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On the fundamental equations of electromagnetoelastic media in variational form with an application to shell/laminae equations

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The fundamental equations of elasticity with extensions to electromagnetic effects are expressed in differential form for a regular region of materials, and the uniqueness of solutions is examined. Alternatively, the fundamental equations are stated as the Euler–Lagrange equations of a unified variational principle, which operates on all the field variables. The variational principle is deduced from a general principle of physics by modifying it through an involutory transformation. Then, a system of two-dimensional shear deformation equations is derived in differential and fully variational forms for the high frequency waves and vibrations of a functionally graded shell. Also, a theorem is given, which states the conditions sufficient for the uniqueness in solutions of the shell equations. On the basis of a discrete layer modeling, the governing equations are obtained for the motions of a curved laminae made of any numbers of functionally graded distinct layers, whenever the displacements and the electric and magnetic potentials of a layer are taken to vary linearly across its thickness. The resulting equations in differential and fully variational, invariant forms account for various types of waves and coupled vibrations of one and two dimensional structural elements as well. The invariant form makes it possible to express the equations in a particular coordinate system most suitable to the geometry of shell (plate) or laminae. The results are shown to be compatible with and to recover some of earlier equations of plane and curved elements for special material, geometry and/or effects.

1. Introduction

Smart (intelligent, adaptive, functional) or responsive materials refer to a class of contemporary materials, which are capable of simultaneously sensing mechanical, electrical, thermal, magnetic and even moisture effects. This type of materials (e.g.,piezoelectric, thermopiezoelectric, hygrothermopiezoelectric, piezomagnetic, electrostrictive, magnetostrictive, electromagnetoelastic and the alike materials) and structures have considerable recent technological applications in sensing, actuation and control, optics and electromagnetics, information processing, and material science (e.g., Wang and Kang, 1998; Tani et al., 1998; Chopra, 2002; Schwartz, 2002; Tzou et al., 2004; Leo, 2007; Reece, 2007). The sensitivity of materials to various fields is a complex case involving with many parameters, and it was modeled by introducing certain simplifying assumptions with desirable accuracy. The principal simplifications are only the retention of the mechanical and electrical fields (piezoelectricity), the piezoelectric and thermal fields (thermopiezoelectricity), the moisture and thermopiezoelectric fields (hygrothermopiezoelectricity), the mechanical and magnetic fields (magnetostrictive), the electrical and magnetic fields (electromagnetism), the piezoelectric and magnetic fields (electromagnetoelasticity). In electromagnetoelastic materials, the mechanical, electrical and magnetic fields interact one with another, for instance, they become magnetized when placed in an electric field and electrically polarized when placed in a magnetic field or a mechanical stress produces an electric and a magnetic field and, conversely, a strain is produced in response to an applied magnetic field and/or an electric potential. Only a few natural materials but some synthetic materials and especially various piezoelectric/magnetostrictive composites were shown to exhibit the magnetoelectric coupling effect (Ryu et al., 2002; Fiebig, 2005). The effect was observed in either single phase materials (e.g, Alshits et al., 1992; Wang and Mai, 2004) or as a result of the product property in composite materials, of which the piezoelectric or magnetostrictive constituents do not exhibit the effect (e.g., Rado and Folen, 1961; van Suchtelen, 1972), as a new class of advanced materials, namely the electromagnetoelastic materials. The materials are a growing area of the basic and applied research due to their unique capability of electromechanical and magnetomechanical energy conversion, which is important in applications of smart and active structural systems. The reader may be referred to the treatises on the interaction processes, including its development, modeling, and

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applications (e.g., Landau and Lifshitz, 1960; Mason, 1966; Freeman and Schmid, 1975; De Lacheisserie, 1993; Parton and Kudryavtsev, 1988; Kudryavtsev et al., 1990; Hutter et al., 2006; Reece, 2007; Bardzokas et al., 2007).

The fundamental equations of electromagnetoelastic interactions, which were corroborated with some experiments (e.g., Ryu et al., 2002) may be expressed, as suggested by Mindlin (1974), by the divergence equations, the gradient equations, the constitutive relations and the boundary and initial conditions to supplement them (e.g., Pao and Yeh, 1973; Pao, 1978; Kiral and Eringen, 1990; Wang and Mai, 2007). The divergence equations (i.e., Newton's and Maxwell's equations) were established in integral (global) form on the basis of the well-known axioms of electromagnetism and mechanics, and they were stated in differential form under certain regularity and local differentiability conditions of the field variables. The gradient equations, which are given in differential form, and the divergence equations are the universal laws within the limits of continuum physics. The constitutive relations, which represent the peculiarities of materials depending on the range of effects, were always stated in differential form under certain rules and requirements of continuum mechanics, except for the non-local case where the nature of long-range intermolecular forces is of special importance. In some high technology materials, namely the functionally graded (FG) materials, the constitutive relations are properly conceived and tailored to vary gradually in certain direction (e.g., Suresh and Mortensen, 1998; Miyamoto et al., 1999; Ichikawa, 2001; Shen, 2009). This type of materials is of a recent design feature and demand so as to improve the working performance and lifetime of the devices operating for sensing, actuating and control purposes. Recently, Birman and Byrd (2007) presented a review of modeling, analysis and developments of functionally graded materials, including a list of extensive references. The initial and boundary conditions were almost always stated in differential form, and they were given so as to ensure the internal consistency of the fundamental equations (i.e., the existence and uniqueness of solutions), that is, the equations well-posed. Some results were reported for the uniqueness of solutions (e.g., Li, 2003; Aouadi, 2007; El-Karamany and Ezzat, 2009), and the existence of solutions and one stability result (Priimenko and Vishnevskii, 2007) in electromagneto elasticity.

In mathematically modeling the interactions, the fundamental equations may be alternatively expressed in variational form in lieu of integral and differential forms, as the Euler–Lagrange equations of some variational principles with their well-known features. Aside an intuitive or a trial-and-error approach, variational principles were systematically obtained either by use of a number of mathematical methods or a general principle of physics and/or by extending it through an involutory transformation. A mathematical method, which leads to an integral type of variational principle with an explicit functional, can be applied only for linear and self-adjoint system of differential equations. On the other hand, a physics principle can be applicable to any system of differential equations, but generates a differential type of variational principle without an explicit functional and with some constraint conditions. The constraint conditions, which are undesirable in computation, can be removed by a variety of methods so as to derive a unified variational principle, that is, a variational principle operating on all the field variables. The unified variational principles were reported in elasticity (Hu, 1954; Washizu, 1955), in piezoelectricity (Dökmeci, 1973; Vekovishcheva, 1974; He, 2001a), thermo- and hygrothermo-piezoelectricity (Dökmeci, 1978; Altay and Dökmeci, 1996 and Altay and Dökmeci, 2008), piezoelectromagnetism (Altay and Dökmeci, 2004; Altay and Dökmeci, 2005), and electromagnetoelasticity (Wang and Shen, 1995; He, 2001b; Yao, 2003; Luo et al., 2006). Some Laplace-transform-type and unconventional Hamilton-type variational principles were presented in electromagnetoelasticity by Wang and Shen (1995) and Luo et al. (2006), respectively. He (2001b) obtained a variational principle by his semi-inverse method so as to describe the static physical behavior of a magneto-electro-elastic medium. Recently, a review of variational principles with an historical account of development and an extensive list of references through the pertinent keywords within the "ISI-Web of Science" was reported (Altay and Dökmeci, 2007; Altay and Dökmeci, 2009), which reveals that a unified variational principle in invariant form or including the discontinuous fields and/ or deduced from a general principle of physics, is unavailable in electro-magneto-elastodynamics.

Structural elements, for instance, a shell or a curved laminae, where one dimension is at least of order of magnitude lower than the other two, was almost always mathematically modeled as a two dimensional continuum. The physical response of a shell may be analyzed by use of either the three dimensional fundamental equations of materials or a system of two dimensional equations, which was deduced from the former with the aid of a method of reduction on the basis of some simplified assumptions (e.g., Ambartsumian, 1964; Kil'chevskiy, 1965; Cicala, 1965; Naghdi, 1972; Koiter and Simmonds, 1973; Librescu, 1975; Gol'denveizer, 1997; Villaggio, 1997; Libai and Simmonds, 1998; Pikul, 2000; Rubin, 2000). Both the three dimensional equations and two dimensional equations of shells inherently contain at least some inevitable as well as unpredictable errors of experimental nature in material properties, which have to be more accurately determined in consistent with efficient numerical techniques and powerful computers nowadays. The relative merit of using either the three or two dimensional equations depends on each specific case, though the former seems to be more accurate and evidently less tractable in numerical computation than the latter. Nevertheless, the two dimensional equations were preferred in the bulk of analysis to predict the physical response of shells due to the complexity of the three dimensional equations. The field variables together with their derivatives up to at least second orders are assumed to be exist, single-valued and piecewise continuous functions of the space coordinates and time in the shell space. In addition, the field variables are assumed not to vary widely, and hence, they can be averaged across the thickness of shell, as in the cross-section of rods introduced first by Leibniz (1684). Other assumptions involve geometrical and/or material linearity, low- and high-frequency motions, especially mathematical modeling of thin, moderately thick or thick shells. Only, a few authors (e.g., Berger, 1973; John, 1965; Dikmen, 1982; Ciarlet, 2000; Libai and Simmonds, 1998; Kaplunov et al., 1998; Kienzler et al., 2004) were concerned with the range of applicability of some assumptions, the internal consistency (i.e.,existence, uniqueness and stability) of solutions, and the type of loadings, which still appeal to much greater elaboration involving intrinsic and especially extrinsic error estimates, as was discussed in a comment (Altay and Dökmeci, 2003a).

A laminae type of plane or curved structural elements was appreciated only recently as a new design feature and demand in electromagnetoelastic devices. The physical response of a laminae can be analyzed either by a micromechanical model of mixture or lattice types (e.g., Li and Dunn, 1998; Chen et al., 2002; Lee et al., 2005; Aboudi, 2007; Tang and Yu, 2008) or a macromechanical model (e.g., Milton, 2002). There basically exist two types of macromechanical models: an effective medium model (e.g., Hashin, 1983; Avellaneda and Harsche, 1994; Nan, 1994; Benveniste, 1995; Tan and Tong, 2002) or a discrete layer modeling (e.g., Dökmeci, 1992; Saravanos and Heyliger, 1999; Delia and Shu, 2007). The former replaces the laminae by a representative medium with the aid of averaged material properties of its constituent elements. This conventional model, although it is relatively simple, omits the mutual coupling of adherent layers, and it is generally suitable in investigating a rather broad class of the static responses of laminae. The discrete layer (or layerwise) modeling incorporates all the essential features of laminae constituents, and it accounts for the dynamic responses of laminae as well. The extension of classical models (e.g., Lagrange's or Karman's model of plates and Love–Kirchhoff's model of shells) to a laminae always leads to a conventional effective medium model, and hence, it disregards the electromagnetoelastic interactions between adjacent layers. Nevertheless, Mindlin's (1955, 1967) or other shear deformable models may be used in a discrete layer modeling so as to account for the interactions. Ghugal and Shimpi (2001, 2002) and Carrera (2002, 2003a,b) reviewed various single layer and layerwise models for laminated structural elements, with an extensive list of references, and they discussed their merits and demerits, as did Qatu (2002a,b) and Alhazza and Alhazza (2004) for laminated shells, and Dökmeci (1980, 1992), Saravanos and Heyliger (1999), Wang and Yang (2000) and Wu et al. (2008a,b) for laminated piezoelectric elements.

The open literature relevant to a static and dynamic analysis of electromagnetoelastic structural elements is rather scanty and directed, in general, toward solutions of specific problems of structural elements. The static analysis includes, for instance, an analytical solution to electromagnetoelastic beams with different boundary conditions (Jiang and Ding, 2004), and an exact three dimensional solution (Pan, 2001; Pan and Han, 2005), a state vector approach (Wang et al., 2003), a discrete layer solution (Heyliger and Pan, 2004; Heyliger et al., 2004), and a partial mixed layerwise finite element model (Garcia Lage et al., 2004) for multilayered magnetoelectroelastic plates. As for the dynamic problems, the vibrations of layered beams (e.g., Annigeri et al., 2007), plates (e.g., Pan and Heyliger, 2002; Qing et al., 2005; Chen et al., 2005; Bhangale and Ganesan, 2006; Chen et al., 2007a,b), hollow cylinders and spheres (Hou and Leung, 2004; Hou et al., 2006; Dai et al., 2007; Wu and Tsai, 2007; Jiangong et al., 2008), and cylindrical and spherical shells (e.g., Buchanan, 2003; Bhangale and Ganesan, 2005; Annigeri et al., 2006; Tsai and Wu, 2008; Daga et al., 2008a,b; Wu et al., 2008a,b) were studied. Pan and Heyliger (2002) solved the vibration problem of a simply supported laminated rectangular plate, and Pan and Han (2005) studied the problem in case of functionally graded material. Buchanan (2004) determined and compared the natural frequencies of layered and different multiphase models of plates using finite element analysis. Ramirez et al. (2006) obtained a discrete layer solution for the static and vibration analysis of a plate having different graded materials. The transient responses of hollow cylinders for symmetric and axisymmetric plain strain problems were investigated by Hou and Leung (2004) and Hou et al. (2006), respectively, and the propagation of longitudinal and torsional harmonic waves in functionally graded hollow cylinders by Jiangong et al. (2008). Using the finite element method, Annigeri et al. (2006) studied the free vibrations of layered and multiphase magnetoelectroelastic shells, and recently, Daga et al. (2008b) made a comparative study of transient response of magneto-electroelastic finite cylindrical shell under constant internal pressure. On the other hand, Green and Naghdi (1983) established the non-linear and linear thermomechanical theories of deformable shell-like bodies in which account is taken of electromagnetic effects by a direct approach with the use of the two-dimensional theory of direct media called Cosserat surfaces.

