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Conservation laws from any conformal transformations and the parameters for a sharp V-notch in plane elasticity

Shi Weichen *, Lu Lin

Department of Mechanical Engineering, Shanghai Maritime University, Haigang Avenue 1550, Shanghai 201306, PR China

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ABSTRACT

Based on the well known complex Kolosov–Muskhelishvili potentials, two new independent Lagrangian functions are presented and their variational problems lead to two independent harmonic equations, which are also the Navier's displacement equations in plane elasticity. By applying Noether's theorem to these Lagrangian functions, it is found that their symmetry-transformation in material space is a conformal transformation in planar Euclidean space. Since any analytic function is a conformal transformation in planar Euclidean space, the conservation law obtained from this kind of symmetry-transformation possesses universality and leads to a path-independent integral. By adjusting the conformal transformation or analytic function, a finite value can be obtained from calculating this kind of path-independent integral around a material point with any order singularity. By applying this path-independent integral to the tip of a sharp V-notch, unlike Rice's *J*-integral, the parameters of Mode I and II problems are found, which remain invariant because of path independence for a fixed notch opening angle. That is, these two parameters are equivalent to the notch stress intensity factors (NSIFs), and two examples are presented to show the application.

1. Introduction

Research on stress singularities in an elastic field plays an important role in estimating macro safety of engineering components. In order to establish the relationship between the stress intensity factor at a material point and the loading at far field and to understand the interaction among the material points with stress singularities, *J*-integral, *L*-integral and *M*-integral discovered independently by Eshelby (1951, 1956, 1970), Rice (1968a,b) and Cherepanov (1967, 1979) have been widely used. This is one of objectives in configuration mechanics since *J*-integral, *L*-integral and *M*-integral are path-independent (Gurtin, 1995, 1999; Kienzler and Herrmann, 2000). However, it is not easy to achieve this objective sometimes. For example, as shown in Fig. 1, Lazzarin et al. (2002) gave a result obtained from calculating Rice's *J*-integral around the tip of a sharp V-notch and we add a limit as follows

$$J = \lim_{r\to 0}\left[\frac{\beta_{1r}^2 \sin\gamma}{2\lambda_1 - 1}\frac{(K_I^N)^2}{E'}r^{2\lambda_1^{(1)}-1} + \frac{\beta_{2r}^2 \sin\gamma}{2\lambda_2 - 1}\frac{(K_{II}^N)^2}{E'}r^{2\lambda_1^{(2)}-1}\right], \qquad (1)$$

where β_{1r}, β_{2r} and γ are the parameters that depend on the notch opening angle $2(\pi-\alpha)$, $\lambda_1^{(1)}$ and $\lambda_1^{(2)}$ are the eigenvalues, E' is given by the Young's modulus E for plane stress or $E/(1-\nu^2)$ for plane strain, in which ν is the Poisson's ratio. Here, the NSIFs K_I^N and K_{II}^N were defined by Gross and Mendelson (1972)

$$K_I^N = \sqrt{2\pi}\lim_{r\to 0} r^{1-\lambda_1^{(1)}}\sigma_\theta(r,\theta=0), \quad K_{II}^N = \sqrt{2\pi}\lim_{r\to 0} r^{1-\lambda_1^{(2)}}\sigma_{r\theta}(r,\theta=0). \qquad (2)$$

Clearly, when the integration path of Rice's *J*-integral is taken to be a circle shrinking to the sharp V-notch tip with radius $r\to 0$, as shown in Fig. 1, the quantity J turns out to be zero because the eigenvalues $1/2 < \lambda_1^{(1)} < 1$ and $1/2 < \lambda_1^{(2)} < 1$ when the notch opening angle $2(\pi-\alpha) > 0$. That is, unlike a crack problem for which Rice's *J*-integral gives an energy release rate, Rice's *J*-integral cannot give a parameter directly for estimating safety of a sharp V-notch. In order to avoid this difficulty, some treatments have been made to define and discuss a new parameter J_V by Lazzarin et al. (2002), Livieri (2008) and Livieri and Tovo (2009).

One can know advances and some problems in the field of sharp V-notches reviewed by Berto and Lazzarin (2009) and Savruk and Kazberuk (2010). Especially, Savruk (1981, 1988) gave an approximate solution with higher accuracy for an infinite elastic plane with two symmetrical sharp V-notches, and its importance is the same as the closed-form solution of a crack in an infinite elastic plane. This is because a lot of essentially physical facts of a crack had been recognized based on this kind of solutions. Apart from this, it should be mentioned that there are a lot of problems with various kinds of elastic singularities reviewed by Carpinteri and Paggi (2009), for example, Flamant's problem (a normal force acting on a straight edge). Also, the multi-layered composites with

* Corresponding author. Tel.: +86 21 58855200x2606; fax: +86 21 58853909.
E-mail address: swc@shmtu.edu.cn (W. Shi).

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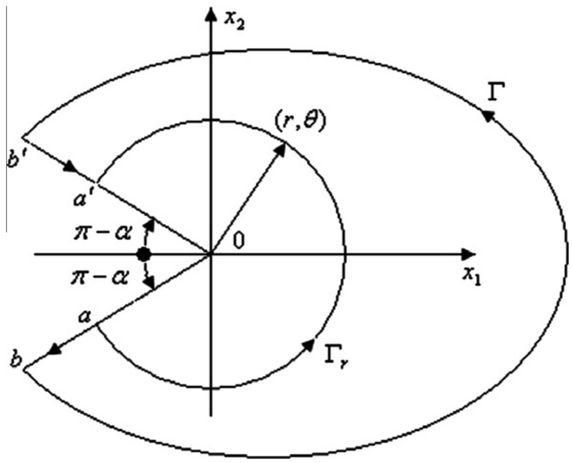

Fig. 1. Integration paths $a \to b$, Γ, $b' \to a'$ and Γ_r for a sharp V-notch and $|b'a'| = |ab| = r_b - r_a$, $|Oa'| = |Oa| = r_a$.

sharp V-notches have been investigated (Carpinteri et al., 2006). Therefore, it is necessary for us to find out new conservation laws or path-independent integrals which can be used for a material point with any order singularity.

Theoretically speaking, both the mathematical form of field equations and the quantity of strain energy density are unchanged if a translation of coordinates is made, which leads to the path independence of Rice's J-integral when Noether's theorem is applied (Fletcher, 1976; Olver, 1984a,b; Shi, 2005; Shi et al., 2006). Moreover, the path independence of Rice's J-integral means the physical and/or mathematical invariance and confirms that a fracture parameter can be obtained in plane elasticity. In terms of convenience, the result of calculating Rice's J-integral around the tip of a crack is a nonzero constant (energy release rate equivalent to SIF), which is unchanged when the two integral endpoints are arbitrarily chosen respectively along the traction-free crack surfaces (Rice, 1968a,b; Cherepanov, 1967, 1979; Honein and Herrmann, 1997; Shi, 2003, 2011). This paper will focus on these two points.

