymbol{y})\,\mathrm{d}V(\boldsymbol{y})\,, \tag{2}$$

$$\sigma_{ij} = C_{ijkl} e_{kl}, \tag{3}$$

$$C_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{jk}\delta_{il}), \tag{4}$$

$$e_{ij} = \frac{1}{2}\big(\beta_{ij} + \beta_{ji}\big)\,, \tag{5}$$

where C_{ijkl} is the tensor of elastic moduli, λ, μ are the Lamé constants, δ_{ij} is the Kronecker delta and e_{ij} denotes the classical elastic strain tensor, which is the symmetric part of the classical elastic distortion tensor. t_{ij} and σ_{ij} are the stress tensor of nonlocal and classical elasticity, respectively.

Within nonlocal elasticity of bi-Helmholtz type, the $\alpha(|\boldsymbol{x}-\boldsymbol{y}|)$ is a nonlocal kernel being a Green function of the linear differential operator of bi-Helmholtz type

$$L = 1 - \epsilon^2\Delta + \gamma^4\Delta\Delta = (1 - c_1^2\Delta)(1 - c_2^2\Delta), \tag{6}$$

where Δ is the Laplacian, $\Delta\Delta$ is the bi-Laplacian and ϵ and γ are parameters of nonlocality. It can be shown that

$$c_1^2 = \frac{\epsilon^2}{2}\Big(1 + \sqrt{1 - 4\frac{\gamma^4}{\epsilon^4}}\Big), c_2^2 = \frac{\epsilon^2}{2}\Big(1 - \sqrt{1 - 4\frac{\gamma^4}{\epsilon^4}}\Big). \tag{7}$$

Consequently, the integral equation (2) reduces to the inhomogeneous bi-Helmholtz equation

$$Lt_{ij} = \sigma_{ij}\,, \tag{8}$$

where the classical stress is the source for the nonlocal stress. The traction (natural) boundary conditions read

$$t_i = t_{ij}n_j = \bar{t}_i\,, \tag{9}$$

where n_i and $\bar{t}_i$ represent the normal to the external boundary and the prescribed boundary tractions, respectively. In Eringen's nonlocal elasticity, no nonlocal strain appears.

The limit from nonlocal elasticity of bi-Helmholtz type to nonlocal elasticity of Helmholtz type is obtained once $\gamma \to 0$ (leading to $c_1^2 \to \epsilon^2$ and $c_2^2 \to 0$). Furthermore, in the classical limit $\epsilon \to 0$ and $\gamma \to 0$ ($c_1^2 \to 0$ and $c_2^2 \to 0$), the classical elasticity is recovered. The nonlocal elasticity of Helmholtz type and classical elasticity can be recovered from the formulation hereafter by applying these limits.

The isotropic two-dimensional nonlocal kernel of (2) reads [9]

$$\alpha(|\boldsymbol{x} - \boldsymbol{x}'|) = \frac{1}{2\pi}\frac{1}{c_1^2 - c_2^2}\Big(K_0(|\boldsymbol{r}|/c_1) - K_0(|\boldsymbol{r}|/c_2)\Big)\,, \tag{10}$$

where $\boldsymbol{r} = \boldsymbol{x} - \boldsymbol{x}'$.

Dislocations are the source of incompatibility. Consequently, in order to study the dislocations, an incompatible elasticity is employed in which the total distortion (β^T_{ij}) (i.e. the gradient of the displacement u_i) is given as a sum of elastic (β_{ij}) and plastic parts β^P_{ij} [9, 29]

$$\beta^T_{ij} = \partial_j u_i = \beta_{ij} + \beta^P_{ij}, \tag{11}$$

for which the dislocation density tensor reads

$$\alpha_{ij} = \epsilon_{jkl}\beta_{il,k} = -\epsilon_{jkl}\beta^{\mathrm{P}}_{il,k}\,. \tag{12}$$

2.2. Gradient elasticity of bi-Helmholtz type

Gradient elasticity is a generalization of linear elasticity which includes higher-grade terms to account for microstructural effects. Within strain gradient elasticity, the strain energy depends on the elastic strain, and higher order strain tensors defined as spatial gradients either of the displacement field, or of the strain field [12]. Here we consider a second strain gradient theory in which the strain energy depends on the elastic strain, first-order strain gradient and second-order strain gradient [13, 14, 16]. Due to the gradient terms, the strain energy contains additional gradient coefficients with the dimension of length.

For an isotropic medium, the strain energy density within the second strain gradient elasticity (or gradient elasticity of bi-Helmholtz type) reads [14, 16],

$$W = \frac{1}{2}\tau_{ij}e_{ij} + \frac{1}{2}\tau_{ijk}e_{ij,k} + \frac{1}{2}\tau_{ijkl}e_{ij,kl}. \tag{13}$$

It should be noted that the strain energy density assumed by Mindlin [13] was written in a different form as a function of the strain and of the second and third gradients of the displacement field.

The stress, double stress and triple stress tensors are

$$\boldsymbol{\tau} = \{\tau_{ij}\} = \{C_{ijkl}e_{kl}\}, \tag{14a}$$

$$\boldsymbol{\tau}^{(1)} = \{\tau_{ijk}\} = \{\epsilon^2\tau_{ij,k}\}, \tag{14b}$$

$$\boldsymbol{\tau}^{(2)} = \{\tau_{ijkl}\} = \{\gamma^4\tau_{ij,kl}\}, \tag{14c}$$

while the expressions for the tensors of elastic moduli (C_{ijkl}), elastic strain (e_{ij}), total distortion (β^T_{ij}) and dislocation density (α_{ij}) are identical to those in nonlocal elasticity and are given by (4), (5), (11), (12), respectively. Here, ϵ and γ are the gradient coefficients which, in line with nonlocal elasticity, are also denoted as parameters of nonlocality. In our notation, it is reminded that σ_{ij} (3) is the classical stress tensor, while τ_{ij} (14) is the stress field within gradient elasticity. We notice that the elastic strain in gradient theory is different from the one in classical (or nonlocal) elasticity. In gradient elasticity, the elastic strain is proportional to τ_{ij}, while in classical and nonlocal elasticity, the elastic strain is proportional to the classical stress, i.e. σ_{ij}.

The equilibrium equations of the gradient elasticity of bi-Helmholtz type read [14],

$$\partial_j(\tau_{ij} - \tau_{ijk,k} + \tau_{ijkl,kl}) = 0. \tag{15}$$

Using a different index rule for stress tensors (14), Polizzotto [16] also derived equilibrium equations of the second strain gradient elasticity. Following a variational approach, the natural boundary conditions were also derived for second strain gradient elasticity [13, 16]. Considering the assumption for the strain energy density (13), we employ those boundary conditions derived in [16]. Adapting the index rule used here, the traction (natural) boundary conditions are given as [16]

$$\boldsymbol{t} = \bar{\boldsymbol{t}} \tag{16a}$$

$$\boldsymbol{t}^{(1)} = \bar{\boldsymbol{t}}^{(1)} \tag{16b}$$

$$\boldsymbol{t}^{(2)} = \bar{\boldsymbol{t}}^{(2)}, \tag{16c}$$

where $\bar{\boldsymbol{t}}$, $\bar{\boldsymbol{t}}^{(1)}$ and $\bar{\boldsymbol{t}}^{(2)}$ are the prescribed generalized traction vectors. The generalized tractions $\boldsymbol{t}$, $\boldsymbol{t}^{(1)}$ and $\boldsymbol{t}^{(2)}$ in (16) read [16]

$$\boldsymbol{t} = \boldsymbol{n}\cdot\boldsymbol{T} - \big(\bar{\boldsymbol{\nabla}}_{(\perp\boldsymbol{n})} + H\boldsymbol{n}\big)\cdot\boldsymbol{\Sigma} \tag{17a}$$

$$\boldsymbol{t}^{(1)} = \boldsymbol{n}\cdot\boldsymbol{\Sigma} - \big(\bar{\boldsymbol{\nabla}}_{(\perp\boldsymbol{n})} + H\boldsymbol{n}\big)\cdot\boldsymbol{\Sigma}^{(1)} \tag{17b}$$

$$\boldsymbol{t}^{(2)} = \boldsymbol{n}\cdot\boldsymbol{\Sigma}^{(1)}. \tag{17c}$$

