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Localization analysis of variationally based gradient plasticity model

Milan Jirásek *, Ondřej Rokoš, Jan Zeman

Department of Mechanics, Faculty of Civil Engineering, Czech Technical University in Prague, Czech Republic

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ABSTRACT

The paper presents analytical or semi-analytical solutions for the formation and evolution of localized plastic zone in a uniaxially loaded bar with variable cross-sectional area. A variationally based formulation of explicit gradient plasticity with linear softening is used, and the ensuing jump conditions and boundary conditions are discussed. Three cases with different regularity of the stress distribution are considered, and the problem is converted to a dimensionless form. Relations linking the load level, size of the plastic zone, distribution of plastic strain and plastic elongation of the bar are derived and compared to another, previously analyzed gradient formulation.

1. One-dimensional softening plasticity model

For many materials, the stress–strain diagrams characterizing their mechanical behavior exhibit the so-called softening branches, with decreasing stress at increasing strain (and thus with a negative tangent stiffness). The physical origin of this intriguing phenomenon is in the initiation, propagation and coalescence of defects such as microcracks or microvoids. Softening-induced localization of inelastic processes into narrow zones often acts as a precursor to failure. Proper modeling of the entire failure process requires an objective description of the localized process zone and its evolution.

Perhaps the most popular class of inelastic material models is represented by the theory of (elasto-)plasticity. The present paper focuses on the localization properties of softening plasticity models. To allow for analytical solutions, all considerations are done in the one-dimensional context, referring to the case of a straight bar under uniaxial loading as the typical paradigm. However, the analysis is nontrivial due to the fact that a variable cross-sectional area is considered, and a regularized formulation of softening plasticity is used.

1.1. Classical formulation

In the small-strain range, classical elastoplasticity is based on the additive split of the total strain into the elastic part and the plastic part. The elastic strain is linked to the stress by Hooke's law, while the plastic strain can grow only if the stress level attains the yield limit, which is mathematically indicated by zero value of the yield function. The oriented direction of the plastic strain rate is specified by the flow rule and the evolution of the yield surface (set of plastic stress states in the stress space) is described by the hardening/softening law. For simplicity, we assume linear softening, i.e., linear dependence of the current yield stress on the cumulative plastic strain. Description of the stress–strain relation by a bilinear diagram is certainly a rough approximation, but it can reflect the main features of elastoplasticity with softening and serve as a prototype model, for which analytical solutions exist.

In the one-dimensional setting, the basic equation can be summarized as follows:

$$\sigma = E\varepsilon_e = E(\varepsilon - \varepsilon_p) \tag{1}$$

$$f(\sigma,\kappa) = \mid \sigma \mid - \sigma_Y(\kappa) \tag{2}$$

$$\sigma_Y(\kappa) = \sigma_0 + H\kappa \tag{3}$$

$$\dot{\varepsilon}_p = \dot{\lambda}\,\mathrm{sgn}\sigma \tag{4}$$

$$\dot{\kappa} = \mid \dot{\varepsilon}_p \mid \tag{5}$$

$$\dot{\lambda} \geqslant 0, \quad f(\sigma,\kappa) \leqslant 0, \quad \dot{\lambda} f(\sigma,\kappa) = 0 \tag{6}$$

Here, σ is the stress, ε is the (total) strain, ε_e and ε_p are its elastic and plastic parts, E is the elastic modulus, f is the yield function, σ_Y is the current yield stress, σ_0 is the initial yield stress, $H < 0$ is the softening modulus, λ is the plastic multiplier and κ is the cumulative plastic strain. The overdot denotes differentiation with respect to time. A more detailed discussion of this specific problem is available in Jirásek et al. (2010) and a broad background of the theory of plasticity e.g. in Lubliner (1990) or Jirásek and Bažant (2001).

If we restrict attention to tensile loading (with possible elastic unloading, but never with a reversal of plastic flow), then the plastic strain ε_p, cumulative plastic strain κ and plastic multiplier λ are

* Corresponding author. Fax: +420 224310775.
E-mail address: Milan.Jirasek@fsv.cvut.cz (M. Jirásek).

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all equal. We will use κ as the primary symbol for (cumulative) plastic strain and rewrite Eqs. (1) and (6) as

$$\sigma = E(\varepsilon - \kappa) \tag{7}$$

$$\dot{\kappa} \geqslant 0, \quad f(\sigma,\kappa) \leqslant 0, \quad \dot{\kappa} f(\sigma,\kappa) = 0 \tag{8}$$

The above equations refer to uniaxial tension, but formally the same framework can be used for the one-dimensional description of a shear problem. Normal stress and strain are then replaced by shear stress and strain, Young's modulus E by the shear modulus G, and the tensile yield stress by the shear yield stress.

1.2. Standard gradient formulation

It is well known that softening is a destabilizing factor that may lead to localization of dissipative processes (in our case of plastic yielding) into narrow zones. For classical continuum formulations with local dependence between stress and strain, the thickness of such localized process zones is undetermined and may become arbitrarily small. The undesired consequence is that the structural response becomes excessively brittle and numerical simulations suffer by pathological sensitivity to the discretization parameters such as the size of elements used by the finite element method. This has to be avoided, e.g. by introducing a regularization technique which enforces a nonzero thickness of the localized process zone and thus nonvanishing dissipation during the failure process.

In the one-dimensional setting, negative plastic modulus H always leads to localization of plastic strain. Consider a straight bar with perfectly uniform properties, subjected to uniaxial tension (induced by applied displacement at one bar end). The response remains uniform in the elastic range and also during plastic yielding with a positive plastic modulus. For a negative (or vanishing) plastic modulus, uniqueness of the solution is lost right at the onset of softening (or of perfectly plastic yielding). Stress distribution along the bar must still remain uniform due to the static equilibrium conditions (in the absence of body forces), but a given stress level can be attained by softening with increasing plastic strain, or by elastic unloading with no plastic strain evolution. Which cross sections unload and which exhibit softening remains completely arbitrary, and there is no lower bound on the total length of the softening region(s). Therefore, infinitely many solutions exist, including solutions with plastic strain evolution localized into extremely small regions. Even if the nonuniqueness of the solution is removed by a slight perturbation of the perfect uniformity of the bar, the problem with localization of softening into arbitrarily small regions (in fact into the weakest cross section) still persists.

Commonly used regularization techniques overcome the problem by suitable enrichments of the governing equations. Such enrichments typically introduce at least one additional parameter with the dimension of length (or a parameter which can be combined with the traditional ones such that the result has the dimension of length). This parameter reflects the intrinsic length scale of the material and is related to the size and spacing of major heterogeneities in the microstructure. The size of the process zone is then controlled by the choice of the length scale parameter.

In principle it is possible to construct regularized models with enriched kinematic and equilibrium equations, e.g. strain-gradient plasticity or Cosserat-type models. From the practical point of view it is more convenient to limit the enrichments to the constitutive equations describing the material behavior and to keep the kinematic and equilibrium equations unchanged. This class of approaches is usually referred to as nonlocal continuum theories in the broad sense. Nonlocality of the stress–strain relation can be introduced by weighted spatial averaging of suitably chosen internal variables, or by incorporation of gradients of such variables into the constitutive description. Here we focus on the latter case, in particular on its typical representative—the second-order explicit gradient model that evolved from the work of Aifantis and colleagues (Aifantis, 1984).

The explicit gradient formulation of elastoplasticity is based on incorporation of a term proportional to the Laplacean of cumulative plastic strain into the softening law (3). In the one-dimensional setting, the Laplacean reduces to the second derivative and the enriched softening law reads

$$\sigma_Y(\kappa) = \sigma_0 + H(\kappa + l^2\kappa'') \tag{9}$$

where l is a new parameter with the dimension of length.

In a bar with perfectly uniform properties (cross section, yield stress, softening modulus, etc.) and in the absence of body forces and inertia forces, the stress is constant along the bar. The plastic zone, $\mathcal{I}_p$, is characterized by growing plastic strain ($\dot{\kappa} > 0$) and vanishing value of the yield function ($f = 0$). Since the yield function is given by (2), we conclude that the yield stress must be constant inside the plastic zone, and then (9) leads to a second-order differential equation with constant coefficients and a constant right-hand side:

$$l^2\kappa''(x) + \kappa(x) = \frac{\sigma - \sigma_0}{H} \tag{10}$$

As shown e.g. in de Borst and Mühlhaus (1992), the (most localized) plastic zone is an interval of length $2\pi l$, arbitrarily placed along the bar.

Analytical solutions for several types of bars with variable cross sections were presented in Jirásek et al. (2010). The governing equation

$$l^2 A(x)\kappa''(x) + A(x)\kappa(x) = \frac{F - \sigma_0 A(x)}{H} \tag{11}$$

was constructed from (10) by setting $\sigma(x) = F/A(x)$, where A is a function describing the distribution of the cross-sectional area along the bar, and F is the axial force transmitted by the bar, which is constant (independent of x) because of static equilibrium.

In the present paper, we will use a modified formulation of the one-dimensional gradient plasticity model, constructed by a variational approach. Analytical or semi-analytical solutions will be derived and compared to the results for the standard gradient formulation based on (11). An important advantage of the variational formulations is that it permits a consistent treatment of problems with discontinuous data, e.g. with a jump in the cross-sectional area (leading to a jump in the stress field).

1.3. Variational gradient formulation

The variational formulation of the second-order explicit gradient plasticity model considered here is inspired by the work of Mühlhaus and Aifantis (1991), Valanis (1996), Svedberg (1996), Svedberg and Runesson (1997, 1998), Polizzotto et al. (1998), Borino et al. (1999), and Liebe and Steinmann (2001). In the one-dimensional case, it is derived from the functional

$$\Pi(u,\kappa) = \int_{\mathcal{L}} \frac{1}{2} EA(u' - \kappa)^2 \,\mathrm{d}x + \int_{\mathcal{L}} \frac{1}{2} HA\left(\kappa^2 - l^2\kappa'^2\right) \mathrm{d}x + \int_{\mathcal{L}} A\sigma_0\kappa \,\mathrm{d}x - \int_{\mathcal{L}} Abu \,\mathrm{d}x \tag{12}$$

in which $\mathcal{L}$ denotes the interval that represents the entire bar, A is the cross sectional area and b is the prescribed body force density in the longitudinal direction (per unit volume), introduced just for the sake of generality but later set equal to zero. The first integral in (12) can be interpreted as the elastic strain energy, the second as the plastic part of free energy, the third as the dissipated energy and the fourth as the potential energy of external forces.

Functional Π is considered in the space of all sufficiently smooth displacement fields u that satisfy the geometric (essential) boundary conditions on the boundary $\partial\mathcal{L}$, and all sufficiently smooth and nonnegative plastic strain fields κ. In formal mathematical language, the domain of definition of functional Π is the space $V = V_u \times V_\kappa$ where

$$V_u = \{u \in H_1(\mathcal{L}) \mid u = \bar{u} \text{ on } \partial\mathcal{L} \text{ in the sense of traces}\} \quad (13)$$

$$V_\kappa = \{\kappa \in H_1(\mathcal{L}) \mid \kappa \geqslant 0 \text{ almost everywhere}\} \quad (14)$$

This means that the functions describing the displacement and the plastic strain must be square-integrable and possess square-integrable generalized first derivatives, but continuity of the first derivatives and existence of the second derivatives are not a priori required.

Due to the lack of convexity, the analysis can hardly rely on global minimization of functional Π. Nevertheless, it is reasonable to expect that stable solutions of the problem are associated with local minima of Π. The subsequent derivations will be based on necessary conditions of a local minimum, in particular, on nonnegative values of the first variation (Gateaux derivative) of functional Π corresponding to all admissible variations of fields u and κ. It will be demonstrated that such an approach leads to a consistent set of conditions that describe the problem and include the equilibrium equation, the complementarity conditions governing the plastic flow, as well as appropriate boundary conditions at the physical boundary and regularity conditions at the elasto-plastic interfaces. A complete analysis should also pay attention to the second variation, which is related to stability issues. Analytical conditions for a non-negative second variation, derived for the simplest case of a bar with uniform properties, are presented in Appendix A.

Strictly speaking, the variational approach should be applied in an incremental fashion, as discussed e.g. by Petryk (2003). However, for the present purpose it is fully sufficient to consider a total formulation. It turns out that, for one-dimensional problems with expanding or stationary plastic zones, the parameterized solutions constructed in this way do not violate the irreversibility constraints and thus represent physically admissible responses to given loading scenarios.