In Section 2, by analyzing the physical and mathematical significance of complex Kolosov–Muskhelishvili potential $\phi(z)$, two new independent Lagrangian functions are presented. In Section 3, by applying Noether's theorem (Noether, 1918) to these Lagrangian functions, it is found that any conformal transformations are symmetry-transformations for getting the conservation laws in material space. Since the conformal transformation in planar Euclidean space can be expressed in terms of an analytic function, the obtained conservation law not only possesses universality but can be changed by adjusting the conformal transformation as well. Section 4 provides applications to the sharp V-notch problem, for which the two parameters of Mode I and II problems are presented for estimating safety based on the elastic theory. A numerical example is also presented for calculating the NSIF.

2. New Lagrangian functions

2.1. The significance of complex Kolosov–Muskhelishvili potential $\phi(z)$

It is well known that the displacements (u_1, u_2), the resultant force functions (X, Y) and the stresses $(\sigma_{11}, \sigma_{22}, \sigma_{12})$ can be expressed in terms of two complex potentials $\phi(z)$ and $\psi(z)$ (Muskhelishvili, 1963)

$$2G(u_1 + iu_2) = \kappa\phi(z) - z\overline{\phi'(z)} - \overline{\psi(z)}, \tag{3}$$

$$-Y + iX = \phi(z) + z\overline{\phi'(z)} + \overline{\psi(z)}, \tag{4}$$

$$\sigma_{11} + \sigma_{22} = 2[\Phi(z) + \overline{\Phi(z)}], \tag{5}$$

$$\sigma_{22} - \sigma_{11} + 2i\sigma_{12} = 2[\bar{z}\Phi'(z) + \Psi(z)], \tag{6}$$

where $\kappa = 3 - 4\nu$ for plane strain, $\kappa = (3-\nu)/(1+\nu)$ for generalized plane stress, G is the shear modulus and ν Poisson's ratio. Here, there are the relations $\Phi(z) = \phi'(z)$ and $\Psi(z) = \psi'(z)$. Conservation integrals in the sense of Noether's theorem for an analytic function have been presented (Shi, 2012) and here we consider a special case for plane elasticity in details.

Apparently, adding Eqs. (3) to (4) and performing some algebraic manipulation, we know

$$\phi(z) = \frac{2G}{\kappa+1}\left[u_1 - \frac{Y}{2G} + i\left(u_2 + \frac{X}{2G}\right)\right]. \tag{7}$$

Moreover, by computing the partial derivative of expression (3) with respect to x_1 and x_2, respectively, it is known that

$$\kappa\phi'(z) - \overline{\phi'(z)} = G[u_{1,1} + u_{2,2} + i(u_{2,1} - u_{1,2})], \tag{8a}$$

$$\kappa\overline{\phi'(z)} - \phi'(z) = G[u_{1,1} + u_{2,2} - i(u_{2,1} - u_{1,2})]. \tag{8b}$$

Solving Eqs. (8a) and (8b) algebraically, we find

$$\Phi(z) = \phi'(z) = G\left[\frac{1}{\kappa-1}(u_{1,1} + u_{2,2}) + \frac{i}{\kappa+1}(u_{2,1} - u_{1,2})\right]. \tag{9}$$

Clearly, the real and imaginary parts of $\Phi(z) = \phi'(z)$ represent the first strain invariant ε_1 and rotation ω in plane elasticity, respectively,

$$\varepsilon_1 = u_{1,1} + u_{2,2}, \quad \omega = \frac{1}{2}(u_{2,1} - u_{1,2}). \tag{10}$$

2.2. Independent harmonic equations

Since the complex potential $\phi(z)$ possesses real and imaginary parts (7), they may be written as

$$\phi(z) = G(U_1 + iU_2), \tag{11}$$

$$U_1 = \frac{2}{\kappa+1}\left(u_1 - \frac{Y}{2G}\right), \quad U_2 = \frac{2}{\kappa+1}\left(u_2 + \frac{X}{2G}\right), \tag{12}$$

which possess the dimension of displacements. Based on Cauchy–Riemann equations $U_{1,1} = U_{2,2}$ and $U_{1,2} = -U_{2,1}$ in the theory of analytic functions, the real and imaginary parts of $\Phi(z) = \phi'(z)$ can be expressed by using (9), (11) and (12) as follows

$$\begin{aligned}\Phi(z) = \phi'(z) &= G(U_{1,1} + iU_{2,1}) = G(U_{2,2} - iU_{1,2}) \\ &= G\left[\tfrac{1}{\kappa-1}(u_{1,1} + u_{2,2}) + \tfrac{i}{\kappa+1}(u_{2,1} - u_{1,2})\right],\end{aligned} \tag{13}$$

which gives

$$U_{1,1} = U_{2,2} = \frac{1}{G}\mathrm{Re}\Phi(z) = \frac{1}{\kappa-1}(u_{1,1} + u_{2,2}), \tag{14a}$$

$$U_{2,1} = -U_{1,2} = \frac{1}{G}\mathrm{Im}\Phi(z) = \frac{1}{\kappa+1}(u_{2,1} - u_{1,2}). \tag{14b}$$

Now, we may define new quantities

$$\Sigma_{11} = \Sigma_{22} = \mathrm{Re}\Phi(z) = GU_{1,1} = GU_{2,2} = \frac{G}{\kappa-1}(u_{1,1} + u_{2,2}), \tag{15a}$$

$$\Sigma_{21} = -\Sigma_{12} = \mathrm{Im}\Phi(z) = GU_{2,1} = -GU_{1,2} = \frac{G}{\kappa+1}(u_{2,1} - u_{1,2}), \tag{15b}$$

which possess the dimension of stress. Then, since Cauchy–Riemann Eqs. (14a) and (14b) lead to independent harmonic equations $U_{1,kk} = U_{2,kk} = 0$, it can be shown by using (15a) and (15b) that

$$\Sigma_{1k,k} = GU_{1,kk} = G\left[\frac{1}{\kappa-1}(u_{1,1}+u_{2,2})_{,1} + \frac{1}{\kappa+1}(u_{1,2}-u_{2,1})_{,2}\right] = 0, \tag{16a}$$

$$\Sigma_{2k,k} = GU_{2,kk} = G\left[\frac{1}{\kappa+1}(u_{2,1}-u_{1,2})_{,1} + \frac{1}{\kappa-1}(u_{1,1}+u_{2,2})_{,2}\right] = 0. \tag{16b}$$

In this paper, the Latin indices run from 1 to 2 for two-dimensional problem and the summation convention for repeated indices is implied. Clearly, Eqs. (16a) and (16b) indicate that not only the independent functions U_1 and U_2 satisfy a harmonic equation, respectively, but also the two harmonic equations are the Navier's displacement equations in plane elasticity.

