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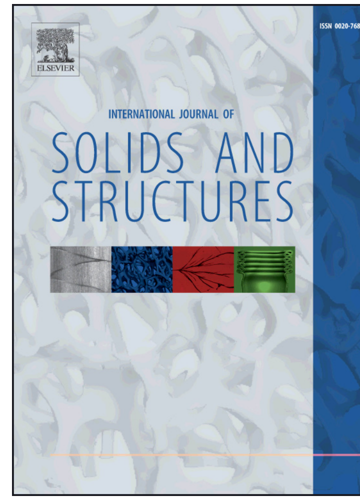

A Theory for Rubber-Like Rods

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A THEORY FOR RUBBER-LIKE RODS

Abstract : A theory for incompressible rubber-like straight rods undergoing finite strains and finite rotations is presented. Strains are expanded asymptotically for transverse coordinate of undeformed rod. The equations of equilibrium and corresponding boundary conditions are derived by implementing minimum total potential energy principle. Necessary conditions for the satisfaction of the stress-free boundary conditions on the top and bottom free surfaces of the rubber-like rods are derived. For the illustration and test of the proposed theory, the flexural buckling problem of Mooney-Rivlin rods under axial compressive loads is considered. Exact solutions corresponding to (i) various alternatives about the perturbation terms of the strain components, (ii) a very rigorous rod theory developed previously, and (iii) the three dimensional elasticity are obtained and compared. Degree of accuracy of the aforementioned approaches is discussed basing on the three dimensional elasticity solution. It is observed that considering all of the second order permutation terms yields very appealing results, almost coinciding with the results corresponding to the three dimensional elasticity for thin and quite thick rods.

Key words : Asymptotic, Constitutive, Flexural buckling, Elastomers, Large deformation, Large strain, Nonlinear elasticity, Perturbation, Polymers, Rubbers, Shear deformation, Stability, Thick rod (bar, beam, column), Thin rod (bar, beam, column)

1. Introduction

Theory of elasticity is a delicate approach to solve the problems of the elastic media. Since it is generally quite difficult to solve the corresponding three dimensional boundary value problems analytically or numerically; rod, plate and shell theories have been proposed and used by researchers and engineers. Obtained through the reduction from three dimensions to one or two dimensions; the rod, plate and shell theories contain assumptions depending on the degree of accuracy demanded by the researchers and engineers. The mentioned assumptions have simplified the problems considerably with the expense of sacrificing from the accuracy of the solution of the problems. Due to the diversity of the degree of accuracy desired in the concerning problems; various rod, plate and shell theories have still been going on to be proposed and used. It has always been a problem haunting the minds of the researchers to establish rod, plate and shell theories approaching the solutions of the theory of elasticity as much as possible. In this context; new theories for rods, plates and shells have unavoidably been tested by comparing the solutions obtained by using them with those corresponding to the theory of elasticity.

Rubber-like materials are hyperelastic materials capable of making very large elastic deformations (e.g., a stretch (Eringen, 1980) of 17.62 was measured by Smith and Chu (1972).). The compressibility of rubber-like materials is generally very small and, therefore, they are generally considered to be incompressible, except a few of them (e.g., polyurethane, foam rubber, hyperelastic soft tissue (Blatz and Ko, 1962; Simmonds, 1987; Yıldırım and Yükseler, 2011; Yükseler, 1996a). The stress-strain relations of rubber-like materials are highly nonlinear (Erman and Mark, 1997; Gent, 2005; Treloar, 1975). Therefore, a study pertaining to rubber-like materials should unavoidably include physical nonlinearity if the study is not restricted with the infinitesimally small deformations.

Rubber-like rods are used in various branches of engineering and science. Although there has been a host of researches about the rods (beams and columns) undergoing (i) infinitesimally small displacements and strains, considering and disregarding the zero traction conditions on the top and bottom free surfaces of the rods (e.g. Baghani et al., 2014; Bickford, 1982; Carrera et al., 2012; Dong et al., 2013; Ghosh et al., 2013; Kang, 2014; Li, 2014; Miranda et al., 2013; Pradhan and Chakraverty, 2014; Reddy, 1984; Salamat-talab et al., 2012; Sarkar and Ganguli, 2013; Thai and Vo, 2012; Timoshenko, 1921; Vlasov, 1961; Wang et al., 2000), (ii) large displacements but infinitesimally small strains (e.g. Chen, 2010; Chen and Hung, 2014; Freno and Cizmas, 2012; Jang, 2013; Karlson and Leamy, 2013; Lin and Lin, 2011; Monsalve et al., 2007; Nallathambi et al., 2010; Neukirch et al., 2012; Posada et al., 2011; Pulngern et al., 2013; Reddy, 2010; Reddy and Borgi, 2014; Sapountzakis and Mokos, 2008; Tari, 2013); there have been only very few studies on the rods undergoing finite strains (e.g., Dias and Audoly, 2014; Hori and Sasagawa, 2000), considering both of the geometrical and physical nonlinearities, and specially rubber-like rods (e.g. Attard and Hunt, 2008; Dai and Li, 2009; Lacarbonara, 2008; Libai and Simmonds, 1998; Makowski and Stumpf, 1988). The author of the present paper has not met any comment on the stress-free conditions on the top and bottom free surfaces of the rubber-like rods in the relevant literature. To the best of the author's knowledge, the most detailed analysis on the rods undergoing finite strains was performed by Makowski and Stumpf (1988). Makowski and Stumpf (1988) proposed a non-simple shearable rod model containing only two independent kinematical variables, namely the displacement and rotation vectors. The model had the assumptions that material fibres initially normal to the reference curve remained straight (but not necessarily normal) during the deformation and that the deformation was isochoric. Not only the strains themselves, but also the gradients of the strains were considered as the constitutive variables and, therefore, additional stress-resultants to the conventional ones were defined in their analysis.

In this study, a theory for incompressible rubber-like straight rods of rectangular cross-sections undergoing finite strains and rotations is presented. No assumption is made about the deformation of the material fibres initially normal to the reference line. In Section 2, the position vectors and base vectors of generic points in the undeformed and deformed rod spaces are defined. Using the base vectors defined in Section 2, the strain components are expressed in terms of deformation variables and expanded asymptotically in Section 3. In Section 4, expression of the transverse coordinate of the deformed rod in terms of the transverse coordinate of the undeformed rod and deformation variables by using the incompressibility condition is presented. In Section 5, the strain components and their perturbation terms are expressed in terms of the displacement components. Constitutive equations and stress resultants are defined in Section 6. In Section 7, equations of equilibrium and natural boundary conditions are obtained by using the calculus of variations (Dikmen, 1979a ; Hildebrand, 1965). Effective stress-resultants corresponding to the proposed theory are presented in this section, as well. In Section 8, general expressions on the satisfaction of the stress-free boundary conditions on the top and bottom free surfaces of the rubber-like rods are presented. In Section 9; as an illustration of the developed approach, the flexural buckling of Mooney-Rivlin straight rods with movable clamped edges is concerned. One dimensional constitutive equations for a Mooney-Rivlin rod are derived. A check of the resulting one dimensional constitutive equations with those corresponding to the elementary beam theory in case of the infinitesimally small deformations is presented. Buckling equations and closed-form solutions for various combinations of the perturbation terms of the strain components for finite deformations are obtained and compared with the rod theory developed by Makowski and Stumpf (1988) and the three dimensional elasticity solution (Nowinski,1969) in this section, as well. In Section 10, concluding remarks are presented.

2. Analysis

Let $\bar{\boldsymbol{r}}$ and $\boldsymbol{r}$ denote the position vectors of the generic points $\bar{H}$ and H in the undeformed and deformed rod spaces, respectively, and

$$\bar{\boldsymbol{r}}(\bar{s},\bar{\xi}) = \bar{\boldsymbol{\rho}}(\bar{s}) + \bar{\xi}\bar{\boldsymbol{e}}_2(\bar{s}) \quad , \tag{1}$$

$$\boldsymbol{r}(\bar{s},\bar{\xi}) = \boldsymbol{\rho}(\bar{s}) + \xi(\bar{s},\bar{\xi})\boldsymbol{A}_2(\bar{s}) \tag{2}$$