Here $\bar{\boldsymbol{\nabla}}_{(\perp\boldsymbol{n})}$ is the surface gradient (which denotes the tangential gradient over a plane of normal $\boldsymbol{n}$), i.e. $\bar{\boldsymbol{\nabla}}_{(\perp\boldsymbol{n})} = \boldsymbol{P}(\boldsymbol{n})\cdot\boldsymbol{\nabla}$, while the projection operator is $\boldsymbol{P}(\boldsymbol{n}) = \boldsymbol{I} - \boldsymbol{n}\boldsymbol{n}$ and $\boldsymbol{I}$ represents the unit dyadic. The symbol $\boldsymbol{\nabla}$ denotes the spatial gradient operator, i.e. $\boldsymbol{\nabla}\boldsymbol{x} = \{\partial_i x_j\}$ and $H = -\bar{\boldsymbol{\nabla}}_{(\perp\boldsymbol{n})}\cdot\boldsymbol{n}$. In (17), the so-called total stress $\boldsymbol{T}$ is defined as

$$\boldsymbol{T} = \{\tau_{ij} - \tau_{ijk,k} + \tau_{ijkl,kl}\} = \left\{\tau_{ij} - \epsilon^2\tau_{ij,kk} + \gamma^4\tau_{ij,kkll}\right\}, \tag{18}$$

and the surface stresses $\mathbf{\Sigma}$ and $\mathbf{\Sigma}^{(1)}$ are

$$\mathbf{\Sigma} = \boldsymbol{S} + \boldsymbol{K} \cdot (\boldsymbol{nn} : \boldsymbol{\tau}^{(2)}), \tag{19a}$$

$$\mathbf{\Sigma}^{(1)} = \boldsymbol{nn} : \boldsymbol{\tau}^{(2)}, \tag{19b}$$

while $\boldsymbol{K} = -\bar{\boldsymbol{\nabla}}_{(\perp \boldsymbol{n})}\boldsymbol{n}$ is the Weingarten curvature tensor and

$$\boldsymbol{S} = \boldsymbol{n} \cdot (\boldsymbol{\tau}^{(1)} - \boldsymbol{\nabla} \cdot \boldsymbol{\tau}^{(2)}) - (\bar{\boldsymbol{\nabla}}_{(\perp \boldsymbol{n})} + H\boldsymbol{n}) \cdot (\boldsymbol{n} \cdot \boldsymbol{\tau}^{(2)}). \tag{20}$$

The non-standard boundary conditions (16) are an important aspect of the gradient theory which should be taken into account, in particular, for crack problems. The boundary conditions (16) are reduced to $\boldsymbol{t} = \boldsymbol{n} \cdot \boldsymbol{\tau} = \bar{\boldsymbol{t}}$ within classical elasticity. In the case of first strain gradient theory (or gradient elasticity of Helmholtz type), the generalized tractions (17), in the index notation, reduce to [31]

$$t_i = (\tau_{ij} - \tau_{ijk,k})n_j - \partial_j(\tau_{ijk}n_k) + n_j\partial_l(\tau_{ijk}n_k n_l), \tag{21a}$$

$$t_i^{(1)} = \tau_{ijk}n_j n_k. \tag{21b}$$

Similar to nonlocal elasticity, in the limit $\gamma \to 0$, the equilibrium equations (15) and the boundary conditions (16) are reduced to those in first strain gradient elasticity.

3. Screw dislocation

In this section, the solution to a screw dislocation within classical, nonlocal and gradient frameworks will be reviewed from the literature.

3.1. Screw dislocation within nonlocal elasticity of bi-Helmholtz type

We consider a straight screw dislocation whose line coincides with the z-axis of a Cartesian coordinate system in an infinitely extended medium. In the framework of nonlocal elasticity of bi-Helmholtz type, the nonlocal stress reads [9]

$$t_{zx} = -\frac{\mu b_z}{2\pi}\frac{y}{r^2}\left\{1 - \frac{1}{c_1^2 - c_2^2}\Big(c_1 r K_1(r/c_1) - c_2 r K_1(r/c_2)\Big)\right\}, \tag{22a}$$

$$t_{zy} = \frac{\mu b_z}{2\pi}\frac{x}{r^2}\left\{1 - \frac{1}{c_1^2 - c_2^2}\Big(c_1 r K_1(r/c_1) - c_2 r K_1(r/c_2)\Big)\right\}, \tag{22b}$$

where b_z is the Burgers vector of the screw dislocation, $r = \sqrt{x^2 + y^2}$ and K_1 is the modified Bessel function of order 1. The modified Bessel functions in (22) regularize the classical $1/r$-singularity of the dislocation. In fact, the nonlocal stress of a screw dislocation is zero at the dislocation core and possesses extremum values (maximum and minimum) near the dislocation core.

Within Eringen's nonlocal elasticity, no nonlocal strain appears. Considering (3), the strain field in nonlocal elasticity is identical to the classical strain, i.e.

$$\mathrm{e}_{zx} = -\frac{b_z}{4\pi}\frac{y}{r^2}, \qquad \mathrm{e}_{zy} = \frac{b_z}{4\pi}\frac{x}{r^2}. \tag{23}$$

In the classical limit $c_1^2 \to 0$ and $c_2^2 \to 0$, the nonlocal stresses (22) reduce to the classical stresses [33]

$$\sigma_{zx} = -\frac{\mu b_z}{2\pi}\frac{y}{r^2}, \qquad \sigma_{zy} = \frac{\mu b_z}{2\pi}\frac{x}{r^2}. \tag{24}$$

Similar to the strain field in nonlocal elasticity (23), the displacement field of the screw dislocation within nonlocal elasticity is identical to the one in classical elasticity, i.e.,

$$w_0 = \frac{b_z}{2\pi}\Big(\arctan\frac{y}{x} - \frac{\pi}{2}\Big) \tag{25}$$

for which the branch cut is considered to be $y < 0$. For $y \to 0$, the displacement field (25) is simplified to

$$w_0(x,0) = -\frac{b_z}{4}\mathrm{sgn}(x). \tag{26}$$

In classical as well as nonlocal elasticity, the dislocation density tensor of screw dislocation reads

$$\alpha_{zz} = b_z \delta(x)\delta(y). \tag{27}$$

Here, δ denotes the Dirac delta function.

3.2. Screw dislocation within gradient elasticity of bi-Helmholtz type

Once again, consider a straight screw dislocation whose line coincides with the z-axis of a Cartesian coordinate system in an infinitely extended medium. The stress field within gradient elasticity of bi-Helmholtz type [14] is

$$\tau_{zx} = -\frac{\mu b_z}{2\pi}\frac{y}{r^2}\left\{1 - \frac{1}{c_1^2 - c_2^2}\Big(c_1 r K_1(r/c_1) - c_2 r K_1(r/c_2)\Big)\right\}, \tag{28a}$$

$$\tau_{zy} = \frac{\mu b_z}{2\pi}\frac{x}{r^2}\left\{1 - \frac{1}{c_1^2 - c_2^2}\Big(c_1 r K_1(r/c_1) - c_2 r K_1(r/c_2)\Big)\right\}, \tag{28b}$$

which are identical to those in nonlocal elasticity of bi-Helmholtz type (22). The double and triple stress tensors can be derived by substituting (28) into (14) [14]. It is observed that the stress, double stress and triple stress components are nonsingular. Having the stress field (28), the total stress tensor is derived using (18). In contrast to stress, double and triple stress tensors, the total stress tensor of gradient elasticity of bi-Helmholtz type behaves as $O(1/r^3)$ and contains singularity at the dislocation core.

The gradient elasticity also provides nonsingular strain fields and non-classical displacement field for a screw dislocation. Considering the constitutive equations (14) and elastic module (4), the elastic strain tensor reads

$$e_{zx} = -\frac{b_z}{4\pi}\frac{y}{r^2}\left\{1 - \frac{1}{c_1^2 - c_2^2}\Big(c_1 r K_1(r/c_1) - c_2 r K_1(r/c_2)\Big)\right\}, \tag{29a}$$

$$e_{zy} = \frac{b_z}{4\pi}\frac{x}{r^2}\left\{1 - \frac{1}{c_1^2 - c_2^2}\Big(c_1 r K_1(r/c_1) - c_2 r K_1(r/c_2)\Big)\right\}. \tag{29b}$$

The displacement field of a screw dislocation within gradient elasticity of bi-Helmholtz type reads [32]

$$w = w_0 + \frac{b_z}{4\pi^2}\frac{1}{c_1^2 - c_2^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{k_1}{k_2}\Big(\frac{c_1^2}{\boldsymbol{k}^2 + \frac{1}{c_1^2}} - \frac{c_2^2}{\boldsymbol{k}^2 + \frac{1}{c_2^2}}\Big)\exp(i\boldsymbol{k}\cdot\boldsymbol{x})\,\mathrm{d}k_1\,\mathrm{d}k_2, \tag{30}$$

while $k^2 = k_1^2 + k_2^2$, $\boldsymbol{k} = (k_1, k_2)$ and $\boldsymbol{x} = (x, y)$. When $y \to 0$, the displacement (30) is simplified to

$$w(x,0) = -\frac{b_z}{4}\mathrm{sgn}(x)\left\{1 - \frac{c_1^2}{c_1^2 - c_2^2}\exp(-|x|/c_1) + \frac{c_2^2}{c_1^2 - c_2^2}\exp(-|x|/c_2)\right\}. \tag{31}$$

The dislocation density tensor of the screw dislocation within gradient elasticity of bi-Helmholtz type reads [32]

$$\alpha_{zz} = \frac{b_z}{2\pi}\frac{1}{c_1^2 - c_2^2}\left\{K_0(r/c_1) - K_0(r/c_2)\right\} \tag{32}$$

which is nonsingular. This is an important aspect of second gradient elasticity comparing to lower order gradient theory. In the limit for gradient theory of Helmholtz type, i.e. $c_1^2 \to \epsilon^2$ and $c_2^2 \to 0$, the dislocation density reads

$$\alpha_{zz} = \frac{b_z}{2\pi}\frac{1}{c_1^2}K_0(r/c_1) \tag{33}$$

which is singular. In the classical limit $c_1^2 \to 0$ and $c_2^2 \to 0$, the expressions for stress (28), strain (29), displacement (30) and dislocation density (32) reduce to those of classical elasticity [33]. Within gradient elasticity of Helmholtz type [15], the stress tensor is nonsingular while the double stress tensor is still singular. Interestingly, within gradient elasticity of bi-Helmholtz type, all the stress tensors (14) including stress, double stress and triple stress are nonsingular [14]. Such full nonsingular framework for dislocation promises a full nonsingular dislocation-based fracture theory.