2.3. Independent Lagrangian functions

We may define new independent Lagrangian functions with the help of (14a), (14b), (15a), and (15b) as follows

$$L_1 = \frac{1}{2}\Sigma_{1k}U_{1,k} = \frac{G}{2}U_{1,k}U_{1,k}, \tag{17a}$$

$$L_2 = \frac{1}{2}\Sigma_{2k}U_{2,k} = \frac{G}{2}U_{2,k}U_{2,k}, \tag{17b}$$

and computing the derivative with respect to $U_{1,k}$ and $U_{2,k}$, we know

$$\Sigma_{1k} = \frac{\partial L_1}{\partial U_{1,k}} = GU_{1,k}, \tag{18a}$$

$$\Sigma_{2k} = \frac{\partial L_2}{\partial U_{2,k}} = GU_{2,k}. \tag{18b}$$

It is well known that for any physical system, when its Lagrangian function is known, its conservation laws can be obtained by using Noether's theorem. Mathematically speaking, according to the theory of analytic functions, both U_1 and U_2 are the independent functions (11), (12), (14a), and (14b), for which there exist their independent variations δU_1 and δU_2. Moreover, variational problems of L_1 and L_2 with the independent functions U_1 and U_2, respectively, lead to the two harmonic equations or Navier's displacement Eqs. (16a) and (16b), and the related nature, homogenous boundary conditions

$$\delta\int_A L_1 dA = \oint_\Gamma (\Sigma_{1k}\delta U_1)n_k d\Gamma - \int_A \Sigma_{1k,k}\delta U_1 dA, \tag{19a}$$

$$\delta\int_A L_2 dA = \oint_\Gamma (\Sigma_{2k}\delta U_2)n_k d\Gamma - \int_A \Sigma_{2k,k}\delta U_2 dA. \tag{19b}$$

This proves that the new Lagrangian functions (17a) and (17b) satisfy the requirement of Noether's theorem (Noether, 1918) although a new energy principle for plane elasticity has not been exploited.

3. Conservation laws

3.1. Symmetry-transformations of the new Lagrangian functions

First of all, we consider the Lagrangian function L_2 in (17b) with the independent harmonic function U_2. Its symmetry-transformation expressed in an infinitesimal form of one parameter transformation group can be written as

$$\hat{U}_2 = U_2 + \varepsilon\eta(x_k, U_2), \quad \hat{x}_i = x_i + \varepsilon\zeta_i(x_k, U_2), \tag{20}$$

where $\eta = \eta(x_k, U_2)$ and $\zeta_i = \zeta_i(x_k, U_2)$ are unknown functions to be determined, and ε is an infinitesimal group parameter. For the first order variational problem (19b), all possible non-trivial conservation laws derived from the invariance of L_2 can be formulated as (Olver, 1993)

$$D_i(\eta\Sigma_{2i} + \zeta_k S_{ik}) = (\eta - \zeta_i U_{2,i})\Sigma_{2k,k} = 0, \tag{21}$$

$$S_{ik} = L_2\delta_{ik} - \Sigma_{2i}U_{2,k} = \frac{G}{2}(U_{2,p}U_{2,p}\delta_{ik} - 2U_{2,i}U_{2,k}), \tag{22}$$

where S_{ik} is the so-called generalized energy–momentum tensor because the function U_2 given in (12) is different from usual displacements in elasticity, $(\eta - \zeta_i U_{2,i})$ represents the characteristic of a conservation law, δ_{ik} is the Kronecker delta and

$$D_i = \frac{\partial}{\partial x_i} + U_{2,i}\frac{\partial}{\partial U_2} + U_{2,pi}\frac{\partial}{\partial U_{2,p}} + \cdots \tag{23}$$

In order to find functions η and ζ_k in Eq. (21), we expand the left-hand side of Eq. (21) as follows

$$G\left\{\frac{\partial\eta}{\partial x_k}U_{2,k} + \left[\left(\frac{\partial\eta}{\partial U_2} + \frac{1}{2}\frac{\partial\zeta_p}{\partial x_p}\right)\delta_{ik} - \frac{\partial\zeta_k}{\partial x_i}\right]U_{2,k}U_{2,i} - \frac{1}{2}\frac{\partial\zeta_k}{\partial U_2}U_{2,k}U_{2,i}U_{2,i}\right\} = 0. \tag{24}$$

Since Noether's theorem (Noether, 1918) demands that Eq. (24) vanish identically, the coefficients of all the independent linear, quadratic and cubic terms of $U_{2,k}$ must be equal to zero. This requirement gives the determining equations

$$U_{2,k}: \quad \frac{\partial\eta}{\partial x_k} = 0; \tag{25}$$

$$U_{2,k}U_{2,i}: \quad 2\left(\frac{\partial\eta}{\partial U_2} + \frac{1}{2}\frac{\partial\zeta_p}{\partial x_p}\right)\delta_{ik} - \frac{\partial\zeta_k}{\partial x_i} - \frac{\partial\zeta_i}{\partial x_k} = 0; \tag{26}$$

$$U_{2,k}U_{2,i}U_{2,i}: \quad \frac{\partial\zeta_k}{\partial U_2} = 0. \tag{27}$$

The elementary analysis of Eqs. (25)–(27) gives

$$\eta = \alpha_0, \tag{28}$$

$$\frac{\partial\zeta_1(x_1,x_2)}{\partial x_1} = \frac{\partial\zeta_2(x_1,x_2)}{\partial x_2}, \frac{\partial\zeta_1(x_1,x_2)}{\partial x_2} = -\frac{\partial\zeta_2(x_1,x_2)}{\partial x_1}, \tag{29}$$

where α_0 is an independent arbitrary constant. Since functions $\zeta_i = \zeta_i(x_1, x_2)$ must satisfy Cauchy–Riemann Eq. (29), they can be expressed in terms of an analytic function

$$\zeta_1 + i\zeta_2 = \zeta(z). \tag{30}$$

This indicates that any conformal transformations (30) in two-dimensional Euclidean space are the symmetry-transformations of Lagrangian function L_2 in (17b).

3.2. Adjustable conservation law

By considering the independence of arbitrary constant (28) and the conformal transformation (30), the conservation laws follow from (21) as follows:

(i) Zero position change of function U_2 ($\alpha_0 \neq 0$)

$$D_i\Sigma_{2i} = 0; \tag{31}$$

(ii) Conformal transformation ($\zeta_1 + i\zeta_2 = \zeta(z) \neq 0$)

$$D_i(\zeta_k S_{ik}) = 0. \tag{32}$$

Also, the conservation law (31) is a harmonic equation which is one of the Navier's displacement equations (16b) in two-dimensional space. The expression (32) belongs to a conservation law in material space according to the classification indicated by Herrmann (1981).