where $\bar{s}$ is the arc length along the reference line $\bar{\Gamma}$ (which is a straight line having a constant angle of $\bar{\varphi}$ with the horizontal) of the undeformed rod of a rectangular cross-section with unit width; $\bar{\xi}$ is the perpendicular distance $\bar{B}\bar{H}$ from the reference line $\bar{\Gamma}$ of the undeformed rod to the point $\bar{H}$ and $\bar{\xi} \in [-\frac{\bar{h}}{2}, \frac{\bar{h}}{2}]$ where $\bar{h}$ is the undeformed rod thickness; ξ is the distance BH from the reference line Γ of the deformed rod to the point H, not being necessarily normal to Γ; $\bar{\boldsymbol{e}}_\alpha$ and $\boldsymbol{e}_\alpha$ form orthonormal biads in the undeformed and deformed configurations, respectively, such that $\bar{\boldsymbol{e}}_1$ is in the direction of the reference line $\bar{\Gamma}$ and $\boldsymbol{e}_1$ is tangent to the reference line Γ; $\boldsymbol{i}_\alpha$ and $\boldsymbol{k}$ are fixed Cartesian base vectors; $\boldsymbol{A}_j$ form an orthonormal triad in the deformed configuration such that $\boldsymbol{A}_3 = \boldsymbol{k}$ and $\boldsymbol{A}_2$ is in the direction of BH having an angle of γ (not being the shear angle at the point B generally) with the normal to Γ; $\bar{\boldsymbol{\rho}}$ and $\boldsymbol{\rho}$ are position vectors of generic points $\bar{B}$ and B on $\bar{\Gamma}$ and Γ, respectively, as shown in Fig. 1. Here and henceforth, the Greek indices represent the numbers 1, 2 and Latin indices represent the numbers 1, 2, 3. Additionally,

$$\bar{\boldsymbol{e}}_1 = \bar{\boldsymbol{\rho}}'(\bar{s}) = \frac{d\bar{\boldsymbol{\rho}}}{d\bar{s}} = \cos\bar{\varphi}\boldsymbol{i}_1 + \sin\bar{\varphi}\boldsymbol{i}_2 \tag{3}$$

$$\boldsymbol{\rho}' = \frac{d\boldsymbol{\rho}}{d\bar{s}} = \lambda(\cos\varphi\boldsymbol{i}_1 + \sin\varphi\boldsymbol{i}_2) = \lambda\boldsymbol{e}_1 \tag{4}$$

where $\bar{\varphi}$ and φ are defined in Fig. 1 and

$$\lambda = \frac{ds}{d\bar{s}} = \sqrt{\boldsymbol{r}'.\boldsymbol{r}'} \tag{5}$$

is the stretch (Eringen, 1980). Here and henceforth, the notation $(.)'$ is used to denote partial derivative with respect to $\bar{s}$. For the base vectors $\bar{\boldsymbol{e}}_\alpha$ and $\boldsymbol{e}_\alpha$,

$$\bar{\boldsymbol{e}}_1' = \bar{\boldsymbol{e}}_2' = \boldsymbol{0} \quad , \tag{6}$$

$$\boldsymbol{e}_2 = \boldsymbol{k} \times \boldsymbol{e}_1 , \ \boldsymbol{e}_2' = -\varphi'\boldsymbol{e}_1 \tag{7}$$

can be written. $\bar{\xi}$ and ξ will be named as transverse coordinates of the generic points $\bar{H}$ and H in the undeformed and deformed rod spaces, respectively. The particle at $\bar{H}$ in the undeformed state (material state or reference state (Eringen and Şuhubi, 1974)) is considered to move to the position of H in the deformed state

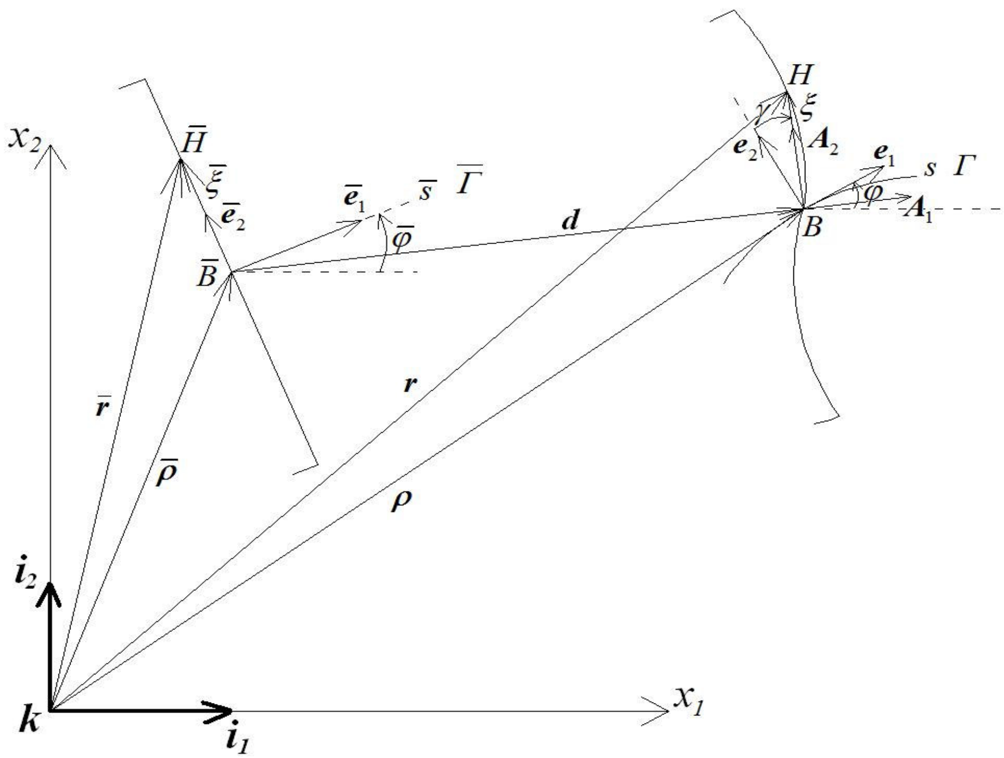

Fig. 1. Undeformed and deformed geometries

(or spatial state (Eringen and Şuhubi, 1974)). Here and henceforth, the suffix ($\bar{\ }$) is used to denote that the related parameter belongs to the undeformed configuration. To be used in the later parts of the analysis,

$$\boldsymbol{A}_1 = cos\gamma\boldsymbol{e}_1 - sin\gamma\boldsymbol{e}_2 \;,\; \boldsymbol{A}_2 = sin\gamma\boldsymbol{e}_1 + cos\gamma\boldsymbol{e}_2 \tag{8}$$

and

$$\boldsymbol{A}_1 = cos\psi\bar{\boldsymbol{e}}_1 - sin\psi\bar{\boldsymbol{e}}_2 \;,\; \boldsymbol{A}_2 = sin\psi\bar{\boldsymbol{e}}_1 + cos\psi\bar{\boldsymbol{e}}_2 \tag{9}$$

can be obtained via Figs. 1,2. ψ, shown in Fig. 2, is the angle of rotation.

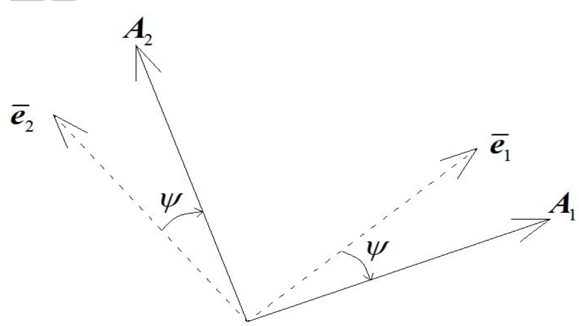

Fig. 2. The angle of rotation

The covariant base vectors (Cinemre, 1989) in the undeformed configuration at $\bar{H}$ can be expressed as

$$\bar{\boldsymbol{g}}_1 = \bar{\boldsymbol{r}}' \,,\;\; \bar{\boldsymbol{g}}_2 = \frac{\partial \bar{\boldsymbol{r}}}{\partial \bar{\xi}} \,,\;\; \bar{\boldsymbol{g}}_3 = \boldsymbol{k} \tag{10}$$

(Başar and Kratzig, 1985; Green and Zerna, 1968). Using Eqs.(1,3,6,10), $\bar{\boldsymbol{g}}_\alpha$ can be rewritten as

$$\bar{\boldsymbol{g}}_1 = \bar{\boldsymbol{e}}_1 \,,\; \bar{\boldsymbol{g}}_2 = \bar{\boldsymbol{e}}_2 \quad . \tag{11}$$

The contravariant base vectors in the undeformed state can be checked to be obtained as

$$\bar{\boldsymbol{g}}^1 = \bar{\boldsymbol{e}}_1 \,,\; \bar{\boldsymbol{g}}^2 = \bar{\boldsymbol{e}}_2 \,,\; \bar{\boldsymbol{g}}^3 = \boldsymbol{k} \;. \tag{12}$$

To be used in Section 9, the contravariant metric tensor at $\bar{H}$ in the undeformed state can be noted to be equal to the Kronecker's delta

$$\bar{g}^{ij} = \bar{\delta}^{ij} \tag{13}$$

where

$$\bar{g}^{ij} = \bar{\boldsymbol{g}}^i \cdot \bar{\boldsymbol{g}}^j \quad . \tag{14}$$