In view of a most recent review with the pertinent key words within the papers recorded by "ISI-Web of Science", this study is addressed (i) to deduce a unified variational principle from a general principle of physics by removing its constraint conditions through an involutory transformation for a regular region and/or stratified region, (ii) to derive a system of two-dimensional shear deformation equations, in invariant and/or fully variational form for a functionally graded thin shell at high frequency motions, including some results for its internal consistency, and (iii) within the frame of a discrete layer modeling, to formulate the system of shear deformation governing equations of a functionally graded laminae, geometrically linear but physically non-linear and satisfying all the interface conditions, in electromagnetoelasticity.

Specifically, the remainder of this section introduces the notation to be used in the paper. First, with reference to a regular electromagnetoelastic region of space, the fundamental equations are summarized in differential invariant form, including a uniqueness result for solutions in the next section. In Section 3, Hamilton's principle is stated for the region, a three-field variational principle is deduced from it, and then, this principle is modified through an involutory transformation so as to derive a unified variational principle operating on all the field variables. Section 4 contains the basic field variables chosen a priori for a shell region, and the gradient equations and the constitutive relations for the shell made of functionally graded materials. In Section 5, a system of two dimensional shear deformation equations is derived for the high frequency motions of the shell. The uniqueness is investigated in solutions of the shell equations, and then, a theorem of uniqueness is given in Section 6, which enumerates the initial and boundary conditions sufficient to the uniqueness. In Section 7, the shell equations are extended to account for the motion of a laminae, which may have any number of perfectly bonded layers, for the case when the basic field variables vary linearly across the thickness of each layer. In Section 8, some cases involving special material, geometry and type of motion are recorded. The last section is devoted to some concluding remarks, future needs of research, numerical results and even technological applications.

Notation. In this paper, standard space and surface tensors are freely used in a Euclidean three-dimensional space Ξ, and Einstein's summation convention is implied for all repeated subscripts or superscript indices, unless they are enclosed with parentheses. Latin indices with the range 1, 2 and 3 are assigned to space tensors and Greek indices with the range 1 and 2 to surface tensors. The θ^i-system in the space is identified with a fixed, right-handed set of geodesic normal coordinates. A superposed dot stands for time differentiation, a comma for partial differentiation with respect to an indicated space coordinate θ^i, and a semicolon and a colon for covariant differentiation with respect to the coordinate using the space and surface metrics, respectively. Further, an asterisk is used to denote a prescribed initial or boundary quantity, and an overbar to indicate a field quantity, which is referred to the base vectors of the middle surface of a shell of uniform thickness 2 h, with its thickness interval $Z = [-h, h]$. A regular electromagnetoelastic region with its boundary surface and closure is denoted by $\Omega \cup \partial\Omega$, $\partial\Omega$, $\overline{\Omega}(= \Omega \cup \partial\Omega)$, respectively, the unit outward vector normal to the boundary surface by n_i, and the region at time t by $\Omega(t)$, the time interval by $T = [t_0, t_1)$ and their Cartesian product by $\Omega \times T$. Also, $C_{\alpha\beta}$ refers to a class of functions with derivatives of order up to and including α and β with respect to the space coordinates and time. The letters (a), (e) and (m) are used to indicate the acoustic (mechanical), electrical and magnetic parts of a quantity, respectively. Essentially, new quantities are defined when they are first introduced, and the following symbols and acronyms are used in the text.

Nomenclature

Ξ	Euclidean three-dimensional space
$\theta^i;\ (\theta^\alpha, \theta^3)$	a fixed right-handed system of geodesic normal coordinates of the space Ξ
$\theta^3 \equiv z$	thickness coordinate
$Z = [-h, h]$	thickness interval
$t, T = [t_o, t_1]$	time, time interval
χ_*	prescribed value of χ
$(\dot{\chi}),\ (\chi)_{,i}$	time differentiation, partial differentiation with respect to θ^i
$(\chi)_{;i},\ (\chi)_{:i}$	covariant differentiations with respect to the space and surface metrics
(a, e, m)	a quantity belonging to mechanical (acoustic), electrical and magnetic fields

(d, g, c, b)	indicates divergence and gradient equations, constitutive relations, and boundary conditions
$\Omega,\ \partial\Omega,\ \overline{\Omega}$	regular region of space, its boundary surface and closure $(= \Omega \cup \partial\Omega)$
$a_{\alpha\beta},\ b_{\alpha\beta},\ c_{\alpha\beta}$	first, second and third fundamental forms of a surface
μ^{α}_{β}	shifters, shell tensor $(\mu^{\alpha}_{\beta} = \delta^{\alpha}_{\beta} - zb^{\alpha}_{\beta})$
V, S, A	shell region and its boundary surface and middle (reference) surface
C	Jordan curve which bounds A
$K_g,\ K_m$	Gaussian and mean curvatures of a surface $(K_g = 1/2\left\|b^{\alpha}_{\beta}\right\|,\ H = 1/2(b^{\alpha}_{\alpha})$
dV, dS, dc	volume, surface and line elements
$2h$	uniform thickness of shell
$2h_n$	uniform thickness of nth layer of laminae
$S_{n,n+1}$	interface between nth and $(n+1)$th layers
$T^{ij},\ T^{i}$	stress tensor and stress vector (traction vector $T^{j} = n_i T^{ij}$)
$T^{ij}_{(n)},\ T^{i}_{(n)}$	stress and traction resultants of order (n)
$u_i,\ a_i$	displacement and acceleration vectors
$A^{i}_{(n)}$	acceleration resultants of order (n)
ρ	mass density
$\bar{u}_i,\ u^{(n)}_i$	shifted components of mechanical displacements, referred to the base vectors of the middle surface of shell A, displacements components of order (n)
$S_{ij},\ S^{(n)}_{ij}$	strain tensor, strain tensor of order (n)
$\varepsilon_{ijk},\ \varepsilon_{\alpha\beta}$	alternating tensor for the shell space and middle surface
$D^{i},\ D^{i}_{(n)}$	electric displacement vector, electric displacement of order (n)
$E_i,\ E^{(n)}_i$	quasi-static electric field vector, electric field vector of order (n)
$\phi,\ \phi_{(n)}$	electric potential, electric potential of order (n)
ρ_e	electric charge density
ρ_m	magnetic charge density
$D,\ D_{(n)}$	surface charge, surface charge of order (n)
$f^{i},\ F^{i}_{(n)}$	body force vector, body force resultants of order (n)
$B^{i},\ B^{i}_{(n)}$	magnetic induction vector, magnetic induction of order (n)
$M^{i},\ M^{i}_{(n)}$	magnetic field vector, magnetic field vector of order (n)
$\psi,\ \psi_{(n)}$	magnetic potential, magnetic potential of order (n)
$n_i,\ v_i$	unit vectors normal to the boundary surface and the Jordan curve C
$S_e,\ S_{lf},\ S_{uf}$	edge boundary surface, lower and upper faces of shell
$\in_s,\ \eta_s,\ \lambda_s$	shell parameters
$c^{ijkl},\ e^{ijk},\ \varepsilon^{ijk},\ \varepsilon^{ij},\ e^{ij},\ c^{ij},\ c^{ijkl}_{(n)},\ e^{ijk}_{(n)},\ \varepsilon^{ijk}_{(n)},\ \varepsilon^{ij}_{(n)},\ e^{ij}_{(n)},\ c^{ij}_{(n)}$	material coefficients, material coefficients of order (n)
$\lambda,\ \lambda^{i},\ \lambda^{ij}$	Lagrange multipliers
k, K; K_S	kinetic energy and aerial kinetic energy densities, kinetic energy of shell
u, U; U_S	internal energy and aerial energy densities, total internal energy of shell
$u_a,\ u_e,\ u_m$	mechanical, electrical and magnetic energy densities
H, W	electromagnetoelastic enthalphy, complementary enthalphy
$\Lambda_a,\ \Lambda_e,\ \Lambda_m$	admissible state of mechanical, electrical and magnetic fields
$C_{\alpha\beta}$	functions with derivatives of order up to and including α and β with respect to the space coordinates and time

2. Fundamental equations in differential form

Now, the time- domain, quasi-static approximate fundamental equations are summarized for the motion of an electromagnetoelastic continuum in which polar, non-local, and thermal, as well as relativistic and quantum effects are excluded. Consider a finite and bounded regular region of continuum in Kellogg's sense (1946), $\Omega + \partial\Omega$ with its smooth boundary surface $\partial\Omega$ and closure $\overline{\Omega}(= \Omega \cup \partial\Omega)$ at time $t = t_0$ in the Euclidean 3-D space Ξ. The fundamental equations are grouped as the divergence and gradient equations, the constitutive relations and the boundary and initial conditions, and stated in invariant differential form at the time interval $T = [t_0, t_1)$, given by (e.g., Berlincourt et al., 1964; Pan, 2001), namely

2.1. Divergence equations (Maxwell's and Newton's equations)

$$L^{j}_{a} = T^{ij}_{;i} + f^{j} - \rho a^{j} = 0 \quad \text{in } \overline{\Omega} \times T \tag{2.1a}$$
$$L^{a}_{i} = \varepsilon_{ijk} T^{ij} = 0 \quad \text{in } \overline{\Omega} \times T \tag{2.1b}$$

and

$$L_e = D^{i}_{;i} - \rho_e = 0 \quad \text{in } \overline{\Omega} \times T \tag{2.2}$$
$$L_m = B^{i}_{;i} - \rho_m = 0 \quad \text{in } \overline{\Omega} \times T \tag{2.3}$$

In these equations, $T^{ij} \in C_{10},\ a^{i}(= \ddot{u}^{i})$ and $u^{i} \in C_{12}$, stand for the symmetric stress tensor, the acceleration and mechanical displacement vectors, respectively, $f^{i} \in C_{00}$ for the body force vector per unit volume of continuum, and ε_{ijk} for the alternating tensor. Also, $D^{i} \in C_{10}$ stands for the electric displacement (i.e., electric flux density or electric induction) vector, $\rho_e \in C_{00}$ for the electric charge density, $B^{i} \in C_{10}$ for the magnetic induction (i.e., magnetic flux) vector, and $\rho_m \in C_{00}$ for the electric current density (or magnetic charge density). In Eq. (2.1), the ponderomotive force (or electromagnetic force) is excluded, electromagnetic couples are omitted and infinitesimal strains are considered (cf., Pao, 1978).

2.2. Gradient equations

$$L^{a}_{ij} = S_{ij} - \frac{1}{2}(u_{i;j} + u_{j;i}) \quad \text{in } \overline{\Omega} \times T \tag{2.4}$$
$$L^{e}_{i} = E_i + \phi_{,i} = 0 \quad \text{in } \overline{\Omega} \times T \tag{2.5}$$
$$L^{m}_{i} = M_i + \psi_{,i} = 0 \quad \text{in } \overline{\Omega} \times T \tag{2.6}$$

Here, $S_{ij} \in C_{00},\ E_i \in C_{00},\ M_i \in C_{00},\ \phi \in C_{10}$ and $\psi \in C_{10}$ denote the linear strain tensor, the electric field vector, the magnetic field vector, the electric potential and the magnetic potential, respectively.

2.3. Constitutive relations of electromagnetoelastic continuum

The principle of conservation of energy is stated in an adiabatic condition of the bounded regular region $\Omega + \partial\Omega$ in the form

$$\frac{\partial}{\partial t}\int_{\Omega}(k+u)\,dV = \int_{\partial\Omega}\left[T^{i}\dot{u}_i - n_i(\phi\dot{D}^{i} + \psi\dot{B}^{i})\right]dS + \int_{\Omega}(f^{i}\dot{u}_i - \dot{\rho}_e\phi - \dot{\rho}_m\psi)\,dV \tag{2.7a}$$

Here, u is the postulated internal energy density and k is the kinetic energy density of the form

$$k = \frac{1}{2}\rho\dot{u}^{i}\dot{u}_i \tag{2.7b}$$

Eq. (2.7) states that the rate of increase of total (kinetic plus internal) energy is equal to the rate at which the work done by the body force, electric charge density and magnetic charge density, and the surface traction $T^{i}(= n_j T^{ji}$, where n_i is the unit outward vector normal to the boundary surface $\partial\Omega$) acting across the boundary surface $\partial\Omega$ less the flux of electric and magnetic energies outward across the surface $\partial\Omega$. Recalling the fact that differentiation and integration operations can be permuted in the integral (2.7) with fixed end points, the differentiation with respect to time is carried out. Also, the surface tractions are replaced with the components of the stress tensor and the rate of the kinetic energy with the components of the mechanical displacements, and then the divergence theorem is applied to the regular region with the result

$$\int_\Omega (\rho \ddot{u}^i \dot{u}_i + \dot{u})\, dV = \int_\Omega \left[(T^{ij}\dot{u}_j)_{;i} - (\phi \dot{D}^i)_{;i} - (\Psi \dot{B}^i)_{;i} \right] dV + \int_\Omega (f^i \dot{u}_i + \dot{\rho}_e \phi + \dot{\rho}_m \psi)\, dV \tag{2.8}$$

where the conservation of mass is considered. Eq. (2.8) is valid for an arbitrary volume of the continuum, and hence it is written as

$$\dot{u} = (T^{ij}_{;i} + f^j - \rho a^j)\dot{u}_j + T^{ij}\dot{u}_{j;i} - \phi_{,i}\dot{D}^i - \phi(\dot{D}^i_{;i} - \dot{\rho}_e) - \psi_{,i}\dot{B}^i - \psi(\dot{B}^i_{;i} - \dot{\rho}_m) \tag{2.9}$$

With the use of the divergence equations (2.1–2.3), Eq. (2.9) is reduced to

$$\dot{u} = T^{ij}\dot{u}_{j;i} - \phi_{,i}\dot{D}^i - \psi_{,i}\dot{M}^i \tag{2.10}$$

Replacing the gradient of the electric and magnetic potentials with the electric field and magnetic field vectors and considering the symmetry of the stress tensor together with Eq. (2.4), one finally has

$$\dot{u} = T^{ij}\dot{S}_{ij} + E_i \dot{D}^i + M_i \dot{B}^i \tag{2.11}$$

This relation is the first law of thermodynamics for the electromagnetoacoustic continuum, and it is an extension of that for the piezoelectric continuum (e.g., Tiersten, 1969).