Now, since the divergence-free expression (32) does not depend on a material coefficient G illustrated in (24), in order to obtain the energy release rate of crack extension, we define a new generalized energy–momentum tensor

$$T_{ik} = \frac{8G}{E'} S_{ik}, \tag{33}$$

where $E' = E$ for plane stress and $E' = E/(1-\nu^2)$ for plane strain. Then, with the help of (14a), (14b), (15a), (15b), (17b), and (22), the generalized energy–momentum tensor T_{ik} and conserved quantities P_i can be written as

$$\begin{aligned} T_{11} &= -T_{22} = \tfrac{4}{E'}[\mathrm{Re}\Phi(z)]^2 = \tfrac{4G^2}{E'}[U_{2,2}^2 - U_{2,1}^2], \\ T_{12} &= T_{21} = -\tfrac{4}{E'}[\mathrm{Im}\Phi(z)]^2 = -\tfrac{8G^2}{E'}U_{2,1}U_{2,2}, \end{aligned} \tag{34}$$

$$\begin{aligned} P_1 - iP_2 &= \zeta_k T_{1k} - i\zeta_k T_{2k} = \tfrac{4}{E'}\zeta(z)[\Phi(z)]^2 \\ &= \tfrac{4G^2}{E'}\{[\zeta_1(U_{2,2}^2 - U_{2,1}^2) - 2\zeta_2 U_{2,1}U_{2,2}] + i[\zeta_2(U_{2,2}^2 - U_{2,1}^2) \\ &\quad + 2\zeta_1 U_{2,1}U_{2,2}]\}. \end{aligned} \tag{35}$$

The conservation law (32) turns out to be

$$D_1P_1 + D_2P_2 = 0. \tag{36}$$

Clearly, the mathematical form of conservation law (36) is one of Cauchy-Riemann equations illustrated in (35), so that any analytic function replacing $[\Phi(z)]^2$ in (35) will lead to a divergence-free expression (36). On the other hand, the integral form of conservation law (36) can be written as

$$\mathrm{Im}\oint_\Gamma \frac{4}{E'}\zeta(z)[\Phi(z)]^2 dz = 0. \tag{37}$$

This expression not only means the path independence but also indicates that one can adjust the conformal transformation $\zeta(z)$ to change the conserved quantities (35) for obtaining a significative result. This is because $\oint_\Gamma z^{-1}dz = 2\pi i$ and $\oint_\Gamma z^n dz = 0$ when $n \neq -1$. It should be mentioned that when applying Noether's theorem (Noether, 1918) to the Lagrange function L_1 in (17a), we will obtain the same conservation law as (36) and the only difference is plus and minus signs of expressions (34) and (35).

3.3. Universal conservation law

According to Liouville theorem (Logan, 1977), there exist only three conformal transformations in three-dimensional Euclidean space, which are the translation, rotation and scale change of coordinates. In two-dimensional Euclidean space, any transformation satisfying Cauchy–Riemann equations in the theory of analytic functions is a conformal transformation. This kind of conformal transformations also includes the translation, rotation and scale change of planar coordinates and can be expressed as

$$\zeta_k = Ax_k + e_{k3p}\Omega_3 x_p + C_k, \tag{38}$$

where A, Ω_3 and C_k are the independent arbitrary constants. By using expression (38) with the help of expressions (30), (34), and (35), because of the independence of A, Ω_3 and C_k, the conservation law (36) is split up into the following conservation laws:

(i) Coordinate translation ($C_k \neq 0$)

$$D_i T_{ik} = 0; \tag{39}$$

(ii) Coordinate rotation ($\Omega_3 \neq 0$)

$$D_i(e_{k3p}x_pT_{ik}) = 0; \tag{40}$$

(iii) Scale change of coordinates ($A \neq 0$)

$$D_i(x_kT_{ik}) = 0. \tag{41}$$

Clearly, these conservation laws (39)–(41) root in the translation, rotation and scale change of coordinates, whose manner is similar to those for getting J-integral, L-integral and M-integral in elasticity (Fletcher, 1976; Maugin, 1993). Therefore, in the sense of that the conservation law (36) or (37) includes any conformal transformations in plane elasticity, it possesses universality.

4. Application

4.1. Path-independent integral for a sharp V-notch

For a sharp V-notch shown in Fig. 1, a path-independent integral can be written by divergence-free expression (36) as follows

$$\begin{aligned} SW = &\lim_{r\to 0}\int_{\Gamma_r} P_k n_k r d\theta = \int_{\text{Point} \quad a\to b} (P_k n_k)^{LS} d\Gamma \\ &+ \int_\Gamma P_k n_k d\Gamma + \int_{\text{Point} \quad b'\to a'} (P_k n_k)^{US} d\Gamma, \end{aligned} \tag{42}$$

where the unit normal vector $\vec{n} = (n_1, n_2)$ points to the right-hand side of the paths $a \to b$, Γ, $b' \to a'$, Γ_r, and $|b'a'| = |ab| = r_b - r_a$ and $|Oa'| = |Oa| = r_a$. Since any conformal transformation is a symmetry-transformation for getting a path-independent integral (42), we use the symbol SW – integral for plane elasticity, which is the same symbol for a longitudinal shear problem (Shi, 2011).

Based on the works of Carpenter (1984), Gross and Mendelson (1972), Karp and Karal (1962) and Williams (1952), due to the traction-free condition along the lower and upper surfaces shown in Fig. 1, the series expansions of complex Kolosov–Muskhelishvili potential $\phi(z)$ for Mode I and II problems near the tip of a sharp V-notch are given in Appendix. Then, we obtain from (A6) and (A7) that

$$\Phi_1(z) = \phi_1'(z) = \sum_{m=1}^{\infty} a_m^{(1)}(\varsigma_m^{(1)} + i\eta_m^{(1)}) z^{\varsigma_m^{(1)} - 1 + i\eta_m^{(1)}}, \tag{43a}$$

$$\Phi_2(z) = \phi_2'(z) = i\sum_{m=1}^{\infty} a_m^{(2)}(\varsigma_m^{(2)} + i\eta_m^{(2)}) z^{\varsigma_m^{(2)} - 1 + i\eta_m^{(2)}}. \tag{43b}$$