The covariant base vectors at H in the deformed configuration can be expressed as

$$\boldsymbol{g}_1 = \boldsymbol{r}' \,,\;\; \boldsymbol{g}_2 = \frac{\partial \boldsymbol{r}}{\partial \bar{\xi}} \,,\; \boldsymbol{g}_3 = \boldsymbol{k} \;, \tag{15}$$

by using the material coordinates (Eringen and Şuhubi, 1974). Using Eqs.(2,4,6,9,15), $\boldsymbol{g}_\alpha$ can be rewritten as

$$\boldsymbol{g}_1 = \boldsymbol{\rho}' + \xi' \boldsymbol{A}_2 + \xi \boldsymbol{A}_2' = \lambda \boldsymbol{e}_1 + \xi' \boldsymbol{A}_2 + \xi \psi' \boldsymbol{A}_1 \;\;,\quad \boldsymbol{g}_2 = \frac{\partial \xi}{\partial \bar{\xi}} \boldsymbol{A}_2 \quad . \tag{16}$$

3. Expressions of Strains

Using the approach proposed by Simmonds and Danielson (1972), the base vectors at the the point H in the deformed rod can be expressed as

$$\boldsymbol{g}_i = (\delta_{ij} + E_{ij})\boldsymbol{A}^j \tag{17}$$

via the orthonormal base vectors $\boldsymbol{A}_i$, where E_{ij} are named as pseudo strains[1] (Reissner, 1970; Taber, 1988). Using Eqs. (8,15,16,17),

[1] Due to being orthonormal of $\boldsymbol{A}_i$, there is no difference between $\boldsymbol{A}_i$ and $\boldsymbol{A}^i$.

$$E_{21} = E_{23} = E_{13} = E_{31} = E_{32} = E_{33} = 0,\ 1 + E_{22} = \frac{\partial \xi}{\partial \bar{\xi}},$$

$$1 + E_{11} = \lambda \cos\gamma + \xi\psi',\ E_{12} = \xi' + \lambda \sin\gamma \tag{18}$$

can be obtained. For the later analysis, the nonzero pseudo strain components are nondimensionalized and expanded asymptotically (Dikmen, 1979b) as

$$E_{11} = E_{11}^{*} = e_{11}^{*} + \varepsilon\kappa_{11}^{(1)*}\bar{\xi}^{*} + \varepsilon^{2}\kappa_{11}^{(2)*}\bar{\xi}^{*2} + O(\varepsilon^{3}), \tag{19}$$

$$E_{22} = E_{22}^{*} = e_{22}^{*} + \varepsilon\kappa_{22}^{(1)*}\bar{\xi}^{*} + \varepsilon^{2}\kappa_{22}^{(2)*}\bar{\xi}^{*2} + O(\varepsilon^{3}), \tag{20}$$

$$E_{12} = E_{12}^{*} = e_{12}^{*} + \varepsilon\kappa_{12}^{(1)*}\bar{\xi}^{*} + \varepsilon^{2}\kappa_{12}^{(2)*}\bar{\xi}^{*2} + O(\varepsilon^{3}) \tag{21}$$

where $e_{\alpha\beta}^{*}$, $\kappa_{11}^{(\alpha)*}$, $\kappa_{22}^{(\alpha)*}$ and $\kappa_{12}^{(\alpha)*}$ are nondimensional perturbation terms of the pseudo strain components. ε is a thickness parameter defined as

$$\varepsilon = \bar{h}/2\bar{L} \tag{22}$$

where $\bar{L}$ is the length of the undeformed rod. Here and henceforth, the notation $(.)^{*}$ is used to denote that the related parameter is nondimensional and the analysis using Eqs.(19-21) with $O(\varepsilon^{3})$ will be named as 'second order analysis'.

4. Relationship Between the Transverse Coordinates

Due to incompressibility, the third invariant of the right Cauchy-Green deformation tensor (Green and Adkins, 1960; Makowski and Stumpf, 1988; Simmonds, 1986; Yükseler, 1996b) must be equal to 1:

$$I_3 = 1 = g / \bar{g} \tag{23}$$

where

$$\bar{g} = \left|\bar{g}_{ij}\right|,\quad g = \left|g_{ij}\right| = g_{11}g_{22} - (g_{12})^{2} \tag{24}$$

and where

$$\bar{g}_{ij} = \bar{\boldsymbol{g}}_{i}.\bar{\boldsymbol{g}}_{j},\ g_{ij} = \boldsymbol{g}_{i}.\boldsymbol{g}_{j}\ . \tag{25}$$

Using Eqs.($17,18,25_2$), the nonzero components of the metric tensor at H in the deformed state can be obtained as

$$g_{11} = (1 + E_{11})^{2} + (E_{12})^{2}\ ,\ g_{12} = E_{12}(1 + E_{22})\ ,\ g_{22} = (\xi_{,\bar{\xi}})^{2} = (1 + E_{22})^{2}\ ,\ g_{33} = 1\ . \tag{26}$$

Via Eqs.(10,11,23-26),

$$(1+E_{11})^2(1+E_{22})^2 = 1 \ . \tag{27}$$

For an asymptotical analysis, nondimensional quantities

$$(\bar{\xi}^*, \xi^*) = (2/\bar{h})(\bar{\xi}, \xi \ ), \qquad \bar{s}^* = \bar{s}/\bar{L} \tag{28}$$

are introduced. The transverse coordinate of any generic point H in the deformed rod space can be expanded asymptotically as

$$\xi^* = \xi^{(0)*} + \varepsilon\xi^{(1)*} + \ O(\varepsilon^2) \quad . \tag{29}$$

Using Eqs.(16-21,28,29) in Eq.(27), equating coefficients of like powers of ε (neglecting $O(\varepsilon^2)$) and integrating give the expression of the transverse coordinate in the deformed configuration in terms of the transverse coordinate in the undeformed configuration and the thickness parameter as

$$\xi^* = B_0\bar{\xi}^* + \varepsilon B_1\bar{\xi}^{*2} + O(\varepsilon^2) \ , \tag{30}$$

where

$$B_0 = (\lambda\cos\gamma)^{-1} \ , \qquad B_1 = -B_0^3\dot{\psi}^*/2 \tag{31}$$

where

$$\psi^* = \psi \ . \tag{32}$$

The notation $\dot{(.)}$ is used to denote partial derivative with respect to $\bar{s}^*$. For the determination of the integration constants, the reference line is assumed to be composed of the same points regardless of the deformation (Erdölen and Yükseler, 2003; Simmonds, 1986; Taber, 1987; Yükseler, 2003; Yükseler, 2005; Yükseler, 2008), i.e.

$$\xi\big|_{\bar{\xi}=0} = 0 \, . \tag{33}$$

5. Kinematics

A displacement vector $\boldsymbol{d}$ of a point $\bar{B}$ on the reference line $\bar{\Gamma}$ is shown in Fig. 1 and can be defined as

$$\boldsymbol{d} = \boldsymbol{\rho} - \bar{\boldsymbol{\rho}} \tag{34}$$

by using the position vectors of a particle on the reference lines $\overline{\Gamma}$ and Γ or by using the components of the displacement vector relative to the base vectors $\boldsymbol{A}_\alpha$:

$$\boldsymbol{d} = u\boldsymbol{A_1} + v\boldsymbol{A_2} \quad . \tag{35}$$

Considering the derivative of Eq. (35) with respect to $\overline{s}$ and using Eqs.(34,8,9,3,4,18),

$$\lambda \cos\gamma = u' + v\psi' + \cos\psi = 1 + E_{11} - \xi\psi' \ , \tag{36}$$

$$\lambda \sin\gamma = v' - u\psi' + \sin\psi = E_{12} - \xi' \tag{37}$$

and consequently,

$$1 + E_{11} = u' + \psi'(v + \xi) + \cos\psi \quad , \quad E_{12} = v' - u\psi' + \sin\psi + \xi' \quad , \tag{38}$$

can be obtained. Using Eqs.(19-21,28,30-32,38) and

$$(u^*, v^*) = (u, v)/\overline{L} \quad , \tag{39}$$

the perturbation terms of E_{11}^* and E_{12}^* can be obtained in terms of the displacement components as

$$e_{11}^* = \dot{u}^* + v^*\dot{\psi}^* + \cos\psi^* - 1 \ , \ \kappa_{11}^{(1)*} = B_0\dot{\psi}^* , \ \kappa_{11}^{(2)*} = B_1\dot{\psi}^* \ , \ e_{12}^* = \dot{v}^* - u^*\dot{\psi}^* + \sin\psi^* ,$$

$$\kappa_{12}^{(1)*} = \dot{B}_0 \ , \ \kappa_{12}^{(2)*} = \dot{B}_1 \quad . \tag{40}$$