Now, the enthalpy density H is introduced by

$$H = u - E_i D^i - M_i B^i \tag{2.12}$$

in electromagnetoelasticity similar to that in piezoelectricity, and after differentiating it with respect to time, one finds the rate of the enthalpy as

$$\dot{H} = \dot{u} - E_i \dot{D}^i - \dot{E}_i D^i - M_i \dot{B}^i - \dot{M}_i B^i \tag{2.13}$$

After using Eq. (2.11), this equation becomes

$$\dot{H} = T^{ij}\dot{S}_{ij} - \dot{E}_i D^i - \dot{M}_i B^i \tag{2.14}$$

and it evidently implies $H = H(S_{ij}, E_i, M_i)$, of which the differentiation with respect to time takes the form

$$\dot{H} = \frac{\partial H}{\partial S_{ij}}\dot{S}_{ij} + \frac{\partial H}{\partial E_i}\dot{E}_i + \frac{\partial H}{\partial M_i}\dot{M}_i \tag{2.15}$$

But by combination of the rate of the enthalpy density in Eq. (2.14) with Eq. (2.15), one finds

$$\left(T^{ij} - \frac{\partial H}{\partial S_{ij}}\right)\dot{S}_{ij} - \left(D^i + \frac{\partial H}{\partial E_i}\right)\dot{E}_i - \left(B^i + \frac{\partial H}{\partial M_i}\right)\dot{M}_i = 0 \tag{2.16}$$

which holds for arbitrary quantities $\dot{E}_i$, $\dot{M}_i$ and $\dot{S}_{ij}$ being consistent with the symmetry condition $\dot{S}_{ij} = \dot{S}_{ji}$ and leads to the constitutive relations of the form

$$L^{ij}_{ac} = T^{ij} - \frac{1}{2}\left(\frac{\partial H}{\partial S_{ij}} + \frac{\partial H}{\partial S_{ji}}\right) = 0,$$
$$L^i_{ec} = D^i + \frac{\partial H}{\partial E_i} = 0, \quad \text{in } \overline{\Omega} \times T \tag{2.17}$$
$$L^i_{mc} = B^i + \frac{\partial H}{\partial M_i} = 0$$

In a reversible processes, then through a Legendre transformation $W\ (T^{ij}, D^i, B^i)$ of the enthalpy density $H = H(S_{ij}, E_i, M_i)$, that is, the complementary enthalpy density is defined by

$$W = (T^{ij}S_{ij} + D^i E_i + B^i M_i) - H(S_{ij}, E_i, M_i) \tag{2.18a}$$

and taking the time derivative as

$$\dot{W} = \frac{\partial W}{\partial T^{ij}}\dot{T}^{ij} + \frac{\partial W}{\partial D^i}\dot{D}^i + \frac{\partial W}{\partial B^i}\dot{B}^i \tag{2.18b}$$

for a case when the Hessian of the enthalpy density function H does not vanish. As before, equating Eqs. (2.18a) and (2.18b), the constitutive relations may be put in the inverted form

$$S_{ij} = \frac{1}{2}\left(\frac{\partial W}{\partial T^{ij}} + \frac{\partial W}{\partial T^{ji}}\right), \quad E_i = -\frac{\partial W}{\partial D^i}, \quad M_i = -\frac{\partial W}{\partial B^i} \quad \text{in } \overline{\Omega} \times T \tag{2.19}$$

and also, other forms of the constitutive relations were reported, for instance, by Soh and Liu (2005). As in elasticity or piezoelectricity, a quadratic homogeneous form of the electromagnetoelastic enthalpy density H is given by

$$H = \frac{1}{2}(c^{ijkl}S_{ij}S_{kl} - e^{ij}E_i E_j - c^{ij}M_i M_j) - e^{ijk}E_i S_{jk} - \varepsilon^{ijk}M_i S_{jk} - \varepsilon^{ij}E_i M_j \tag{2.20}$$

which, in view of Eq. (2.17) leads to the linear constitutive relations of the form

$$L^{ij}_{ac} = T^{ij} - (c^{ijkl}S_{kl} - e^{ijk}E_k - \varepsilon^{ijk}M_k) = 0 \quad \text{in } \overline{\Omega} \times T \tag{2.21}$$
$$L^i_{ec} = D^i - (e^{ijk}S_{jk} + e^{ij}E_j + \varepsilon^{ij}M_j) = 0 \quad \text{in } \overline{\Omega} \times T \tag{2.22}$$
$$L^i_{mc} = B^i - (\varepsilon^{ijk}S_{jk} + c^{ij}M_j + \varepsilon^{ij}E_j) = 0 \quad \text{in } \overline{\Omega} \times T \tag{2.23}$$

where c^{ijkl}, e^{ijk}, ε^{ijk} and ε^{ij} stand for the elastic, piezoelectric, piezomagnetic and magnetoelectric moduli, and e^{ij} and c^{ij} for the dielectric permittivity and the magnetic permeability, respectively. The material coefficients are taken to be dependent only to the space coordinate, that is, the functionally graded materials, excluding their dependency on time, temperature and the alike.

They satisfy the usual symmetry conditions of the form

$$c^{ijkl} = c^{jikl} = c^{ijlk}, \quad e^{ijk} = e^{ikj}, \quad \varepsilon^{ijk} = \varepsilon^{ikj}, \quad e^{ij} = e^{ji}, \quad \varepsilon^{ij} = \varepsilon^{ji}, \quad c^{ij} = c^{ji} \quad \text{in } \overline{\Omega} \times T \tag{2.24}$$

and the positive-semidefinite conditions, namely

$$e^{ij}\eta_i\eta_j \geqslant 0, \quad c^{ij}\eta_i\eta_j \geqslant 0, \quad c^{ijkl}\eta_{ij}\eta_{kl} \geqslant 0 \quad \text{in } \overline{\Omega} \times T \tag{2.25}$$

for a non-zero vector η_i and a non-zero tensor η_{ij} as well. In view of the symmetry conditions, an electromagneoelastic material can have 75 independent coefficients, of which the elastic coefficients c^{ijkl} refer to free coefficients since they describe the strain-displacement relations when the electric and magnetic fields are absent, while the remaining coefficients refer to clamped coefficients (e.g., Venkataraman et al., 1975).

2.4. Boundary and initial conditions

$$L^j_{a^*} = T^j_* - n_i T^{ij} = 0 \quad \text{on } \partial\Omega_t \times T, \quad L^*_{ai} = u_i - u^*_i = 0 \quad \text{on } \partial\Omega_u \times T \tag{2.26}$$

$$L_{e^*} = D_* - n_i D^i = 0 \quad \text{on } \partial\Omega_d \times T, \quad L^e_{\phi^*} = \phi - \phi_* = 0 \quad \text{on } \partial\Omega_\phi \times T \tag{2.27}$$

$$L_{m^*} = B_* - n_i B^i = 0 \quad \text{on } \partial\Omega_b \times T, \quad L^m_{\psi^*} = \psi - \psi_* = 0 \quad \text{on } \partial\Omega_\psi \times T \tag{2.28}$$

and

$$L_i^{a^*} = u_i(\theta^j, t_0) - v_i^*(\theta^j) = 0, \quad L_{ia}^* = \dot{u}_i(\theta^j, t_o) - w_i^*(\theta^j) = 0 \quad \text{in } \Omega(t_o) \tag{2.29}$$

$$L_*^e = \phi(\theta^i, t_o) - \alpha_*(\theta^i) = 0 \quad \text{in } \Omega(t_o) \tag{2.30}$$

$$L_*^m = \psi(\theta^i, t_o) - \beta_*(\theta^i) = 0 \quad \text{in } \Omega(t_o) \tag{2.31}$$

Here, an asterisk indicates the prescribed quantities. Besides, the outside of the boundary surfaces in (2.27) and (2.28) is a point of importance, which was discussed, for instance, by Lee (1991) and Altay and Dökmeci (2005). The fundamental equations (2.1)–(2.6) and (2.17) or (2.21)–(2.23) and (2.26)–(2.30), comprise the 29 equations governing the 29 dependent variables, that is, $\Lambda = \left\{u_i, S_{ij}, T^{ij}; \phi, E_i, D^i; \psi, M_i, B^i\right\}$, and hence, they are deterministic and define an initial and mixed boundary value problem for the electromagnetoacoustic continuum.

2.5. Uniqueness of solutions

In a mathematical modeling of the physical response of a continuum, the internal consistency, that is, the uniqueness and especially the existence of solutions, is of special importance. The internal consistency was well established in solutions of the three-dimensional fundamental equations (e.g., Gurtin, 1972; Fichera, 1972), and only some uniqueness results were reported in solutions of one or two dimensional equations (e.g, Green and Naghdi, 1971 for shells; Dökmeci, 1975 for rods), in linear elastodynamics. As for other types of materials, the uniqueness of solutions was most often investigated, and some results were reported for the existence of solutions only in a few cases. Mindlin (1974) proved the uniqueness in solutions of both the three dimensional fundamental equations of thermopiezoelectricity and the two dimensional equations of thermopiezoelectric plates, as did Altay and Dökmeci (2004, 2005) in piezoelectromagnetism and piezoelectromagnetic plates. Some results involving with the uniqueness and existence of solutions were reported on Maxwell's equations (e.g., Duvaut and Lions, 1979; Santos and Sheen, 2000), and with the uniqueness of solutions in electro-magnetoelasticity (e.g., Li, 2003), including the thermal effects. Nevertheless, the uniqueness was completely overlooked in solutions of lower order equations of electromagnetoelastic structural elements. To investigate the uniqueness of solutions in the governing equations of materials, various devices are available, for instance, the classical energy argument, the logarithmic convexity argument, Protter's technique, the method of analyticity, and the alike (e.g., Knops and Payne, 1972). The energy argument, which is based on the positive-definiteness of stored energies (e.g., Mindlin, 1967, 1974; see also, Deresiewicz et al., 1989) and the logarithmic convexity argument which does not impose a definiteness conditions on the material elasticity's (e.g., Altay and Dökmeci, 1998, Altay and Dökmeci, 2006) were most frequently used in studying the uniqueness of solutions in structural elements. The energy argument due to its physical nature and relative simplicity is used herein so as to prove a theorem, which enumerates the boundary and initial conditions sufficient to the uniqueness in solutions of the fundamental equations of electromagnetoelastic materials.

As usual, the possibility of two distinct sets of admissible solutions is assumed, namely

$$\Lambda^{(\alpha)} = \left\{u_i \in C_{12}, S_{ij} \in C_{00}, \ldots; \phi \in C_{10}, E_i \in C_{00}, D^i \in C_{10}; \psi \in C_{10}, M_i \in C_{00}, B^i \in C_{10}\right\}^{(\alpha)} \quad \text{in } \overline{\Omega} \times T \tag{2.32}$$

with both satisfying the divergence and gradient equations and the constitutive relations, Eqs. (2.1)–(2.6) and Eqs. (2.21)–(2.23), for the same boundary and initial data at each point of the regular region of a electromagnetoelastic continuum. Since the system of the fundamental equations is linear, the difference set of the two solutions given by

$$\Lambda = \Lambda^{(2)} - \Lambda^{(1)} = \left\{u_i = u_i^{(2)} - u_i^{(1)}, \ldots, D^i = D^i_{(2)} - D^i_{(1)}, \ldots, B^i = B^i_{(2)} - B^i_{(1)}\right\} \tag{2.33}$$

is a solution of the same system of equations in their homogeneous form. In other words, they satisfy the homogeneous divergence and gradient equations and the linear constitutive relations in the form

$$L_a^i = L_i^a = L_e = L_m = 0, \quad L_{ij}^a = L_i^e = L_i^m = 0, \quad L_{ac}^{ij} = L_{ec}^i = L_{mc}^i = 0 \quad \text{in } \overline{\Omega} \times T \tag{2.34}$$

the homogeneous boundary conditions as

$$L_{a^*}^j = n_i T^{ij} = 0 \quad \text{on } \partial\Omega_t \times T, \quad L_{ai}^* = u_i = 0 \quad \text{on } \partial\Omega_u \times T \tag{2.35}$$

$$L_{e^*} = n_i D^i = 0 \quad \text{on } \partial\Omega_d \times T, \quad L_{e^*}^{\phi} = \phi = 0 \quad \text{on } \partial\Omega_\phi \times T \tag{2.36}$$

$$L_{m^*} = n_i B^i = 0 \quad \text{on } \partial\Omega_b \times T, \quad L_{m^*}^{\psi} = \psi = 0 \quad \text{on } \partial\Omega_\psi \times T \tag{2.37}$$

and the homogeneous initial conditions as

$$L_{ai}^* = u_i(\theta^i, t_o) = 0, \quad L_{ia}^* = \dot{u}_i(\theta^j, t_0) = 0 \quad \text{in } \Omega(t_o) \tag{2.38a}$$

and

$$L_e^* = \phi(\theta^i, t_o) = 0, \quad L_m^* = \psi(\theta^i, t_o) = 0 \quad \text{in } \Omega(t_o) \tag{2.38b}$$