It should be mentioned that only positive $\varsigma_m^{(1)}$ and $\varsigma_m^{(2)}$ are physically meaningful. According to the definitions of NSIFs in (2), when $m = 1$, $\eta_1^{(1)} = \eta_1^{(2)} = 0$, and K_I^N and K_{II}^N are associated with $a_1^{(1)}$ and $a_1^{(2)}$ as follows

$$a_1^{(1)} = \frac{1}{\lambda_1^{(1)}[1 + \lambda_1^{(1)} - \cos(2\lambda_1^{(1)}\alpha) - \lambda_1^{(1)}\cos(2\alpha)]}\frac{K_I^N}{\sqrt{2\pi}}, \tag{44a}$$

$$a_1^{(2)} = \frac{1}{\lambda_1^{(2)}[-1 + \lambda_1^{(2)} + \cos(2\lambda_1^{(2)}\alpha) - \lambda_1^{(2)}\cos(2\alpha)]}\frac{K_{II}^N}{\sqrt{2\pi}}, \tag{44b}$$

where $\lambda_1^{(1)} = \varsigma_1^{(1)}$ and $\lambda_1^{(2)} = \varsigma_1^{(2)}$ are the lowest eigenvalues for Mode I and II problems, respectively. By substituting two expressions (43a) and (43b) into expression (35), respectively, and taking $\zeta(z) = z^{1-2\varsigma_1^{(1)}}$ for Mode I problem and $\zeta(z) = -z^{1-2\varsigma_1^{(2)}}$ for Mode II problem, the conserved quantities (35) may be written as

$$\begin{aligned} P_1^{(I)} - iP_2^{(I)} &= \frac{4}{E'}\zeta(z)[\Phi_1(z)]^2 \\ &= \frac{4}{E'}\left(H_{11}^{(1)}z^{-1} + \sum_{m+n=3}^{\infty} H_{mn}^{(1)} z^{\varsigma_m^{(1)} + \varsigma_n^{(1)} - 2\varsigma_1^{(1)} - 1}\right), \end{aligned} \tag{45a}$$

$$\begin{aligned} P_1^{(II)} - iP_2^{(II)} &= \frac{4}{E'}\zeta(z)[\Phi_2(z)]^2 \\ &= \frac{4}{E'}\left(H_{11}^{(2)}z^{-1} + \sum_{m+n=3}^{\infty} H_{mn}^{(2)} z^{\varsigma_m^{(2)} + \varsigma_n^{(2)} - 2\varsigma_1^{(2)} - 1}\right), \end{aligned} \tag{45b}$$

where

$$H^{(1)}_{mn} = a^{(1)}_m a^{(1)}_n (\varsigma^{(1)}_m + i\eta^{(1)}_m)(\varsigma^{(1)}_n + i\eta^{(1)}_n) z^{i(\eta^{(1)}_m + \eta^{(1)}_n)}, \tag{45c}$$

$$H^{(2)}_{mn} = a^{(2)}_m a^{(2)}_n (\varsigma^{(2)}_m + i\eta^{(2)}_m)(\varsigma^{(2)}_n + i\eta^{(2)}_n) z^{i(\eta^{(2)}_m + \eta^{(2)}_n)}. \tag{45d}$$

It is worth noting that $\varsigma^{(1)}_m + \varsigma^{(1)}_n - 2\varsigma^{(1)}_1 \geqslant 0$ and $\varsigma^{(2)}_m + \varsigma^{(2)}_n - 2\varsigma^{(2)}_1 \geqslant 0$ with $(m, n = 1, 2, \ldots)$. Since $z^{i(\eta^{(k)}_m + \eta^{(k)}_n)} = e^{-(\eta^{(k)}_m + \eta^{(k)}_n)\theta} \{\cos[(\eta^{(k)}_m + \eta^{(k)}_n)\ln r] + i\sin[(\eta^{(k)}_m + \eta^{(k)}_n)\ln r]\}$ with $(k = 1, 2)$, and $\eta^{(k)}_m \geqslant 0$ and $\eta^{(k)}_n \geqslant 0$ are always taken, we know that

$$|H^{(k)}_{mn}| \leqslant |H^{(k)}_{mn}|_{\max} = |a^{(k)}_m a^{(k)}_n| e^{(\eta^{(k)}_m + \eta^{(k)}_n)\alpha} \sqrt{\varsigma^2_{m^{(k)}} + \eta^2_{m^{(k)}}} \sqrt{\varsigma^2_{n^{(k)}} + \eta^2_{n^{(k)}}}, \quad (k = 1, 2). \tag{45e}$$

Firstly, when calculating the integral (42) along paths $a \to b$ and $b' \to a'$, as shown in Fig. 1, it can be derived from (45a) and (45b) that

$$\int_{\text{Point} \quad a \to b} (P^{(I)}_k n_k)^{LS} d\Gamma + \int_{\text{Point} \quad b' \to a'} (P^{(I)}_k n_k)^{US} d\Gamma = \mathrm{Im}\left[\int_{r_a}^{r_b} (P^{(I)}_1 - iP^{(I)}_2) dz + \int_{r_b}^{r_a} (P^{(I)}_1 - iP^{(I)}_2) dz\right], \tag{46a}$$

$$\int_{\text{Point} \quad a \to b} (P^{(II)}_k n_k)^{LS} d\Gamma + \int_{\text{Point} \quad b' \to a'} (P^{(II)}_k n_k)^{US} d\Gamma = \mathrm{Im}\left[\int_{r_a}^{r_b} (P^{(II)}_1 - iP^{(II)}_2) dz + \int_{r_b}^{r_a} (P^{(II)}_1 - iP^{(II)}_2) dz\right], \tag{46b}$$

and the differential forms $dz = dx_1 + itg\beta dx_2 = dx_1 + itg(\pi - \alpha)dx_2$ and $dz = dx_1 - itg\beta dx_2 = dx_1 - itg(\pi - \alpha)dx_2$ along the lower and upper surfaces should be used, respectively. Considering $\cos\beta dr = \cos(\pi - \alpha)dr = dx_1$ and substituting $\theta = \alpha$ and/or $\theta = -\alpha$ into expressions (45a) and (45b) with $z = re^{i\theta}$, respectively, we know from (46a), (46b), and (45e) that

$$\left|\int_{\text{Point} \quad a \to b} (P^{(I)}_k n_k)^{LS} d\Gamma + \int_{\text{Point} \quad b' \to a'} (P^{(I)}_k n_k)^{US} d\Gamma\right| \leqslant \frac{8}{E'} \sum_{m+n=3}^{\infty} \frac{|H^{(1)}_{mn}|_{\max} \Delta r^{\varsigma^{(1)}_m + \varsigma^{(1)}_n - 2\varsigma^{(1)}_1}}{\varsigma^{(1)}_m + \varsigma^{(1)}_n - 2\varsigma^{(1)}_1} |\sin(\varsigma^{(1)}_m + \varsigma^{(1)}_n - 2\varsigma^{(1)}_1)\alpha|, \tag{47a}$$

$$\left|\int_{\text{Point} \quad a \to b} (P^{(II)}_k n_k)^{LS} d\Gamma + \int_{\text{Point} \quad b' \to a'} (P^{(II)}_k n_k)^{US} d\Gamma\right| \leqslant \frac{8}{E'} \sum_{m+n=3}^{\infty} \frac{|H^{(2)}_{mn}|_{\max} \Delta r^{\varsigma^{(2)}_m + \varsigma^{(2)}_n - 2\varsigma^{(2)}_1}}{\varsigma^{(2)}_m + \varsigma^{(2)}_n - 2\varsigma^{(2)}_1} |\sin(\varsigma^{(2)}_m + \varsigma^{(2)}_n - 2\varsigma^{(2)}_1)\alpha|, \tag{47b}$$

$$\Delta r^{\varsigma^{(1)}_m + \varsigma^{(1)}_n - 2\varsigma^{(1)}_1} = r_b^{\varsigma^{(1)}_m + \varsigma^{(1)}_n - 2\varsigma^{(1)}_1} - r_a^{\varsigma^{(1)}_m + \varsigma^{(1)}_n - 2\varsigma^{(1)}_1}, \tag{48a}$$

$$\Delta r^{\varsigma^{(2)}_m + \varsigma^{(2)}_n - 2\varsigma^{(2)}_1} = r_b^{\varsigma^{(2)}_m + \varsigma^{(2)}_n - 2\varsigma^{(2)}_1} - r_a^{\varsigma^{(2)}_m + \varsigma^{(2)}_n - 2\varsigma^{(2)}_1}. \tag{48b}$$