Within gradient elasticity of bi-Helmholtz type, the plastic distortion along $y = 0$ is given by [32]

$$\beta^P_{xz}(x,0) = -\frac{b_z}{4}\frac{1}{c_1^2 - c_2^2}\left\{c_1 \exp(-|x|/c_1) - c_2 \exp(-|x|/c_2)\right\}, \tag{34}$$

while within gradient elasticity of Helmholtz type ($c_2^2 \to 0$), the plastic distortion (34) reduces to

$$\beta^P_{xz}(x,0) = -\frac{b_z}{4}\frac{1}{c_1}\exp(-|x|/c_1). \tag{35}$$

In the limit for classical elasticity, the plastic distortion reads

$$\beta^P_{xz}(x,y) = -b_z\delta(x)H(-y) \tag{36}$$

where $H(y)$ is the Heaviside step function

$$H(y) = \begin{cases} 0, & y < 0, \\ 1, & y > 0. \end{cases} \tag{37}$$

It is understood that nonlocal and gradient theories share common features, but it should be emphasized that they differ in some aspects, in particular, regarding the displacement field, elastic strain, plastic distortion and dislocation density. Moreover, the natural boundary conditions within gradient elasticity differ from those of nonlocal elasticity.

Once the plane is assumed to contain a screw dislocation situated at a point with coordinates (η, ζ), the stress fields of the plane may be deduced by transforming (x, y) to $(x-\eta, y-\zeta)$.

4. Dislocation-based antiplane fracture mechanics within nonlocal elasticity of bi-Helmholtz type

Dislocation is an elementary defect in solids which is able to build composite defects (Weertman [2]). Using the distributed dislocation technique (DDT), the arbitrary configuration of cracks can be modelled (Hills et al. [3]). The basic idea of DDT is that a crack can be represented by a distribution of dislocations. In other words, the field tensor of cracks is determined by the convolution of the field tensor of a dislocation with a distribution function. This distribution function is determined using the crack face boundary conditions. In general, DDT is capable of the analysis of multiple curved cracks. Here, for simplicity, we consider one straight crack. Employing this analysis for multiple inclined cracks is straight forward.

Consider a plane weakened by one straight crack of length $2a$ along x-axis (Fig. 1). The parametric form of the crack is

$$x = \alpha(s) = as\,, \qquad -1 < s < 1 \tag{38a}$$

$$y = 0 \tag{38b}$$

Within nonlocal elasticity, the traction boundary conditions are simply given by (9). Consequently, the antiplane traction (t_z) on the surface of the crack (for which $n_x = 0$, $n_y = 1$, Fig. 1) is

$$t_z = t_{yz} = \bar{t}_z\,, \tag{39}$$

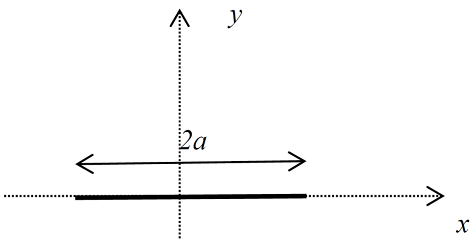

Figure 1: Plane weakened by one crack

The crack is modelled by the convolution of screw dislocation with a distribution function $B^n(\xi)$ [2] while superscript 'n' denotes nonlocal elasticity. A so-called dislocation density $B^n_z(\xi)$ can be written in terms of the distribution function $B^n(\xi)$ as [3]

$$B^n_z(\xi) = b_z B^n(\xi) \qquad -1 < \xi < 1 \tag{40}$$

which depicts the density of the dislocations along the crack surface. Employing the principle of superposition, the antiplane traction on the surface of the crack due to the presence of the above-mentioned distribution of dislocations is

$$t_z\big(\alpha(s)\big) = \int_{-1}^{1} aK^n(s,\xi)B^n_z(\xi)\,\mathrm{d}\xi\,, \tag{41}$$

in which the kernel of the integral is

$$K^n(s,\xi) = \frac{1}{b_z} t_{zy}\big(\alpha(s),\alpha(\xi)\big) = \frac{\mu}{2\pi}\frac{1}{X}\left\{1 - \frac{1}{c_1^2 - c_2^2}\Big(c_1|X|K_1\big(|X|/c_1\big) - c_2|X|K_1\big(|X|/c_2\big)\Big)\right\}\,, \tag{42}$$

where $X = a(s-\xi)$. In comparison with classical elasticity, due to the presence of Bessel functions, the integral equation of nonlocal elasticity of bi-Helmholtz type (41) is nonsingular.

The unknown dislocation density $B^n_z(\xi)$ or distribution function $B^n(\xi)$ should be determined using the crack face boundary conditions (39).

In order to study the antiplane shear problem (Mode III), consider a plane subjected to the loadings

$$t^{\infty}_{yz} = t_{yz0}\,, \qquad t^{\infty}_{xz} = 0\,. \tag{43}$$

According to Bueckner's superposition principle [34], the solution of the crack problem (Fig. 2a) can be obtained by the superposition of two constituent problems. The first problem deals with an un-cracked body under remote shear field (Fig. 2b). The second problem is an un-cracked body which is subjected to equal and opposite tractions present along the line of the crack in the first problem, but in the absence of remote loading (Fig. 2c). The crack faces must be traction free. The superposition of these two problems will enforce the traction-free condition along the crack face. Now by inducing the stress field arising in the second problem via distribution of dislocations, the equations for determining the unknown dislocation density can be derived.

In the first problem, the un-cracked body under loading (43) is in a state of antiplane shear with following uniform stress field

$$t_{yz}(x,y) = t_{yz0}\,, \qquad t_{xz}(x,y) = 0\,. \tag{44}$$

Therefore, the traction at the location of the crack in the un-cracked plane reads

$$\bar{t_z} = t_{yz0}\,. \tag{45}$$

Considering the second problem and using the Bueckner's superposition principle [34], the left-hand side of the integral equation (41) is identical to the traction in (45) with opposite sign and gives the 'corrective' solution

$$\int_{-1}^{1} aK^n(s,\xi)B^n_z(\xi)\,\mathrm{d}\xi = -t_{yz0}\,. \tag{46}$$

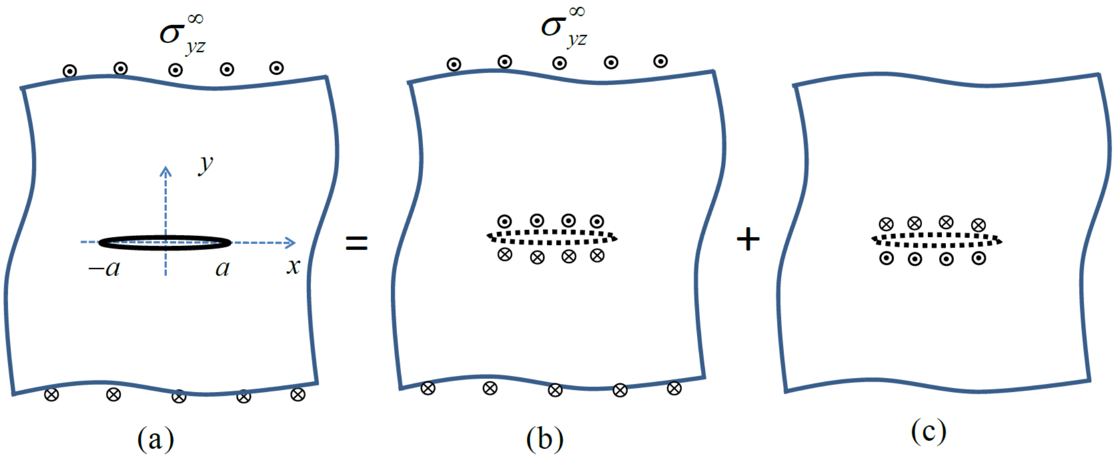

Figure 2: Bueckner's superposition principle

The closure requirement, i.e.,

$$\int_{-1}^{1} B_z^n(\xi)\,\mathrm{d}\xi = 0 \tag{47}$$

should be applied to the integral equation of an embedded crack to reach single-valued displacement field out of crack surfaces. The unknown dislocation density $B_z^n(\xi)$ and also distribution function $B^n(\xi)$ (40) can now be obtained by solving the integral equation (46) together with the closure condition (47). Once the distribution function is evaluated, we can apply it to the solution of the screw dislocation to reach all field quantities of a plane weakened by a crack, i.e. to determine effective dislocation density, crack opening displacement, and strain and stress fields of the crack.