Due to the incompressibility condition, E_{22}^* is not independent from E_{11}^*. Therefore; substituting Eqs.(19-21) into Eq.(27) and equating the like powers of ε, the perturbation terms of E_{22}^* can be obtained in terms of the perturbation terms of E_{11}^* as

$$e_{22}^* = (1 + e_{11}^*)^{-1} - 1 \ , \ \kappa_{22}^{(1)*} = -\kappa_{11}^{(1)*}(1 + e_{11}^*)^{-2} \ , \ \kappa_{22}^{(2)*} = (1 + e_{11}^*)^{-2}[(1 + e_{11}^*)^{-1}\kappa_{11}^{(1)*2} - \kappa_{11}^{(2)*}] . \tag{41}$$

6. Constitutive Equations

The three-dimensional constitutive equations are

$$\sigma^{ij} = \partial\Phi \ / \partial E_{ij} \tag{42}$$

(Reissner, 1975; Taber, 1988) where σ^{ij} are the components of the first Piola-Kirchhoff stress tensor (or pseudo stress tensor) (Eringen, 1962; Eringen, 1980; Malvern, 1969; Piola, 1833) and Φ is the three-dimensional strain energy density function depending on E_{11}, E_{12} and E_{22}, which are the only nonzero pseudo strain components, with an important note that E_{22} is a function of E_{11} through the incompressibility condition, Eq.(27)[2] (or Eqs.(41) in terms of the perturbations). Stress and moment resultant vectors acting on the reference line $\overline{\Gamma}$ of the undeformed rod with unit width can be defined as

$$N=\int_{-\bar{h}/2}^{\bar{h}/2}\sigma^{11}d\bar{\xi} \quad , \quad M=\int_{-\bar{h}/2}^{\bar{h}/2}\sigma^{11}\bar{\xi}d\bar{\xi} \quad , \quad Q=\int_{-\bar{h}/2}^{\bar{h}/2}\sigma^{12}d\bar{\xi} \tag{43}$$

or in nondimensional form as

$$N^{*}=\frac{1}{2}\int_{-1}^{1}\sigma^{11*}d\bar{\xi}^{*} \; , \quad M^{*}=\frac{1}{4}\int_{-1}^{1}\sigma^{11*}\bar{\xi}^{*}d\overline{\xi^{*}} \; , \quad Q^{*}=\frac{1}{2}\int_{-1}^{1}\sigma^{12*}d\bar{\xi}^{*} \tag{44}$$

where

$$\sigma^{ij*}=\sigma^{ij}/C \; , \quad (N^{*},Q^{*})=(1/C\bar{h})(N,Q) \; , \quad M^{*}=M/C\bar{h}^{2} \tag{45}$$

by additionally using Eqs. (28). C is a material constant. The constitutive equations can, also, be checked to be written in terms of one-dimensional strain energy density function ϕ as

$$N^{*}=\frac{\partial\phi^{*}}{\partial e_{11}^{*}} \quad , \quad M^{*}=\frac{1}{2\varepsilon}\frac{\partial\phi^{*}}{\partial\kappa_{11}^{*}} \quad , \quad Q^{*}=\frac{\partial\phi^{*}}{\partial e_{12}^{*}} \tag{46}$$

where

$$\phi^{*}=\frac{\phi}{C\bar{h}}=\frac{1}{2}\int_{-1}^{1}\Phi^{*}d\bar{\xi}^{*} \quad , \qquad \Phi^{*}=\frac{\Phi}{C} \tag{47}$$

via Eqs. (19-21,28,43-45). Only the conventional stress-resultants are concerned in the present study.

7. Equations of Equilibrium and Natural Boundary Conditions

The virtual strain energy δS of the rod with unit width can be expressed as

$$\delta S=\int_{0}^{\bar{L}}\delta\phi d\bar{s} \tag{48}$$

where

[2] If the incompressibility condition had not been enforced *a priori* in the strain energy density function Φ, through Eq.(27), the strain energy density function should have been modified as

$$\Phi_{m}=\Phi+p\mu$$

where p is a Lagrange multiplier (hydrostatic pressure) and μ is the incompressibility constraint, obtained from Eq.(27) as

$$\mu=(1+E_{11})^{2}(1+E_{22})^{2}-1 \quad .$$

In that case, there would be an additional term in Eq.(42), as well (See e.g. pages 120-121 of (Hildebrand, 1965) for the mathematical background of the problem.).

$$\delta\phi = \int_{-\bar{h}/2}^{\bar{h}/2} \sigma^{ij} \delta E_{ij} d\bar{\xi} \tag{49}$$

(Wang et al., 2000) or in nondimensional form as

$$\delta\phi^* = \frac{1}{2}\int_{-1}^{1} \sigma^{ij*} \delta E_{ij}^* d\bar{\xi}^* \tag{50}$$

using Eqs.(19-21, 28_1,45_1) . Neglecting the contributions of σ^{22*} on the strain energy and using Eqs.(19,21,42, 45-47) in Eq.(50),

$$\delta\phi^* = \frac{\partial\phi^*}{\partial e_{11}^*}\delta e_{11}^* + \frac{\partial\phi^*}{\partial\kappa_{11}^*}\delta\kappa_{11}^{(1)*} + \frac{\partial\phi^*}{\partial e_{12}^*}\delta e_{12}^* \tag{51}$$

and

$$\delta\phi^* = N^*\delta e_{11}^* + 2\varepsilon M^*\delta\kappa_{11}^{(1)*} + Q^*\delta e_{12}^* \tag{52}$$

can be obtained. It can be noted that no higher order stress-resultant is considered, and, therefore, there is no work conjugate of $\kappa_{11}^{(2)*}$, $\kappa_{12}^{(1)*}$ and $\kappa_{12}^{(2)*}$. If, alternatively, $\delta\phi^*$ is expressed in the following form

$$\delta\phi^* = U_0\delta u^* + U_1\delta\dot{u}^* + V_0\delta v^* + V_1\delta\dot{v}^* + \Psi_0\delta\psi^* + \Psi_1\delta\dot{\psi}^* \quad ; \tag{53}$$

considering the variations of e_{11}^*, e_{12}^* and $\kappa_{11}^{(1)*}$ through Eqs.(40) in Eq.(52) and equating the multipliers of δu^*, $\delta\dot{u}^*$, δv^*, $\delta\dot{v}^*$, $\delta\psi^*$ and $\delta\dot{\psi}^*$,

$$U_0 = -\dot{\psi}^* Q^* + O(\varepsilon^2)\,, \quad U_1 = N^* - 2\varepsilon\dot{\psi}^* B_0^2 M^* + O(\varepsilon^2) \quad , \quad V_0 = \dot{\psi}^* N^* - 2\varepsilon\dot{\psi}^{*2} B_0^2 M^* + O(\varepsilon^2)\,,$$

$$V_1 = Q^* + O(\varepsilon^2)\,, \Psi_0 = (-N^* + 2\varepsilon\dot{\psi}^* B_0^2 M^*)\sin\psi^* + Q^*\cos\psi^* + O(\varepsilon^2)\,,$$

$$\Psi_1 = N^* v^* - Q^* u^* + 2\varepsilon B_0 (1 - B_0\dot{\psi}^* v^*) M^* + O(\varepsilon^2) \tag{54}$$

can be obtained. Using Green's theorem (Hildebrand, 1965) in Eq.(53), the nondimensional virtual strain energy $\delta S^* (= \delta S / C\bar{h}\bar{L})$ (Wang et al., 2000) can be expressed as

$$\delta S^* = \int_0^1 [(U_0 - \dot{U}_1)\delta u^* + (V_0 - \dot{V}_1)\delta v^* + (\Psi_0 - \dot{\Psi}_1)\delta\psi^*] d\bar{s}^* + [U_1\delta u^* + V_1\delta v^* + \Psi_1\delta\psi^*]_0^1 \quad . \tag{55}$$

The virtual potential energy of loads $\delta\Lambda^* (= \delta\Lambda / C\bar{h}\bar{L})$ is

$$\delta\Lambda^* = -\int_0^1 (p_u^*\delta u^* + p_v^*\delta v^* + l^*\delta\psi^*) d\bar{s}^* \tag{56}$$

where the external force and moment vectors per unit length of $\overline{\Gamma}$ of the undeformed rod with unit width are

$$\boldsymbol{p}^* = p_u^* \boldsymbol{A_1} + p_v^* \boldsymbol{A_2} \ , \ \boldsymbol{l}^* = l^* \boldsymbol{k} \ , \tag{57}$$

respectively, where

$$p_u^* = \frac{p_u \overline{L}}{C\overline{h}} \ , \ p_v^* = \frac{p_v \overline{L}}{C\overline{h}} \ , \ l^* = \frac{l}{C\overline{h}} \ . \tag{58}$$