in the denotations defined in Eqs. (2.1)–(2.6), (2.21)–(2.23) and Eqs. (2.26)–(2.31). With the help of Eq. (2.34), one forms an integral by non-zero (u_i, ϕ, ψ) as follows:

$$I = \int_T dt \int_\Omega \left[(T^{ij}_{;i} - \rho\ddot{u}^j)\dot{u}_j - \dot{D}^i_{;i}\phi - \dot{B}^i_{;i}\psi\right] dV = 0 \tag{2.39}$$

which may be written as

$$I = \int_T dt \int_\Omega \left[-T^{ij}\dot{u}_{j;i} + \dot{D}^i\phi_{,i} + \dot{B}^i\psi_{,i} + (T^{ij}\dot{u}_j - \dot{D}^i\phi - \dot{B}^i\psi)_{;i} - \rho\ddot{u}^i\dot{u}_i\right] dV = 0 \tag{2.40}$$

and applying the divergence theorem to the regular region $\Omega + \partial\Omega$ as

$$I = I_\Omega - I_{\partial\Omega} = 0 \tag{2.41}$$

with the denotations of the form

$$I_\Omega = \int_T dt \int_\Omega (T^{ij}\dot{S}_{ij} + E_i\dot{D}^i + M_i\dot{B}^i + \dot{k})\, dV, \quad I_{\partial\Omega} = \int_T dt \int_{\partial\Omega} n_i(T^{ij}\dot{u}_j - \dot{D}^i\phi - \dot{B}^i\psi)\, dS \tag{2.42}$$

where the rate of the kinetic energy from Eqs. (2.7b) and (2.34) or Eqs. (2.4)–(2.6) for the difference system is considered. Inserting Eq. (2.11) in terms of Eq. (2.33) and performing the integration with respect to time in the second integral of Eq. (2.42), one obtains

$$I_\Omega = \int_T dt \int_\Omega (\dot{u} + \dot{k})\, dV = 0 \tag{2.43}$$

$$I_{\partial\Omega} = \int_T dt \int_{\partial\Omega} n_i(T^{ij}\dot{u}_j + D^i\dot{\phi} + B^i\dot{\psi})\, dS - \int_{\partial\Omega} n_i(D^i\phi + B^i\psi)|_T\, dS \tag{2.44}$$

The boundary and initial conditions (2.35)–(2.38) evidently render the integrands in Eq. (2.44) to zero, and hence, one writes

$$I = I_\Omega = \int_T dt \int_\Omega (\dot{u} + \dot{k})\, dV = \int_T (\dot{U} + \dot{K})\, dt = 0 \tag{2.45}$$

which, after integration with respect to time, takes the form

$$I = U(t_1) + K(t_1) - U(t_0) - K(t_0) = E(t_1) - E(t_0) = 0 \tag{2.46}$$

Here, U, K and $E(=U+K)$ are the internal, kinetic and total energy of the electromagnetoelastic region with no singularities of any type, and they are calculated from the densities k and u by integration. The densities are positive definite, by definition, and initially zero, so that the total energy E calculated from the difference set of solutions has the same property, namely

$$E(t_1) = E(t_0) = 0 \tag{2.47}$$

This clearly indicates a trivial solution for the difference set of solutions as

$$\Lambda = \Lambda^{(2)} - \Lambda^{(1)} = 0; \quad \Lambda^{(2)} = \Lambda^{(1)} \tag{2.48}$$

and the uniqueness is assured in solutions of the fundamental equations of the continuum and a theorem which enumerates the conditions sufficient to the uniqueness is concluded as follows.

Theorem. *Given a regular region of space $\Omega + \partial\Omega$ with its boundary surface $\partial\Omega$ and closure $\overline{\Omega}$, then there exist at most one set of single-valued state of solutions, namely*

$$\begin{aligned}\Lambda = \Big\{ & u_i \in C_{12}, S_{ij} \in C_{00}, T^{ij} \in C_{10}; \phi \in C_{10}, E_i \in C_{00}, D^i \in C_{10}; \psi \in C_{10}, M_i \in C_{00}, \\ & B^i \in C_{10} \Big\} \quad \text{in } \overline{\Omega} \times T \end{aligned} \tag{2.49}$$

which satisfies the divergence equations (2.1)–(2.3), the gradient equations (2.4)–(2.6), the constitutive relations (2.19)–(2.21), and the boundary and initial conditions (2.24)–(2.29), of the magnetoelectroacoustic continuum, under the symmetry of the stress tensor (2.1b) and the material coefficients (2.24) as well as the usual existence and continuity conditions of the field variables in the region.

3. Hamilton's principle and a unified variational principle

In formulating variational principles of mechanics, a general principle of physics (e.g., Hamilton's principle, the principle of virtual work or power) was most often taken as a point of of departure. Hamilton (1834, 1835) presented a general principle of physics for the dynamics of a discrete mechanical system. The principle was originally deduced from D'Alembert's principle by means of integration over time, and later, was extended by Kirchhoff (1876) to a continuous medium. The application of the principle to a finite region of continuum always leads to a variational principle, which has, as its Euler–Lagrange equations, the divergence equations and the associated natural boundary conditions only. The variational principle is either an integral type in case of conservative forces or a differential type otherwise. In the variational principle, the variations of each of the field variables are independent or unconstrained within the region and are constrained to vanish at the time t_0 and t_1 in the region and its boundary surface. Hamilton's principle, as a powerful and elegant tool, was extensively used in systematically deriving some integral and differential types of variational principles in mechanics (e.g., Miloh, 1984; Dökmeci, 1988 and references cited therein). Especially, the principle was used in deriving some unified variational principle by modifying it through an involutory transformation. Hamilton's principle is now applied in order to derive a unified variational principle operating on all the field variables for an electromagnetoelastic continuum.

Hamilton's principle states that the action integral is stationary between two instants of time t_o and t_1, and it is expressed for the regular region $\Omega + \partial\Omega$ at the time interval T (e.g., Lanczos, 1986), by

$$\delta L_H = -\delta \int_T L\,dt - \int_T \delta F\,dt + \int_T \delta^* W\,dt = 0 \tag{3.1}$$

Here, L is the Lagrangian function and $\delta^* W$ is the virtual work done by the external mechanical, electrical and magnetic forces as

$$\begin{aligned} &\delta L = \delta \int_\Omega [H(S_{ij}, E_i, M_i) - k]\,dV, \quad \delta^* W = \int_{\partial\Omega} (T^i_* \delta u_i + D_* \delta\phi + B_* \delta\psi)\,dS \\ &\delta F = \int_\Omega (-f^i \delta u_i + \rho_e \delta\phi + \rho_m \delta\psi)\,dS \end{aligned} \tag{3.2}$$

where an asterisk placed upon δ^* in Eqs. (3.1) and (3.2) is used to distinguish it from the variation operator δ. After inserting Eq. (3.2) into Eq. (3.1), carrying out the variations, using Eq. (2.17), and integrating the kinetic energy term by parts with respect to time t, one obtains

$$\begin{aligned} \delta L_H = \int_T dt \int_\Omega \Big\{ & -\rho \ddot{u}^i \delta u_i - \frac{1}{2}\left(\frac{\partial H}{\partial S_{ij}} \delta S_{ij} + \frac{\partial H}{\partial S_{ji}} \delta S_{ji}\right) - \frac{\partial H}{\partial E_i} \delta E_i \\ & - \frac{\partial H}{\partial M_i} \delta M_i \Big\} dV + \int_\Omega \rho \dot{u}^i \delta u_i |_T \,dV - \int_T \delta F\,dt + \int_T \delta^* W\,dt = 0 \end{aligned} \tag{3.3}$$

In this equation and henceforth, the variation and integration operations are permuted for the time and volume integrals with fixed end points, and the principle of conservation of mass is considered [i.e., $\delta(\rho\,dV) = 0$]. The gradient equations (2.4)–(2.6) and the constitutive relations (2.17) are inserted into Eq. (3.3), and the symmetry condition of the stress tensor (2.1b) is used with the result

$$\begin{aligned} \delta L_H = \int_T dt \int_\Omega \Big\{ & -\rho \ddot{u}^i \delta u_i - (T^{ij} \delta u_i)_{;j} + T^{ij}_{;i} \delta u_j - (D^i \delta\phi)_{;i} + D^i_{;i} \delta\phi \\ & -(B^i \psi)_{;i} + B^i_{;i} \delta\psi \Big\} dV - \int_T \delta F\,dt + \int_T \delta^* W\,dt = 0 \end{aligned} \tag{3.4}$$

Here, the constraint conditions of the form

$$\delta u_i = 0 \quad \text{in } \Omega(t_0) \& \Omega(t_1) \tag{3.5}$$

is imposed, which is customary in the use of Hamilton's principle. Applying the divergence theorem to the regular region $\Omega + \partial\Omega$ in Eq. (3.4), and combining the terms in the surface and volume integrals, one finally obtains a three-field variational principle of the form

$$\begin{aligned} \delta L_H\{\Lambda_H = u_i, \phi, \psi\} = \int_T dt \int_\Omega & \Big[(T^{ij}_{;i} + f^i - \rho \ddot{u}^j) \delta u_j + (D^i_{;i} - \rho_e) \delta\phi \\ & + (B^i_{;i} - \rho_m) \delta\psi \Big] dV + \int_T dt \int_{\partial\Omega} \Big[(T^i_* - n_j T^{ji}) \delta u_i \\ & + \Big(D_* - n_i D^i) \delta\phi + (B_* - n_i B^i) \delta\psi \Big] dS = 0 \end{aligned} \tag{3.6}$$

and in a compact form as

$$\begin{aligned} \delta L_H\{\Lambda_H\} = \int_T dt \int_\Omega (L^i_a \delta u_i + L_e \delta\phi + L_m \delta\psi)\,dV + \int_T dt \int_{\partial\Omega} (L^i_a \delta u_i \\ + L_e \delta\phi + L_m \delta\psi)\,dS = 0 \end{aligned} \tag{3.7}$$

in terms of the denotations defined in Eqs. (2.1)–(2.3), and Eqs. (2.26)–(2.28). The variational principle (3.6) or (3.7) leads to the divergence equations and the natural boundary conditions, as its Euler–Lagrange equations. The rest of the fundamental equations remain as its constraint (subsidiary) conditions. The three-field variational principle operating on the admissible state Λ_H is an augmented version of that operating on the two fields, $\Lambda_P = \{u_i, \phi\}$ in piezoelectricity (e.g., Tiersten, 1969).

The constraint conditions prevent a free and simple choice of approximating (or trial, shape, coordinate) functions of which the choice is crucial (Strang, 1975), and hence, the constraint conditions of the variational principle (3.7) are now removed through an involutory transformation (Friedrichs, 1929) so as to derive a unified variational principle for the continuum. The transformation was widely used in relaxing both holonomic and nonholonomic conditions in deriving the unified variational principle of a continuum due to its versatility in application. To apply the transformation, the dislocation potentials (de Veubeke, 1973), each constraint as a zero times a Lagrange multiplier, for each of the constraint conditions as

$$\Delta_1^1 = \int_T dt \int_\Omega (\lambda^{ij} L_{ij}^a + \eta^i L_i^e + \mu^i L_i^m)\, dV \tag{3.8}$$

$$\Lambda_2^2 = \int_T dt \left[\int_{\partial\Omega_u} \lambda^i L_{ai}^* \, dS + \int_{\partial\Omega_\phi} \eta L_{\phi^*}^e \, dS + \int_{\partial\Omega_\psi} \mu L_{\psi^*}^m \right] dS \tag{3.9}$$

are introduced in terms of the denotations (2.4)–(2.6) and (2.26)–(2.28), and the Lagrange parameters ($\lambda^{ij}, \ \eta^i, \ \mu^i, \ \lambda^i, \ \eta$ and μ), which are to be determined, and they are added into the variational integral (3.1) as

$$\delta L_G = \delta L_H + \delta \Delta_\alpha^\alpha = 0 \tag{3.10}$$

As before, taking the variations, considering the Lagrange multipliers as independent variables and integrating by parts with respect to time, one obtains Eq. (3.10) in the form

$$\delta L_G = \delta L_a \left\{ u_i, S_{ij}, \lambda^{ij}, \lambda^i \right\} + \delta L_e \{\phi, E_i, \eta^i, \eta\} + \delta L_m \{\psi, M_i, \mu^i, \mu\} = 0 \tag{3.11}$$

with the denotations of the form

$$\begin{aligned} \delta L_a = & \int_T dt \int_\Omega \left\{ \delta\lambda^{ij} L_{ij}^a + \lambda^{ij}(\delta S_{ij} - \delta u_{i;j}) + (f^i - \rho \ddot{u}^i)\delta u_i - \frac{\partial H}{\partial S_{ij}} \delta S_{ij} \right\} dV \\ & + \int_T dt \int_{\partial\Omega_u} (\delta\lambda^i L_{ai}^* + \lambda^i \delta u_i)\, dS + \int_T dt \int_{\partial\Omega} T_*^i \delta u_i \, dS \end{aligned} \tag{3.12}$$

$$\begin{aligned} \delta L_e = & \int_T dt \int_\Omega \left\{ \delta\eta^i L_i^e + \eta^i(\delta E_i + \delta\phi_{,i}) - \rho_e \delta\phi - \frac{\partial H}{\partial E_i} \delta E_i \right\} dV \\ & + \int_T dt \int_{\partial\Omega_\phi} (\delta\eta L_{\phi^*}^e + \eta\delta\phi)\, dS + \int_T dt \int_{\partial\Omega} D^* \delta\phi \, dS \end{aligned} \tag{3.13}$$

$$\begin{aligned} \delta L_m = & \int_T dt \int_\Omega \left\{ \delta\mu^i L_i^m + \mu^i(\delta M_i + \delta\psi_{,i}) - \rho_m \delta\psi - \frac{\partial H}{\partial M_i} \delta M_i \right\} dV \\ & + \int_T dt \int_{\partial\Omega_\psi} (\delta\mu L_{\psi^*}^m + \mu\delta\psi)\, dS + \int_T dt \int_{\partial\Omega} B^* \delta\psi \, dS \end{aligned} \tag{3.14}$$