Secondly, by using the conserved quantities (45a) and (45b), respectively, it can be derived from (42), (44a), and (44b) that

$$SW_I = \lim_{r \to 0} \int_{\Gamma_r} P^{(I)}_i n_i r d\theta = \frac{4\alpha}{\pi[1 + \lambda^{(1)}_1 - \cos(2\lambda^{(1)}_1 \alpha) - \lambda^{(1)}_1 \cos(2\alpha)]^2} \frac{(K^N_I)^2}{E'} \tag{49a}$$

for Mode I problem, and

$$SW_{II} = \lim_{r \to 0} \int_{\Gamma_r} P^{(II)}_i n_i r d\theta = \frac{4\alpha}{\pi[-1 + \lambda^{(2)}_1 + \cos(2\lambda^{(2)}_1 \alpha) - \lambda^{(2)}_1 \cos(2\alpha)]^2} \frac{(K^N_{II})^2}{E'} \tag{49b}$$

for Mode II problem. When the sharp V-notch becomes a crack $\alpha \to \pi$, $\lambda^{(1)}_1 \to 1/2$ and $\lambda^{(2)}_1 \to 1/2$, expressions (49a) and (49b) are reduced to the energy release rates of Mode I and II crack extensions

$$J_I = SW_I = \lim_{r \to 0} \int_{\Gamma_r} P^{(I)}_i n_i r d\theta = \frac{(K_I)^2}{E'}, \tag{50a}$$

$$J_{II} = SW_I = \lim_{r \to 0} \int_{\Gamma_r} P^{(II)}_i n_i r d\theta = \frac{(K_{II})^2}{E'}. \tag{50b}$$

Moreover, the relationships between the parameters (49a) and (50a) or (49b) and (50b) can be written as follows

$$SW_I = \frac{4\alpha}{\pi[1 + \lambda^{(1)}_1 - \cos(2\lambda^{(1)}_1 \alpha) - \lambda^{(1)}_1 \cos(2\alpha)]^2} \left(\frac{K^N_I}{K_I}\right)^2 J_I, \tag{50c}$$

$$SW_{II} = \frac{4\alpha}{\pi[-1 + \lambda^{(2)}_1 + \cos(2\lambda^{(2)}_1 \alpha) - \lambda^{(2)}_1 \cos(2\alpha)]^2} \left(\frac{K^N_{II}}{K_{II}}\right)^2 J_{II}. \tag{50d}$$

Apparently, SW – integral (49a) and (49b) possess a kind of physical invariance since their path independence always holds within a planar elastic field for any given fixed notch opening angle $2(\pi - \alpha)$. Physical meaning is expressed by (50c) and (50d) since J_I and J_{II} are the crack extension forces. Therefore, SW – integral (49a) and (49b) are the physical parameters for application to a sharp V-notch in Mode I and II problems based on the elastic theory. Moreover, expressions (47a)-(48b) indicate that the theoretical accuracy will be very higher when the integration path of SW – integral (49a) and (49b) is taken to be a circle closing to the sharp V-notch tip.

4.2. An elastic plane with two symmetrical sharp V-notches

Let's consider an elastic plane with two symmetrical sharp V-notches subjected to uniform tension at infinity, as shown in Fig. 2. Savruk (1981, 1988) had solved this problem by using the singular integral equations method and the NSIF K^N_I can be estimated by

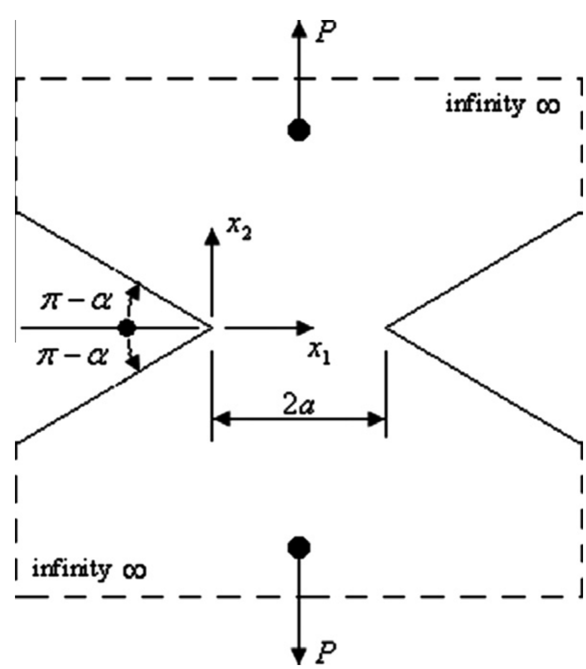

Fig. 2. An infinite elastic plane with two symmetrical sharp V-notches subjected to uniform tension at infinity.

$$K_I^N = \frac{F_I^V P}{\sqrt{\pi} a^{1-\lambda_1}} = \frac{1 - 0.9134\lambda_1}{1 + 0.4138\lambda_1}\sqrt{\frac{\pi}{2}}\frac{P}{a^{1-\lambda_1}}, \tag{51}$$

$$\lambda_1 \approx 1.247\cos\beta - 1.312\cos^2\beta + 0.8532\cos^3\beta - 0.2882\cos^4\beta, \quad (\beta = \pi - \alpha), \tag{52}$$

where $\lambda_1^{(1)} = 1 - \lambda_1$, F_I^V is the numerical value of dimensionless NSIF, P is the force per unite thickness and $2a$ is the width of cross connection between two notch vertexes. Savruk and Kazberuk (2010) also pointed out that relative error of (51) is less than 0.5% for the entire range of the parameter λ_1 and the maximum absolute error of the numerical values from Eq. (52) is below 0.1%. When the notch opening angle $2(\pi - \alpha) \to 0$, $F_I^V \to 1$ and expression (51) becomes a stress intensity factor (Tada et al., 1973)

$$K_I = P/\sqrt{\pi a}. \tag{53}$$

In order to describe the usability of SW – integral and to compare the result (49a) with the energy release rate of a crack extension in (50a), we introduce a dimensionless quantity γ and let

$$\gamma = a/L, \tag{54}$$

where L is a reference length. Then, the dimensionless comparison with the help of (49a) (50a) (51)-(54) can be written as

$$L^{2\lambda_1^{(1)}-1}\frac{[SW]}{[J]} = \frac{2\pi\alpha(0.0866 + 0.9134\lambda_1^{(1)})^2}{\gamma^{2\lambda_1^{(1)}-1}(1.4138 - 0.4138\lambda_1^{(1)})^2[1 + \lambda_1^{(1)} - \cos(2\lambda_1^{(1)}\alpha) - \lambda_1^{(1)}\cos(2\alpha)]^2}, \tag{55}$$

where [SW] and [J] are given in (49a) and (50a), respectively, and $2(\pi - \alpha)$ is the notch opening angles, in which α changes from 90°to 180° shown in Fig. 2.