For a rod in equilibrium, the total virtual potential energy $\delta W^* (= \delta W / C\overline{h}\overline{L})$ is equal to zero:

$$\delta W^* = \delta S^* + \delta \Lambda^* = 0 \tag{59}$$

or using Eqs. (55,56,59),

$$\delta W^* = \int_0^1 [(U_0 - \dot{R}_N - p_u{}^*)\delta u^* + (V_0 - \dot{R}_Q - p_v{}^*)\delta v^* + (\Psi_0 - \dot{R}_M - l^*)\delta\psi^*] d\overline{s}^* +$$

$$[R_N \delta u^* + R_Q \delta v^* + R_M \delta \psi^*]_0^1 = 0 \tag{60}$$

where U_1, V_1 and Ψ_1 are re-denoted as R_N, R_Q and R_M, respectively, in Eq.(60):

$$R_N = U_1 \quad , \quad R_Q = V_1 \quad , \quad R_M = \Psi_1 \tag{61}$$

due to their physical meanings. R_N, R_Q and R_M are nondimensional effective normal force, effective shear force and effective bending moment, respectively. Accordingly; the equations of equilibrium, Euler's equations (Hildebrand, 1965), are

$$U_0 - \dot{R}_N - p_u^* = 0 \quad , \quad V_0 - \dot{R}_Q - p_v^* = 0 \quad , \quad \Psi_0 - \dot{R}_M - l^* = 0 \quad , \tag{62}$$

and the natural boundary conditions are

$$R_N = 0 \ \text{ or } \ u^* = 0 \quad , \quad R_Q = 0 \ \text{ or } \ v^* = 0 \quad , \quad R_M = 0 \ \text{ or } \ \psi^* = 0 \quad . \tag{63}$$

For the natural boundary conditions; u^* or R_N, v^* or R_Q and ψ^* or R_M should be defined.

8. Satisfaction of the Zero-Traction Conditions

In addition to the natural boundary conditions; if the vanishing of the traction (stress) vector on the top and bottom free surfaces of the rod is desired to be satisfied, then

$$\sigma^{\alpha\beta*}\big|_{\overline{\xi}^* = \mp 1} \overline{n}_\alpha = 0 \tag{64}$$

should be satisfied in the problem, with the note that $\sigma^{l3} = \sigma^{3l} = 0$ (l=1,2,3)[3]. $\bar{\boldsymbol{n}}$ is the unit outward normal vector of the top free surface of the rod in the material state and is equal to $\bar{\boldsymbol{e}}_2$ which can be expressed in terms of $\boldsymbol{A}_\alpha$ as

$$\bar{\boldsymbol{n}} = \bar{\boldsymbol{e}}_2 = -\sin\psi \boldsymbol{A}_1 + \cos\psi \boldsymbol{A}_2 \tag{65}$$

by using Eqs.(9). The unit outward normal vector belonging to the bottom free surface of the undeformed rod can be noted to be equal to $-\bar{\boldsymbol{n}}$. Recalling that σ^{22^*} is neglected and σ^{21^*} is always vanishing due to not being included of E_{21} in the metric tensor of the deformed rod, Eqs.(26) (and, therefore, in the strain energy of any rubber-like material (Alexander, 1968; Demiray and Vito; 1976; Makowski and Stumpf, 1988)), considered in the present theory, Eq.(64) can be rewritten as

$$\sigma^{11^*}\Big|_{\bar{\xi}^*=\mp 1} \sin\psi^* = 0 \quad , \quad \sigma^{12^*}\Big|_{\bar{\xi}^*=\mp 1} \sin\psi^* = 0 \tag{66}$$

by using Eqs.(32,65). Eqs. (66) can be noted to be satisfied (i) if the strains or rotations are vanishing, (ii) approximately if the strains and rotations are infinitesimally small.

9. Illusrative Problem

9.1. Flexural Buckling of a Mooney-Rivlin straight rod subjected to axial compressive loads

The buckling of a Mooney-Rivlin straight rod, $\bar{\varphi}^* = 0$, subjected to an axial compressive force P, as shown in Fig.3, is concerned. The forces P are considered to be applied through rigid plates which are well-lubricated and constrained such that they remain perpendicular to the axis of the rod during the deformation

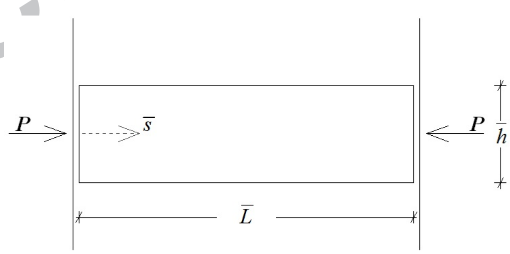

Fig. 3. A Mooney-Rivlin rod with movable clamped edges under axial compressive forces

[3] For the physical meaning and derivation of Eq.(64), pages 108-110 of (Eringen, 1962) or pages 113-115 of (Eringen, 1980) or pages 220-223 of (Malvern, 1969) can be referred.

(Makowski and Stumpf, 1988). The rod ends can slide in the vertical direction. Referring to the corresponding natural boundary conditions (63);

$$\psi^* = 0 \ , \quad R_Q = 0 \ , \quad R_N = -P^* \tag{67}$$

at $\bar{s}^* = 0$ and $\bar{s}^* = 1$ where

$$P^* = P / C\bar{h} \quad . \tag{68}$$

9.2. One-dimensional constitutive equations

For the Mooney-Rivlin material, the three-dimensional strain energy density function Φ is expressed as

$$\Phi = C(I-2) \tag{69}$$

(Makowski and Stumpf, 1988) where

$$I = \bar{g}^{\alpha\beta} g_{\alpha\beta} \quad . \tag{70}$$

Substituting Eqs.(70,13,26) into Eq.(69), the three-dimensional strain energy density function can be expressed in terms of the strain components as

$$\Phi = C[(1+E_{11})^2 + (E_{12})^2 + (1+E_{22})^2 - 2] \quad . \tag{71}$$

Using Eqs.(46_1,47,71), the constitutive equation for N^* can expessed as

$$N^* = \int_{-1}^{1} [(1+E_{11}^*)\frac{\partial E_{11}^*}{\partial e_{11}^*} + (1+E_{22}^*)\frac{\partial E_{22}^*}{\partial e_{11}^*}] d\overline{\xi^*} \quad . \tag{72}$$

Using Eqs. (19,20,41), additionally,

$$N^* = 2[(1+e_{11}^*) - (1+e_{11}^*)^{-3}] + 2\varepsilon^2(1+e_{11}^*)^{-4}[\kappa_{11}^{(2)*} - 2\kappa_{11}^{(1)*2}(1+e_{11}^*)^{-1}/3] + O(\varepsilon^4) \quad . \tag{73}$$

Following similar procedures,

$$M^* = \frac{1}{2\varepsilon}\int_{-1}^{1} [(1+E_{11}^*)\frac{\partial E_{11}^*}{\partial \kappa_{11}^{(1)*}} + (1+E_{22}^*)\frac{\partial E_{22}^*}{\partial \kappa_{11}^{(1)*}}] d\overline{\xi^*}$$

$$= \varepsilon\kappa_{11}^{(1)*}[\frac{1}{3} + (1+e_{11}^*)^{-4}] + 6\varepsilon^3(1+e_{11}^*)^{-5}\kappa_{11}^{(1)*}[(1+e_{11}^*)^{-1}\kappa_{11}^{(1)*2} - \kappa_{11}^{(2)*}]/5 + O(\varepsilon^4) \, ,$$

$$Q^* = \int_{-1}^{1} E_{12}^* d\overline{\xi^*} = e_{12}^* + \frac{2}{3}\varepsilon^2\kappa_{12}^{(2)*} + O(\varepsilon^4) \tag{74}$$

can be obtained via Eqs. (46,47,19-21,41).

In case of the infinitesimally small deformations, it is possible to check two of the one-dimensional constitutive equations, namely Eq.(73) and the first of Eqs.(74), with those corresponding to the linear beam theory (see e.g. Case et al., 1999 or Hearn, 2000) for the linearly elastic material and straight rods, $\bar{\varphi}^* = 0$, of rectangular section with unit width. Neglecting the nonlinear terms in Eq.(73) and using Eq.(45_2),

$$\tilde{N} = 2C\bar{h}\tilde{e}_{11} \tag{75}$$

can be written. Here and henceforth, the suffix $\tilde{()}$ is used to denote that the related parameter belongs to the infinitesimally small strains. In the linear beam theory, the normal force can be noted to be expressed as

$$\tilde{N} = \Xi\bar{h}\tilde{e} \tag{76}$$

where Ξ is the modulus of elasticity and $\tilde{e}$ is the linear longitudinal strain. Since

$$\tilde{e}_{11} = \tilde{e} \tag{77}$$

in the case of the infinitesimally small deformations,

$$C = \Xi / 2 \tag{78}$$

via Eqs. (75-77). Neglecting the nonlinear terms in Eq.(74_1) and the effect of E_{22}, and using Eqs.($45_3, 40_2, 31_1, 36, 32, 22, 28_2, 78$),

$$\tilde{M} = \frac{\Xi\bar{h}^3}{12}\frac{d\tilde{\psi}}{d\bar{s}} \tag{79}$$

can be obtained. Eq.(79) can be noted to coincide with the corresponding expression for the bending moment in the linear beam theory.