The first variation of functional Π defined in (12) can be expressed as

$$\begin{aligned}\delta\Pi(\delta u, \delta\kappa; u, \kappa) = & \int_{\mathcal{L}} EA(u' - \kappa)(\delta u' - \delta\kappa)\,\mathrm{d}x \\ & + \int_{\mathcal{L}} HA\left(\kappa\delta\kappa - l^2\kappa'\delta\kappa'\right)\mathrm{d}x + \int_{\mathcal{L}} A\sigma_0\delta\kappa\,\mathrm{d}x \\ & - \int_{\mathcal{L}} Ab\delta u\,\mathrm{d}x \end{aligned} \quad (15)$$

where δu is the displacement variation (difference between two admissible displacement fields taken from V_u) and $\delta\kappa$ is the variation of plastic strain (difference between two admissible plastic strain fields taken from V_κ). Integration by parts of the terms with $\delta u'$ and $\delta\kappa'$ leads to

$$\begin{aligned}\delta\Pi(\delta u, \delta\kappa; u, \kappa) = & -\int_{\mathcal{L}} \left[(EA(u' - \kappa))' + Ab\right]\delta u\,\mathrm{d}x \\ & + \int_{\mathcal{L}} \left[HA\kappa + (HAl^2\kappa')' + A\sigma_0 - EA(u' - \kappa)\right]\delta\kappa\,\mathrm{d}x \\ & + \sum_{\partial\mathcal{L}} EA(u' - \kappa)n\delta u - \sum_i [[EA(u' - \kappa)\delta u]]_{x_i} \\ & - \sum_{\partial\mathcal{L}} HAl^2\kappa' n\delta\kappa + \sum_i [[HAl^2\kappa'\delta\kappa]]_{x_i} \end{aligned} \quad (16)$$

where $\sum_{\partial\mathcal{L}}$ stands for the boundary integral, in the one-dimensional setting reduced to the sum over two points at the boundary of the interval $\mathcal{L}$, n is the "unit normal", equal to -1 at the left boundary and to 1 at the right boundary, and the sums with subscript i are taken over all points at which the quantity in the double square brackets has a discontinuity. The double brackets denote the jump of the quantity inside the brackets, defined as

$$[[f]]_{x_i} = \lim_{x \to x_i^+} f(x) - \lim_{x \to x_i^-} f(x) \quad (17)$$

As already explained, the first variation $\delta\Pi$ must be nonnegative for all admissible variations δu and $\delta\kappa$. By admissible variations we mean arbitrary changes of u and κ for which $u + \delta u \in V_u$ and $\kappa + \delta\kappa \in V_\kappa$.

Variations δu are arbitrary inside $\mathcal{L}$ but vanishing on the boundary. The expression multiplying δu in the integral on the first line of (16) must be identically equal to zero, which provides the equilibrium conditions

$$(EA(u' - \kappa))' + Ab = 0 \quad (18)$$

Of course, u' corresponds to the strain, $u' - \kappa$ is the elastic strain, $E(u' - \kappa)$ is the stress and $EA(u' - \kappa)$ is recognized as the axial force. Since δu is arbitrary inside $\mathcal{L}$, the jumps of $EA(u' - \kappa)$ must vanish, i.e., the axial force (not the stress) must remain continuous.

The variation of κ is not completely arbitrary, because $\kappa + \delta\kappa$ must remain nonnegative. If we define the plastic zone $\mathcal{I}_p = \{x \in \mathcal{L} \mid \kappa(x) > 0\}$, $\delta\kappa$ can have an arbitrary sign inside $\mathcal{I}_p$ but must be nonnegative outside $\mathcal{I}_p$. Therefore, the expression multiplying $\delta\kappa$ in the integral on the second line of (16) must vanish in $\mathcal{I}_p$ but outside $\mathcal{I}_p$ it is only constrained to be nonnegative. We recognize the resulting equation

$$HA\kappa + (HAl^2\kappa')' + A\sigma_0 = EA(u' - \kappa) \quad \text{for } x \in \mathcal{I}_p \quad (19)$$

as the yield condition, and the resulting inequality

$$HA\kappa + (HAl^2\kappa')' + A\sigma_0 \geqslant EA(u' - \kappa) \quad \text{for } x \in \mathcal{L} \setminus \mathcal{I}_p \quad (20)$$

as the plastic admissibility condition. The advantage of the variational formulation is that the cases of variable sectional area A, softening modulus H or internal length l are covered in a systematic way, even in cases when some of these quantities exhibit discontinuities. From the last two lines of (16) we obtain the corresponding jump conditions and also the boundary conditions.

On the physical boundary $\partial\mathcal{L}$, we get $HAl^2\kappa' n = 0$ if the boundary point belongs to $\mathcal{I}_p$, or $HAl^2\kappa' n \leqslant 0$ if this point does not belong to $\mathcal{I}_p$. The first condition means that if the plastic zone contains a point of the physical boundary, the homogeneous Neumann condition $\kappa' = 0$ should be imposed at that point. The second condition means that if the boundary point remains elastic ($\kappa = 0$ at the boundary), the spatial derivative of plastic strain could, in principle, be nonzero. Since $H < 0$ and $Al^2 > 0$, at the right end of the bar ($n = 1$) the derivative κ' must not be negative. However, if $\kappa = 0$ at the right boundary and $\kappa \geqslant 0$ everywhere, κ' cannot be strictly positive at the boundary and its only admissible value is again zero.

The jump conditions imply that if the quantity $HAl^2\kappa'\delta\kappa$ is discontinuous at some point, its jump should be nonnegative for any admissible $\delta\kappa$.

- For points inside the plastic zone $\mathcal{I}_p$, $\delta\kappa$ can be positive as well as negative, and therefore $HAl^2\kappa'$ must remain continuous. So in general we should not enforce continuous differentiability of plastic strain, but rather continuity of the product $HAl^2\kappa'$. This is important when the spatial distribution of the softening modulus or of the sectional area is discontinuous.
- For points outside the plastic zone, the variation $\delta\kappa$ is nonnegative and so the jump of $HAl^2\kappa'$ is in principle admissible but only if it is positive. Inside the elastic zone, κ' vanishes and

$HAl^2\kappa'$ has no jump at all. However, at the elasto-plastic interface (which is located at the boundary of the plastic zone), the derivative of plastic strain could exhibit a jump from zero value in the elastic zone to nonzero value in the plastic zone. At the left boundary of the plastic zone, the jump is equal to the value in the plastic zone. Since H is negative and Al^2 is positive, the limit of κ' (as we approach the boundary of the plastic zone from inside) is allowed to be negative (or zero). However, $\kappa' < 0$ would mean that $\kappa < 0$ at some point inside the plastic zone (because $\kappa = 0$ at the boundary of that zone), which is not admissible. Similar arguments can be applied at the right boundary of the plastic zone, where the jump is minus the value in the plastic zone, and therefore κ' is allowed to be positive (or zero), but a positive slope of the plastic strain profile is impossible to achieve without generating negative plastic strains inside the plastic zone near the boundary. So this discussion leads to the conclusion that the condition $\kappa' = 0$ should be imposed at the boundary of the plastic zone.

In the absence of body forces, we set $b = 0$ and Eq. (18) implies that $EA(u' - \kappa)$ is constant along the bar (independent of the spatial coordinate x). Physically, this constant represents the axial force transmitted by the bar and therefore will be denoted as F. Eq. (19) is then rewritten as

$$HA(x)\kappa(x) + (HA(x)l^2\kappa'(x))' + A(x)\sigma_0 = F \quad \text{for } x \in \mathcal{I}_p \tag{21}$$

2. Bar with piecewise constant stress distribution

Having presented the governing equations, we can proceed to localization analysis of a tensile bar with variable cross section. As the first case, consider a bar with piecewise constant sectional area (Fig. 1(a)). Suppose that the bar contains a weak segment of length $2l_g$ and sectional area A_c, while the remaining parts have a larger sectional area $A_c/(1-\beta)$ where $\beta \in [0, 1)$ is a dimensionless parameter. The origin of the spatial coordinate system can be placed into the center of the weak segment. The solution is then expected to exhibit symmetry with respect to the origin.

Let us emphasize that the present analysis is strictly focused on one-dimensional modeling. Therefore, the stress distribution across each section is considered as uniform. Of course, for a real three-dimensional body containing notches, the stress field would have a singularity at the notch tip and the stress distribution across sections near that singularity would be highly nonuniform. However, we use the case of variable cross section as a paradigmatic example of the localization properties of a gradient plasticity model in cases with non-smooth and sometimes even discontinuous data. Therefore, the stress is expressed simply as the normal force divided by the sectional area.

For the bar with a weak segment of length $2l_g$, the stress distribution is described by

$$\sigma(x) = \begin{cases} F/A_c = \sigma_c & \text{for } \mid x \mid < l_g \\ F/[A_c/(1-\beta)] = (1-\beta)\sigma_c & \text{for } \mid x \mid > l_g \end{cases} \tag{22}$$

and has discontinuities at sections $x = \pm l_g$; see Fig. 2. As long as the stress remains below the yield limit, the response is purely elastic. The onset of yielding can be expected when the yield limit is attained, which happens at the elastic limit force $F_0 = A_c\sigma_0$. For a softening plasticity model without any regularization, plastic yielding would localize into one cross section, the dissipation would vanish (because the plastic zone has zero volume) and the response would be extremely brittle. Regularization by the additional gradient term leads to a finite size of the plastic zone.

2.1. Plastic zone contained in weak segment

Let us first assume that the plastic zone $\mathcal{I}_p$ is fully contained in the weak segment, i.e., $\mathcal{I}_p \subset (-l_g, l_g)$. The yield condition (21) with $A = A_c$ and $F = A_c\sigma_c$ can be rewritten as

$$l^2\kappa''(x) + \kappa(x) = \frac{\sigma_c - \sigma_0}{H} \quad \text{for } x \in \mathcal{I}_p \tag{23}$$

This is a linear second-order differential equation with constant coefficients and a constant right-hand side, and its general solution is

$$\kappa(x) = \frac{\sigma_c - \sigma_0}{H} + C_1 \cos\frac{x}{l} + C_2 \sin\frac{x}{l} \tag{24}$$

where C_1 and C_2 are arbitrary constants. Let L_p denote the length of the plastic zone and suppose that the plastic zone is centered at the origin, i.e., $\mathcal{I}_p = (-L_p/2, L_p/2)$. If this was not the case, the origin could simply be shifted to the center of the plastic zone. As explained at the end of Section 1.3, the plastic strain κ must remain continuous and its derivative must vanish at the elasto-plastic interface, $\partial\mathcal{I}_p$, which consists of two points, $x = \pm L_p/2$. Conditions $\kappa(-L_p/2) = 0$, $\kappa(L_p/2) = 0$, $\kappa'(-L_p/2) = 0$ and $\kappa'(L_p/2) = 0$ lead to $C_2 = 0$, $C_1 = (\sigma_0 - \sigma_c)/(H\cos(L_p/2l))$ and $\sin(L_p/2l) = 0$. The last condition means that the length of the plastic zone, L_p, must be an integer multiple of $2\pi l$. The shortest positive length of plastic zone is obtained if $L_p = 2\pi l$. However, such a solution is admissible only if the plastic zone is indeed fully contained in the weak segment of length $2l_g$, i.e., if $l_g > \pi l$. In that case, the solution is given by

$$\kappa(x) = \begin{cases} \frac{\sigma_c - \sigma_0}{H}\left(1 + \cos\frac{x}{l}\right) & \text{for } x \in \mathcal{I}_p = (-\pi l, \pi l) \\ 0 & \text{for } x \notin \mathcal{I}_p \end{cases} \tag{25}$$

This is the classical solution that describes localization in a bar with perfectly uniform properties (de Borst and Mühlhaus, 1992). Analysis of the second variation of functional Π, presented in detail in Appendix A, reveals that this solution corresponds to a local minimum of Π.

If the weak segment is sufficiently long with respect to the characteristic length of the material (longer than $2\pi l$), the plastic zone will form inside that segment and the solution will not be affected by stronger parts of the bar. However, if the weak segment is shorter than the plastic zone in a perfectly uniform bar, solution (25) is not admissible and the derivation must be modified.

It is convenient and elegant to convert the problem to a dimensionless format. For this purpose, we introduce the dimensionless

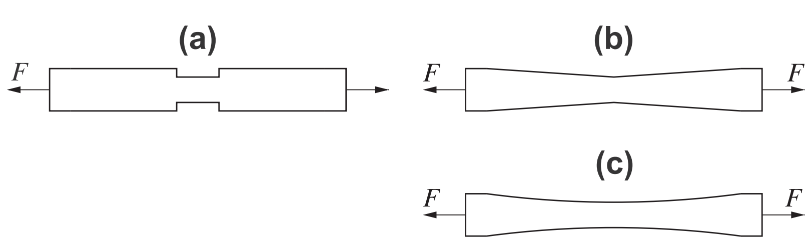

Fig. 1. Tensile bars with (a) discontinuous distribution of stress, (b) continuous distribution of stress, and (c) smooth distribution of stress.