Within classical elasticity for which the plane is subjected to the loading

$$\sigma_{yz}^{\infty} = \sigma_{yz0}\,, \qquad \sigma_{xz}^{\infty} = 0\,, \tag{48}$$

the integral equations (46) is simplified to

$$-2\pi\frac{\sigma_{yz0}}{\mu} = \int_{-1}^{1} \frac{1}{s-\xi} B_z^c(\xi)\,\mathrm{d}\xi\,. \tag{49}$$

for which the analytical solution, satisfying the classical closure requirement, is [2]

$$B_z^c(s) = b_z B^c(s) = \frac{2\sigma_{yz0}}{\mu}\frac{s}{\sqrt{1-s^2}}\,, \qquad -1 < s < 1. \tag{50}$$

Here, superscript 'c' denotes classical elasticity. In the limit $\gamma \to 0$ (i.e. $c_2 \to 0$ and $c_1 \to \epsilon$), the integral equation (46) is simplified to the one for nonlocal elasticity of Helmholtz type [20].

4.1. Effective dislocation density within nonlocal elasticity of bi-Helmholtz type

The distribution function $B^n(\xi)$ depicts the proper distribution of single dislocations for representing the crack (Fig 1). Having determined this distribution function, we can apply it (as a convolution) to various fields of a single dislocation and determine effective fields of a plane weakened by a crack.

The effective nonlocal dislocation density of the plane weakened by the crack (B_{eff}^n) is defined as the convolution of the dislocation density of a single dislocation (27) with the nonlocal distribution function $B^n(\xi)$, i.e.

$$B_{eff}^n(x,y) = a\int_{-1}^{1} b_z\delta(X)\delta(Y)B^n(\xi)\,\mathrm{d}\xi, \tag{51}$$

while $X = x - a\xi$ and $Y = y$. Considering $x = as, y = 0$ and using $\delta(ax) = \delta(x)/a$, the effective dislocation density of crack reads

$$B^n_{cr}(s,0) = b_z B^n(s)\,, \qquad -1 < s < 1 \tag{52}$$

Here, subscript 'cr' denotes crack. From physical point of view, the effective dislocation density of crack demonstrates the effect of the generalized continuum theory on the distribution of dislocations. It is concluded that the effective dislocation density of a crack B^n_{cr} is simply the distribution function $B^n(\xi)$ multiplied by the Burgers vector b_z. This result has also been reported for nonlocal elasticity of Helmholtz type ($c_2 \to 0$) [20]. Considering (40), it is noticed that the effective dislocation density of a crack (B^n_{cr}) is identical to the dislocation density (B^n_z),

$$B^n_{cr}(s,0) = B^n_z(s)\,, \qquad -1 < s < 1. \tag{53}$$

In classical elasticity, the effective dislocation density of crack reads

$$B^c_{cr}(s,0) = B^c_z(s) = \frac{2\sigma_{yz0}}{\mu}\frac{s}{\sqrt{1-s^2}}\,, \qquad -1 < s < 1. \tag{54}$$

4.2. Crack opening displacement within nonlocal elasticity of bi-Helmholtz type

The strain and displacement field within nonlocal elasticity (of Helmholtz and bi-Helmholtz types) are identical to those in classical elasticity. By convolution of the displacement field of classical discrete dislocation (25) with the classical distribution function ($B^c(\xi)$), the displacement field within nonlocal (and classical) elasticity reads

$$w^n_{cr}(x,y) = w^c_{cr}(x,y) = \frac{a}{2\pi}\int_{-1}^{1}\Big(\arctan\frac{Y}{X} - \frac{\pi}{2}\Big)B^c_z(\xi)\,\mathrm{d}\xi, \tag{55}$$

while $X = x - a\xi$ and $Y = y$, and B^c_z is given in (50). The nonlocal (and classical) crack opening displacement (COD) is defined as

$$g^n(x) = g^c(x) = 2w^c_{cr}(x,0). \tag{56}$$

Considering (26), the COD for $x = as$ reduces to

$$g^n(s) = g^c(s) = -\frac{a}{2}\int_{-1}^{1}\mathrm{sgn}(s-\xi)B^c_z(\xi)\,\mathrm{d}\xi\,, \qquad -1 < s < 1. \tag{57}$$

In view of classical closure requirement, the COD within nonlocal (and classical) elasticity takes the form

$$g^n(s) = g^c(s) = -a\int_{-1}^{s}B^c_z(\xi)\,\mathrm{d}\xi\,, \qquad -1 < s < 1. \tag{58}$$

Therefore, within nonlocal and classical elasticity, the dislocation density represents the negative of the slope at any point between the crack faces. Using the classical dislocation density (50), the COD is simplified to

$$g^n(s) = g^c(s) = 2a\frac{\sigma_{yz0}}{\mu}\sqrt{1-s^2}\,, \qquad -1 < s < 1. \tag{59}$$

According to (59), within nonlocal and classical elasticity, the crack opening is limited to the crack surface ($-a < x < a$).

4.3. Plastic distortion within nonlocal elasticity of bi-Helmholtz type

The plastic distortion within nonlocal elasticity of bi-Helmholtz type is identical to the one in classical elasticity which is given by

$$\beta^{P,cr}_{zx}(x,0^-) = -B^c_z(x/a). \tag{60}$$

The plastic distortion (60) is derived by convolution of the classical plastic distortion of a single dislocation (36) with the classical distribution function.

4.4. Strain field within nonlocal elasticity of bi-Helmholtz type

The strain field within Eringen's nonlocal elasticity is identical to the one in classical elasticity. By convolution of the strain field of a discrete dislocation (23) with the classical distribution function, the displacement field within nonlocal (and classical) elasticity reads

$$e^{cr}_{zx}(x,y) = -\frac{a}{4\pi}\int_{-1}^{1}\frac{Y}{R^2}B^c_z(\xi)\,\mathrm{d}\xi, \tag{61a}$$

$$e^{cr}_{zy}(x,y) = \frac{\sigma_{yz0}}{2\mu} + \frac{a}{4\pi}\int_{-1}^{1}\frac{X}{R^2}B^c_z(\xi)\,\mathrm{d}\xi, \tag{61b}$$

while $X = x - a\xi$ and $Y = y$.

4.5. Stress field within nonlocal elasticity of bi-Helmholtz type

By convolution of the nonlocal stress field of a discrete dislocation (22) with the nonlocal distribution function, the nonlocal stress field reads

$$t^{cr}_{zx}(x,y) = -\frac{a\mu}{2\pi}\int_{-1}^{1}\frac{Y}{R^2}\left\{1 - \frac{1}{c_1^2 - c_2^2}\Big(c_1 R K_1(R/c_1) - c_2 R K_1(R/c_2)\Big)\right\}B^n_z(\xi)\,\mathrm{d}\xi, \tag{62a}$$

$$t^{cr}_{zy}(x,y) = t_{yz0} + \frac{a\mu}{2\pi}\int_{-1}^{1}\frac{X}{R^2}\left\{1 - \frac{1}{c_1^2 - c_2^2}\Big(c_1 R K_1(R/c_1) - c_2 R K_1(R/c_2)\Big)\right\}B^n_z(\xi)\,\mathrm{d}\xi, \tag{62b}$$

while $X = x - a\xi$ and $Y = y$. The nonlocal stress components (62) are nonsingular. Within nonlocal elasticity of Helmholtz type ($c_2 \to 0$), the stress field (62) is simplified to the nonsingular stress field reported by Mousavi and Lazar [20]. In the limit $c_1 \to 0$ and $c_2 \to 0$, the classical singular stress field, i.e.

$$\tau^{cr}_{zx}(x,y) = -\frac{a\sigma_{yz0}}{\pi}\int_{-1}^{1}\frac{Y}{R^2}\frac{\xi}{\sqrt{1-\xi^2}}\,\mathrm{d}\xi, \tag{63a}$$

$$\tau^{cr}_{zy}(x,y) = \sigma_{yz0} + \frac{a\sigma_{yz0}}{\pi}\int_{-1}^{1}\frac{X}{R^2}\frac{\xi}{\sqrt{1-\xi^2}}\,\mathrm{d}\xi \tag{63b}$$

is recovered.

5. Dislocation-based antiplane fracture mechanics within gradient elasticity of bi-Helmholtz type

Within gradient elasticity of bi-Helmholtz type, the generalized traction boundary conditions are given by (16). On the surface of crack (Fig.1), the normal vector is $\boldsymbol{n} = \{0,1,0\}$. Therefore, for a Mode III crack, the non-zero generalized tractions (17) of the crack surface are simplified to

$$t_z = \tau_{yz} - \epsilon^2(\tau_{zy,pp} + \tau_{zx,xy}) + \gamma^4(\tau_{zy,ppqq} + \tau_{zx,xypp} + \tau_{zx,xxxy} + \tau_{zy,xxyy})\,, \tag{64a}$$

$$t^{(1)}_z = \epsilon^2\tau_{zy,y} - \gamma^4(\tau_{zy,ypp} + \tau_{zy,xxy} + \tau_{zx,xyy})\,, \tag{64b}$$

$$t^{(2)}_z = \gamma^4\tau_{zy,yy}\,, \tag{64c}$$

while $p, q \in \{x, y\}$. Consequently, the crack face boundary conditions of gradient elasticity of bi-Helmholtz type read

$$t_z = \bar{t}_z\,, \tag{65a}$$

$$t^{(1)}_z = \bar{t}^{(1)}_z\,, \tag{65b}$$

$$t^{(2)}_z = \bar{t}^{(2)}_z\,. \tag{65c}$$

Various models of the crack-tip region lead to different solutions for the crack problem. Here, the crack face boundary conditions (65) are in terms of generalized tractions and no further explicit condition is imposed on the displacement field. In general, the crack face model may include additional conditions on the displacement field ahead of the crack tip [22].