Applying the divergence theorem to the regular region $\Omega + \partial\Omega$, and then, combining the terms in the integrands of the surface and volume integrals, one finally arrives at the variational equations of the form

$$\begin{aligned} \delta L_a = & \int_T dt \int_\Omega \left[(\lambda^{ij}_{;i} + f^j - \rho \ddot{u}^j)\delta u_j + \left(\lambda^{ij} - \frac{\partial H}{\partial S_{ij}} \right) \delta S_{ij} + \delta\lambda^{ij} L_{ij}^a \right] dV \\ & + \int_T dt \int_{\partial\Omega_u} \left[(\lambda^i - n_j \lambda^{ji})\delta u_i + \delta\lambda^i L_{ai}^* \right] dS \\ & + \int_T dt \int_{\partial\Omega} (T_*^i - n_j \lambda^{ji}) \delta u_i \, dS \end{aligned} \tag{3.15}$$

$$\begin{aligned} \delta L_e = & \int_T dt \int_\Omega \left[-(\eta^i_{;i} + \rho_e)\delta\phi + \left(\eta^i - \frac{\partial H}{\partial E_i} \right) \delta E_i + \delta\eta^i L_i^e \right] dV \\ & + \int_T dt \int_{\partial\Omega_\phi} \left[(\eta + n_i \eta^i)\delta\phi + \delta\eta L_{e\phi}^* \right] dS \\ & + \int_T dt \int_{\partial\Omega_d} (D^* + n_i \eta^i) \delta\phi \, dS \end{aligned} \tag{3.16}$$

$$\begin{aligned} \delta L_m = & \int_T dt \int_\Omega \left[-(\mu^i_{;i} + \rho_m)\delta\psi + \left(\mu^i - \frac{\partial H}{\partial M_i} \right) \delta M_i + \delta\mu^i L_i^m \right] dV \\ & + \int_T dt \int_{\partial\Omega_\psi} \left[(\mu + n_i \mu^i)\delta\psi + \delta\mu L_{m\psi}^* \right] dS \\ & + \int_T dt \int_{\partial\Omega_b} (B^* + n_i \mu^i) \delta\phi \, dS \end{aligned} \tag{3.17}$$

Substituting Eqs. (3.15)–(3.17) in Eq. (3.11), the Lagrangian multipliers are identified by the physical quantities of the form

$$\begin{aligned} & \lambda^{ij} - \frac{\partial H}{\partial S_{ij}} = 0, \ \ \lambda^{ij} = \frac{\partial H}{\partial S_{ij}} = T^{ij}; \ \ \lambda^i - n_j \lambda^{ji} = 0, \ \ \lambda^i = n_j \lambda^{ji} = n_j T^{ji} \\ & \eta^i - \frac{\partial H}{\partial E_i} = 0, \ \ \eta^i = \frac{\partial H}{\partial E_i} = -D^i; \ \ \eta + n_i \eta^i = 0, \ \ \eta = -n_i \eta^i = n_i D^i \\ & \mu^i - \frac{\partial H}{\partial M_i} = 0, \ \ \mu^i = \frac{\partial H}{\partial M_i} = -B^i; \ \ \mu + n_i \mu^i = 0, \ \ \mu = -n_i \mu^i = n_i B^i \end{aligned} \tag{3.18}$$

since the volumetric variations are arbitrary in the volume and the surface variations on the boundary surface. Lastly, the Lagrange multipliers in Eq. (3.18) are inserted in Eqs. (3.15)–(3.17), and then into Eq. (3.11) to obtain a unified variational principled of the form

$$\delta L_A\{\Lambda_A\} = \delta L_a\{\Lambda_a\} + \delta L_e\{\Lambda_e\} + \delta L_m\{\Lambda_m\} = 0\} \tag{3.19a}$$

with its admissible state by

$$\begin{aligned} & \Lambda_A = \Lambda_a \cup \Lambda_e \cup \Lambda_m; \Lambda_a = \{u_i, S_{ij}, T^{ij}\}, \Lambda_e = \{\phi, E_i, D^i\}, \\ & \Lambda_m = \{\psi, M_i, B^i\} \end{aligned} \tag{3.19b}$$

and the denotations of the form

$$\begin{aligned} \delta L_a\{\Lambda_a\} = & \int_T dt \int_\Omega \left\{ (T^{ij}_{;i} + f^j - \rho a^j)\delta u_j + \left[T^{ij} - \frac{1}{2}\left(\frac{\partial H}{\partial S_{ij}} + \frac{\partial H}{\partial S_{ji}} \right) \right] \delta S_{ij} \right. \\ & \left. + \left[S_{ij} - \frac{1}{2}(u_{i;j} + u_{j;i}) \right] \delta T^{ij} \right\} dV + \int_T dt \int_{\partial\Omega_u} (u_j - u_j^*) n_i \delta T^{ij} dS \\ & + \int_T dt \int_{\partial\Omega_t} (T_*^j - n_i T^{ij}) \delta u_j \, dS \end{aligned} \tag{3.20}$$

$$\begin{aligned} \delta L_e\{\Lambda_e\} = & \int_T dt \int_\Omega \left\{ (D^i_{;i} - \rho_e)\delta\phi - \left(D^i + \frac{\partial H}{\partial E_i} \right) \delta E_i - (E_i + \phi_{,i})\delta D^i \right\} dV \\ & + \int_T dt \int_{\partial\Omega_\phi} n_i(\phi - \phi_*)\delta D^i dS + \int_T dt \int_{\partial\Omega_d} (D^* - n_i D^i)\delta\phi dS \end{aligned} \tag{3.21}$$

$$\begin{aligned} \delta L_m\{\Lambda_m\} = & \int_T dt \int_\Omega \left\{ (B^i_{;i} - \rho_m)\delta\psi - \left(B^i + \frac{\partial H}{\partial M_i} \right) \delta M_i - (M_i + \psi_{,i})\delta B^i \right\} dV \\ & + \int_T dt \int_{\partial\Omega_\psi} n_i(\psi - \psi_*)\delta B^i dS + \int_T dt \int_{\partial\Omega_b} (B^* - n_i B^i)\delta\psi dS \end{aligned} \tag{3.22}$$

The variational principle is written in compact form by

$$\begin{aligned} \delta L_A\{\Lambda_A\} = & \int_T dt \int_\Omega (L_a^i \delta u_i + L_{ij}^a \delta T^{ij} + L_{ac}^{ij} S_{ij})\, dV \\ & + \int_T dt \int_{\partial\Omega_u} L_{ai}^* n_j \delta T^{ji} \, dS + \int_T dt \int_{\partial\Omega_t} L_{a^*}^i \delta u_i \, dS \\ & + \int_T dt \int_\Omega (L_e \delta\phi - L_i^e \delta D^i - L_{ec}^i \delta E_i)\, dV \\ & + \int_T dt \int_{\partial\Omega_\phi} L_{\phi^*}^e n_i \delta D^i \, dS + \int_T dt \int_{\partial\Omega_d} L_{e*} \delta\phi \, dS \\ & + \int_T dt \int_\Omega (L_m \delta\psi - L_i^m \delta B^i - L_{em}^i \delta M_i)\, dV \\ & + \int_T dt \int_{\partial\Omega_\psi} L_{\psi^*}^m n_i \delta B^i \, dS + \int_T dt \int_{\partial\Omega_m} L_{m^*} \delta\psi \, dS \end{aligned} \tag{3.23a}$$

and

$$\delta L_A\{\Lambda_A\} = \delta L_d\{\Lambda_d\} + \delta L_g\{\Lambda_g\} + \delta L_c\{\Lambda_c\} + \delta L_b\{\Lambda_b\} = 0\} \tag{3.23b}$$

in terms of Eqs. (2.1)–(2.6), (2.17) or (2.21)–(2.23), and (2.26)–(2.28). This equation is expressed in the form

$$\delta L_d\{\Lambda_d\} = \int_T dt \int_\Omega (L_a^i \delta u_i + L_e \delta\phi + L_m \delta\psi)\, dV, \Lambda_d = \{u_i, \phi, \psi\} \tag{3.24}$$

$$\delta L_g\{\Lambda_g\} = \int_T dt \int_\Omega (L_{ij}^a \delta T^{ij} - L_i^e \delta D^i - L_i^m \delta B^i)\, dV, \Lambda_g = \{T^{ij}, D^i, B^i\} \tag{3.25}$$

$$\delta L_c\{\Lambda_c\} = \int_T dt \int_\Omega (L_{ac}^{ij} \delta S_{ij} - L_{ec}^i \delta E_i - L_{mc}^i \delta M_i)\, dV, \Lambda_d = \{S_{ij}, E_i, M_i\} \tag{3.26}$$

$$\begin{aligned}\delta L_b\{\Lambda_b\} = {} & \int_T dt \int_{\Omega_u} L_{ai}^* n_j \delta T^{ji}\, dS + \int_T dt \int_{\Omega_t} L_{a^*}^i \delta u_i\, dS \\ & + \int_T dt \int_{\Omega_\phi} L_{\phi^*}^e n_i \delta D^i\, dS + \int_T dt \int_{\Omega_d} L_{e^*} \delta\phi\, dS + \int_T dt \int_{\Omega_\psi} L_{\psi^*}^m n_i \delta B^i\, dS \\ & + \int_T dt \int_{\Omega_m} L_{m^*} \delta\psi\, dS, \Lambda_b = \{u_i, T^{ij}, \phi, D^i, \psi, B^i\}\end{aligned} \tag{3.27}$$

for the subsequent development in the next sections. The nine-field variational principle is the counterpart of the Hu–Washizu variational principle of elasticity, in electromagnetoelasticity, and it is expressed by the following theorem.

3.1. Unified variational principle

Let $\Omega + \partial\Omega$ denote a regular, finite and bounded region of the electromagnetoelastic continuum, with its piecewise smooth boundary surface $\partial\Omega$ as

$$\begin{aligned}\partial\Omega &= \partial\Omega_u \cup \partial\Omega_t = \partial\Omega_\phi \cup \partial\Omega_d = \partial\Omega_\psi \cup \partial\Omega_b; \quad \partial\Omega_u \cap \partial\Omega_t \\ &= \partial\Omega_\phi \cap \partial\Omega_d = \partial\Omega_\psi \cap \partial\Omega_b = \emptyset\end{aligned}$$

and its closure $\overline{\Omega}$. Then, of all the admissible states that satisfy the initial conditions, (2.27)–(2.29), the symmetry of the stress tensor (2.1) and also the usual existence, continuity and differentiability conditions of the field variables, *if and only if*, that admissible state which satisfies the divergence equations (2.1)–(2.3), the gradient equations (2.4)–(2.6), the constitutive relations, (2.17), and the natural boundary conditions, (2.26)–(2.28), is determined by the variational principle, $\delta L_A\{\Lambda_A\} = 0$ in Eq. (3.23), as its Euler–Lagrange equations. Conversely, if these equations are identically met, the nine-field unified variational principle is evidently satisfied.

3.2. Discontinuous field variables

Now, consider the regular region $\Omega + \partial\Omega$ with a fixed internal surface of discontinuity S, which splits the region into the subregions $\Omega_\alpha + \partial\Omega_\alpha + S$ with their boundary surface $\partial\Omega_\alpha + S$ (see Fig. 1). The field variables undergo a jump across the internal surface S as

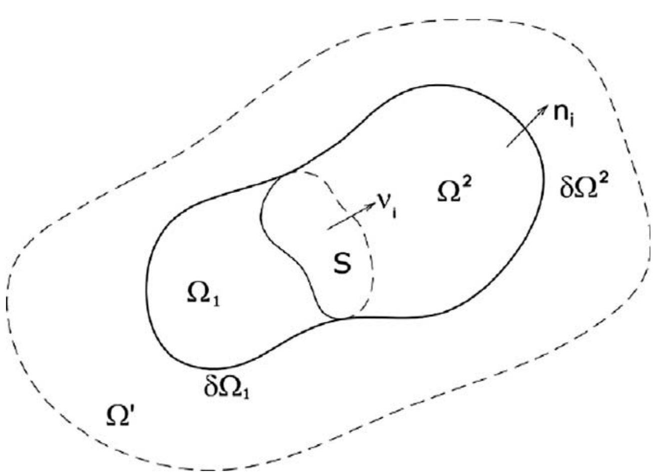

Fig. 1. Regular region of electromagnetoelastic materials with a discontinuity surface.