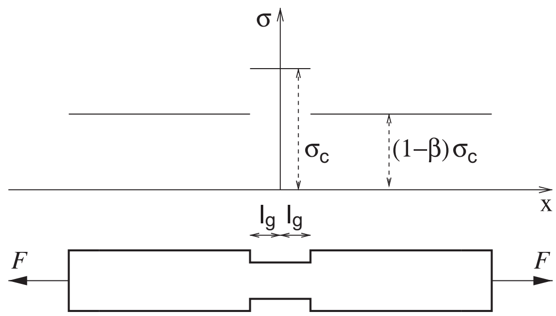

Fig. 2. Bar with a weak segment and the corresponding stress distribution.

spatial coordinate, $\xi = x/l$, normalized plastic strain, $\kappa_n = -H\kappa/\sigma_0$, and dimensionless stress $\phi = \sigma_c/\sigma_0$. Note that ϕ, defined as the ratio between the stress in the weak segment and the yield stress, is at the same time the ratio between the axial force, F, and its elastic limit value, F_0, and thus will be referred to as the load parameter. The distribution of plastic strain in a uniform bar, given by (25), can be described in terms of the dimensionless quantities as

$$\kappa_n(\xi) = \begin{cases} (1-\phi)(1+\cos\xi) & \text{for } \xi \in (-\pi, \pi) \\ 0 & \text{for } \xi \notin (-\pi, \pi) \end{cases} \tag{26}$$

This solution is valid for a bar with a weak segment of length $2l_g$ provided that $2l_g > 2\pi l$, i.e., $\lambda_g > \pi$ where $\lambda_g = l_g/l$ is a dimensionless parameter describing the ratio between the "geometric" characteristic length, l_g, and the material characteristic length, l.

2.2. Plastic zone extending to strong segments

Let us proceed to the case when $2l_g < 2\pi l$, i.e., $\lambda_g < \pi$. Eq. (23) is now valid only for $|x| < l_g$. In terms of the dimensionless quantities, we rewrite it as

$$\kappa_n''(\xi) + \kappa_n(\xi) = 1 - \phi, \quad |\xi| < \lambda_g \tag{27}$$

For simplicity, the derivatives with respect to ξ will still be denoted by primes, despite possible confusion with derivatives with respect to x. For the parts of the plastic zone surrounding the weak segment, a similar equation with a modified right-hand side can be derived from (21):

$$\kappa_n''(\xi) + \kappa_n(\xi) = 1 - \phi + \beta\phi, \quad \lambda_g < |\xi| < \lambda_p \tag{28}$$

Here, $\lambda_p = L_p/2l$ is a dimensionless parameter characterizing the ratio between one half of the plastic zone length, $L_p/2$, and the material characteristic length, l. Since the solution is again expected to be symmetric with respect to the origin, we construct it only in the weak segment and in one of the stronger segments adjacent to it:

$$\kappa_n(\xi) = \begin{cases} 1 - \phi + C_1\cos\xi + C_2\sin\xi & \text{for } -\lambda_g \leqslant \xi \leqslant \lambda_g \\ 1 - \phi + \beta\phi + C_3\cos\xi + C_4\sin\xi & \text{for } \lambda_g \leqslant \xi \leqslant \lambda_p \end{cases} \tag{29}$$

Integration constant C_2 must vanish due to symmetry. Constants C_1, C_3 and C_4 and the dimensionless plastic zone size λ_p can be determined from four conditions: continuity of κ_n and of $A\kappa_n'$ at $\xi = \lambda_g$ and at $\xi = \lambda_p$. These conditions lead to the set of four equations

$$C_1\cos\lambda_g - C_3\cos\lambda_g - C_4\sin\lambda_g = \beta\phi \tag{30}$$

$$-C_1(1-\beta)\sin\lambda_g + C_3\sin\lambda_g - C_4\cos\lambda_g = 0 \tag{31}$$

$$C_3\cos\lambda_p + C_4\sin\lambda_p = (1-\beta)\phi - 1 \tag{32}$$

$$-C_3\sin\lambda_p + C_4\cos\lambda_p = 0 \tag{33}$$

which are linear in terms of C_1, C_3 and C_4 but nonlinear in terms of λ_p. Since the load parameter, ϕ, enters the equations in a linear fashion, it is of advantage to reformulate the problem and solve for the integration constants and ϕ in terms of λ_p. Another parameter that affects the solution is λ_g, and so we can parameterize the solution by λ_g and λ_p and write

$$C_1(\lambda_g, \lambda_p) = \beta\phi\,\frac{\sin(\lambda_p - \lambda_g)}{D(\lambda_g, \lambda_p)} \tag{34}$$

$$C_3(\lambda_g, \lambda_p) = (\beta - 1)\beta\phi\,\frac{\sin\lambda_g\cos\lambda_p}{D(\lambda_g, \lambda_p)} \tag{35}$$

$$C_4(\lambda_g, \lambda_p) = (\beta - 1)\beta\phi\,\frac{\sin\lambda_g\sin\lambda_p}{D(\lambda_g, \lambda_p)} \tag{36}$$

$$\phi(\lambda_g, \lambda_p) = \frac{1}{1-\beta}\,\frac{1}{1 + \frac{\beta\sin\lambda_g}{D(\lambda_g,\lambda_p)}} \tag{37}$$

where

$$\begin{aligned} D(\lambda_g, \lambda_p) &= (1 - \beta\sin^2\lambda_g)\sin\lambda_p - \beta\sin\lambda_g\cos\lambda_g\cos\lambda_p \\ &= \left(1 - \frac{1}{2}\beta\right)\sin\lambda_p + \frac{1}{2}\beta\sin(\lambda_p - 2\lambda_g) \end{aligned} \tag{38}$$

Recall that parameter ϕ, defined as the ratio σ_c/σ_0, can also be interpreted as the ratio between the axial force transmitted by the bar, $F = A_c\sigma_c$, and its limit elastic value, $F_0 = A_c\sigma_0$. For a fixed λ_g, the dimensionless size of the plastic zone λ_p can be varied as a parameter describing various stages of plastic zone evolution. The range in which the solution makes physical sense can be determined from the conditions that the initial value of the load parameter at the onset of yielding is $\phi = 1$ and that $\phi = 0$ corresponds to complete failure. The relation between the load parameter ϕ and the dimensionless plastic zone size λ_p is graphically illustrated in Fig. 3 for $\beta = 0.5$ and for several values of parameter λ_g ranging from 0.01 to 2.5. The interesting range of λ_g is between zero and π because for $\lambda_g > \pi$ the weak zone is long enough to allow formation of the complete plastic zone in its interior, and the solution is the same as for a bar with a perfectly uniform section A_c.

When the axial force reaches the limit elastic value (i.e., when $\phi = 1$), the entire weak segment of length $2l_g$ starts yielding and the initial value of the dimensionless plastic zone size is $\lambda_p = \lambda_g$. Subsequently, the plastic zone expands continuously, and this initially happens at an increasing load level, provided that $\lambda_g < \pi/2$, i.e., that the length of the weak segment $2l_g$ is smaller than πl. The maximum force

$$F_{max} = \frac{F_0}{1-\beta}\,\frac{1}{1 + \frac{\beta\sin\lambda_g}{\sqrt{1-\beta(2-\beta)\sin^2\lambda_g}}} \tag{39}$$

is attained when the dimensionless size of the plastic zone is

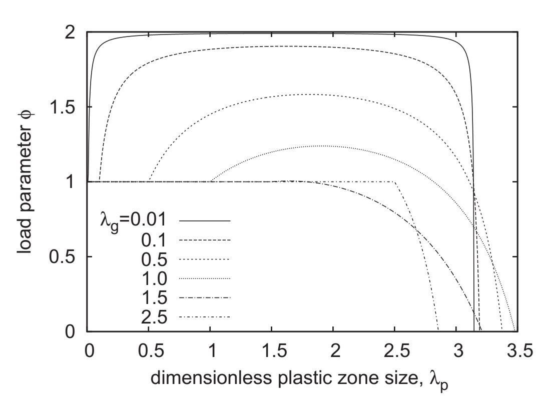

Fig. 3. Relation between load parameter and plastic zone size for a piecewise constant stress distribution (bar with a weak segment).

$$\lambda_{p,peak} = \pi - \arctan\frac{1-\beta\sin^2\lambda_g}{\beta\sin\lambda_g\cos\lambda_g} \tag{40}$$

After that, the axial force decreases to zero as the size of the plastic zone approaches the limit

$$\lambda_{p,max} = \lambda_{p,peak} + \frac{1}{2}\pi \tag{41}$$

If the size of the weak segment exceeds πl, the global response is softening right from the onset of plastic yielding.

Substituting from (34)–(37) into (29), we obtain the distribution of plastic strain parameterized by λ_p, and integrating over the plastic zone we get the plastic elongation. An example of the evolution of plastic strain profile, calculated for $\beta = 0.5$ and $\lambda_g = 0.5$, is shown in Fig. 4. The plastic zone expands and, at each section, the plastic strain grows monotonically.

It is also of interest to construct the load–displacement diagram for the entire bar. The elastic elongation is proportional to the axial force and the proportionality factor (bar compliance) depends on the bar length. Therefore, it is convenient to consider the contribution of plastic strain to the elongation separately, since the bar length does not affect it (provided that the bar is sufficiently long such that the full plastic zone can develop). The plastic part of bar elongation,

$$u_p = \int_{-L_p/2}^{L_p/2} \kappa(x)\,\mathrm{d}x \tag{42}$$

can be computed by integrating the plastic strain along the plastic zone. In the context of dimensionless description, it is natural to deal with the dimensionless plastic elongation

$$\int_{-\lambda_p}^{\lambda_p} \kappa_n(x)\,\mathrm{d}\xi = \int_{-L_p/2}^{L_p/2} -\frac{H\kappa(x)}{\sigma_0}\frac{\mathrm{d}x}{l} = -\frac{Hu_p}{\sigma_0 l} = \frac{u_p}{\kappa_f l} \tag{43}$$

where $\kappa_f = -\sigma_0/H$ is a material parameter that corresponds to the plastic strain at complete failure if the gradient terms are ignored.

The plastic part of the load–displacement diagram, obtained by plotting the dimensionless load parameter ϕ against the dimensionless plastic elongation $u_p/\kappa_f l$, is shown in Fig. 5(a) for fixed $\beta = 0.5$ and different dimensionless sizes of the weak segment, λ_g, and in Fig. 5(b) for fixed $\lambda_g = 0.5$ and different values of β. These graphs explain the influence of both parameters on the shape of the load–displacement diagram. The weak segment can be considered as an imperfection in a uniform bar, parameter λ_g refers to the length of that imperfection and β to its "magnitude" (in the sense that larger β means a more dramatic reduction of the sectional area). For short imperfections, the initial structural hardening is very steep and the maximum axial force is close to the value calculated for a perfect bar. For somewhat longer imperfections, the structural hardening is more gradual and the maximum force is between the values that would correspond to the weaker and stronger sections, and for imperfection lengths exceeding πl (which is exactly one half of the plastic zone that would form in a perfectly uniform bar) the maximum load is dictated exclusively by the weak section and the load–displacement diagram is softening from the onset of yielding.

3. Bar with piecewise linear stress distribution

3.1. General solution

Now consider a bar with variable sectional area described by a function that is continuous but not continuously differentiable at the weakest section; see Fig. 1(b). To allow for analytical solutions, we define the specific distribution of sectional area such that the corresponding stress distribution becomes piecewise linear. This is achieved by setting

$$A(x) = \frac{A_c l_g}{l_g - |x|} \tag{44}$$

where l_g is a parameter that sets the geometric length scale of the problem. Substituting (44) into (21) and dividing by $HA(x)$, we obtain

$$l^2\kappa''(x) + \frac{l^2\,\mathrm{sgn}x}{l_g - |x|}\kappa'(x) + \kappa(x) = \frac{\sigma_c(l_g - |x|)}{Hl_g} - \frac{\sigma_0}{H} \quad \text{for } x \in \mathcal{I}_p - \{0\} \tag{45}$$

Eq. (45) must be satisfied at all points inside the plastic zone, with the exception of point $x = 0$ at which the sectional area is not differentiable. At that point, conditions of continuity of κ and $A\kappa'$ have to be imposed. Since A is continuous, the latter condition actually means continuity of κ'. After conversion to dimensionless form, the governing equation reads

$$\kappa_n''(\xi) + \frac{\mathrm{sgn}\xi}{\lambda_g - |\xi|}\kappa_n'(\xi) + \kappa_n(\xi) = 1 - \phi\left(1 - \frac{|\xi|}{\lambda_g}\right) \tag{46}$$

This is a second-order differential equation, but in contrast to (27) or (28), it contains on the left-hand side an additional term with the first-order derivative which has a non-constant coefficient, and the right-hand side is not constant. Still, an analytical solution in terms of special functions can be constructed. The derivation is presented in detail in Appendix B. The resulting general solution has the form

$$\kappa_n(\xi) = (\lambda_g - |\xi|)C_1^{\pm}\mathrm{J}_1(\lambda_g - |\xi|) + (\lambda_g - |\xi|)C_2^{\pm}\mathrm{Y}_1(\lambda_g - |\xi|) + 1 - \frac{\phi\pi(\lambda_g - |\xi|)}{2\lambda_g}\mathbf{H}_1(\lambda_g - |\xi|) \quad \text{for } \xi \in \mathcal{I}_p \tag{47}$$

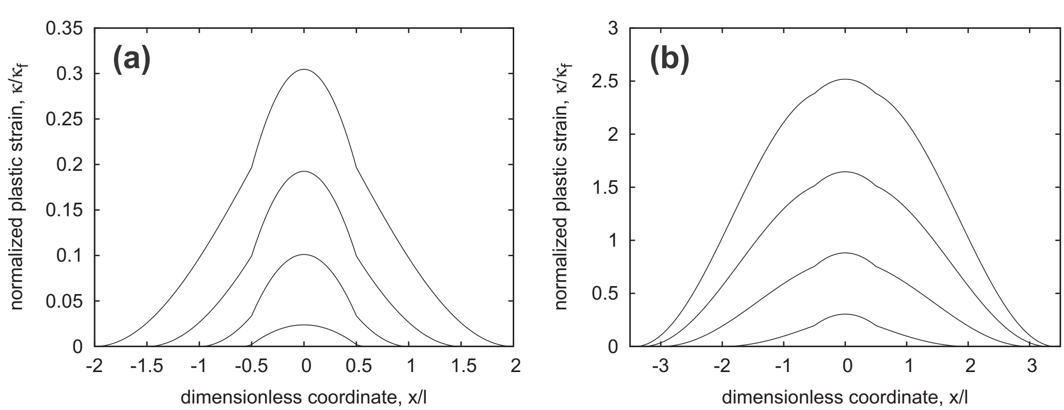

Fig. 4. (a) Early stage and (b) late stage of evolution of plastic strain profile for a piecewise constant stress distribution.