Within gradient elasticity, similar to previous section, a crack is constructed by a continuous distribution of dislocations along the crack surface with density $B_z^g(\xi)$ for which [2, 3]

$$B_z^g(\xi) = b_z B^g(\xi) \qquad -1 < \xi < 1. \tag{66}$$

Here, superscript 'g' denotes gradient elasticity, and $B^g(\xi)$ is the distribution function within gradient elasticity. Employing the principle of superposition, the generalized tractions on the surface of the crack due to the presence of the distribution of dislocations (66) are

$$t_z\big(\alpha(s), \beta(s)\big) = \int_{-1}^{1} aK^g(s,\xi) B_z^g(\xi)\,\mathrm{d}\xi\,, \tag{67a}$$

$$t_z^{(1)}\big(\alpha(s), \beta(s)\big) = \int_{-1}^{1} aK^{g1}(s,\xi) B_z^g(\xi)\,\mathrm{d}\xi\,, \tag{67b}$$

$$t_z^{(2)}\big(\alpha(s), \beta(s)\big) = \int_{-1}^{1} aK^{g2}(s,\xi) B_z^g(\xi)\,\mathrm{d}\xi\,, \tag{67c}$$

in which the kernels of the integrals are

$$K^g(s,\xi) = \frac{1}{b_z}\{\tau_{yz} - \epsilon^2(\tau_{zy,pp} + \tau_{zx,xy}) + \gamma^4(\tau_{zy,ppqq} + \tau_{zx,xypp} + \tau_{zx,xxxy} + \tau_{zy,xxyy})\}\,, \tag{68a}$$

$$K^{g1}(s,\xi) = \frac{1}{b_z}\{\epsilon^2\tau_{zy,y} - \gamma^4(\tau_{zy,ypp} + \tau_{zy,xxy} + \tau_{zx,xyy})\}\,, \tag{68b}$$

$$K^{g2}(s,\xi) = \frac{1}{b_z}\{\gamma^4\tau_{zy,yy}\}\,. \tag{68c}$$

Here the stress components (after taking derivatives) are written in terms of (X,Y) where $X = a(s-\xi)$, $Y = 0$ and $R = |X|$. Considering $Y = 0$, the kernels (68) are simplified to

$$K^g(s,\xi) = \frac{1}{b_z}\{1 - \epsilon^2\partial^2_{xx} + \gamma^4(\partial^4_{xxxx} - \partial^4_{xxyy})\}\tau_{yz}\,, \tag{69a}$$

$$K^{g1}(s,\xi) = 0\,, \tag{69b}$$

$$K^{g2}(s,\xi) = \frac{1}{b_z}\{\gamma^4\partial^2_{yy}\}\tau_{zy}\,. \tag{69c}$$

The kernels (69), after substituting the stress component τ_{zy} (28), are simplified to

$$\begin{aligned}
K^g(s,\xi) &= \frac{\mu}{2\pi}\frac{1}{X}\left\{1 - \frac{1}{c_1^2 - c_2^2}\Big(c_1|X|K_1\big(\frac{|X|}{c_1}\big) - c_2|X|K_1\big(\frac{|X|}{c_2}\big)\Big)\right\} \\
&- \frac{\epsilon^2\mu}{2\pi}\frac{1}{X^3}\left\{2 - \frac{X^2}{c_1^2 - c_2^2}\Big(\frac{|X|}{c_1}K_1\big(\frac{|X|}{c_1}\big) - \frac{|X|}{c_2}K_1\big(\frac{|X|}{c_2}\big) + K_2\big(\frac{|X|}{c_1}\big) - K_2\big(\frac{|X|}{c_2}\big)\Big)\right\} \\
&+ \frac{\gamma^4\mu}{2\pi}\frac{1}{c_1^3c_2^3(c_1^2 - c_2^2)}\frac{|X|}{X^6}\bigg\{48\frac{|X|}{X}c_1^3c_2^3(c_1^2 - c_2^2) - 24X|X|c_1^3c_2^3\Big(K_0\big(\frac{|X|}{c_1}\big) - K_0\big(\frac{|X|}{c_2}\big)\Big) \\
&- 3X^3|X|c_1c_2\Big(c_2^2K_0\big(\frac{|X|}{c_1}\big) - c_1^2K_0\big(\frac{|X|}{c_2}\big)\Big) - 12X^3c_1^2c_2^2\Big(c_2K_1\big(\frac{|X|}{c_1}\big) - c_1K_1\big(\frac{|X|}{c_2}\big)\Big) \\
&- 48Xc_1^3c_2^3\Big(c_1K_1\big(\frac{|X|}{c_1}\big) - c_2K_1\big(\frac{|X|}{c_2}\big)\Big) - X^5\Big(c_2^3K_1\big(\frac{|X|}{c_1}\big) - c_1^3K_1\big(\frac{|X|}{c_2}\big)\Big)\bigg\},
\end{aligned} \tag{70a}$$

$$K^{g1}(s,\xi) = 0\,, \tag{70b}$$

$$K^{g2}(s,\xi) = -\frac{\mu\gamma^4}{2\pi}\frac{1}{X^3}\left\{2 - X^2\frac{1}{c_1^2-c_2^2}\Big(K_2\big(\frac{|X|}{c_1}\big) - K_2\big(\frac{|X|}{c_2}\big)\Big)\right\}. \tag{70c}$$

The unknown dislocation density $B_z^g(\xi)$ should be determined using the crack face boundary conditions (65).

In order to study the antiplane shear problem (Mode III), consider a plane subjected to the loading

$$\tau_{yz}^{\infty} = \tau_{yz0}\,, \qquad \tau_{xz}^{\infty} = 0, \tag{71a}$$

$$\boldsymbol{\tau}_{\infty}^{(1)} = \{\tau_{ijk}^{\infty}\} = 0\,, \qquad \boldsymbol{\tau}_{\infty}^{(2)} = \{\tau_{ijkl}^{\infty}\} = 0. \tag{71b}$$

Similar to the previous section, the crack problem is decomposed into two companion problems (Fig. 2). The un-cracked body under loading (71) is in a state of pure uniform antiplane shear. Due to uniform stress and strain fields, no gradient of strain exists and the double and triple stress tensors vanish. Consequently the stress field of the un-cracked body (Fig. 2a) reduces to

$$\tau_{yz}(x,y) = \tau_{yz0}\,, \qquad \tau_{xz}(x,y) = 0, \tag{72a}$$

$$\boldsymbol{\tau}^{(1)} = \{\tau_{ijk}\} = 0\,, \qquad \boldsymbol{\tau}^{(2)} = \{\tau_{ijkl}\} = 0. \tag{72b}$$

Therefore, the tractions at the location of the crack in the un-cracked plane read

$$\bar{t}_z = \tau_{yz0}\ , \qquad \bar{t}_z^{(1)} = 0\ , \qquad \bar{t}_z^{(2)} = 0\,. \tag{73}$$

Using the Bueckner's superposition principle [34], the left-hand side of the integral equations (67) are identical to the corresponding tractions in (73) with opposite sign, resulting in

$$\int_{-1}^{1} aK^g(s,\xi)B_z^g(\xi)\,\mathrm{d}\xi = -\tau_{yz0}\,, \tag{74a}$$

$$\int_{-1}^{1} K^{g2}(s,\xi)B_z^g(\xi)\,\mathrm{d}\xi = 0\,. \tag{74b}$$

Being the convolution of nonsingular dislocations, the integral equations (74) are nonsingular. The closure requirement

$$\int_{-1}^{1} B_z^g(\xi)\,\mathrm{d}\xi = 0 \tag{75}$$

should also accompany the integral equations of an embedded crack (74) to reach single-valued displacement field out of crack surfaces. In other words, the unknown dislocation density $B_z^g(\xi)$ should be obtained by solving the integral equations (74) together with the closure condition (75). In the limit $c_2 \to 0$, the integral equations (74) are simplified to those for gradient elasticity of Helmholtz type [27].

Similar to nonlocal elasticity, once the distribution function is evaluated, all field quantities of the plane weakened by the crack can be determined.