$$\begin{aligned}&L_{as}^j = v_i[\![T^{ij}]\!] = 0, L_i^{as} = [\![u_i]\!] = 0; \quad L_{se} = v_i[\![D^i]\!] = 0, L_{es} = [\![\phi]\!] = 0 \\ &L_{sm} = v_i[\![B^i]\!] = 0, L_{ms} = [\![\psi]\!] = 0 \quad \text{on } S \times T\end{aligned} \tag{3.28}$$

where the subregions are taken to have different electromagnetoelastic materials, v_i is the unit outward vector normal to S and pointing to Ω_2 , and the bold-face brackets, $[\![\chi]\!] = \chi_2 - \chi_1$, introduced by Christoffel (1877), is adopted in order to indicate the jump of a quantity across a surface of discontinuity S. The variational principle (3.23) is expressed for the regular region as an eighteen-field variational principle in the form

$$\delta L_D\{\Lambda_D\} = \sum_{\alpha=1}^{2} \delta L_A\{\Lambda_A\}; \quad \Lambda_D = \Lambda_A^{(1)} \cup \Lambda_A^{(2)} \tag{3.29}$$

which leads to all the fundamental equations of each subregion except the initial conditions, the symmetry of the stress tensor and the jump or continuity conditions on the interface S. To remove the interface (jump) conditions from the variational principle, the dislocation potential with the Lagrange parameters λ_a^i, λ_e and λ_m is introduced as follows:

$$\Delta = \int_T dt \int_S (\lambda_a^i L_i^{as} + \lambda_e L_{es} + \lambda_m L_{ms})\, dS \tag{3.30}$$

and it is added into Eq. (3.1) as

$$\delta L_D = \sum_{\alpha=1}^{2} \left\{ -\delta \int_T L\, dt - \int_T \delta F\, dt + \int_T \delta^* W\, dt + \delta\Delta_\beta^\beta \right\}^{(\alpha)} + \delta\Delta = 0 \tag{3.31}$$

in terms of Eqs. (3.8) and (3.9). As before in going from Eqs. (3.1) to (3.23), the variations are executed in Eq. (3.31), the generalized version of Green's theorem of the form

$$\int_{\Omega - S} \chi_{;i}^i\, dV = \oint_{\partial\Omega} n_i \chi^i\, dS - \int_S v_i [\![\chi^i]\!]\, dS \tag{3.32}$$

is applied and then the volumetric terms in Ω_α and the surface terms on $\partial\Omega_\alpha$ and S are collected with the result

$$\delta L_D\{\Lambda_D\} = \sum_{\alpha=1}^{2} \delta L_G^{(\alpha)}\{\Lambda_G\} + \delta L_S\{\Lambda_S\} = 0 \tag{3.33}$$

where

$$\begin{aligned}\delta L_S = {} & \int_T dt \int_S \Big\{ \delta\lambda_a^i [\![u_i]\!] + \lambda_a^i [\![\delta u_i]\!] + \delta\lambda_e [\![\phi]\!] + \lambda_e [\![\delta\phi]\!] + \delta\lambda_m [\![\psi]\!] \\ & + \lambda_m [\![\delta\psi]\!] \Big\}\, dS + \int_T dt \int_S v_i \{ [\![\lambda^{ij} \delta u_i]\!] - [\![\eta^i \delta\phi]\!] - [\![\mu^i \delta\psi]\!] \}\, dS\end{aligned} \tag{3.34a}$$

which can be expressed in an appropriate form

$$\begin{aligned}\delta L_S = {} & \int_T dt \int_S \{ \delta\lambda_a^i [\![u_i]\!] + \delta\lambda_e [\![\phi]\!] + \delta\lambda_m [\![\delta\psi]\!] \} \\ & + \int_T dt \int_S \sum_{\alpha=1}^{2} \Big\{ (\lambda_a^j + v_i \lambda_{(\alpha)}^{ij}) \delta u_j^{(\alpha)} + (\lambda_e - v_i \eta_{(\alpha)}^j) \delta\phi^{(\alpha)} \\ & + (\lambda_m - v_i \mu_{(\alpha)}^j) \delta\psi^{(\alpha)} \Big\} (-1)^{(\alpha)}\, dS\end{aligned} \tag{3.34b}$$

The Euler–Lagrange equations of Eq. (3.33) are those of the variational equation (3.11) together with Eq. (3.18), the jump conditions (3.28), and the equations of the form

$$\lambda_a^j + v_i \lambda_{(\alpha)}^{ij} = 0, \quad \lambda_e - v_i \eta_{(\alpha)}^i = 0, \quad \lambda_m - v_i \mu_{(\alpha)}^i = 0 \tag{3.35}$$

This equation with the result of Eq. (3.18) identifies the Lagrange parameters in an appropriate form

$$\begin{aligned}&\lambda_a^j = -v_i < \lambda^{ij} >= -v_i < T^{ij} >, \quad \lambda_e = v_i < \eta^i >= -v_i < D^i > \\ &\lambda_m = v_i < \mu^i >= -v_i < B^i >\end{aligned} \tag{3.36}$$

in terms of the physical quantities. Substituting Eq. (3.36) into Eq. (3.33), one arrives at a unified variational principle in the form

$$\delta L_D\{\Lambda_D = \Lambda_A\} = \sum_{\alpha=1}^{2}(\delta L_a\{\Lambda_a\} + \delta L_e\{\Lambda_e\} + \delta L_m\{\Lambda_m\}) + \int_T dt \int_S \nu_i \Big\{ [\![T^{ij}]\!] < \delta u_j > - < \delta T^{ij} > [\![u_j]\!] + [\![D^i]\!] < \delta\phi > - < \delta D^i > [\![\phi]\!] + [\![B^i]\!] < \delta\psi > - < \delta B^i > [\![\psi]\!] \Big\} dS = 0 \tag{3.37}$$

in terms of the denotations introduced in Eqs. (3.20)–(3.22). This unified variational principle leads, as the Euler–Lagrange equations, to all the fundamental equations, including the interface conditions on their bonding surface, except the symmetry of the stress tensor and initial conditions, for each of the electromagnetoelastic subregions.

3.3. Unified variational principle for a stratified region

Paralleling to the derivation of variational principle (3.37), a unified variational principle can be readily formulated for a stratified regular region $\Omega + \partial\Omega$ of electromagnetoelastic materials. The region contains a number of fixed internal surface of discontinuity $S_{n,n+1}$ or perfectly bonded dissimilar subregions (or layers) $\Omega_n + \partial\Omega_n$ with its boundary surface $\partial\Omega_n$ and closure $\overline{\Omega}_n$, where $n = 1,2,\ldots,N$. The interface or bonding surface between the (n)th and $(n+1)$th layers is denoted by $S_{n,n+1}$ with a unit outward vector $\nu_i^{(n,n+1)}$ pointing the $(n+1)$th layer. The motion of the electromagnetoelastic region is governed by the divergence equations (2.1)–(2.3), the gradient equations (2.4)–(2.6), and the constitutive relations (2.17) given by

$$L_a^{j(n)} = 0;\ L_e^{(n)} = 0;\ L_m^{(n)} = 0 \ \text{in}\ \overline{\Omega}^{(n)} \times T,\ n = 1,2,\ldots,N \tag{3.38}$$

$$L_i^{a(n)} = 0 \ \text{in}\ \overline{\Omega}^{(n)} \times T,\ n = 1,2,\ldots,N \tag{3.39}$$

$$L_{ij}^{a(n)} = 0,\ L_i^{e(n)} = 0,\ L_i^{m(n)} = 0 \ \text{in}\ \overline{\Omega}^{(n)} \times T,\ n = 1,2,\ldots,N \tag{3.40}$$

$$L_{ac}^{ij(n)} = 0,\ L_{ec}^{i(n)} = 0,\ L_{mc}^{i(n)} = 0 \ \text{in}\ \overline{\Omega}^{(n)} \times T,\ n = 1,2,\ldots,N \tag{3.41}$$

the boundary conditions (2.26)–(2.28) by

$$L_{a^*}^{j(n)} = 0, \quad \text{on}\ \partial\Omega_t^{(n)} \times T; \quad L_{ai}^{*(n)} = 0 \quad \text{on}\ \partial\Omega_u^{(n)} \times T; \quad n = 1,2,\ldots,N \tag{3.42}$$

$$L_{e^*}^{(n)} = 0; \quad \text{on}\ \partial\Omega_d^{(n)} \times T; \quad L_{\phi^*}^{e(n)} = 0 \quad \text{on}\ \partial\Omega_\phi^{(n)} \times T; \quad n = 1,2,\ldots,N \tag{3.43}$$

$$L_{m^*}^{(n)} = 0; \quad \text{on}\ \partial\Omega_b^{(n)} \times T; \quad L_{\psi^*}^{m(n)} = 0 \quad \text{on}\ \partial\Omega_\psi^{(n)} \times T; \quad n = 1,2,\ldots,N \tag{3.44}$$

the initial conditions (2.29)–(2.31) by

$$L_{ai}^{*(n)} = 0, \quad L_{ia}^{*(n)} = 0, \quad L_e^{*(n)} = 0, \quad L_m^{*(n)} = 0 \quad \text{in}\ \Omega^{(n)}(t_0) \times T; \quad n = 1,2,\ldots,N \tag{3.45}$$

and the interface conditions (3.28) by

$$\{L_{as}^j = \nu_i[\![T^{ij}]\!]\}^{(n)} = 0,\ \{L_i^{as} = [\![u_i]\!]\}^{(n)} = 0;\ \{L_{se} = \nu_i[\![D^i]\!]\}^{(n)} = 0,\ \{L_{es} = [\![\phi]\!]\}^{(n)} = 0 \\ \{L_{sm} = \nu_i[\![B^i]\!]\} = 0,\ \{L_{ms} = [\![\psi]\!] = 0\}^{(n)} \ \text{on}\ S_{(n,n+1)} \times T;\ n = 1,2,\ldots,(N-1) \tag{3.46}$$

at the time interval T for the electromagnetoelastic region. The system of Eqs. (3.38)–(3.41) comprises the twenty-nine N equations governing the 29 N dependent variables

$$\Lambda = \Big\{ u_i, S_{ij}, T^{ij}; \phi, E_i, D^i; \psi, M_i, B^i \Big\}^{(n)} \quad \text{with}\ n = 1,2,\ldots,N \tag{3.47}$$

The boundary and initial conditions (3.42)–(3.45) and the interface conditions (3.46) ensure the uniqueness in solutions of the governing equations. The conditions may be shown to be sufficient for the uniqueness as in the previous section.

Now, Hamilton's principle (3.1) is expressed in a modified form by

$$\delta L_{HL} = \sum_{n=1}^{N} \Big\{ -\delta \int_T L\,dt - \int_T \delta F\,dt + \int_T \delta^* W\,dt + \delta\Delta_\alpha^\alpha \Big\}^{(n)} = 0 \tag{3.48}$$

in terms of the dislocation potentials (3.8) and (3.9) for each constitutents of the stratified region. Proceeding as before, an evaluation of Eq. (3.48) yields a unified variational principle in terms of Eq. (3.23) as

$$\delta L_{DL}\{\Lambda_{DL}\} = \sum_{n=1}^{N} (\{\delta L_A\{\Lambda_A\})^{(n)} = 0 \tag{3.49}$$

which has the fundamental equations of all the subregions under the constraint conditions of the principle (3.19) or (3.23), that is, the symmetry of the stress tensor (2.1b) and the initial conditions (2.29)–(2.31) and the interface conditions (3.46). To incorporate the latter conditions into the variational principle, the dislocation potential is introduced as

$$\Delta_L = \sum_{n=1}^{N-1} \Big\{ \int_T dt \int_S (\lambda_a^i [\![u_i]\!] + \lambda_e [\![\phi]\!] + \lambda_m [\![\psi]\!])\,dS \Big\}^{(n)} \tag{3.50}$$

and it is added into Eq. (3.48), namely

$$\delta L_L\{\Lambda_L\} = \sum_{n=1}^{N} \Big\{ -\delta \int_T L dt - \int_T \delta F dt + \int_T \delta^* W dt + \delta\Delta_\alpha^\alpha \Big\}^{(n)} + \sum_{n=1}^{N-1} \delta \Big\{ \int_T dt \int_S (\lambda_a^i [\![u_i]\!] + \lambda_e [\![\phi]\!] + \lambda_m [\![\psi]\!])\,dS \Big\}^{(n)} = 0 \tag{3.51}$$

As in the derivation of the variational principle (3.37), the evaluation of Eq. (3.51) leads to Eq. (3.18) for each layer, and Eq. (3.36) for adjacent layers, namely

$$\lambda_{(n)}^{ij} = T_{(n)}^{ij};\ \lambda_{(n)}^{i} = (n_j T^{ji})^{(n)}; \quad n = 1,2,\ldots,N \tag{3.52}$$

and

$$\eta_{(n)}^i = -D_{(n)}^i; \quad \eta_{(n)} = (n_i D^i)^{(n)} \\ \mu_{(n)}^i = -B_{(n)}^i; \quad \mu_{(n)} = (n_i B^i)^{(n)}; \quad n = 1,2,\ldots,N \tag{3.53}$$

which readily leads to

$$\lambda_a^{j(n)} = \{-\nu_i < T^{ij} > \}^{(n)}; \quad \lambda_e^{(n)} = \{-\nu_i < D^i > \}^{(n)}; \\ \lambda_m^{(n)} = \{-\nu_i < B^i > \}^{(n)} \quad n = 1,2,\ldots,(N-1) \tag{3.54}$$

With the use of Eq. (3.54) in Eq. (3.51), one obtains a unified variational principle as

$$\delta L_L\{\Lambda_L\} = \sum_{n=1}^{N} (\delta L_a\{\Lambda_a\} + \delta L_e\{\Lambda_e\} + \delta L_m\{\Lambda_m\})^{(n)} + \delta L_{SL} = 0 \tag{3.55}$$

Here, the last term represents the interface conditions of the form

$$\delta L_{SL} = \int_T dt \sum_{n=1}^{N-1} \Big(\int_S \nu_i \Big\{ [\![T^{ij}]\!] < \delta u_j > - < \delta T^{ij} > [\![u_j]\!] + [\![D^i]\!] < \delta\phi > - < \delta D^i > [\![\phi]\!] + [\![B^i]\!] < \delta\psi > - < \delta B^i > [\![\psi]\!] \Big\} dS \Big)^{(n)} = 0 \tag{3.56}$$

Also, the variational principle may be expressed in an appropriate form by

$$\delta L_L\{\Lambda_L\} = \sum_{n=1}^{N} \langle \delta L_d\{\Lambda_d\} + \delta L_g\{\Lambda_g\} + \delta L_c\{\Lambda_c\} + \delta L_b\{\Lambda_b\} \rangle^{(n)} + \delta L_{SL} = 0 \tag{3.57}$$

in terms of the denotations of Eq. (3.24)–(3.27).