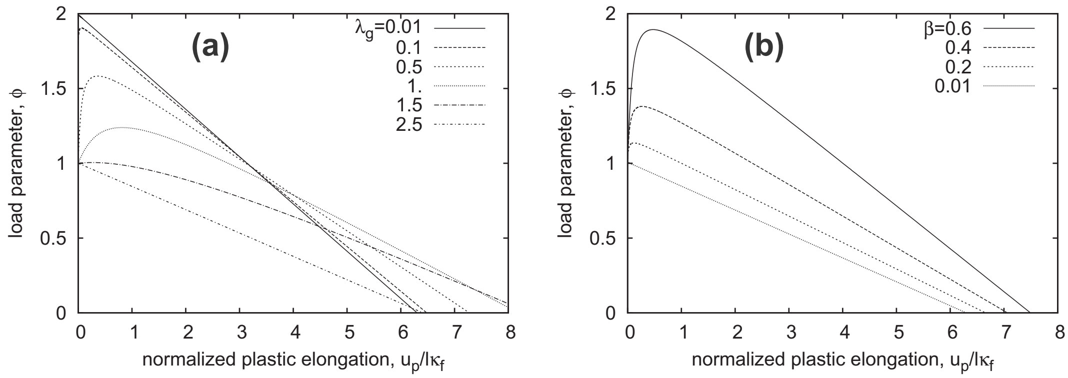

Fig. 5. Plastic part of load–displacement diagram for a piecewise constant stress distribution for (a) different values of relative imperfection size $\lambda_g = l_g/l$ and (b) different values of imperfection "magnitude" β.

where $C_1^{\pm}$ and $C_2^{\pm}$ are integration constants, J_1 and Y_1 are the Bessel functions of the first and second kind, resp., and $\mathbf{H}_1$ is the Struve function (Korenev, 2002).

3.2. Particular solution

For simplicity, we have written (47) in a compact form, but the integration constants have different values in the "positive part" of the plastic zone, where $\xi > 0$, and in the "negative part" of the plastic zone, where $\xi < 0$. Therefore, we deal with four integration constants, C_1^+, C_2^+, C_1^- and C_2^-, and with two additional unknowns that determine the position of the right and left boundary of the plastic zone. This makes a total of six unknowns, which can be determined from appropriate jump and boundary conditions. According to the foregoing analysis, the solution must remain continuously differentiable at the origin and at the two boundary points, which leads to six equations for the six unknowns.

Due to symmetry, the problem can be simplified and it is sufficient to restrict attention to the positive part of the plastic zone, $\mathcal{I}_p^+ = (0, \lambda_p)$. The three relevant unknowns, C_1^+, C_2^+ and λ_p, can be determined from three conditions,

$$\kappa_n'(0) = 0 \tag{48}$$

$$\kappa_n(\lambda_p) = 0 \tag{49}$$

$$\kappa_n'(\lambda_p) = 0 \tag{50}$$

Substituting the general solution (47) into (48)–(50), we obtain a set of three equations,

$$-2\lambda_g\left[C_1^+\mathrm{J}_0(\lambda_g) + C_2^+\mathrm{Y}_0(\lambda_g)\right] + \phi\pi\mathbf{H}_0(\lambda_g) = 0 \tag{51}$$

$$C_1^+\mathrm{J}_1(\lambda_g - \lambda_p) + C_2^+\mathrm{Y}_1(\lambda_g - \lambda_p) - \frac{\phi\pi}{2\lambda_g}\mathbf{H}_1(\lambda_g - \lambda_p) + \frac{1}{\lambda_g - \lambda_p} = 0 \tag{52}$$

$$-2\lambda_g\left[C_1^+\mathrm{J}_0(\lambda_g - \lambda_p) + C_2^+\mathrm{Y}_0(\lambda_g - \lambda_p)\right] + \phi\pi\mathbf{H}_0(\lambda_g - \lambda_p) = 0 \tag{53}$$

which are linear in terms of C_1^+ and C_2^+ but nonlinear in terms of λ_p. In (51) and (53), we exploited identities $(\xi\mathrm{J}_1(\xi))\prime = \xi\mathrm{J}_0(\xi)$, $(\xi\mathrm{Y}_1(\xi))\prime = \xi\mathrm{Y}_0(\xi)$ and $(\xi\mathbf{H}_1(\xi))\prime = \xi\mathbf{H}_0(\xi)$, where J_m and Y_m are Bessel functions of order m and $\mathbf{H}_m$ are Struve functions of order m; see e.g. Andrews (1992) and Korenev (2002). Again, it is of advantage to reformulate the problem and solve for C_1^+, C_2^+ and ϕ in terms of λ_p and λ_g:

$$C_1^+(\lambda_g, \lambda_p) = \frac{\pi\left[-\mathrm{Y}_0(\lambda_g - \lambda_p)\mathbf{H}_0(\lambda_g) + \mathrm{Y}_0(\lambda_g)\mathbf{H}_0(\lambda_g - \lambda_p)\right]}{D^+(\lambda_g, \lambda_p)} \tag{54}$$

$$C_2^+(\lambda_g, \lambda_p) = \frac{\pi\left[\mathrm{J}_0(\lambda_g - \lambda_p)\mathbf{H}_0(\lambda_g) - \mathrm{J}_0(\lambda_g)\mathbf{H}_0(\lambda_g - \lambda_p)\right]}{D^+(\lambda_g, \lambda_p)} \tag{55}$$

$$\phi(\lambda_g, \lambda_p) = \frac{2\lambda_g\left[\mathrm{J}_0(\lambda_g - \lambda_p)\mathrm{Y}_0(\lambda_g) - \mathrm{J}_0(\lambda_g)\mathrm{Y}_0(\lambda_g - \lambda_p)\right]}{D^+(\lambda_g, \lambda_p)} \tag{56}$$

where

$$\begin{aligned} D^+(\lambda_g, \lambda_p) = {} & 2\mathbf{H}_0(\lambda_g) + \pi(\lambda_g - \lambda_p)\{[-\mathrm{J}_1(\lambda_g - \lambda_p)\mathrm{Y}_0(\lambda_g) \\ & + \mathrm{J}_0(\lambda_g)\mathrm{Y}_1(\lambda_g - \lambda_p)]\mathbf{H}_0(\lambda_g - \lambda_p) + [\mathrm{J}_0(\lambda_g \\ & - \lambda_p)\mathrm{Y}_0(\lambda_g) - \mathrm{J}_0(\lambda_g)\mathrm{Y}_0(\lambda_g - \lambda_p)]\mathbf{H}_1(\lambda_g - \lambda_p)\} \end{aligned} \tag{57}$$

From symmetry with respect to the origin, we obtain integration constants corresponding to the negative part of the plastic zone as $C_1^- = C_1^+$ and $C_2^- = C_2^+$. The assumption of symmetry may seem to be restrictive, but it can be verified by (tedious) analysis of the complete problem with six equations and six unknowns that no nonsymmetric admissible solution exists.

3.3. Results and discussion

Based on the solution constructed in the previous subsection, the evolution of the plastic zone can be analyzed, and the corresponding load–displacement diagram can be constructed. The plastic part of the load–displacement diagram is shown in Fig. 6, where the dimensionless load parameter ϕ is plotted against the dimensionless plastic elongation, $u_p/l\kappa_f$. Recall that $\phi = \sigma_c/\sigma_0 = F/F_0$ is the ratio between the axial force F and its value at the onset of yielding, $F_0 = A_c\sigma_0$. The plastic elongation, u_p, is normalized by a reference value $l\kappa_f$, which would correspond to the plastic elongation of a bar of length l at complete failure if the solution remained uniform.

During the elastic stage of loading, the axial force increases from 0 to F_0 and no plastic strain evolves. Thus the initial part of the diagram in Fig. 6, up to $\phi = 1$, is vertical. For a bar with a uniform section, the continuation of that diagram would be a straight line that corresponds to linear softening, and the plastic elongation

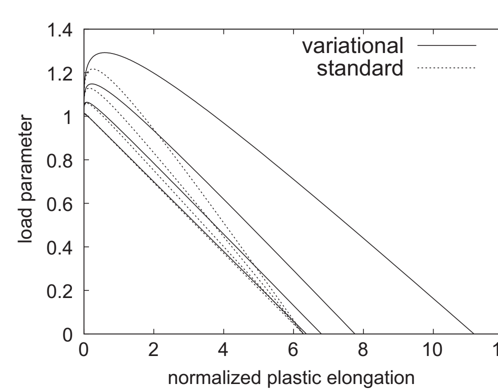

Fig. 6. Plastic part of normalized load–displacement diagram for piecewise linear stress distribution.

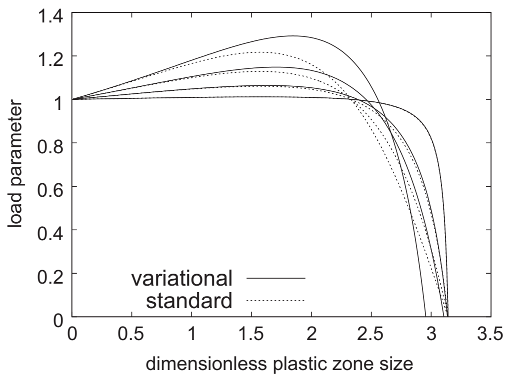

Fig. 7. Evolution of plastic zone size for piecewise linear stress distribution.

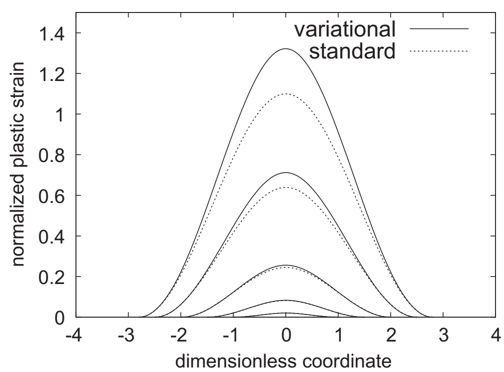

Fig. 8. Evolution of plastic strain profile for piecewise linear stress distribution.

at complete failure ($\phi = 0$) would be $2\pi l\kappa_f$. This is in fact a limit case of the present solution with $\lambda_g \to \infty$. For finite λ_g, i.e., for a bar with a variable section, the load parameter first increases and only later decreases. Complete failure is attained at larger elongations than for the uniform bar. This is represented in Fig. 6 by the solid curves, which correspond to different values of parameter $\lambda_g = 3.2$, 5, 10 and 50 (from top to bottom). Lower values of λ_g correspond to stronger variation of the sectional area and lead to higher peak loads and higher elongations at failure. For comparison, the solutions constructed in Jirásek et al. (2010) using a "standard" formulation, with the second term in (21) replaced by $HAl^2\kappa''$, are plotted by the dotted curves. The present, variationally based formulation leads to qualitatively similar solutions, but with higher peak loads and higher elongations at complete failure. Note that, for the standard formulation, the elongation at failure is always the same as for a uniform bar, independently of the value of parameter λ_g.