5.1. Effective dislocation density within gradient elasticity of bi-Helmholtz type

The effective dislocation density of the plane (B_{eff}^g) is defined as the convolution of the dislocation density of a single dislocation (32) with the distribution function $B^g(\xi)$, resulting in

$$B_{eff}^g(x,y) = \frac{a}{2\pi}\frac{1}{c_1^2-c_2^2}\int_{-1}^{1}\{K_0(R/c_1) - K_0(R/c_2)\}\,B_z^g(\xi)\,\mathrm{d}\xi \tag{76}$$

while $X = x - a\xi$, $Y = y$ and $R^2 = X^2 + Y^2$. Assuming $x = as, y = 0$, the effective dislocation density of crack reads

$$B_{cr}^g(x,0) = \frac{a}{2\pi}\frac{1}{c_1^2-c_2^2}\int_{-1}^{1}\{K_0(|X|/c_1) - K_0(|X|/c_2)\}\,B_z^g(\xi)\,\mathrm{d}\xi. \tag{77}$$

Here, subscript 'cr' denotes crack. It is noticed that the effective dislocation density of a crack within nonlocal elasticity B^n_{cr} is simply identical to B^n_z (53) and is limited to $-a < x < a$. In contrast, within gradient elasticity of bi-Helmholtz type, B^g_{cr} is given by (77) and is not limited to the crack faces. Within gradient elasticity of Helmholtz type, considering $c_2 \to 0$, the effective dislocation density reads

$$B^g_{eff}(x,y) = \frac{a}{2\pi}\frac{1}{c_1^2}\int_{-1}^{1} K_0(R/c_1) B^g_z(\xi)\,\mathrm{d}\xi. \tag{78}$$

5.2. Crack opening displacement within gradient elasticity of bi-Helmholtz type

By convolution of the displacement field of a discrete dislocation (30) with the distribution function and using (66), the displacement field within gradient elasticity can be determined. Similarly, using the displacement field of a discrete dislocation on the x-axis (31), the crack opening displacement (COD) within second gradient elasticity reads

$$g^g(x) = -\frac{a}{2}\int_{-1}^{1} \mathrm{sgn}(X)\left\{1 - \frac{c_1^2}{c_1^2-c_2^2}\exp(-|X|/c_1) + \frac{c_2^2}{c_1^2-c_2^2}\exp(-|X|/c_2)\right\} B^g_z(\xi), \tag{79}$$

while $X = x - a\xi$. Considering the closure requirement (75), the COD within second gradient elasticity takes the form

$$\begin{aligned} g^g(x) &= -a\int_{-1}^{x/a} B^g_z(\xi)\,\mathrm{d}\xi \\ &+ \frac{a}{2}\int_{-1}^{x/a}\left\{\frac{c_1^2}{c_1^2-c_2^2}\exp(-|x-a\xi|/c_1) - \frac{c_2^2}{c_1^2-c_2^2}\exp(-|x-a\xi|/c_2)B^g_z(\xi)\right\}\mathrm{d}\xi \\ &- \frac{a}{2}\int_{x/a}^{1}\left\{\frac{c_1^2}{c_1^2-c_2^2}\exp(-|x-a\xi|/c_1) - \frac{c_2^2}{c_1^2-c_2^2}\exp(-|x-a\xi|/c_2)B^g_z(\xi)\right\}\mathrm{d}\xi. \end{aligned} \tag{80}$$

In the limit $c_2 \to 0$, the COD within gradient elasticity of Helmholtz type is given by

$$g^g(x) = -\frac{a}{2}\int_{-1}^{1} \mathrm{sgn}(X)\left\{1 - \exp(-|X|/c_1)\right\} B^g_z(\xi). \tag{81}$$

In contrast to classical and nonlocal elasticity, (80) and (81) inform that within gradient elasticity, the crack opening is not limited to the crack surface. This interesting feature of the gradient theory is illustrated in the next section.

5.3. Plastic distortion within gradient elasticity of bi-Helmholtz type

By convolving the plastic distortion of a discrete dislocation (34) with the distribution function $B^g(\xi)$, the plastic distortion of the plane weakened by a crack ($\beta^{P,cr}_{zx}$) along $y = 0$ is obtained as

$$\beta^{P,cr}_{zx}(x,0) = -\frac{a}{4}\frac{1}{c_1^2-c_2^2}\int_{-1}^{1}\left\{c_1\exp(-|x-a\xi|/c_1) - c_2\exp(-|x-a\xi|/c_2)\right\} B^g_z(\xi)\,\mathrm{d}\xi. \tag{82}$$

In the limit, for gradient elasticity of Helmholtz type, the plastic distortion along the crack line reads

$$\beta^{P,cr}_{zx}(x,0) = -\frac{a}{4c_1}\int_{-1}^{1}\exp(-|x-a\xi|/c_1) B^g_z(\xi)\,\mathrm{d}\xi, \tag{83}$$

and within classical elasticity, it is simplified to (60). According to (60), the classical plastic distortion vanishes out of the crack line while within gradient elasticity, the plastic distortion (82, 83) appears also beyond the crack tip, giving rise to crack tip plasticity.

5.4. Strain field within gradient elasticity of bi-Helmholtz type

By convolution of the strain field of a discrete dislocation (29) with the distribution function, the strain field within gradient elasticity of bi-Helmholtz type reads

$$e_{zx}^{cr}(x,y) = -\frac{a}{4\pi}\int_{-1}^{1}\frac{Y}{R^2}\left\{1-\frac{1}{c_1^2-c_2^2}\Big(c_1RK_1(R/c_1)-c_2RK_1(R/c_2)\Big)\right\}B_z^g(\xi)\,\mathrm{d}\xi, \tag{84a}$$

$$e_{zy}^{cr}(x,y) = \frac{\tau_{yz0}}{2\mu}+\frac{a}{4\pi}\int_{-1}^{1}\frac{X}{R^2}\left\{1-\frac{1}{c_1^2-c_2^2}\Big(c_1RK_1(R/c_1)-c_2RK_1(R/c_2)\Big)\right\}B_z^g(\xi)\,\mathrm{d}\xi, \tag{84b}$$

while $X = x-\alpha(\xi)$, $Y = y$ and $R^2 = X^2+Y^2$.

5.5. Stress field within gradient elasticity of bi-Helmholtz type

Convolution of the stress field of a discrete dislocation (28) with the distribution function gives the stress field as

$$\tau_{zx}^{cr}(x,y) = -\frac{a\mu}{2\pi}\int_{-1}^{1}\frac{Y}{R^2}\left\{1-\frac{1}{c_1^2-c_2^2}\Big(c_1RK_1(R/c_1)-c_2RK_1(R/c_2)\Big)\right\}B_z^g(\xi)\,\mathrm{d}\xi, \tag{85a}$$

$$\tau_{zy}^{cr}(x,y) = \tau_{yz0}+\frac{a\mu}{2\pi}\int_{-1}^{1}\frac{X}{R^2}\left\{1-\frac{1}{c_1^2-c_2^2}\Big(c_1RK_1(R/c_1)-c_2RK_1(R/c_2)\Big)\right\}B_z^g(\xi)\,\mathrm{d}\xi, \tag{85b}$$

while $X = x-\alpha(\xi)$, $Y = y$ and $R^2 = X^2+Y^2$. The double and triple stress components can be derived by substituting (85) in (14). As mentioned earlier, for a single dislocation, the double stress and triple stress tensors are nonsingular within gradient elasticity of bi-Helmholtz type. Consequently, the double and triple stress fields of a plane weakened by a crack, being the convolution of the discrete dislocations, are nonsingular. On the other hand, the total stress tensor can be determined by substituting (85) in (18). Since the total stress of a discrete dislocation is singular, it results in singular total stress at crack tips.

Within gradient elasticity of Helmholtz type ($c_2 \to 0$) the stress field (85) is simplified to those given by Mousavi and Aifantis [27]. In the following section, numerical results will be presented for a crack of Mode III.

6. Numerical results

In this section, a horizontal crack is studied within nonlocal and gradient elasticity of bi-Helmholtz type. Employing the dislocation-based formulation presented in the previous section, we are able to determine the stress, strain, crack opening and effective dislocation density of the plane weakened by a crack, and study results within classical elasticity and nonlocal and gradient elasticity (of Helmholtz and bi-Helmholtz types).

In the following, the plane (Fig.1) is assumed to be under uniform antiplane shear loading (48), (43) or (71) (for classical, nonlocal or gradient elasticity, respectively) for which $\sigma_{yz0} = t_{yz0} = \tau_{yz0} = \mu$. The parameters of nonlocality (or gradient coefficients) are assumed to be $\epsilon = 2\gamma = 0.1a$, while a is the half-crack length. In the case of gradient and nonlocal elasticity of Helmholtz type, the parameter of nonlocality is $\epsilon = 0.1a$.

6.1. Dislocation density

The dislocation densities within classical, nonlocal and gradient elasticity of bi-Helmholtz type (B_z^c, B_z^n and B_z^g, respectively) are depicted in Fig. 3. It is observed that the dislocation density is singular for all three cases including classical, nonlocal and gradient elasticity. The dislocation density of gradient elasticity approximately coincides with the classical density, while the dislocation density of nonlocal elasticity is distinctive. This difference between gradient and nonlocal theory stems from the difference of the crack face boundary conditions.