This variational principle generates the divergence equations (3.38), the gradient equations (3.40), the constitutive relations (3.41), the boundary conditions (3.42)–(3.44), and the interface conditions (3.46) of the stratified region, as its Euler–Lagrange equations, under the symmetry of the stress tensor (3.39) and the initial conditions (3.45), which can be relaxed as well (see, e.g., Dökmeci, 1994).

4. Shell geometry. Basic field variables. Constitutive and gradient equations

This section introduces the method of reduction in deducing the system of two-dimensional approximate equations governing the motions of a thin shell from the system of three- dimensional fundamental equations of electromagnetoelasticity.The shell is made of a functionally graded material, geometrically linear but physically non-linear in the elastic range, and responding at the short-wave length high-frequency motions. The long-wave length low-frequency approximation was almost always employed in the vibrations of shells (plates), where the wave length is large as compared with the thickness. However, by piezoelectric excitation, it is widespread to excite shells at both low and high frequency vibrations, where the wavelength is of the order of magnitude or smaller than the thickness. Perhaps, Ekstein (1945) was the first to investigate the vibrations of thin crystal plates at the high frequency vibrations. Mindlin (1961), Tiersten and Mindlin (1962) and Mindlin (1972), and then Mindlin (1974) studied the high frequency vibrations of crystal plates, piezoelectric and thermopiezoelectric plates, respectively. In addition, mention is made of the papers which treat the high frequency vibrations of shells (e.g., Berdichevskii and Khan'Chau, 1982; Langley and Bardell, 1998; Altay and Dökmeci, 2001; and Altay and Dökmeci, 2006; Lee and Hodges, 2009a,b), piezoelectric shells (Dökmeci, 1974; Le, 1999) and thermopiezoelectric shells (Altay and Dökmeci, 2002a). The low- and high-frequency motions of electromagnetoelastic structural elements are treated herein by chosing the mechanical displacements and the electric and magnetic potentials as the basic field variables. The resultants of stress, electric displacements and magnetic inductions are defined, and then the constitutive relations and the gradient equations are obtained using the unified variational principle in the previous section.

4.1. Geometry of shell

In the Euclidean three-dimensional space Ξ, consider the region of a thin shell $V+S$ (see Fig. 2), called the shell space with its boundary surface S and closure $\overline{V}(=V+S)$. The shell of uniform thickness 2 h is bounded by its lateral surface S_e, lower face S_{lf} and an upper face S_{uf}. The lateral or edge surface $S_e(=S\cap S_f, S_f=S_{uf}\cup S_{lf})$ is a right cylindrical surface with generators perpendicular to the middle (reference) surface A of shell, and it intersects the reference surface along a Jordan curve C. An outward unit vector normal to the edge surface is denoted by v_i and to the faces by n_i. The shell space is referred to by a fixed right-handed system of geodesic normal coordinates θ^i situated on the reference surface, and the $\theta^3(\equiv z)$ axis is taken upward so that

$$\theta^3(=z)=-h, \quad z=h, \quad f(\theta^1,\theta^2)=0 \tag{4.1}$$

define the lower and upper faces and the edge boundary surface, respectively. The region of thin shell is defined mathematically by the assumptions of the form

$$\in_s=\frac{2h}{R_0}\ll 1, \quad \eta_s=\frac{2h}{L}\ll 1 \tag{4.2}$$

and physically by

$$\lambda_s=\frac{\max|u_i|}{2h}\ll 1 \tag{4.3}$$

Here, $(\in_s,\ \eta_s$ and $\lambda_s)$, R_0 and L represent the shell parameters, the least principal radius of curvature and the smallest structural dimension of the middle surface of shell, respectively, and u_i is the mechanical displacement components of shell. The thinness of shell (not moderately thick or thick) is characterized by Eq. (4.2), and the range of geometrical linearity primarily by Eq. (4.3). The first restriction in Eq. (4.2) is a sufficient condition to ensure the existence of shell tensor (or shifters) μ^{α}_{β}, which plays an important role in the relationships between the space and surface tensors (e.g., Naghdi, 1963).

Certain results from the differential geometry of a surface, including some relations and identities between space and surface tensors and their derivatives are freely used in the derivation of shell equations in the next sections (see, e.g., Naghdi, 1963 and Librescu, 1975).

4.2. Method of reduction

The two-dimensional equations of shells were deduced from the three-dimensional fundamental equations by means of some methods of reduction (e.g., the direct method, the method of series expansions, the asymptotic method, Mindlin's method of reduction, the variational-asymptotic method). The methods of reduction were essentially based upon the approximation of a field variable, which is chosen as a basis of the derivation of shell equations at the outset. The approximation has no unique choice, but cannot be arbitrary, and it has to be mathematically complete in representing the field variable. A choice of kinematical type was widely used for the basis, it involves with differentiation operations, which are generally, simpler than integration operation, and also, with no additional equations of compatibility types arise in the derivation. Nevertheless, any other type of field quantities, for instance, strains, stresses and energy may be selected as a basis of the derivation in place of deformation. An approximation of kinematic type was expressed by some series expansions (e.g., power series, trigonometric series, series of Legendre and Jacobi polynomials) of the thickness coordinate and extensively used in the formulation of a large number of two-dimensional equations of shells with various shape, material and type of vibrations. The method of series expansions for a field of kinematic type, which was introduced by Cauchy (1829) and Poisson (1829) for plates and Basset (1890) for shells, allows taking into account of all the significant effects of higher orders. Mindlin (1967) recapitulated and used the method of series expansions with the integral method of Kirchhoff (1850) in investigating the high frequency of vibrations of beams and plates (see his collected works, Deresiewicz et al., 1989). Motivated by Mindlin's method of reduction, a number of authors systematically established the system of one or two-dimensional equations for the high frequency vibrations of structural elements (e.g., Altay and Dökmeci, 2002a, Altay and Dökmeci, 2002b, Altay and Dökmeci, 2005; Altay and Dökmeci, 2006; Altay and Dökmeci, 2007, and the references cited therein); but there was no paper available on the high frequency motions of electromagnetoelastic plates or shells within the publications recorded by "ISI-Web of Science".

4.3. Basic field variables

The displacements and the electric and magnetic potentials are chosen as a basic in deriving a system of two-dimensional approximate equations of electromagnetoelastic functionally graded shells. In view of the fundamental assumptions of shells (4.2) and (4.3) together with their pertinent existence, single-valued, suitable regularity, smoothness and continuity conditions, the shifted components of mechanical displacements and the electric and magnetic potentials are chosen as the basic field variables, and they are represented by a power series expansions of the thickness coordinate in the form

$$\bar{u}_i(\theta^i,t)=\sum_{n=0}^{N=\infty}X_{(n)}^{a(i)}(\theta^3\equiv z)\bar{u}_i^{(n)}(\theta^\alpha,t)\quad \text{in } \overline{\Omega}\times T \tag{4.4a}$$

with

$$\begin{aligned}&\bar{u}_\alpha=(\mu^{-1})_\alpha^\beta u_\alpha,\quad u_\alpha=\mu_\alpha^\beta\bar{u}_\beta,\quad u_3=\bar{u}_3\\ &\mu_\beta^\alpha=\delta_\beta^\alpha-zb_\beta^\alpha,\quad \mu_\alpha^\beta(\bar{\mu}^1)_\beta^\sigma=\delta_\alpha^\sigma\end{aligned} \tag{4.4b}$$

and

$$\phi(\theta^i,t)=\sum_{n=0}^{M=\infty}X_{(n)}^{e(i)}(z)\phi_{(n)}(\theta^\alpha,t)\quad \text{in } \overline{\Omega}\times T \tag{4.5}$$

$$\psi(\theta^i,t)=\sum_{n=0}^{P=\infty}X_{(n)}^{m(i)}(z)\psi_{(n)}(\theta^\alpha,t)\quad \text{in } \overline{\Omega}\times T \tag{4.6}$$

Here, u_i and $\bar{u}_i$ represent the components of mechanical displacement and its shifted components, which are referred to the base vectors of the shell space and the shell reference surface, respectively, μ_α^β is the shell tensor, $b_{\alpha\beta}$ denotes the second fundamental form of the middle surface of shell. The functions $(\bar{u}_i^{(n)}\in C_{12},\ \phi_{(n)}\in C_{10},\ \psi_{(n)}\in C_{10})$ are unknown a priori and independent functions of order (n) to be determined. From the mathematical point of view, a separation of variables solution is sought for the shell equations using Eqs. (4.6) and (4.7). The approximation functions are consistently taken as

$$X_{(n)}^{a(i)}=X_{(n)}^{e(i)}=X_{(n)}^{m(i)}=z^n,\quad \text{with } M=N=P \tag{4.7}$$

due to Weierstrass's theorem. The order of approximation is indicated by (N), and $N=1$ is the closest to the classical theory of elastic shells (cf., The Kirchhoff–Love hypotheses). The series expansions of field variables, (4.5)–(4.7) have sufficient freedom so as to incorporate as many higher order mechanical, electrical and magnetic effects as deemed desirable for a case of interest. The reader may be referred to Qatu (2004), Altay and Dökmeci (2003a) and Karlash (2008) for a discussion of the choice of the basic field variables.

4.4. Gradient equations

The choice of the basic field variables as in Eqs. (4.4)–(4.7) leads to similar series expansions for the rest of the field variables, namely

$$\begin{aligned}&\{S_{ij}(\theta^i,t),\ldots,D^i,\ldots,M_i(\theta^i,t),\ldots\}\\ &\quad=\sum_{n=0}^{N=\infty}\left\{S_{ij}^{(n)}(\theta^\alpha,t),\ldots,D_{(n)}^i,\ldots,M_i^{(n)}(\theta^\alpha,t),\ldots\right\}z^n\end{aligned} \tag{4.8}$$

Here, $S_{ij}^{(n)},\ D_{(n)}^i,\ldots,M_i^{(n)}$ represent the components of strain tensor, electric displacement and magnetic field vector of order (n), respectively. To obtain the quantities of higher orders, the gradient part (3.25) of the variational principle (3.23) is written as

$$\delta L_g=\int_T dt\int_Z\int_A(L_{ij}^a\delta T^{ij}-L_i^e\delta D^i-L_i^m\delta B^i)\mu\, dA\, dz \tag{4.9}$$

together with Eqs. (2.4)–(2.6). The strain–mechanical displacements relations (2.4) are expressed by

$$\begin{aligned}&S_{\alpha\beta}=\frac{1}{2}\left[\mu_\alpha^\nu(\bar{u}_{\nu:\beta}-b_{\nu\beta}\bar{u}_3)+\mu_\beta^\nu(\bar{u}_{\nu:\alpha}-b_{\nu\alpha}\bar{u}_3)\right],\\ &S_{\alpha3}=\frac{1}{2}\left[(\mu_\alpha^\beta\bar{u}_{\beta,3}+\bar{u}_{3,\alpha}+b_\alpha^\beta\bar{u}_\beta)\right],S_{33}=\bar{u}_{3,3}\end{aligned} \tag{4.10}$$

where the relations of the form

$$\begin{aligned}&u_{\alpha;\beta}=\mu_\alpha^\nu(\bar{u}_{\nu:\beta}-b_{\nu\beta}\bar{u}^3),\quad u_{\alpha;3}=\mu_\alpha^\nu\bar{u}_{\nu,3},\quad u_{3;\alpha}=\bar{u}_{3,\alpha}+b_\alpha^\beta\bar{u}_\beta,\\ &u_{3;3}=\bar{u}_{3,3}\end{aligned} \tag{4.11}$$

are used and the strain components are obtained in terms of the shifted components of the displacements. Thus, the variational integral (3.25) is expressed in the form

$$\begin{aligned}\delta L_g=\int_T dt\int_A dA\int_Z\Big\langle&\{S_{\alpha\beta}-\frac{1}{2}\left[\mu_\alpha^\nu(\bar{u}_{\nu:\beta}-b_{\nu\beta}\bar{u}_3)+\mu_\beta^\nu(\bar{u}_{\nu:\alpha}-b_{\nu\alpha}\bar{u}_3)\right]\}\delta T^{\alpha\beta}\\ &+2\left\{S_{\alpha3}-\frac{1}{2}\left[(\mu_\alpha^\beta\bar{u}_{\beta,3}+\bar{u}_{3,\alpha}+b_\alpha^\beta\bar{u}_\beta)\right]\right\}\delta T^{\alpha3}\ +(S_{33}-\bar{u}_{3,3})T^{33}\\ &+(E_i+\phi_{,i})\delta D^i+(M_i+\psi_{,i})\delta B^i\Big\rangle\mu dz\end{aligned} \tag{4.12}$$