The evolution of the plastic zone size is illustrated in Fig. 7 by plotting the load parameter ϕ against the dimensionless plastic zone size $\lambda_p = L_p/2l$, again for $\lambda_g = 3.2$, 5, 10 and 50. For a uniform bar ($\lambda_g \to \infty$), the plastic zone would attain its full size $L_p = 2\pi l$ (corresponding to $\lambda_p = \pi$) immediately at the onset of yielding. In contrast to that, for the bar with a variable section, the plastic zone evolves continuously from the weakest section up to its full size, attained at complete failure. The fact that the plastic zone expands monotonically is important, because it justifies our tacit assumption that the material outside the current plastic zone has not experienced any plastic straining before. The full size of the plastic zone turns out to be somewhat smaller than for a uniform bar, which is different from the standard solution constructed in Jirásek et al. (2010), plotted for comparison by dotted curves.

The evolution of the plastic strain profile is plotted in Fig. 8 for $\lambda_g = 5$. Again, solid curves represent the variational formulation and dotted curves the standard formulation from Jirásek et al. (2010). At early stages of plastic strain evolution, both formulations give almost the same result, but later the differences grow.

4. Bar with smooth stress distribution

As the most regular case, we consider a smooth distribution of sectional area, with continuous derivatives of an arbitrary order. The origin of the spatial coordinate is again placed at the weakest section, where plastic yielding is expected to start. In Jirásek et al. (2010), the function describing the sectional area was selected such that the resulting stress distribution was quadratic. For the present purpose, it turns out to be more convenient to consider another special case in which the stress distribution is given by a Gaussian function,

$$\sigma(x) = \sigma_c\, \mathrm{e}^{-x^2/2l_g^2} \tag{58}$$

The sectional area is thus

$$A(x) = A_c\, \mathrm{e}^{x^2/2l_g^2} \tag{59}$$

Since, for this case, the solution based on the standard formulation has not been published yet, let us present it before turning attention to the variational formulation.

4.1. Solution based on standard formulation

According to the standard formulation used in Jirásek et al. (2010), the yield condition is considered in the form

$$HAl^2\kappa''(x) + HA\kappa(x) + A\sigma_0 = F \quad \text{for } x \in \mathcal{I}_p \tag{60}$$

which differs from (21) by the second term. The corresponding differential equation for the plastic strain can be converted into the dimensionless form

$$\kappa_n''(\xi) + \kappa_n(\xi) = 1 - \phi\, \mathrm{e}^{-\xi^2/2\lambda_g^2} \tag{61}$$

The general solution of the corresponding homogeneous equation is a linear combination of $\cos\xi$ and $\sin\xi$, and the particular solution for the given right-hand side can be constructed by variation of constants. For the particular solution represented by

$$\tilde{\kappa}_\mathrm{n}(\xi) = \widetilde{C}_1(\xi)\cos\xi + \widetilde{C}_2(\xi)\sin\xi \tag{62}$$

we obtain a set of two equations

$$\widetilde{C}_1'(\xi)\cos\xi + \widetilde{C}_2'(\xi)\sin\xi = 0 \tag{63}$$

$$-\widetilde{C}_1'(\xi)\sin\xi + \widetilde{C}_2'(\xi)\cos\xi = 1 - \phi\, \mathrm{e}^{-\xi^2/2\lambda_g^2} \tag{64}$$

from which

$$\widetilde{C}_1(\xi) = \cos\xi + \phi\int \mathrm{e}^{-\xi^2/2\lambda_g^2}\sin\xi\, \mathrm{d}\xi \tag{65}$$

$$\widetilde{C}_2(\xi) = \sin\xi - \phi\int \mathrm{e}^{-\xi^2/2\lambda_g^2}\cos\xi\, \mathrm{d}\xi \tag{66}$$

The integrals can be conveniently expressed by switching to complex functions. As shown in Appendix C, the resulting general solution of Eq. (61) can be expressed as

$$\kappa_\mathrm{n}(\xi) = C_1\cos\xi + C_2\sin\xi + 1 - \phi\lambda_g \mathrm{e}^{-\xi^2/2\lambda_g^2} \times \frac{\sqrt{2}}{2}\left[F\left(\frac{\lambda_g^2 - i\xi}{\sqrt{2}\lambda_g}\right) + F\left(\frac{\lambda_g^2 + i\xi}{\sqrt{2}\lambda_g}\right)\right] \tag{67}$$

where C_1 and C_2 are arbitrary constants, i denotes the imaginary unit, and

$$\mathrm{F}(x) = \mathrm{e}^{-x^2}\int_0^x \mathrm{e}^{t^2}\, \mathrm{d}t \tag{68}$$

is the so-called Dawson function; see e.g. Olver (1997).

In general, one should impose two boundary conditions (of vanishing value and vanishing derivative) at each boundary point and solve for two integration constants C_1 and C_2 and two unknown coordinates of the boundary points. Due to symmetry, the problem can be simplified. Integration constant C_2 must vanish, and the plastic zone $\mathcal{I}_p = (-\lambda_p, \lambda_p)$ is characterized by one unknown parameter, λ_p. Boundary conditions $\kappa_n(\lambda_p) = 0$ and $\kappa'_n(\lambda_p) = 0$ lead to a set of two equations,

$$-C_1\cos\lambda_p + \phi\frac{\lambda_g}{\sqrt{2}}e^{-\lambda_p^2/2\lambda_g^2}\left[F\left(\frac{\lambda_g^2 - i\lambda_p}{\sqrt{2}\lambda_g}\right) + F\left(\frac{\lambda_g^2 + i\lambda_p}{\sqrt{2}\lambda_g}\right)\right] = 1 \quad (69)$$

$$C_1\sin\lambda_p + \phi\frac{i\lambda_g}{\sqrt{2}}e^{-\frac{\lambda_p^2}{2\lambda_g^2}}\left[F\left(\frac{\lambda_g^2 - i\lambda_p}{\lambda_g\sqrt{2}}\right) - F\left(\frac{\lambda_g^2 + i\lambda_p}{\lambda_g\sqrt{2}}\right)\right] = 0 \quad (70)$$

which can be written as

$$-c_pC_1 + EF_R\phi = 1 \quad (71)$$

$$s_pC_1 + EF_I\phi = 0 \quad (72)$$

where $c_p = \cos\lambda_p$, $s_p = \sin\lambda_p$, $E = \sqrt{2}\lambda_g e^{-\lambda_p^2/2\lambda_g^2}$ and

$$F_R = \frac{1}{2}\left[F\left(\frac{\lambda_g^2 + i\lambda_p}{\lambda_g\sqrt{2}}\right) + F\left(\frac{\lambda_g^2 - i\lambda_p}{\lambda_g\sqrt{2}}\right)\right] \quad (73)$$

$$F_I = -\frac{i}{2}\left[F\left(\frac{\lambda_g^2 + i\lambda_p}{\lambda_g\sqrt{2}}\right) - F\left(\frac{\lambda_g^2 - i\lambda_p}{\lambda_g\sqrt{2}}\right)\right] \quad (74)$$

are the real and imaginary parts of $F\left(\frac{\lambda_g^2+i\lambda_p}{\lambda_g\sqrt{2}}\right)$. Eq. (70) follows from the condition $\kappa'_n(\lambda_p) = 0$ with the derivative of F expressed as $F'(x) = 1 - 2xF(x)$, which easily follows from the definition of Dawson function (68).

Solving for C_1 and ϕ, we obtain

$$C_1(\lambda_p, \lambda_g) = -\frac{F_I}{F_Rs_p + F_Ic_p} = \frac{F\left(\frac{\lambda_g^2+i\lambda_p}{\lambda_g\sqrt{2}}\right) - F\left(\frac{\lambda_g^2-i\lambda_p}{\lambda_g\sqrt{2}}\right)}{e^{-i\lambda_p}F\left(\frac{\lambda_g^2-i\lambda_p}{\lambda_g\sqrt{2}}\right) - e^{i\lambda_p}F\left(\frac{\lambda_g^2+i\lambda_p}{\lambda_g\sqrt{2}}\right)} \quad (75)$$

$$\phi(\lambda_p, \lambda_g) = \frac{s_p}{E(F_Rs_p + F_Ic_p)} = \frac{\sqrt{2}}{\lambda_g}\frac{i\sin\lambda_p e^{\lambda_p^2/2\lambda_g^2}}{e^{i\lambda_p}F\left(\frac{\lambda_g^2+i\lambda_p}{\lambda_g\sqrt{2}}\right) - e^{-i\lambda_p}F\left(\frac{\lambda_g^2-i\lambda_p}{\lambda_g\sqrt{2}}\right)} \quad (76)$$

As usual, the solution is parameterized by the dimensionless size of the plastic zone, λ_p. For a given value of λ_p, the corresponding load parameter ϕ is given by (76) and the distribution of plastic strain is obtained by substituting C_1 given by (75) and $C_2 = 0$ into (67).

4.2. Solution based on variational formulation

For the variational formulation, the yield condition is used in the form (21) instead of (60), and Eq. (61) is replaced by

$$e^{-\xi^2/2\lambda_g^2}\left(e^{\xi^2/2\lambda_g^2}\kappa_n(\xi)'\right)' + \kappa_n(\xi) = 1 - \phi e^{-\xi^2/2\lambda_g^2} \quad (77)$$

As shown in Appendix D, the general solution of the corresponding homogeneous equation is a linear combination of functions

$$\kappa_{n,1}(\xi) = e^{-\frac{\xi^2}{2\lambda_g^2}}\,{}_1F_1\left(\frac{1-\lambda_g^2}{2}; \frac{1}{2}; \frac{\xi^2}{2\lambda_g^2}\right) \quad (78)$$

$$\kappa_{n,2}(\xi) = e^{-\frac{\xi^2}{2\lambda_g^2}}\frac{\xi}{\sqrt{2}\lambda_g}\,{}_1F_1\left(\frac{2-\lambda_g^2}{2}; \frac{3}{2}; \frac{\xi^2}{2\lambda_g^2}\right) \quad (79)$$

where ${}_1F_1$ denotes the so-called confluent hypergeometric function of the first kind (Sneddon, 1956), defined by formulae (D.12) and (D.13) in Appendix D. A particular solution of the non-homogeneous Eq. (77) could be constructed by variation of constants. Following the same procedure as in Section 4.1, we obtain a particular solution

$$\tilde{\kappa}_n(\xi) = \widetilde{C}_1(\xi)\kappa_{n,1}(\xi) + \widetilde{C}_2(\xi)\kappa_{n,2}(\xi) \quad (80)$$

with

$$\widetilde{C}_1(\xi) = \int\frac{\kappa_{n,2}(\xi)}{\kappa'_{n,1}(\xi)\kappa_{n,2}(\xi) - \kappa_{n,1}(\xi)\kappa'_{n,2}(\xi)}\left(1 - \phi e^{-\frac{\xi^2}{2\lambda_g^2}}\right)d\xi \quad (81)$$

$$\widetilde{C}_2(\xi) = -\int\frac{\kappa_{n,1}(\xi)}{\kappa'_{n,1}(\xi)\kappa_{n,2}(\xi) - \kappa_{n,1}(\xi)\kappa'_{n,2}(\xi)}\left(1 - \phi e^{-\frac{\xi^2}{2\lambda_g^2}}\right)d\xi \quad (82)$$

Unfortunately, these integrals cannot be evaluated analytically.

An alternative approach can be based on the Green function of the differential operator on the left-hand side of (77). Formally, the Green function represents the solution of the differential equation with the right-hand side replaced by Dirac distribution centered at a point η. At points ξ different from η, the right-hand side is zero and the solution is a linear combination of functions (78) and (79). Due to the singularity at $\xi = \eta$, different coefficients of linear combination must be used for $\xi \leqslant \eta$ and for $\xi \geqslant \eta$. Moreover, these coefficients depend on the specific value of η. Therefore, we can write the Green function as

$$G(\xi, \eta) = \begin{cases} A_1(\eta)\kappa_{n,1}(\xi) + A_2(\eta)\kappa_{n,2}(\xi) & \text{for } -\lambda_p \leqslant \xi \leqslant \eta \\ B_1(\eta)\kappa_{n,1}(\xi) + B_2(\eta)\kappa_{n,2}(\xi) & \text{for } \eta \leqslant \xi \leqslant \lambda_p \end{cases} \quad (83)$$

and impose at $\xi = \eta$ the continuity condition for the value and the unit jump condition for the first derivative. This leads to two equations,

$$A_1(\eta)\kappa_{n,1}(\eta) + A_2(\eta)\kappa_{n,2}(\eta) = B_1(\eta)\kappa_{n,1}(\eta) + B_2(\eta)\kappa_{n,2}(\eta) \quad (84)$$

$$A_1(\eta)\kappa'_{n,1}(\eta) + A_2(\eta)\kappa'_{n,2}(\eta) = B_1(\eta)\kappa'_{n,1}(\eta) + B_2(\eta)\kappa'_{n,2}(\eta) - 1 \quad (85)$$

In usual problems solved on a fixed interval, the Green function should also satisfy two boundary conditions (one at each boundary point). However, in our case the exact position of the boundary points is not specified and the number of conditions to be satisfied at each boundary point is two (vanishing value and vanishing first derivative). Making use of symmetry, we can restrict attention to non-negative values of ξ and η and impose only one condition of vanishing derivative at $\xi = 0$, while at $\xi = \lambda_p$ we still need to satisfy two conditions. One of them can be incorporated into the Green function, and the other will be imposed aposteriori on the resulting solution and will provide a link between the load parameter ϕ and the dimensionless size of the plastic zone, λ_g.