9.3. Buckling equations

Let $(\breve{u}, \breve{v}, \breve{\psi})$ and $(\hat{u}, \hat{v}, \hat{\psi})$ be two adjacent equilibrium configurations (Brush and Almroth, 1975) such that

$$u = \breve{u} + \hat{u} \quad , \quad v = \breve{v} + \hat{v} \quad , \quad \psi = \breve{\psi} + \hat{\psi} \tag{80}$$

where $(\breve{u}, \breve{v}, \breve{\psi})$ are considered to belong to the prebuckling state and $(\hat{u}, \hat{v}, \hat{\psi})$ are arbitrarily small incremental buckling displacements to $(\breve{u}, \breve{v}, \breve{\psi})$. Correspondingly,

$$N = \breve{N} + \hat{N} \quad , \quad Q = \breve{Q} + \hat{Q} \quad , \quad M = \breve{M} + \hat{M} \tag{81}$$

where $\hat{N}$, $\hat{Q}$, $\hat{M}$ can be obtained by dropping the quadratic and higher order terms of the displacements with the subscript $\hat{()}$ in the constitutive equations (73,74).

In the prebuckling equlibrum configuration, the rod is considered to remain straight and

$$\breve{\lambda} = L/\bar{L}\ ,\ \breve{B}_1 = \breve{\gamma} = \breve{\kappa}_{11}^{(1)*} = \breve{\kappa}_{11}^{(2)*} = \breve{e}_{12}^{*} = \breve{\kappa}_{12}^{(1)*} = \breve{\kappa}_{12}^{(2)*} = \breve{v}^{*} = \breve{M}^{*} = \breve{R}_M = 0\ ,\ \breve{B}_0 = \breve{\lambda}^{-1}\ ,$$

$$\breve{e}_{11}^{*} = \dot{\breve{u}}^{*} = \breve{\lambda} - 1 = \textit{constant},\quad \breve{R}_N = \breve{N}^{*} = 2(\breve{\lambda} - \breve{\lambda}^{-3}) = -P^{*} = \textit{constant}\ ,\ \breve{Q}^{*} = \breve{R}_Q = \breve{\psi}^{*} = 0 \tag{82}$$

can be checked to be valid. L is the length of the deformed rod. From Eq.(82_4)[4] and due to the symmetry of the deformation,

$$\breve{u}^{*}(1/2) = 0\ ,\qquad \breve{u}^{*} = (\breve{\lambda} - 1)\bar{s}^{*} + (1 - \breve{\lambda})/2 \tag{83}$$

can easily be obtained.

Using Eqs.(80-83,40,73,74) and neglecting the quadratic and higher order small incremental terms, expressions pertaining to the buckling state can be obtained as

$$\hat{e}_{11}^{*} = \dot{\hat{u}}^{*}\ ,\ \hat{\kappa}_{11}^{(1)*} = \dot{\hat{\psi}}^{*}/\breve{\lambda}\ ,\ \hat{\kappa}_{11}^{(2)*} = 0\ , \hat{B}_0 = -\breve{\lambda}^{-2}\dot{\hat{u}}^{*}\ , \hat{B}_1 = -\hat{\psi}/(2\breve{\lambda}^{3})\ ,\quad \hat{e}_{12}^{*} = \dot{\hat{v}}^{*} - \breve{u}^{*}\dot{\hat{\psi}}^{*} + \hat{\psi}^{*}\ ,$$

$$\hat{\kappa}_{12}^{(1)*} = -\ddot{\hat{u}}^{*}/\breve{\lambda}^{2}\quad ,\ \hat{\kappa}_{12}^{(2)*} = -\ddot{\hat{\psi}}^{*}/(2\breve{\lambda}^{3})\ ,\quad \hat{M}^{*} = \varepsilon\dot{\hat{\psi}}^{*}\breve{\lambda}^{-1}(\breve{\lambda}^{-4} + \frac{1}{3})\ ,$$

$$\hat{R}_Q = \hat{Q}^{*} = 2\dot{\hat{v}}^{*} - 2\dot{\hat{\psi}}^{*}\breve{u}^{*} + 2\hat{\psi}^{*} - \varepsilon^{2}\ddot{\hat{\psi}}^{*}/(3\breve{\lambda}^{3})\ ,\quad \hat{R}_N = \hat{N}^{*} = 2\dot{\hat{u}}^{*}. \tag{84}$$

9.4. Solutions for various alternatives

Substituting Eqs.(80-82,84) into the equations of equilibrium (62),

$$2\varepsilon^{2}\breve{\lambda}^{-4}\dddot{\hat{u}}^{*}/3 - 2\ddot{\hat{u}}^{*} = 0\ ,\ 2\dot{\hat{\psi}}^{*}(\breve{\lambda} - \breve{\lambda}^{-3}) - \dot{\hat{Q}}^{*} = 0\ ,\ \hat{Q}^{*} - \breve{N}^{*}\hat{\psi}^{*} - (\breve{N}^{*}\hat{v}^{*} + 2\varepsilon\breve{B}_0\hat{M}^{*} - \breve{u}^{*}\hat{Q}^{*})^{\dot{}} = 0 \tag{85}$$

can be obtained via Eqs.(61,54).

In order to satisfy the last of the natural boundary conditions (67); $\hat{u}^{*}$ should be taken to be equal to a constant, recalling Eqs.(84_{11},81_1,82_5). Therefore, for the concerning problem

$$\hat{e}_{11}^{*} = \hat{B}_0 = \hat{\kappa}_{12}^{(1)*} = \hat{R}_N = \hat{N}^{*} = 0 \tag{86}$$

by using Eqs.(84_1,84_4,84_7,84_{11}). Correspondingly, Eq.(85_1) is automaticly satisfied.

[4] The subscripts in the equation numbers are used in the series of equations with common equation numbers. The concerning equations are considered to be seperated by commas and numbered accordingly.

In order to see the effects of some of the perturbation terms in the expansions of the strain components, given in Eqs.(19,21), the solutions corresponding to several combinations of the perturbation terms are presented below:

(a) Keeping all of the aforementioned perturbation terms (the second order analysis), if $\hat{\psi}^*$ is chosen as the fundamental variable,

$$\ddot{\hat{\psi}}^* + [\frac{2(1-\breve{\lambda}^4)}{\varepsilon^2(\breve{\lambda}^4 + 5/3)}]\hat{\psi}^* = 0 \tag{87}$$

can be obtained from Eqs.(84_{10},86,85_2,85_3). The solution of the differential equation (87), satisfying $\hat{\psi}^*(0) = \hat{\psi}^*(1) = 0$ which allow the satisfaction of the first of the natural boundary conditions (67) together with Eq.(80_3) and the last equation of Eqs.(82), yields

$$\frac{\overline{L}}{\overline{h}} = \frac{1}{2\varepsilon} = \frac{\pi}{2}\sqrt{\frac{(\breve{\lambda}^4 + 5/3)}{2(1-\breve{\lambda}^4)}} \quad . \tag{88}$$

Alternatively; if $\hat{Q}^*$ is chosen as the fundamental variable, the same result can be obtained, satisfying $\hat{Q}^*(0) = \hat{Q}^*(1) = 0$ or equivalently $\hat{R}_Q(0) = \hat{R}_Q(1) = 0$ which allow the satisfaction of the second of the natural boundary conditions (67) together with Eqs.(81_2,82_6).

(b) If $\kappa_{12}^{(2)*}$ in the expansion of E_{12}^* is neglected, then the constitutive equation for $\hat{M}^*$ is same as Eq.(84_9) but the constitutive equation for $\hat{Q}^*$ is changed as

$$\hat{Q}^* = 2e_{12}^* = 2\dot{\hat{v}}^* - 2\dot{\hat{\psi}}^*\breve{u}^* + 2\hat{\psi}^* \quad . \tag{89}$$

Following the same procedure mentioned above,

$$\frac{\overline{L}}{\overline{h}} = \frac{\pi}{2}\sqrt{\frac{(1+\breve{\lambda}^4/3)}{(1-\breve{\lambda}^4)}} \tag{90}$$

can be obtained.