Since an infinite elastic plane means the infinite deepness of two sharp V-notches shown in Fig. 2, the dimensionless quantity $\gamma \geqslant 1$ is meaningful for engineering in practice. Yosibash et al. (2006) also pointed out that the notch opening angles $2(\pi - \alpha)$ up to $4\pi/3$ or 240^0 are of greatest importance. Under these considerations, the dimensionless comparison (55) is presented in Fig. 3, which shows that the numerical values decrease when the notch opening angle $2(\pi - \alpha)$ increases from 0° to 240°. This is interesting and means that when the order of singularity decreases, the dimensionless comparison (55) decreases.

4.3. Numerical values of NSIFs of a finite V-notched plate under uniaxial tension load

Here, we consider a finite V-notched plate with h = 200 mm and w = 40 mm under uniaxial tension load, as shown in Fig. 4. The plate is in plane stress state with Young's modulus $E = 3.9 \times 10^9$ Pa, Poisson's ratio $\nu = 0.25$ and load $\sigma = 1$MPa. There

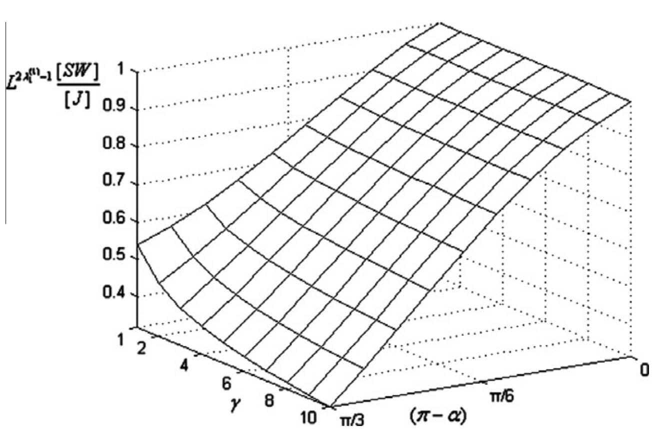

Fig. 3. The dimensionless comparison of the parameter for Mode I problem of a sharp V-notch with the energy release rate of crack extension.

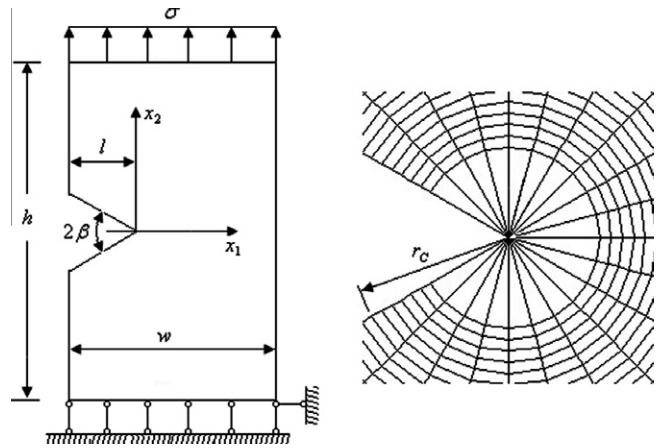

Fig. 4. (a) A finite V-notched plate and (b) the finite element mesh closing to the tip and r_C is the radius of integration path.

are the opening angle 2β and the depth l of the notch. For this Mode I problem with letting $\zeta(z) = z^{1-2\varsigma_1^{(1)}}$, by using expressions (9) and (35), the path-independent integral (42) can be rewritten as

$$\begin{aligned} SW &= \lim_{r\to 0}\int_{\Gamma_r} P_k n_k r d\theta = \mathrm{Im}\left[\lim_{r\to 0}\int_{\Gamma_r}(P_1^{(I)} - iP_2^{(I)})dz\right] \\ &= \mathrm{Im}\left\{\lim_{r\to 0}\int_{\Gamma_r}\frac{4}{E'}\zeta(z)[\Phi_1(z)]^2 dz\right\} \\ &= \lim_{r\to 0}\left\{\frac{4G^2}{E'}r^{2(1-\lambda_1^{(1)})}\int_{-\alpha}^{\alpha}[A\cos 2(1-\lambda_1^{(1)})\theta - B\sin 2(1-\lambda_1^{(1)})\theta]d\theta\right\}, \end{aligned} \tag{56a}$$

$$\text{where} \quad A = \left(\frac{u_{1,1} + u_{2,2}}{\kappa - 1}\right)^2 - \left(\frac{u_{2,1} - u_{1,2}}{\kappa + 1}\right)^2, \quad B = \frac{2(u_{1,1} + u_{2,2})(u_{2,1} - u_{1,2})}{\kappa^2 - 1}. \tag{56b}$$

Clearly, expression (56a) is equal to (49a), so that we have

$$K_I^N = \lim_{r\to 0}\sqrt{\frac{\pi}{\alpha}}[1 + \lambda_1^{(1)} - \cos(2\lambda_1^{(1)}\alpha) - \lambda_1^{(1)}\cos(2\alpha)]Gr^{(1-\lambda_1^{(1)})}\sqrt{I^{(I)}}, \tag{57a}$$

$$I^{(I)} = \int_{-\alpha}^{\alpha}[A\cos 2(1-\lambda_1^{(1)})\theta - B\sin 2(1-\lambda_1^{(1)})\theta]d\theta. \tag{57b}$$

In order to obtain NSIFs, expression (57a) is calculated for the problem shown in Fig. 4(a), for which ANSYS is used and the finite element mesh is shown in Fig. 4(b). Two points should be mentioned: (i) in order to reduce error due to the stress singularities, many finite elements are enclosed by the circle with radius r_C of integration path; (ii) as indicated in (46a) and (47a), when a very small r_C in (57b) is taken, the theoretical accuracy will be very higher. So, the calculation is carried out by

$$K_I^N \approx \sqrt{\frac{\pi}{\alpha}}[1 + \lambda_1^{(1)} - \cos(2\lambda_1^{(1)}\alpha) - \lambda_1^{(1)}\cos(2\alpha)]Gr_C^{(1-\lambda_1^{(1)})}\sqrt{I^{(I)}}, \tag{58}$$

and the obtained results are listed in Table 1. Here, when the relative difference is calculated by

$$\Delta(\%) = \frac{\text{Solution}_{\text{Pr esent}} - \text{Solution}_{\text{Niu}}}{\text{Solution}_{\text{Niu}}} \times 100, \tag{59}$$

the maximum relative difference in Table 1 is 6.5%. It should be mentioned that Niu et al. (2009) also gave numerical results calculated by BEM.