Based on the foregoing discussion, we constrain the Green function by conditions of vanishing derivative at $\xi = 0$ and vanishing value at $\xi = \lambda_p$, which gives two additional equations,

$$A_1(\eta)\kappa'_{n,1}(0) + A_2(\eta)\kappa'_{n,2}(0) = 0 \quad (86)$$

$$B_1(\eta)\kappa_{n,1}(\lambda_p) + B_2(\eta)\kappa_{n,2}(\lambda_p) = 0 \quad (87)$$

Since $\kappa'_{n,1}(0) = 0$ (see Appendix D) and $\kappa'_{n,2}(0) \neq 0$, Eq. (86) gives $A_2(\eta) = 0$. Combining Eqs. (84), (85) and (87), we can express

$$A_1(\eta) = \frac{\kappa_{n,2}(\eta)\kappa_{n,1}(\lambda_p) - \kappa_{n,1}(\eta)\kappa_{n,2}(\lambda_p)}{D(\eta)} \quad (88)$$

$$B_1(\eta) = -\frac{\kappa_{n,2}(\lambda_p)\kappa_{n,1}(\eta)}{D(\eta)} \quad (89)$$

$$B_2(\eta) = \frac{\kappa_{n,1}(\lambda_p)\kappa_{n,1}(\eta)}{D(\eta)} \quad (90)$$

with the auxiliary function D given by

$$D(\eta) = \left[\kappa_{n,1}(\eta)\kappa'_{n,2}(\eta) - \kappa'_{n,1}(\eta)\kappa_{n,2}(\eta)\right]\kappa_{n,1}(\lambda_p) \quad (91)$$

and construct the Green function by substituting this back into (83). The solution of (77) is then given by

$$\kappa_n(\xi) = \int_0^{\lambda_p} G(\xi,\eta)\left(1 - \phi\, e^{-\eta^2/2\lambda_g^2}\right) d\eta$$
$$= \int_0^{\lambda_p} G(\xi,\eta)\, d\eta - \phi \int_0^{\lambda_p} G(\xi,\eta)\, e^{-\eta^2/2\lambda_g^2}\, d\eta \tag{92}$$

This solution always satisfies conditions $\kappa_n'(0) = 0$ and $\kappa_n(\lambda_p) = 0$, which have been incorporated into the Green function. The remaining condition to be satisfied, $\kappa_n'(\lambda_p) = 0$, provides a link between the load parameter and the size of the plastic zone. As usual, it is more convenient to express the load parameter ϕ in terms of the dimensionless plastic zone size λ_p than vice versa. Indeed, we can formally write

$$\kappa_n'(\xi) = \int_0^{\lambda_p} G'(\xi,\eta)\, d\eta - \phi \int_0^{\lambda_p} G'(\xi,\eta)\, e^{-\eta^2/2\lambda_g^2}\, d\eta \tag{93}$$

where, for simplicity,

$$G'(\xi,\eta) = \frac{\partial G(\xi,\eta)}{\partial \xi} = \begin{cases} A_1(\eta)\kappa_{n,1}'(\xi) & \text{for } 0 \leqslant \xi < \eta \\ B_1(\eta)\kappa_{n,1}'(\xi) + B_2(\eta)\kappa_{n,2}'(\xi) & \text{for } \eta < \xi < \lambda_p \end{cases} \tag{94}$$

denotes the partial derivative of the Green function G with respect to its first argument. From condition $\kappa_n'(\lambda_p) = 0$ we obtain

$$\phi(\lambda_p, \lambda_g) = \frac{\int_0^{\lambda_p} G'(\lambda_p,\eta)\, d\eta}{\int_0^{\lambda_p} G'(\lambda_p,\eta)\, e^{-\eta^2/2\lambda_g^2}\, d\eta} \tag{95}$$

4.3. Results and discussion

For illustration, the solution has been evaluated and plotted for a range of values of parameter λ_g. Fig. 9 indicates that the plastic zone evolves continuously from the weakest section and its size grows monotonically, which confirms admissibility of the solution. Evolution of the plastic zone profile is shown in Fig. 10 for $\lambda_g = 1$ and 10, and the plastic part of the normalized load–displacement diagram is in Fig. 11.

In all these figures, solid curves correspond to the variational formulation from Section 4.2 and dotted curves to the standard one from Section 4.1. For large values of λ_g, e.g. 10, both formulations give almost identical results. Initially, the plastic zone quickly expands and the load parameter ϕ increases only slightly from its value 1 at the onset of plastic yielding. The softening part of the load–displacement diagram is almost linear and as the load approaches zero, the plastic zone size tends to $2\pi l$ (for the standard formulation) or to a value very close to $2\pi l$ (for the variational formulation). For intermediate values of λ_g, e.g. 2 or 1.15, the hardening due to structural effects is more pronounced for the variational formulation than for the standard one, and the load–displacement diagram becomes more ductile. Finally, for small values of λ_g, e.g. 1.01 or 0.95, the variational formulation gives unlimited hardening and unlimited expansion of the plastic zone, while the standard formulation still leads to limited hardening up to a finite peak load, followed by softening and expansion of the plastic zone to its maximum size $2\pi l$.

The strong hardening effect of the variational formulation for small values of λ_g can be attributed to the fact that the stabilizing second term in (21) can be expanded into $Hl^2A(x)\kappa''(x) + Hl^2A'(x)\kappa'(x)$. The second part, dependent on the derivative of the sectional area, is neglected by the standard formulation but taken into account by the variational one. For the problem considered here, the product $A'(x)\kappa'(x)$ is always negative (because $A(x)$ is decreasing in the left half of the plastic zone and increasing in the right half, while $\kappa(x)$ does the opposite), and so the additional term

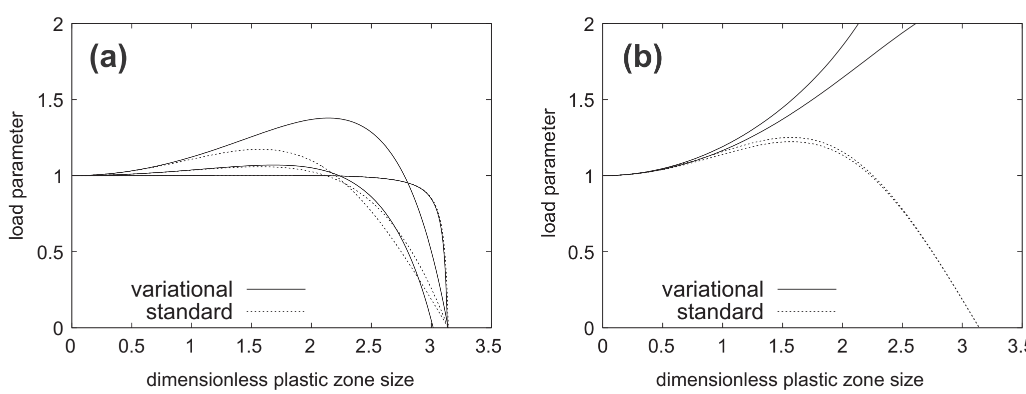

Fig. 9. Relation between load parameter and plastic zone size for a smooth (Gaussian) stress distribution: (a) $\lambda_g = 1.15$, 2, 10 and (b) $\lambda_g = 0.95$, 1.01 (from top to bottom).

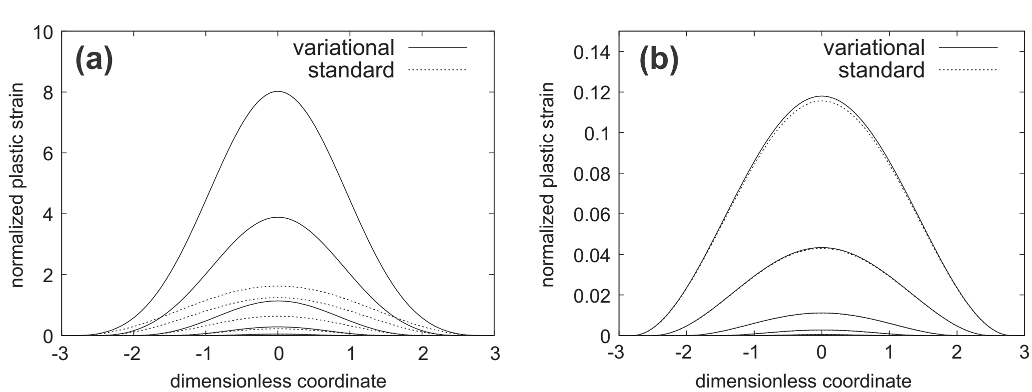

Fig. 10. Evolution of plastic strain profile for a smooth (Gaussian) stress distribution: (a) $\lambda_g = 1$ and (b) $\lambda_g = 10$.

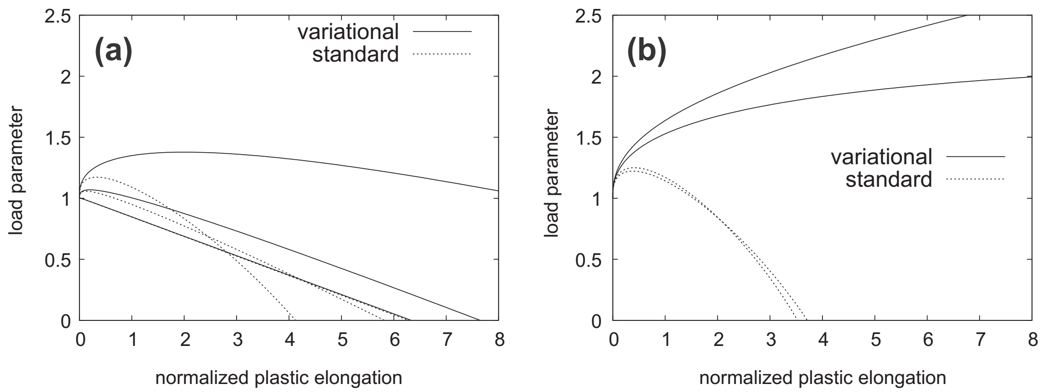

Fig. 11. Plastic part of normalized load–displacement diagram for a smooth (Gaussian) stress distribution: (a) $\lambda_g = 1.15$, 2, 10 and (b) $\lambda_g = 0.95$, 1.01 (from top to bottom).

has a similar effect as negative $\kappa''(x)$. Such a term increases the yield force and thus causes structural hardening accompanied by an expansion of the plastic zone.

It is important to note that cases shown in Figs. 9(b) and Fig. 11(b) correspond to an extreme nonuniformity of sectional area. Indeed, $\lambda_g = 1$ means that the sectional area at point $x = \pi l$, which would be the right boundary of the plastic zone, is $e^{\pi^2/2} \approx 139$ times the area of the weakest section at the center of the plastic zone. So this case is only of academic interest, and is included here for completeness.

5. Summary and conclusions

In this paper, localization of plastic strain induced by a negative plastic modulus has been studied using a variational formulation of a gradient-enriched plasticity model. The main points can be summarized as follows:

1. Mathematical description of one-dimensional gradient plasticity has been derived using a consistent variational approach, which naturally provides not only the differential equation that represents the yield condition and needs to be satisfied inside the plastic zone, but also the appropriate form of the boundary and jump conditions.
2. Taking into account the jump conditions following from a variational principle, a prototype problem with discontinuous data (in this specific case, with discontinuous distribution of the sectional area) has been handled successfully. It has been shown that the problem has a physically reasonable solution, which can be constructed analytically.
3. Two additional prototype problems, one with continuous but non-smooth data and the other with smooth data, have been analyzed, and analytical or semi-analytical solutions have been provided. The results have been compared to an alternative model that does not have a variational format.
4. The influence of various parameters on the evolution of the plastic zone, on the shape of the plastic strain profile and on the resulting load–displacement diagram has been investigated for all three cases mentioned above. It has been shown that the plastic zone evolves from the weakest section and monotonically expands, which is in most cases initially accompanied by an increase of the axial force over its elastic limit value. This means that the structural response exhibits hardening despite the softening character of the material model, which is related to the expansion of the yielding process into stronger parts of the structure induced by the gradient enrichment.
5. The variational approach adopted here is based on the condition of non-negative first variation of a certain energetic functional. States that satisfy this condition are considered as valid solutions of the problem. It is expected that stable solutions correspond to local minima of the functional. A rigorous analysis of the second variation, leading to an explicit stability criterion, is presented in Appendix A for the simplest case of a bar with uniform properties.