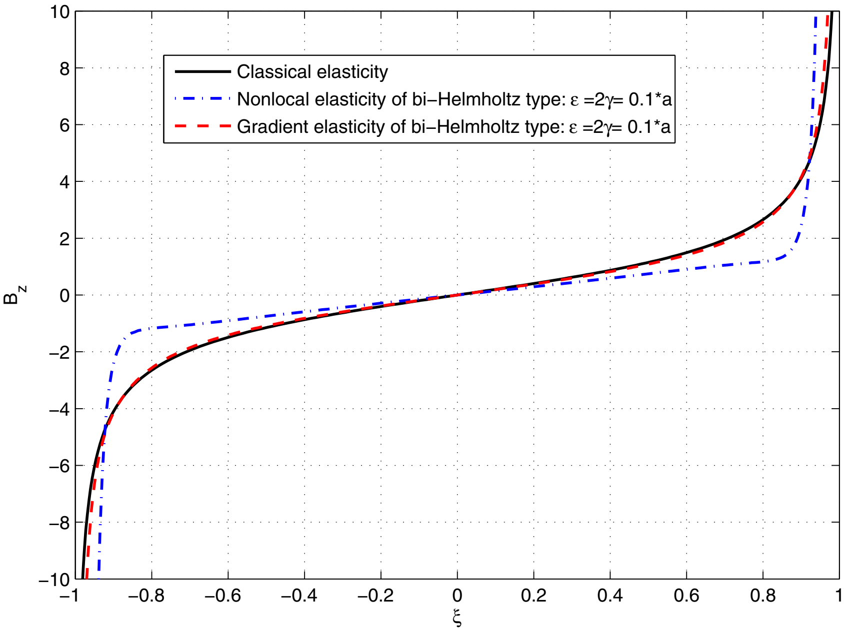

Figure 3: Dislocation density for classical elasticity and nonlocal and gradient elasticity of bi-Helmholtz type

For completeness, the results are also given for nonlocal and gradient elasticity of Helmholtz type (Fig. 4). No qualitative difference is observed between dislocation densities of gradient elasticity of Helmholtz and bi-Helmholtz types. In contrast, nonlocal elasticity of bi-Helmholtz type predicts dislocation density with qualitative difference comparing to the one in nonlocal elasticity of Helmholtz type. The singularity of the nonlocal dislocation density of bi-Helmholtz type (which occurs at crack tip) is in the same direction as the classical elasticity, while the singularity of the nonlocal dislocation density of Helmholtz type is in the opposite direction [20].

It is also interesting to study the effective dislocation density of a crack. It is emphasized that the effective dislocation density of the crack, in general, is different from the dislocation density given in Fig. 3. In particular, as given by (53) and (54), within classical and nonlocal elasticity, the effective dislocation density of a crack is identical to the dislocation density. This is not the case in gradient elasticity (77). Fig. 5 compares the effective dislocation density of crack within gradient elasticity of bi-Helmholtz type (77) and Helmholtz type (78) with the classical effective dislocation density (54).

As mentioned in the previous sections, classical and gradient elasticity of Helmholtz type give singular density of discrete dislocation (27 and 33). In contrast, the gradient elasticity of bi-Helmholtz type predicts nonsingular dislocation density (32). Fig. 5 depicts that the classical effective dislocation density of a crack is singular at crack tips. Within gradient elasticity of Helmholtz type, this singularity is regularized, but still there exist a discontinuity of density at crack tips. Interestingly, within gradient elasticity of bi-Helmholtz type, the effective dislocation density of the crack is nonsingular and continuous. In this case, in the vicinity of the crack tip and along the crack surface, there is a peak for the effective dislocation density of the crack.

6.2. Crack opening displacement

The crack opening displacement is demonstrated in Fig. 6. The classical profile of the crack confirms the classical strain singularity at the crack tip. As expected, this singularity is regularized within gradient elasticity. A distinctive characteristic for the gradient COD is non-zero opening beyond the crack surface. This is due to the fact that unlike the jump occurring in the classical displacement field of a discrete screw

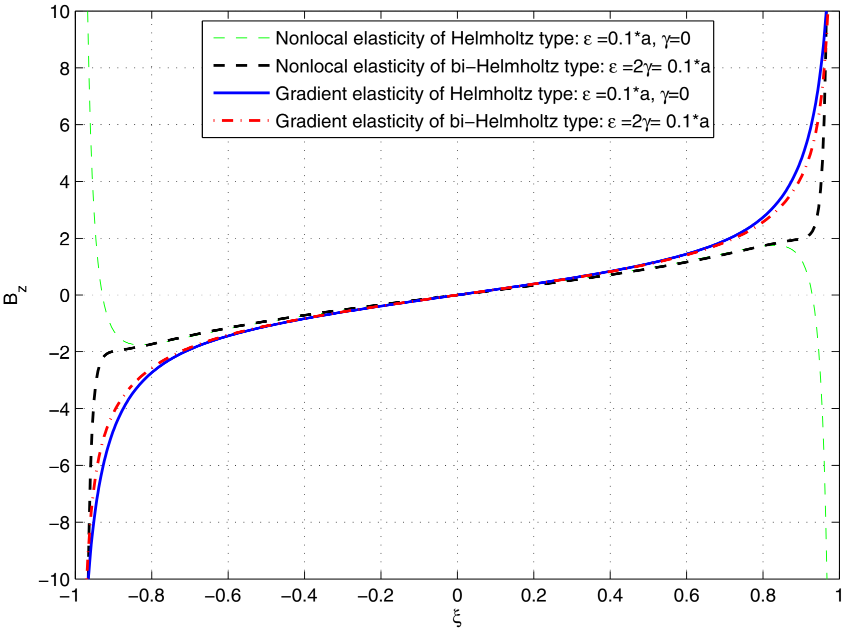

Figure 4: Dislocation density for nonlocal and gradient elasticity of Helmholtz and bi-Helmholtz types

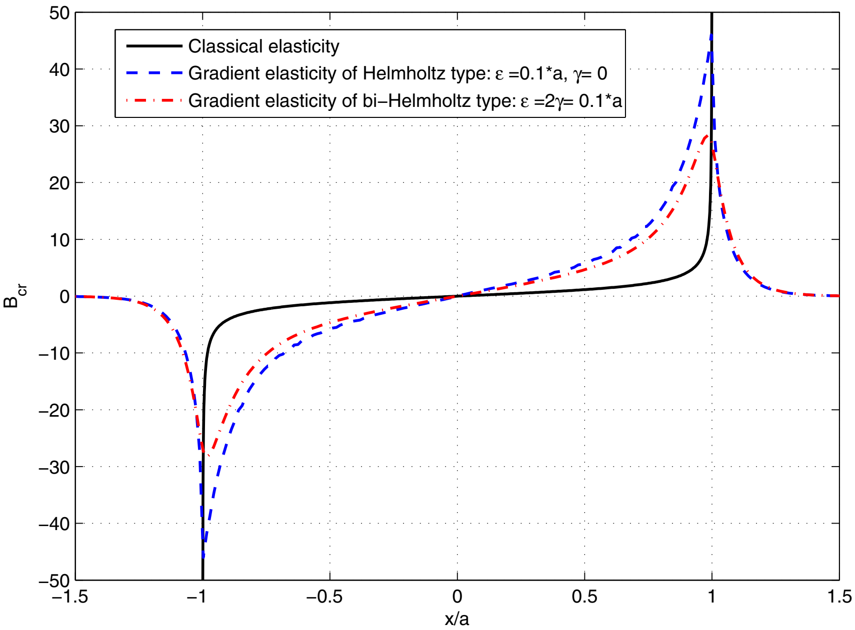

Figure 5: Effective dislocation density of a crack within classical and gradient elasticity

dislocation, the gradient elasticity lead to a smoothing of the displacement field of the screw dislocation (31). Accordingly, the convolution of the displacement field of a discrete dislocation within gradient elasticity gives

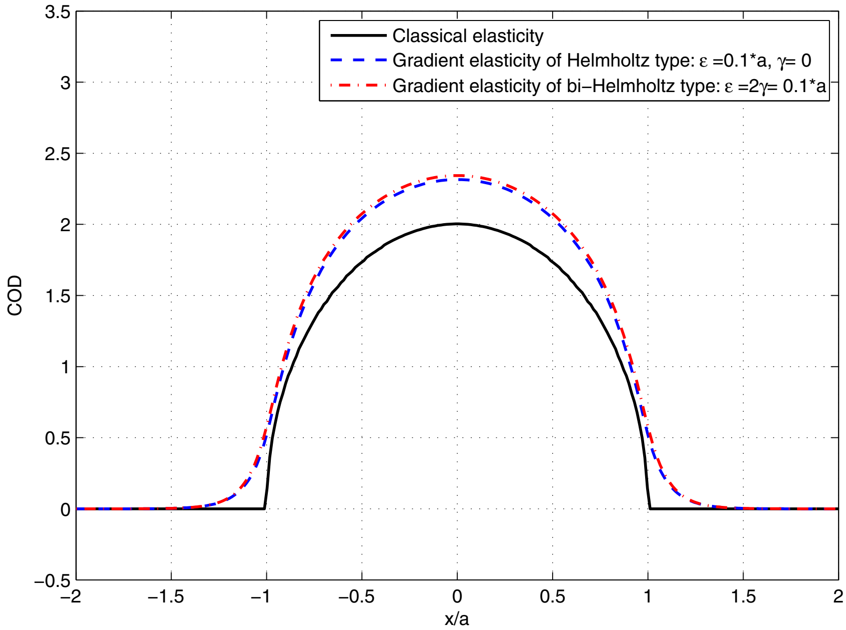

Figure 6: COD within classical and gradient elasticity

rise to crack opening profile which includes displacement across the crack plane ahead of the crack tip. In other words, the crack does not completely close at the tip (x/a = 1) of the physical region for which the traction is specified. This feature has been reported for the first strain gradient elasticity for a different set of boundary conditions [35]. The condition of zero displacement out of crack surface has also been employed within first gradient theory [22, 36] resulting in the cusp-like closure of the crack faces.