Table 1
Notch stress intensity factors and circle radii r_C of integration path ($\beta = \pi - \alpha$).

l/w	$2\beta = 30^0$		$2\beta = 60^0$	
	Present (r_C)	Niu (2009)	Present (r_C)	Niu (2009)
0.1	4.1834(0.0008)	4.2668	4.3218(0.002)	4.4294
0.2	6.7103(0.001)	6.8740	6.8634(0.0043)	7.0501
0.3	9.7157(0.001)	10.1131	9.68(0.0065)	10.3576

Table 2
The first three eigenvalues $\lambda_m^{(1)} = \varsigma_m^{(1)} + i\eta_m^{(1)}$ for Mode I problem ($\beta = \pi - \alpha$).

$2\beta(^0)$		$\varsigma_1^{(1)}$	$\eta_1^{(1)}$	$\varsigma_2^{(1)}$	$\eta_2^{(1)}$	$\varsigma_3^{(1)}$	$\eta_3^{(1)}$
30^0	Present	0.5015	0	1.2030	0	1.4904	0
	Cheng et al. (2009)	0.5020	0	1.2060	0	1.4860	0
60^0	Present	0.5122	0	1.4710	0.1419	2.6776	0.2849
	Niu et al. (2009)	0.5122	0	1.4710	0.1419	2.6777	0.2850

5. Concluding remarks

Under consideration of the physical and mathematical significance of complex Kolosov–Muskhelishvili potential $\phi(z)$, the independent Lagrangian functions (17a) and (17b) are presented. Their variational problem leads to a harmonic equation and the related nature, homogenous boundary conditions when U_1 and U_2 are considered as independent functions (16a), (16b), (19a), and (19b), respectively, based on the theory of analytic functions. Actually, the obtained harmonic equations are also the Navier's displacement equations in plane elasticity (16a) and (16b), so that the requirement of Noether's theorem (Noether, 1918) is satisfied.

It is found that any conformal transformation or analytic function is a symmetry- transformation (30) for obtaining the conservation law (36) or (37) in material space. Since the conformal transformation in two-dimensional Euclidean space includes the translation, rotation and scale change of planar coordinates, the conservation law (36) or (37) possesses universality. On the other hand, by adjusting the conformal transformation or analytic function (30), a finite value can be obtained from calculating the path-independent integral (37) or (42) around a material point with any order singularity. Especially, it is expectative that the interaction among several material points with singularities may be investigated by using the path-independent integral (37) based on the residue theorem in the theory of analytic functions. Also, multi-material junctions may be investigated by using the path-independent integral (37).

The path-independent integral (37) or (42) is applied to a sharp V-notch and the obtained finite values (49a) and (49b) are directly related to the NSIFs. Actually, SW – integral (49a) and (49b) possess a kind of physical invariance because of the path independence (49a) and (49b). Since this invariance always holds for any given fixed notch opening angle, the obtained finite values (49a) and (49b) are equivalent to the NSIFs, which are the physical parameters for application to a sharp V-notch in elastic fracture field. In Section 4, Fig. 3 shows the feature of parameter (49a) and Table 1 gives numerical results of NSIFs by using parameter (49a).

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Appendix A

In polar coordinates, the stresses $\sigma_{\theta\theta}$ and $\sigma_{r\theta}$ can be written by transforming expressions (5) and (6) as follows

$$\sigma_{\theta\theta} - i\sigma_{r\theta} = \phi'(z) + \overline{\phi'(z)} + \bar{z}[\overline{\phi''(z)} + z^{-1}\overline{\psi'(z)}]. \tag{A1}$$

The complex potentials are assumed to have the form

$$\phi(z) = (a^{(1)} + ia^{(2)})z^{\lambda}, \quad \psi(z) = (b^{(1)} + ib^{(2)})z^{\lambda}, \quad \lambda = \varsigma + i\eta, \tag{A2}$$

where $a^{(1)}$, $a^{(2)}$, $b^{(1)}$, $b^{(2)}$, ς and η are the real constants. Under the consideration of the independence of the constants $a^{(1)}$ and $a^{(2)}$, the traction-free conditions of expression (A1) along $\theta = \alpha$ and $\theta = -\alpha$, as shown in Fig. 1, gives

$$a^{(1)}: \begin{cases} (e^{-2\eta\alpha} - e^{2\eta\alpha})\cos(2\varsigma\alpha) - 2\eta\sin(2\alpha) = 0, \\ (e^{-2\eta\alpha} + e^{2\eta\alpha})\sin(2\varsigma\alpha) + 2\varsigma\sin(2\alpha) = 0; \end{cases} \tag{A3}$$

$$a^{(2)}: \begin{cases} (e^{-2\eta\alpha} - e^{2\eta\alpha})\cos(2\varsigma\alpha) + 2\eta\sin(2\alpha) = 0, \\ (e^{-2\eta\alpha} + e^{2\eta\alpha})\sin(2\varsigma\alpha) - 2\varsigma\sin(2\alpha) = 0; \end{cases} \tag{A4}$$

$$\begin{Bmatrix} b^{(1)} \\ b^{(2)} \end{Bmatrix} = \begin{bmatrix} -e^{-2\eta\alpha}\cos 2\varsigma\alpha - \varsigma\cos 2\alpha + \eta\sin 2\alpha & e^{-2\eta\alpha}\sin 2\varsigma\alpha - \varsigma\sin 2\alpha - \eta\cos 2\alpha \\ e^{-2\eta\alpha}\sin 2\varsigma\alpha + \varsigma\sin 2\alpha + \eta\cos 2\alpha & e^{-2\eta\alpha}\cos 2\varsigma\alpha - \varsigma\cos 2\alpha + \eta\sin 2\alpha \end{bmatrix} \begin{Bmatrix} a^{(1)} \\ a^{(2)} \end{Bmatrix}. \tag{A5}$$

The complex eigenvalues $\lambda = \varsigma + i\eta$ for Mode I and II problems can be obtained by solving the eigen equations. (A3) and (A4), respectively. By using Matlab software, the first three complex eigenvalues are listed in Table 2. Then, the complex potential $\phi(z)$ for Mode I and II problems can be rewritten as

$$\phi_1(z) = \sum_{m=1}^{\infty} a_m^{(1)} z^{\lambda_m^{(1)}}, \quad (\lambda_m^{(1)} = \varsigma_m^{(1)} + i\eta_m^{(1)}), \tag{A6}$$

$$\phi_2(z) = i\sum_{m=1}^{\infty} a_m^{(2)} z^{\lambda_m^{(2)}}, \quad (\lambda_m^{(2)} = \varsigma_m^{(2)} + i\eta_m^{(2)}). \tag{A7}$$

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