Finally, let us note that the approach elaborated here for the second-order gradient model and for variable cross section can be extended to the fourth-order gradient model and to variable material properties (such as the yield stress or plastic modulus). The latter extension could be useful in studies of the development of plastic zone near the interface of two materials with different properties.

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Appendix A. Second variation and stability conditions

All analyses presented in the main body of the paper have been based on the condition of non-negative first variation of functional Π. Of course, this is only a necessary but not a sufficient condition for a local minimum. Intuitively it can be expected that solutions corresponding to a non-negative first variation but not to a local minimum are unstable. Therefore, it is useful to investigate in more detail the changes of Π in the immediate neighborhood of a "point" (u, κ) that represents a valid solution, i.e., leads to a non-negative first variation $\delta\Pi$ for all admissible variations δu and $\delta\kappa$.

If $\delta\Pi > 0$ for given δu and $\delta\kappa$, the value of Π increases at least locally (for sufficiently small variations), and the second variation does not need to be evaluated. However, for those δu and $\delta\kappa$ that give a vanishing first variation $\delta\Pi$, the sign of the increment is determined by the second variation. So, as a first step, we restrict attention to those combinations δu and $\delta\kappa$ for which $\delta\Pi = 0$. Since the solution (u, κ) satisfies the equilibrium condition (18), which is an equality, the first integral in (16) vanishes for an arbitrary displacement variation δu. On the other hand, the yield condition (19) is an equality satisfied in the plastic zone $\mathcal{I}_p$ only. Outside the plastic zone, the plastic admissibility condition (20) becomes an inequality. Leaving aside some degenerated cases of neutral loading, this condition is typically satisfied as a strict inequality. To make sure that the first variation $\delta\Pi$ vanishes, the plastic strain variation $\delta\kappa$ must be set to zero outside the plastic zone. By similar arguments based on the jump terms, it can be shown that the values of $\delta\kappa$ and of its derivative on the boundary of the plastic zone must vanish.

Since functional Π given by (12) is quadratic, its second variation is easily expressed as

$$\delta^2\Pi(\delta u, \delta\kappa; u, \kappa) = \int_{\mathcal{L}} EA(\delta u' - \delta\kappa)^2\,\mathrm{d}x + \int HA\left(\delta\kappa^2 - l^2\delta\kappa'^2\right)\mathrm{d}x \tag{A.1}$$

Taking into account that $\delta\kappa$ vanishes outside $\mathcal{I}_p$, as justified above, the integration domains can be reduced. After simple rearrangements (with moduli E and H considered as constants), the resulting stability condition can be written as

$$\int_{\mathcal{L}\setminus\mathcal{I}_p} A\delta u'^2\,\mathrm{d}x + \int_{\mathcal{I}_p} A(\delta u' - \delta\kappa)^2\,\mathrm{d}x - \frac{H}{E}\int_{\mathcal{I}_p} A\left(l^2\delta\kappa'^2 - \delta\kappa^2\right)\mathrm{d}x \geqslant 0 \tag{A.2}$$

In a loading test performed under displacement control, the displacements on the physical boundary $\partial\mathcal{L}$ are prescribed and the variations δu vanish on $\partial\mathcal{L}$. Condition (A.2) should be satisfied for all such δu and for all variations $\delta\kappa$ with vanishing values and vanishing first derivatives on the elasto-plastic boundary $\partial\mathcal{I}_p$.

General analysis of condition (A.2) for variable sectional area A would be very tedious. However, the case of a uniform bar (with A = const.) is manageable, because (A.2) simplifies to

$$\int_{\mathcal{L}\setminus\mathcal{I}_p} \delta u'^2\,\mathrm{d}x + \int_{\mathcal{I}_p} (\delta u' - \delta\kappa)^2\,\mathrm{d}x - \frac{H}{E}\int_{\mathcal{I}_p} \left(l^2\delta\kappa'^2 - \delta\kappa^2\right)\mathrm{d}x \geqslant 0 \tag{A.3}$$

In this case, the plastic zone $\mathcal{I}_p$ is an interval of length $L_p = 2\pi l$ (see Section 2.1), and the elastically unloading zone $\mathcal{L}\setminus\mathcal{I}_p$ is a union of two intervals of total length $L - L_p$. Individual integrals in (A.3) can be estimated from below. For the second integral, we can exploit the Cauchy–Schwarz inequality, which implies that

$$\int_{\mathcal{I}_p} (\delta u' - \delta\kappa)^2\,\mathrm{d}x \geqslant \frac{1}{L_p}\left(\int_{\mathcal{I}_p} (\delta u' - \delta\kappa)\,\mathrm{d}x\right)^2 = L_p(\delta\bar{\varepsilon} - \delta\bar{\kappa})^2 \tag{A.4}$$

where

$$\delta\bar{\varepsilon} = \frac{1}{L_p}\int_{\mathcal{I}_p} \delta u'\,\mathrm{d}x, \quad \delta\bar{\kappa} = \frac{1}{L_p}\int_{\mathcal{I}_p} \delta\kappa\,\mathrm{d}x \tag{A.5}$$

are constants that represent the mean values of $\delta u'$ and $\delta\kappa$ over the plastic zone. The first integral in (A.3) can be estimated in a similar fashion, taking into account that the mean value of $\delta u'$ in $\mathcal{L}\setminus\mathcal{I}_p$ is directly related to $\delta\bar{\varepsilon}$, because of the compatibility constraint $\int_{\mathcal{L}} \delta u'\,\mathrm{d}x = 0$:

$$\int_{\mathcal{L}\setminus\mathcal{I}_p} \delta u'\,\mathrm{d}x = -\int_{\mathcal{I}_p} \delta u'\,\mathrm{d}x = -L_p\delta\bar{\varepsilon} \tag{A.6}$$

$$\int_{\mathcal{L}\setminus\mathcal{I}_p} \delta u'^2\,\mathrm{d}x \geqslant \frac{1}{L - L_p}\left(\int_{\mathcal{L}\setminus\mathcal{I}_p} \delta u'\,\mathrm{d}x\right)^2 = \frac{L_p^2}{L - L_p}\delta\bar{\varepsilon}^2 \tag{A.7}$$

Finally, the last integral in (A.3) can be estimated using the Wirtinger inequality, which is a one-dimensional case of the Poincaré inequality, with explicitly known optimal constant. The theorem is applicable to the zero-mean part of $\delta\kappa$ and implies that

$$\left(\frac{L_p}{2\pi}\right)^2\int_{\mathcal{I}_p} \delta\kappa'^2\,\mathrm{d}x \geqslant \int_{\mathcal{I}_p} (\delta\kappa - \delta\bar{\kappa})^2\,\mathrm{d}x = \int_{\mathcal{I}_p} \delta\kappa^2\,\mathrm{d}x - L_p\delta\bar{\kappa}^2 \tag{A.8}$$

Since $L_p/2\pi = l$, we get

$$\int_{\mathcal{I}_p} \left(l^2\delta\kappa'^2 - \delta\kappa^2\right)\mathrm{d}x \geqslant -L_p\delta\bar{\kappa}^2 \tag{A.9}$$

Substituting (A.4), (A.7) and (A.9) into (A.3), we obtain condition

$$\frac{L_p^2}{L - L_p}\delta\bar{\varepsilon}^2 + L_p(\delta\bar{\varepsilon} - \delta\bar{\kappa})^2 + \frac{H}{E}L_p\delta\bar{\kappa}^2 \geqslant 0 \tag{A.10}$$

which means that a certain quadratic form of variables $\delta\bar{\varepsilon}$ and $\delta\bar{\kappa}$ should be positive semidefinite. This leads to the following restrictions on the parameters:

$$\frac{L_p^2}{L - L_p} + L_p \geqslant 0 \;\Rightarrow\; L \geqslant L_p \tag{A.11}$$

$$L_p + \frac{H}{E}L_p \geqslant 0 \;\Rightarrow\; E + H \geqslant 0 \tag{A.12}$$

$$\left(\frac{L_p^2}{L - L_p} + L_p\right)\left(L_p + \frac{H}{E}L_p\right) - L_p^2 \geqslant 0 \;\Rightarrow\; \frac{L_p}{L} \geqslant -\frac{H}{E} \tag{A.13}$$

The first restriction corresponds to our tacit assumption that bar is longer than L_p, so that the full plastic zone can develop (for shorter bars, the analysis would have to be modified). The second restriction excludes snapback of the stress–strain diagram with no gradient effects. The third restriction is the most stringent one. It guarantees stability of the localized solution under displacement control and can be interpreted e.g. as a constraint on the maximum length of the bar (with respect to the characteristic length l, reflected by the plastic zone size $L_p = 2\pi l$). It is reassuring that this condition exactly coincides with the condition of negative slope of the post-peak load–displacement diagram. Indeed, the total elongation of the bar after the onset of plastic yielding can be expressed as a sum of the elastic and plastic parts:

$$u_{tot} = \int_{\mathcal{L}} \varepsilon\,\mathrm{d}x = \int_{\mathcal{L}} \frac{F}{EA}\,\mathrm{d}x + \int_{\mathcal{I}_p} \kappa\,\mathrm{d}x = \frac{FL}{EA} + u_p \tag{A.14}$$

Substituting the plastic strain distribution according to (25), which refers to the plastic zone $\mathcal{I}_p = (-\pi l, \pi l)$, the plastic part of elongation turns out to be

$$u_p = \int_{-\pi l}^{\pi l} \frac{F/A - \sigma_0}{H}\left(1 + \cos\frac{x}{l}\right)\mathrm{d}x = \frac{F/A - \sigma_0}{H}2\pi l = \frac{FL_p}{HA} - \frac{\sigma_0 L_p}{H} \tag{A.15}$$

The slope of the post-peak part of load–displacement diagram is the reciprocal value of the tangent structural compliance $L/EA + L_p/HA$. Snapback occurs if the tangent compliance (and thus also the tangent stiffness) is positive, i.e., if

$$\frac{L}{E} + \frac{L_p}{H} > 0 \tag{A.16}$$

This is of course equivalent to L_p/L, which holds if and only if the stability condition (A.13) is violated.

To illustrate the difference between the condition of nonnegative first variation and the (stronger) condition of a local minimum, let us recall that the localization problem for a uniform bar admits not only solutions with the plastic zone of size $L_p = 2\pi l$, but also solutions with L_p equal to integer multiples of $2\pi l$; see Section 2.1. For instance, a plastic strain distribution given by

$$\kappa(x) = \begin{cases} \frac{\sigma - \sigma_0}{H}\left(1 - \cos\frac{x}{l}\right) & \text{for } x \in \mathcal{I}_p = (-2\pi l, 2\pi l) \\ 0 & \text{for } x \notin \mathcal{I}_p \end{cases} \tag{A.17}$$

satisfies differential equation (23) as well as conditions $\kappa = 0$ and $\kappa' = 0$ at the boundary of $\mathcal{I}_p$, i.e., at points $\pm 2\pi l$. The length of the plastic zone, $L_p = 4\pi l$, is now the double of the minimum plastic zone length. Consider an admissible variation of κ in the form

$$\delta\kappa(x) = \begin{cases} c\,\mathrm{sgn}(x)\left(1 - \cos\frac{x}{l}\right) & \text{for } x \in \mathcal{I}_p \\ 0 & \text{for } x \notin \mathcal{I}_p \end{cases} \tag{A.18}$$

where c is an arbitrary constant, not exceeding in magnitude the positive constant $(\sigma - \sigma_0)/H$ (for larger $|c|$, the function $\kappa + \delta\kappa$ would not be non-negative and would not belong to the space of admissible functions V_κ defined in (14)). A simple calculation reveals that

$$\int_{\mathcal{I}_p} \left(l^2 \delta\kappa'^2 - \delta\kappa^2\right) \mathrm{d}x = \int_{-2\pi l}^{2\pi l} c^2 \left[\sin^2 \frac{x}{l} - \left(1 - \cos\frac{x}{l}\right)^2\right] \mathrm{d}x = -4c^2\pi l \tag{A.19}$$

Since the mean value of $\delta\kappa$ is zero, the displacement variation δu can be selected such that $\delta u' = \delta\kappa$, and then the first two integrals in (A.3) vanish. The third integral, evaluated in (A.19), is negative and is multiplied by a positive constant $-H/E$, and so the stability condition (A.3) is always violated. This proves that solutions with plastic zone exceeding the minimum length $2\pi l$ would be unstable, independently of the total bar length L.