(c) If all of the second order terms in the expansions of the pseudo normal strains, namely $\kappa_{11}^{(2)*}$ and $\kappa_{22}^{(2)*}$, are neglected, then the constitutive equation for $\hat{Q}^*$ is same as Eq.(84_{10}) but the constitutive equation for $\hat{M}^*$ is changed as

$$\hat{M}^* = \varepsilon\dot{\psi}^* \breve{\lambda}^{-1}(\breve{\lambda}^{-4}+1)/3 \quad . \tag{91}$$

Following the same procedure mentioned above,

$$\frac{\overline{L}}{\overline{h}} = \frac{\pi}{2}\sqrt{\frac{(1+3\breve{\lambda}^4)}{6(1-\breve{\lambda}^4)}} \tag{92}$$

can be obtained.

(d) If all of the second order terms in the expansions of the pseudo normal strains and the pseudo transverse shear strain, namely $\kappa_{11}^{(2)*}$, $\kappa_{22}^{(2)*}$ and $\kappa_{12}^{(2)*}$, are neglected, then the constitutive equation for $\hat{Q}^*$ is same as Eq.(89) and the constitutive equation for $\hat{M}^*$ is same as Eq.(91). Following the same procedure mentioned above,

$$\frac{\overline{L}}{\overline{h}} = \frac{\pi}{2}\sqrt{\frac{(1+\breve{\lambda}^4)}{3(1-\breve{\lambda}^4)}} \tag{93}$$

can be obtained. This analysis using Eqs.(19-21) with $O(\varepsilon^2)$ will be named as 'first order analysis'.

For the cases *(b)* and *(d)* where $\kappa_{12}^{(2)*}$ is neglected commonly; since $\kappa_{12}^{(1)*}$ has been noted to be vanishing as well for the concerning problem, Eq.(86), the distribution of the pseudo transverse shear strain (E_{12}^*) is constant ($=e_{12}^*$) through the thickness, Eq.(21)[5]. Therefore; the cases *(b)* and *(d)* will be considered as the cases where the pseudo transverse shear strain is assumed to be constant through the thickness in the second order analysis and the first order analysis, respectively.

Considering the finite strains and rotations in the concerning problem, the satisfaction of the stress-free boundary conditions on the top and bottom free surfaces of the rod is impossible, as mentioned in Section 8 (see Eqs.(66)).

The values of $\breve{\lambda}$ satisfying Eqs.(88,90,92,93) are the critical values of $\breve{\lambda}$ for the given values of $\overline{L}/\overline{h}$ and will be denoted as λ_c. The λ_c versus $\overline{L}/\overline{h}$ curves corresponding to the *(a)* second order analysis considering all of the perturbation terms, given in Eqs.(19-21), *(b)* neglition of $\kappa_{12}^{(2)*}$ in the second order analysis *(c)* neglition of $\kappa_{11}^{(2)*}$ and $\kappa_{22}^{(2)*}$, *(d)* first order analysis (neglition of $\kappa_{12}^{(2)*}$, $\kappa_{11}^{(2)*}$ and $\kappa_{22}^{(2)*}$), *(e)* results

[5] In other words, plane sections remain plane during the deformation but not necessarily perpendicular to the reference line Γ. The angle γ is the shear angle in that case.

corresponding to (Makowski and Stumpf, 1988)[6], and *(f)* results corresponding to the three dimensional elasticity solution (Nowinski, 1969)[7] are drawn and shown in Fig.4. Due to the importance of the results, the detailed numerical values of $\bar{L}/\bar{h}$ versus λ_c for the aforementioned approaches are tabulated and shown in Table 1, as well. Results corresponding to *(e)* and *(b)* are seen to be almost equal to each other (Results corresponding to *(b)* are somewhat nearer to the results of *(f)*.). The results corresponding to *(e)* and *(b)* are departing from those of *(f)* as $\bar{L}/\bar{h}$ is decreased, especially for $(\bar{L}/\bar{h}) \le 2.0$. The results corresponding to *(a)* are almost coinciding with those of *(f)*, especially for $(\bar{L}/\bar{h}) \ge 2.5$, and are much better than those corresponding *(e)* and *(b)*. The results corresponding to *(c)* and *(d)* are quite far from those of *(f)*.

[6] The numerical results corresponding to the rod theory proposed by Makowski and Stumpf (1988) are obtained by using Eqs.(12,13) of (Makowski and Stumpf, 1988) for *n*=1.
[7] The numerical results corresponding to the three dimensional elasticity solution of Nowinski (1969) are obtained by adapting the general solution of (Nowinski,1969), Eqs.(1.18,1.19) of (Nowinski,1969), to the Mooney-Rivlin material for asymmetric buckling.