6.3. Plastic distortion

The classical plastic distortion (60) and plastic distortion within gradient elasticity (82, 83) of the plane weakened by a crack is shown in Fig. 7. In classical fracture mechanics, a model incorporating crack tip plasticity was initially introduced by Dugdale [4] for cracks in metal sheet specimens pulled in tension (Mode I). Barenblatt [5] proposed a local fracture theory that is known as Barenblatt's cohesive fracture theory. This theory is based on the idea that cohesive forces must be distributed in such a way as to be able to close the crack faces smoothly and remove the stress singularity at the crack tip.

Fig. 7 depicts that within gradient elasticity and using a dislocation-based approach, plasticity appears at the crack plane ahead of the crack tip. In other words, without any assumption of cohesive forces, the crack tip plasticity is captured.

6.4. Stress field

The stress component of classical, nonlocal and gradient elasticity of bi-Helmholtz type (σ_{yz}, t_{yz} and τ_{yz} derived in (63), (62) and (85), respectively) are demonstrated in Fig. 8. The classical singularity is regularized in both nonlocal and gradient theories. As imposed by the nonlocal boundary conditions (9), the nonlocal stress t_{yz} vanishes at the crack surface, while due to gradient boundary conditions (16), the stress τ_{yz} is non-zero along the crack. In fact, due to the boundary conditions (16), the generalized tractions t_z, t_z^1 and t_z^2 vanish along the crack face.

Finally, Fig. 9 demonstrates the stress fields for nonlocal and gradient elasticity of Helmholtz and bi-Helmholtz types. For both nonlocal and gradient theories, it is observed that the application of bi-Helmholtz

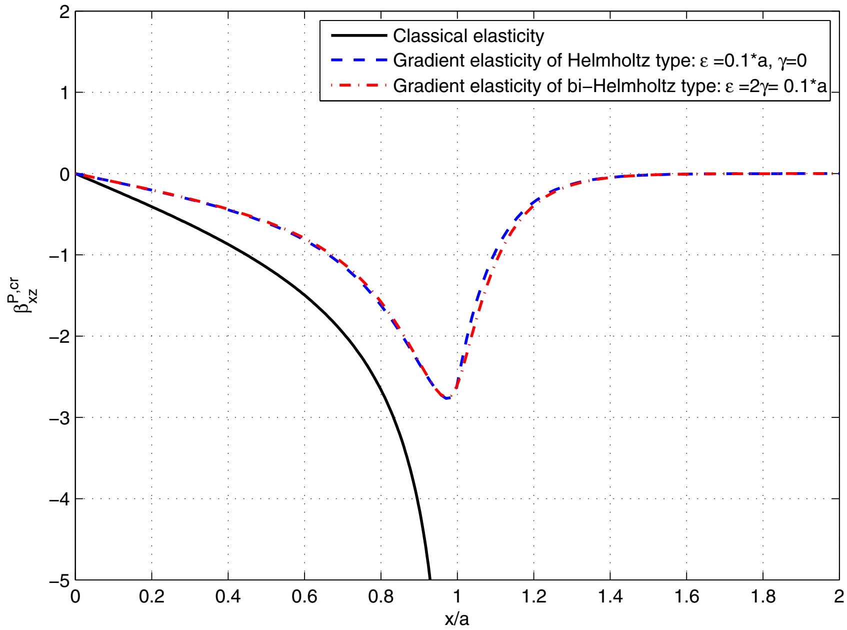

Figure 7: Plastic distortion $\beta^{P,cr}_{zx}(x, 0)$ within classical and gradient elasticity

type elasticity does not contribute to significant change comparing to Helmholtz type elasticity. Within bi-Helmholtz type elasticity, the peak value of the stress field is lower than the peak value within Helmholtz type elasticity. Moreover, the stress peak position is closer to the crack tip in Helmholtz type elasticity.

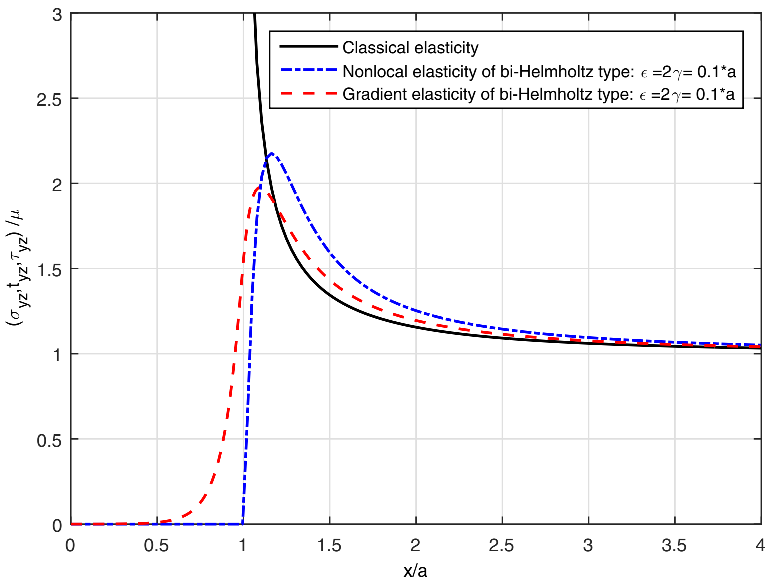

Figure 8: Stress fields for classical, nonlocal and gradient elasticity of bi-Helmholtz type

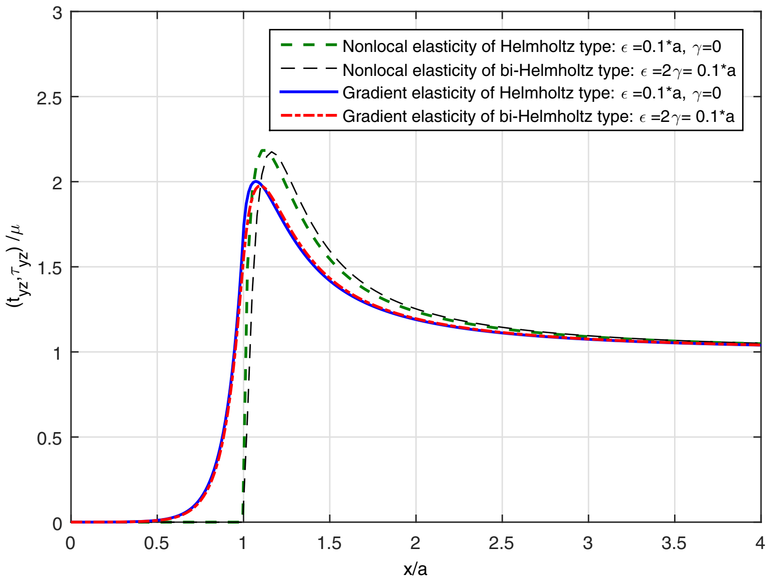

Figure 9: Stress fields for nonlocal and gradient elasticity of Helmholtz and bi-Helmholtz types

7. Conclusion

Employing a dislocation-based approach, nonsingular fracture theory is derived within nonlocal and gradient elasticity of bi-Helmholtz type. It is concluded that, for a plane weakened by a crack and under antiplane loading, stress fields (including higher order stresses) are regularized in nonlocal elasticity, while the strain fields remain singular. Gradient elasticity of bi-Helmholtz type gives rise to nonsingular stress and strain fields for the plane weakened by a crack while the total stress is singular.

The boundary conditions within gradient elasticity include non-standard higher order relations while boundary conditions in nonlocal elasticity are as simple as in classical elasticity. Consequently, the formulation of the dislocation-based fracture mechanics is much simpler for nonlocal theory than gradient elasticity. However, gradient elasticity eliminates the singularities of the strain and dislocation density tensors while nonlocal elasticity keeps these tensors singular (identical to those in classical elasticity). Therefore, the crack opening displacement (COD) within nonlocal elasticity is identical to the classical COD while gradient elasticity leads to a modified COD.

Interestingly, crack tip plasticity is captured within gradient elasticity. Having distributed the dislocations just along the crack surface (and not ahead of the crack tip), plastic distortion occurs along the crack and also ahead of the crack tip out of crack plane. Consequently, without any assumption of a cohesive zone, the classical singularities are regularized while the crack tip plasticity is modelled.

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