Appendix B. General solution of Eq. (46)

For negative ξ, Eq. (46) can be written as

$$\kappa_n''(\xi) - \frac{1}{\lambda_g + \xi}\kappa_n'(\xi) + \kappa_n(\xi) = 1 - \phi - \frac{\phi}{\lambda_g}\xi \tag{B.1}$$

Using substitutions

$$\xi = \eta - \lambda_g \tag{B.2}$$

$$\kappa(\xi) = (\lambda_g + \xi) g(\lambda_g + \xi) = \eta g(\eta) \tag{B.3}$$

where g is a new unknown function and η is a shifted dimensionless spatial coordinate, we can convert (B.1) into

$$\eta^2 g''(\eta) + \eta g'(\eta) + (\eta^2 - 1) g(\eta) = \eta - \frac{\phi}{\lambda_g}\eta^2 \tag{B.4}$$

where primes denote derivatives with respect to η. The homogeneous counterpart of (B.4) is the Bessel equation of order $v = 1$, and its general solution can be written as

$$g_h(\eta) = C_1 \mathrm{J}_1(\eta) + C_2 \mathrm{Y}_1(\eta) \tag{B.5}$$

where C_1 and C_2 are arbitrary constants, and J_1 and Y_1 are respectively the Bessel functions of the first and second kind; see Korenev (2002).

Now we need to find a particular solution for the given right-hand side. It turns out that the expression on the left-hand side of (B.4) gives a multiple of η if g is set simply to $1/\eta$, and a multiple of η^2 if g is set to the Struve function $\mathbf{H}_1(x)$; see Korenev (2002). Therefore, we look for the particular solution in the form

$$\tilde{g}(\eta) = \frac{k_1}{\eta} + k_2 \mathbf{H}_1(\eta) \tag{B.6}$$

and substituting into the left-hand side of (B.4) we get the condition

$$k_1\eta + \frac{2k_2\eta^2}{\pi} = \eta - \frac{\phi}{\lambda_g}\eta^2 \tag{B.7}$$

from which $k_1 = 1$ and $k_2 = -\pi\phi/2\lambda_g$. Combining the particular solution $\tilde{g}$ given by (B.6) with the general solution of the homogeneous equation g_h given by (B.5) and substituting this back into (B.3), we get the general solution of (B.1) in the form

$$\kappa(\xi) = (\lambda_g + \xi)[C_1 \mathrm{J}_1(\lambda_g + \xi) + C_2 \mathrm{Y}_1(\lambda_g + \xi)] + 1 - \frac{\pi\phi}{2\lambda_g}(\lambda_g + \xi)\mathbf{H}_1(\lambda_g + \xi) \tag{B.8}$$

Recall that (B.1) is the specific form of (46) valid for $\xi < 0$. For $\xi > 0$, it is sufficient to replace $\lambda_g + \xi$ by $\lambda_g - \xi$. The resulting expression valid for both positive and negative ξ is given in (47).

Appendix C. General solution of Eq. (61)

The integrals in (65) and (66) are most conveniently evaluated if one introduces an auxiliary complex function

$$\widetilde{C}(\xi) = \widetilde{C}_1(\xi) - i\widetilde{C}_2(\xi) \tag{C.1}$$

Substituting from (65) and (66), one gets

$$\widetilde{C}(\xi) = \cos\xi - i\sin\xi + \phi \int \mathrm{e}^{-\xi^2/2\lambda_g^2}(\sin\xi + i\cos\xi)\,\mathrm{d}\xi = \mathrm{e}^{-i\xi} + i\phi \int \mathrm{e}^{-\xi^2/2\lambda_g^2}\mathrm{e}^{-i\xi}\,\mathrm{d}\xi \tag{C.2}$$

Since

$$-\frac{\xi^2}{2\lambda_g^2} - i\xi = -\frac{(\xi + i\lambda_g^2)^2 + \lambda_g^4}{2\lambda_g^2} = -\frac{(\xi + i\lambda_g^2)^2}{2\lambda_g^2} - \frac{\lambda_g^2}{2} \tag{C.3}$$

the last integral in (C.2) can be written as

$$\int \mathrm{e}^{-\xi^2/2\lambda_g^2}\mathrm{e}^{-i\xi}\,\mathrm{d}\xi = \mathrm{e}^{-\lambda_g^2/2}\int \exp\left(-\frac{(\xi + i\lambda_g^2)^2}{2\lambda_g^2}\right)\mathrm{d}\xi \tag{C.4}$$

and substitution $\xi = \sqrt{2}\lambda_g t - i\lambda_g^2$ leads to

$$\int \exp\left(-\frac{(\xi + i\lambda_g^2)^2}{2\lambda_g^2}\right)\mathrm{d}\xi = \sqrt{2}\lambda_g \int \mathrm{e}^{-t^2}\mathrm{d}t = \lambda_g\sqrt{\frac{\pi}{2}}\mathrm{erf}(t) = \lambda_g\sqrt{\frac{\pi}{2}}\mathrm{erf}\left(\frac{\xi + i\lambda_g^2}{\sqrt{2}\lambda_g}\right) \tag{C.5}$$

where erf is the "error function" defined by

$$\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}}\int \mathrm{e}^{-x^2}\,\mathrm{d}x, \quad \mathrm{erf}(0) = 0 \tag{C.6}$$

Combining (C.4) and (C.5) and substituting back into (C.2), we obtain

$$\widetilde{C}(\xi) = \mathrm{e}^{-i\xi} + i\phi\lambda_g\sqrt{\frac{\pi}{2}}\mathrm{e}^{-\lambda_g^2/2}\mathrm{erf}\left(\frac{\xi + i\lambda_g^2}{\sqrt{2}\lambda_g}\right) \tag{C.7}$$

Functions $\widetilde{C}_1$ and $\widetilde{C}_2$ could now be extracted from the real and imaginary part of the expression in (C.7). But this is not even necessary, because the particular solution of equation (61) given by formula (62) can be presented as

$$\tilde{\kappa}_n(\xi) = \mathrm{Re}\left[(\widetilde{C}_1(\xi) - i\widetilde{C}_2(\xi))(\cos\xi + i\sin\xi)\right] = \mathrm{Re}\left[\widetilde{C}(\xi)\mathrm{e}^{i\xi}\right] \tag{C.8}$$

where Re stands for the real part. Substituting for $\widetilde{C}(\xi)$ according to (C.7), we get the particular solution in the form

$$\tilde{\kappa}_n(\xi) = 1 + \phi\lambda_g\sqrt{\frac{\pi}{2}}\mathrm{e}^{-\lambda_g^2/2}\mathrm{Re}\left[i\,\mathrm{erf}\left(\frac{\xi + i\lambda_g^2}{\sqrt{2}\lambda_g}\right)\mathrm{e}^{i\xi}\right] \tag{C.9}$$

Recalling the definition of the Dawson function (Olver, 1997),

$$\mathrm{F}(x) = e^{-x^2}\int_0^x e^{t^2}\,\mathrm{d}t = -\frac{i\sqrt{\pi}}{2}e^{-x^2}\mathrm{erf}(ix) \tag{C.10}$$

and taking into account that $\overline{F(x)} = F(\bar{x})$ (with the bar denoting the complex conjugate), we can rewrite the result as

$$\tilde{\kappa}_n(\xi) = 1 - \phi\lambda_g \mathrm{e}^{-\xi^2/2\lambda_g^2}\frac{\sqrt{2}}{2}\left[F\left(\frac{\lambda_g^2 - i\xi}{\sqrt{2}\lambda_g}\right) + F\left(\frac{\lambda_g^2 + i\xi}{\sqrt{2}\lambda_g}\right)\right] \tag{C.11}$$

The general solution (67) of Eq. (61) is then obtained by adding a linear combination of functions $\cos\xi$ and $\sin\xi$ with arbitrary coefficients.

Appendix D. General solution of homogeneous form of Eq. (77)

The homogeneous counterpart of Eq. (77) can be written as

$$\kappa_n''(\xi) + \frac{\xi}{\lambda_g^2}\kappa_n'(\xi) + \kappa_n(\xi) = 0 \tag{D.1}$$

Defining a rescaled spatial variable $\eta = \xi/\lambda_g\sqrt{2}$ and expressing the unknown function as

$$\kappa_n(\xi) = \mathrm{e}^{-\frac{\xi^2}{2\lambda_g^2}} u\left(\frac{\xi}{\lambda_g\sqrt{2}}\right) = \mathrm{e}^{-\eta^2}u(\eta) \tag{D.2}$$

where u is a transformed unknown function, we can convert (D.1) into the so-called Hermite differential equation,

$$u''(\eta) - 2\eta u'(\eta) + 2\nu u(\eta) = 0 \tag{D.3}$$

where $\nu = \lambda_g^2 - 1$ is a real parameter (in our case larger than -1) and primes denote differentiation with respect to η.

The solution of the Hermite equation (D.3) can be constructed in terms of infinite power series

$$u(\eta) = \sum_{r=0}^{\infty} a_r\eta^r \tag{D.4}$$

Expressing the derivatives

$$u'(\eta) = \sum_{r=1}^{\infty} a_r r\eta^{r-1} \tag{D.5}$$

$$u''(\eta) = \sum_{s=2}^{\infty} a_s s(s-1)\eta^{s-2} \tag{D.6}$$

and substituting them into (D.3), we obtain

$$\sum_{s=2}^{\infty} a_s s(s-1)\eta^{s-2} - 2\sum_{r=1}^{\infty} a_r r\eta^r + 2\nu\sum_{r=0}^{\infty} a_r\eta^r = 0 \tag{D.7}$$

In the first sum, s can be replaced by $r+2$, with r running from 0 to infinity, and in the second sum, r can also run from 0 without changing the result (because of the presence of the factor r which annihilates the term with $r = 0$). The equation can thus be rewritten as

$$\sum_{r=0}^{\infty}[a_{r+2}(r+2)(r+1) - 2a_r r + 2\nu a_r]\eta^r = 0 \tag{D.8}$$

and the term in the square brackets must vanish for each individual value of $r = 0, 1, 2, \ldots$. This condition results into the recursive formula

$$a_{r+2} = \frac{2(r-\nu)}{(r+1)(r+2)}a_r, \quad r = 0, 1, 2, \ldots \tag{D.9}$$

Note that a_{r+2} is expressed in terms of a_r. It is therefore possible to select arbitrary values of a_0 and a_1, and then express all other coefficients with even subscripts in terms of a_0 and all other coefficients with odd subscripts in terms of a_1. Setting $a_0 = 1$ and $a_1 = 0$, or $a_0 = 0$ and $a_1 = 1$, leads to two linearly independent functions

$$u_1(\eta) = 1 - \frac{2\nu}{2!}\eta^2 + \frac{2^2\nu(\nu-2)}{4!}\eta^4 - \frac{2^3\nu(\nu-2)(\nu-4)}{6!}\eta^6 + \cdots \tag{D.10}$$

$$u_2(\eta) = \eta - \frac{2(\nu-1)}{3!}\eta^3 + \frac{2^2(\nu-1)(\nu-3)}{5!}\eta^5 + \cdots \tag{D.11}$$

and every solution of Eq. (D.3) can be expressed as their linear combination. The fundamental solutions can be conveniently expressed in terms of the so-called confluent hypergeometric function of the first kind, denoted as ${}_1F_1(\alpha;\gamma;x)$ (note that the symbol F has a left subscript and a right subscript), which is defined by the infinite series (Sneddon, 1956)

$${}_1F_1(\alpha;\gamma;x) = \sum_{r=0}^{\infty}\frac{(\alpha)_r}{(\gamma)_r}\frac{x^r}{r!} \tag{D.12}$$

Here, $(\bullet)_r$ is the so-called Pochhammer symbol, which can be expressed in terms of Euler's gamma function:

$$(\alpha)_r = \alpha(\alpha+1)\cdots(\alpha+r-1) = \frac{\Gamma(\alpha+r)}{\Gamma(\alpha)} \tag{D.13}$$

It is easy to verify by simple substitution that the fundamental solutions u_1 and u_2 from (D.10) and (D.11) can be rewritten as

$$u_1(\eta) = {}_1F_1(-\nu/2; 1/2; \eta^2) \tag{D.14}$$

$$u_2(\eta) = \eta\ {}_1F_1((1-\nu)/2; 3/2; \eta^2) \tag{D.15}$$

Substituting this into (D.2) and replacing ν by $\lambda_g^2 - 1$ and η by $\xi/\lambda_g\sqrt{2}$, we finally obtain two linearly independent solutions of (D.1) in the form

$$\kappa_{n,1}(\xi) = \mathrm{e}^{-\frac{\xi^2}{2\lambda_g^2}}\ {}_1\mathrm{F}_1\left(\frac{1-\lambda_g^2}{2}; \frac{1}{2}; \frac{\xi^2}{2\lambda_g^2}\right) \tag{D.16}$$

$$\kappa_{n,2}(\xi) = \mathrm{e}^{-\frac{\xi^2}{2\lambda_g^2}}\frac{\xi}{\sqrt{2}\lambda_g}\ {}_1\mathrm{F}_1\left(\frac{2-\lambda_g^2}{2}; \frac{3}{2}; \frac{\xi^2}{2\lambda_g^2}\right) \tag{D.17}$$

Note that function u_1 is even and u_2 is odd, and so $\kappa_{n,1}$ is even and $\kappa_{n,2}$ is odd. One useful consequence is that $\kappa_{n,1}'(0) = 0$.

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