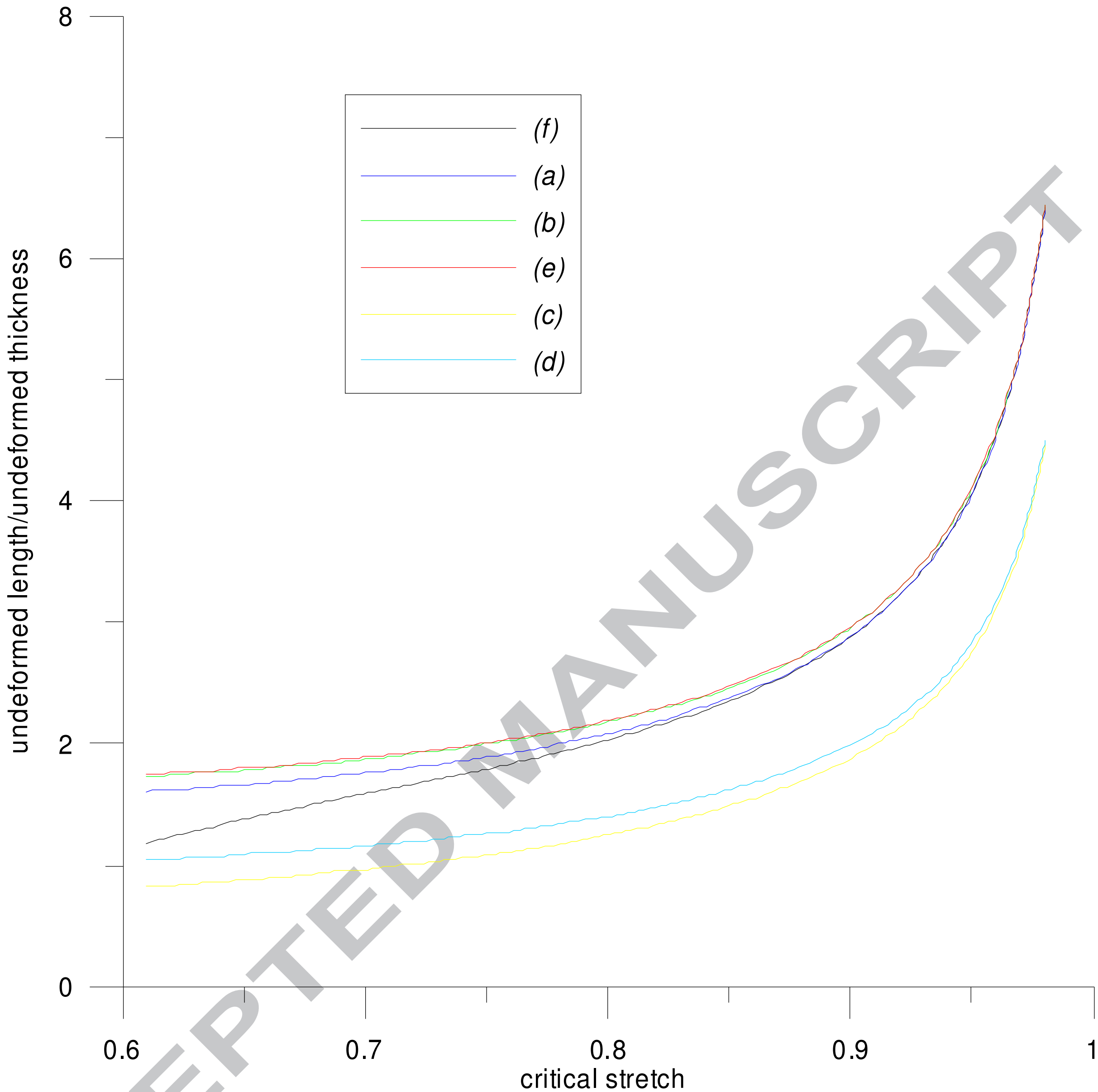

Fig. 4. Comparison of the results

λ_c	*(f)*	*(a)*	*(b)*	*(e)*	*(c)*	*(d)*
0.98	6.4200	6.4144	6.4463	6.4496	4.4671	4.5129
0.97	5.2454	5.2390	5.2781	5.2821	3.6203	3.6767
0.96	4.5454	4.5391	4.5842	4.5888	3.1121	3.1775
0.95	4.0679	4.0623	4.1126	4.1178	2.7630	2.8364
0.94	3.7150	3.7110	3.7660	3.7717	2.5038	2.5846
0.93	3.4420	3.4386	3.4979	3.5041	2.3011	2.3888
0.92	3.2208	3.2197	3.2829	3.2896	2.1369	2.2310
0.91	3.0385	3.0390	3.1059	3.1130	2.0002	2.1005
0.90	2.8831	2.8866	2.9570	2.9646	1.8840	1.9902
0.89	2.7504	2.7561	2.8298	2.8377	1.7836	1.8954
0.88	2.6337	2.6429	2.7195	2.7279	1.6957	1.8129
0.87	2.5305	2.5434	2.6230	2.6317	1.6179	1.7404
0.86	2.4386	2.4554	2.5377	2.5468	1.5484	1.6759
0.85	2.3556	2.3768	2.4617	2.4711	1.4858	1.6183
0.84	2.2793	2.3061	2.3936	2.4033	1.4291	1.5664
0.83	2.2093	2.2423	2.3322	2.3422	1.3773	1.5193
0.82	2.1464	2.1843	2.2765	2.2868	1.3299	1.4764
0.81	2.0860	2.1314	2.2258	2.2364	1.2863	1.4373
0.80	2.0308	2.0829	2.1794	2.1903	1.2460	1.4013
0.79	1.9781	2.0384	2.1369	2.1481	1.2086	1.3682
0.78	1.9286	1.9974	2.0978	2.1093	1.1739	1.3376
0.77	1.8810	1.9595	2.0618	2.0735	1.1415	1.3093
0.76	1.8360	1.9244	2.0284	2.0404	1.1112	1.2830
0.75	1.7914	1.8918	1.9975	2.0098	1.0829	1.2585
0.74	1.7499	1.8615	1.9689	1.9813	1.0563	1.2357
0.73	1.7074	1.8333	1.9422	1.9549	1.0313	1.2144
0.72	1.6674	1.8070	1.9174	1.9303	1.0078	1.1946
0.71	1.6270	1.7824	1.8943	1.9074	0.9857	1.1760
0.70	1.5864	1.7594	1.8727	1.8860	0.9649	1.1585
0.69	1.5467	1.7379	1.8525	1.8660	0.9452	1.1422
0.68	1.5053	1.7178	1.8336	1.8473	0.9266	1.1269
0.67	1.4645	1.6989	1.8159	1.8298	0.9090	1.1125
0.66	1.4220	1.6812	1.7994	1.8135	0.8924	1.0989
0.65	1.3782	1.6646	1.7839	1.7981	0.8767	1.0862
0.64	1.3329	1.6491	1.7694	1.7837	0.8619	1.0743
0.63	1.2849	1.6344	1.7557	1.7703	0.8478	1.0630
0.62	1.2344	1.6207	1.7429	1.7576	0.8345	1.0525
0.61	1.1802	1.6078	1.7309	1.7457	0.8219	1.0425

Table 1. Values of $\frac{\overline{L}}{\overline{h}}$ for the given values of λ_c.

10. Concluding Remarks

A shear deformable model for rubber-like rods undergoing finite strains and rotations including the pseudo transverse normal strain and pseudo transverse shear strain is presented. The pseudo transverse normal stress is assumed to be negligible. Main emphasis is given to the perturbation terms in the asymptotic expansions of the strain components. The expressions for the satisfaction of the stress-free boundary conditions on the top and bottom free surfaces of the rubber-like rods are derived and it is concluded that the zero-traction conditions

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can be satisfied (i) if the strains or rotations are vanishing, (ii) approximately if the strains and rotations are infinitesimally small. The proposed theory is applied to the flexural buckling problem of the Mooney-Rivlin rods with movable clamped edges. Unavoidably disregarding the mentioned zero-traction conditions, the exact solutions for the various alternatives about the aforementioned perturbation terms of the strains are obtained and compared with those corresponding to a previously established rod theory and the theory of elasticity. Basing on the solution obtained by the theory of elasticity, the following deductions are made:

(i) The second order analysis yields excellent results especially for thin and quite thick rods.

(ii) The term $\kappa_{12}^{(2)*}$ has a remarkable contribution to the accuracy of the results.

(iii) The assumption of the constant pseudo transverse shear strain through the thickness in the second order analysis yields a solution very near to the solution obtained by Makowski and Stumpf (1988).

(iv) Neglition of $\kappa_{11}^{(2)*}$ and $\kappa_{22}^{(2)*}$ yields incorrect results.

The paper is restricted with the straight rods where $\bar{\varphi}$ is constant. It can be extended to the curved rods by considering $\bar{\varphi}$ as a function of $\bar{s}$ in a later study.

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FIGURE CAPTIONS

Figure 1. Undeformed and deformed geometries

Figure 2. The angle of rotation

Figure 3. A Mooney-Rivlin rod with movable clamped edges under axial compressive forces

Figure 4. Comparison of the results

TABLE CAPTIONS

Table 1. Values of $\frac{\overline{L}}{\overline{h}}$ for the given values of λ_c.

λ_c	*(f)*	*(a)*	*(b)*	*(e)*	*(c)*	*(d)*
0.98	6.4200	6.4144	6.4463	6.4496	4.4671	4.5129
0.97	5.2454	5.2390	5.2781	5.2821	3.6203	3.6767
0.96	4.5454	4.5391	4.5842	4.5888	3.1121	3.1775
0.95	4.0679	4.0623	4.1126	4.1178	2.7630	2.8364
0.94	3.7150	3.7110	3.7660	3.7717	2.5038	2.5846
0.93	3.4420	3.4386	3.4979	3.5041	2.3011	2.3888
0.92	3.2208	3.2197	3.2829	3.2896	2.1369	2.2310
0.91	3.0385	3.0390	3.1059	3.1130	2.0002	2.1005
0.90	2.8831	2.8866	2.9570	2.9646	1.8840	1.9902
0.89	2.7504	2.7561	2.8298	2.8377	1.7836	1.8954
0.88	2.6337	2.6429	2.7195	2.7279	1.6957	1.8129
0.87	2.5305	2.5434	2.6230	2.6317	1.6179	1.7404
0.86	2.4386	2.4554	2.5377	2.5468	1.5484	1.6759
0.85	2.3556	2.3768	2.4617	2.4711	1.4858	1.6183
0.84	2.2793	2.3061	2.3936	2.4033	1.4291	1.5664
0.83	2.2093	2.2423	2.3322	2.3422	1.3773	1.5193
0.82	2.1464	2.1843	2.2765	2.2868	1.3299	1.4764
0.81	2.0860	2.1314	2.2258	2.2364	1.2863	1.4373
0.80	2.0308	2.0829	2.1794	2.1903	1.2460	1.4013
0.79	1.9781	2.0384	2.1369	2.1481	1.2086	1.3682
0.78	1.9286	1.9974	2.0978	2.1093	1.1739	1.3376
0.77	1.8810	1.9595	2.0618	2.0735	1.1415	1.3093
0.76	1.8360	1.9244	2.0284	2.0404	1.1112	1.2830
0.75	1.7914	1.8918	1.9975	2.0098	1.0829	1.2585
0.74	1.7499	1.8615	1.9689	1.9813	1.0563	1.2357
0.73	1.7074	1.8333	1.9422	1.9549	1.0313	1.2144
0.72	1.6674	1.8070	1.9174	1.9303	1.0078	1.1946
0.71	1.6270	1.7824	1.8943	1.9074	0.9857	1.1760
0.70	1.5864	1.7594	1.8727	1.8860	0.9649	1.1585
0.69	1.5467	1.7379	1.8525	1.8660	0.9452	1.1422
0.68	1.5053	1.7178	1.8336	1.8473	0.9266	1.1269
0.67	1.4645	1.6989	1.8159	1.8298	0.9090	1.1125
0.66	1.4220	1.6812	1.7994	1.8135	0.8924	1.0989
0.65	1.3782	1.6646	1.7839	1.7981	0.8767	1.0862
0.64	1.3329	1.6491	1.7694	1.7837	0.8619	1.0743
0.63	1.2849	1.6344	1.7557	1.7703	0.8478	1.0630
0.62	1.2344	1.6207	1.7429	1.7576	0.8345	1.0525
0.61	1.1802	1.6078	1.7309	1.7457	0.8219	1.0425

Table 1. Values of $\frac{\overline{L}}{\overline{h}}$ for the given values of λ_c.

HIGHLIGHTS

- A theory for rubber-like rods undergoing finite strains and rotations is presented.
- The pseudo strains are expanded asymptotically for the transverse coordinate.
- Flexural buckling problem of Mooney-Rivlin rods under compressive loads is considered.
- Basing on three dimensional elasticity solution, very appealing results are obtained.