Inserting Eqs. (4.4)–(4.6) with Eq. (4.7) into Eq. (4.12), and integrating with respect to the thickness coordinate (z), one obtains

$$\begin{aligned}\delta L_{Sg}=\int_T dt\int_A\sum_{n=0}^{N}\Big\langle&2\left\{S_{\alpha3}^{(n)}-\frac{1}{2}\left[(n+1)u_\alpha^{(n+1)}+u_{3,\alpha}^{(n)}-(n-1)b_\alpha^\nu u_\nu^{(n)}\right]\right\}\\ &\delta T_{(n)}^{\alpha3}\ +\left\{S_{\alpha\beta}^{(n)}-\frac{1}{2}\left[u_{\alpha:\beta}^{(n)}+u_{\beta:\alpha}^{(n)}-2b_{\alpha\beta}u_3^{(n)}-\left(b_\alpha^\nu u_{\nu:\beta}^{(n-1)}+b_\beta^\nu u_{\nu:\alpha}^{(n-1)}\right.\right.\right.\\ &\left.\left.\left.-2c_{\alpha\beta}u_3^{(n-1)}\right)\right]\right\}\delta T_{(n)}^{\alpha\beta}+[(S_{33}^{(n)}-(n+1)u_3^{(n+1)}]\delta T_{(n)}^{33}\\ &-(E_\alpha^{(n)}+\phi_{,\alpha}^{(n)})\delta D_{(n)}^\alpha-[(E_3^{(n)}+(n+1)\phi^{(n+1)}]\delta D_{(n)}^3\\ &-(M_\alpha^{(n)}+\psi_{,\alpha}^{(n)})\delta B_{(n)}^\alpha-[(M_3^{(n)}+(n+1)\psi^{(n+1)}]\delta B_{(n)}^3\Big\rangle dA\end{aligned} \tag{4.13}$$

Here, $c_{\alpha\beta}(=a_{\alpha\sigma}b_\beta^\sigma)$ and $a_{\alpha\sigma}$ are the third and first fundamental form of the reference surface A, and the stress, electric displacement and magnetic induction resultants of order (n) are defined in the form

$$(T_{(n)}^{ij},D_{(n)}^i,B_{(n)}^i)=\int_Z(T^{ij},D^i,B^i)\mu z^n\,dz,\quad n=1,2,\ldots,N \tag{4.14}$$

Also, they can be expressed by

$$T_{(n)}^{ij}=t_{(n)}^{ij}-2K_m t_{(n+1)}^{ij}+K_g t_{(n+2)}^{ij}\quad n=1,2,\ldots,N \tag{4.15}$$

and

$$\begin{aligned}&D_{(n)}^i=d_{(n)}^i-2K_m d_{(n+1)}^i+K_g d_{(n+2)}^i,\quad n=1,2,\ldots,N\\ &B_{(n)}^i=b_{(n)}^i-2K_m b_{(n+1)}^i+K_g b_{(n+2)}^i,\quad n=1,2,\ldots,N\end{aligned} \tag{4.16}$$

in terms of the mean and Gaussian curvatures, $K_m(=1/2b_\alpha^\alpha)$ and $K_g(=|b_\beta^\alpha|)$, of the reference surface, and the quantities of the form

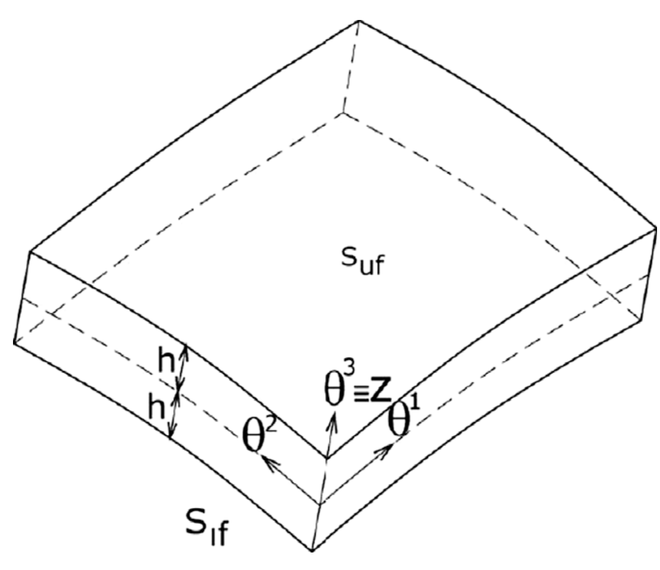

Fig. 2. An element of electromagnetoelastic shell.

$$(t^{ij}_{(n)}, d^i_{(n)}, b^i_{(n)}) = \int_Z (T^{ij}, D^i, B^i) z^n \, dz, \quad n = 1, 2, \ldots, N \tag{4.17}$$

The quantities of order (n) are evidently the functions of the surface coordinates and time t.

4.5. Constitutive relations

The constitutive coefficients of a material are known to be dependent on position, orientation and time as well as on temperature, moisture, stress, acceleration or other effects. All the effcets are taken to be negligible but position for the material of electromagnetoelastic shell (i.e., functionally graded). The numerical values of material coefficients are usually reported in a specific coordinate system, namely, the material coordinate system, where the coefficients have maximum symmetry, and they can be readily obtained from the material coordinate system to any system of coordinates through the usual tensor transformation. The shell material is assumed to be slowly and continuously varying and graded across the thickness direction (e.g., Markworth et al., 1995; Wu et al., 1996). The material coefficients are given in the fixed right-handed system of geodesic normal coordinates θ^i, to which the shell region is referred. A graded material was usually modeled either by a power law or an exponential law, which may be considered as a power law from the standpoint of numerical computation. The mass density, the body force and the material coefficients are expressed by a power law

$$\{\rho(z), f^i(z)\} = \sum_{n=0}^{N} \{\rho_n, f^i_n\} z^n \tag{4.18}$$

$$\{c^{ijkl}(z), e^{ijk}(z), \varepsilon^{ijk}(z), e^{ij}(z), \varepsilon^{ij}(z), c^{ij}(z)\} = \sum_{n=0}^{N} \{c^{ijkl}_{(n)}, e^{ijk}_n, \varepsilon^{ijk}_n, e^{ij}_n, \varepsilon^{ij}_n, c^{ij}_n\} z^n \tag{4.19}$$

in consistent with the series expansions of the basic field variables (4.4)–(4.7). To obtain the constitutive relations of shell, consider the constitutive part (3.26) of the variational integral (3.23) together with Eq. (2.17) in the form

$$\delta L_c = \int_T dt \int_A dA \int_Z \left[\left(T^{ij} - \frac{\partial H}{\partial S_{ij}} \right) \delta S_{ij} - \left(D^i + \frac{\partial H}{\partial E_i} \right) \delta E_i - (B^i + \frac{\partial H}{\partial M_i}) \delta M_i) \right] \mu \, dz \tag{4.20}$$

and inserting Eq. (4.19) into Eq. (4.20) and integrating with respect to the thickness coordinate, one obtains

$$\delta L_{Sc} = \int_T dt \int_A \sum_{n=1}^{N} \left[\left(T^{ij}_{(n)} - T^{ij}_{(n)c} \right) \delta S^{(n)}_{ij} - (D^i_{(n)} - D^i_{(n)c}) \delta E^{(n)}_i - (B^i_{(n)} - B^i_{(n)c}) \delta M^{(n)}_i \right] dA \tag{4.21}$$

with

$$T^{ij}_{(n)c} = \frac{1}{2} \left(\frac{\partial H_{(n)}}{\partial S^{(n)}_{ij}} + \frac{\partial H_{(n)}}{\partial S^{(n)}_{ji}} \right), \quad -D^i_{(n)c} = -\frac{\partial H_{(n)}}{\partial E^{(n)}_i},$$
$$B^i_{(n)c} = -\frac{\partial H_{(n)}}{\partial M^{(n)}_i}, \quad n = 1, 2, \ldots, N, \quad \text{on } A \times T \tag{4.22a}$$

where the enthalpy function per unit area of the reference surface A is given by

$$H_{(n)} = \int_Z H z^n \mu \, dz \tag{4.22b}$$

and the linear version of the constitutive equations (4.21) is obtained as

$$T^{ij}_{(n)c} = \sum_{p=0}^{N} \sum_{q=0}^{N} (c^{ijkl}_{(p)} S^{(q)}_{kl} - e^{ijk}_{(p)} E^{(q)}_k - \varepsilon^{ijk}_{(p)} M^{(q)}_k) \mu_{(p+q+n)},$$
$$n = 1, 2, \ldots, N; \quad \text{on } A \times T \tag{4.23}$$

$$D^i_{(n)c} = \sum_{p=0}^{N} \sum_{q=0}^{N} (e^{ijk}_{(p)} S^{(q)}_{jk} + e^{ij}_{(p)} E^{(q)}_k + \varepsilon^{ij}_{(p)} M^{(q)}_j) \mu_{(p+q+n)},$$
$$n = 1, 2, \ldots, N; \quad \text{on } A \times T \tag{4.24}$$

$$B^i_{(n)c} = \sum_{p=0}^{N} \sum_{q=0}^{N} (\varepsilon^{ijk}_{(p)} S^{(q)}_{jk} + c^{ij}_{(p)} M^{(q)}_k + \varepsilon^{ij}_{(p)} E^{(q)}_j) \mu_{(p+q+n)},$$
$$n = 1, 2, \ldots, N; \quad \text{on } A \times T \tag{4.25}$$

in which Eqs. (2.21)–(2.23) are used,and they are reduced to the form

$$T^{ij}_{(n)c} = \sum_{p=0}^{N} (c^{ijkl} S^{(p)}_{kl} - e^{ijk} E^{(p)}_k - \varepsilon^{ijk} M^{(p)}_k) \mu_{(p+n)},$$
$$n = 1, 2, \ldots, N; \quad \text{on } A \times T \tag{4.26}$$

$$D^i_{(n)c} = \sum_{p=0}^{N} (e^{ijk} S^{(p)}_{jk} + e^{ij} E^{(p)}_j + \varepsilon^{ij} M^{(p)}_j) \mu_{(p+n)},$$
$$n = 1, 2, \ldots, N; \quad \text{on } A \times T \tag{4.27}$$

$$B^i_{(n)c} = \sum_{p=0}^{N} (\varepsilon^{ijk} S^{(p)}_{jk} + c^{ij} M^{(p)}_j + \varepsilon^{ij} E^{(p)}_j) \mu_{(p+n)},$$
$$n = 1, 2, \ldots, N; \quad \text{on } A \times T \tag{4.28}$$

for a homogeneous material of shell. In the equations above, a quantity of the form

$$\mu_n = \int_Z \mu z^n \, dz = I_n - 2K_m I_{(n+1)} + K_g I_{(n+2)}, \quad n = 1, 2, \ldots, N \tag{4.29a}$$

in terms of the curvatures of the reference surface and the moment of inertia of order (n) by

$$I_n = \int_Z z^n \, dz = \frac{2h^{(n+1)}}{(n+1)} \delta_{(n,2p)}, \quad n = 1, 2, \ldots, N \tag{4.29b}$$

is introduced.

5. Equations of an electromagnetoelastic shell

This section deals with the derivation of the divergence equations and the natural boundary conditions for the high frequency vibrations of electromagneoelastic shell in variational form. The tractions are taken to be prescribed on the faces $S_f (= S_{lf} \cup S_{uf})$ and on some part C_t of the Jordan curve C on the edge surface S_e, and the mechanical displacements on the remaining part C_u of the curve C. The electric and magnetic potentials are given on the faces and the surface charge and the magnetic charge on the edge surface.

5.1. Divergence equations

Consider the divergence part (3.24) of the variational principle (3.23) in the absence of free charge and free current densities, namely

$$\delta L_d\{u_i, \phi, \psi\} = \int_T dt \int_A dA \int_Z \left\{ \left(T^{ij}_{;i} + f^j - \rho a^j \right) \delta u_j + D^i_{;i} \delta \phi + B^i_{;i} \delta \psi \right\} \mu \, dz \tag{5.1}$$

which may be written in terms of the shifted components of the mechanical displacements as

$$\delta L_d\{\bar{u}_i,\phi,\psi\} = \int_T dt \int_A dA \int_Z \Big\{ (T^{\beta\sigma}_{;\beta} + T^{3\sigma}_{;3} + f^\sigma)\mu^\alpha_\sigma \delta\bar{u}_\alpha + (T^{\alpha 3}_{;\alpha} + T^{33}_{;3} + f^3)\delta\bar{u}_3 - \rho\ddot{\bar{u}}^i\delta\bar{u}_i + D^i_{;i}\delta\phi + B^i_{;i}\delta\psi \Big\}\mu\, dz \tag{5.2}$$

After using the identities (e.g., Naghdi, 1963) of the form

$$\mu D^\alpha_{;\alpha} = (\mu D^\alpha)_{:\alpha} - \mu(\mu^{-1})^\alpha_\beta b^\beta_\alpha D^3 \tag{5.3}$$

for a vector field D^i and those by

$$\begin{aligned} &\mu\mu^\nu_\alpha T^{\alpha\beta}_{;\beta} = (\mu\mu^\nu_\eta T^{\eta\beta})_{:\beta} - \mu\mu^\nu_\alpha(\mu^{-1})^\beta_\lambda b^\lambda_\beta T^{\alpha 3} - \mu b^\nu_\beta T^{3\beta}\\ &\mu T^{3\alpha}_{;\alpha} = (\mu T^{3\alpha})_{:\alpha} + \mu\mu^\nu_\alpha b_{\nu\beta}T^{\alpha\beta} - \mu(\mu^{-1})^\gamma_\nu b^\nu_\beta T^{33}\\ &\mu^\beta_\alpha T^{\alpha 3}_{;3} = (\mu^\beta_\alpha T^{\alpha 3})_{,3} \end{aligned} \tag{5.4}$$

for a tensor field T^{ij}, together with the relations as

$$\mu_{,3} = -\mu(\mu^{-1})^\alpha_\beta b^\beta_\alpha \tag{5.5}$$

the covariant differentiations are expressed with respect to the metric tensor of the reference surface A in place of those of the space. Thus, Eq. (5.2) is expressed in the form

$$\begin{aligned} \delta L_d\{\bar{u}_i,\phi,\psi\} = \int_T dt \int_A dA \int_Z \Big\{ &\Big[(\mu\mu^\alpha_\nu T^{\beta\nu})_{:\beta} - \mu\mu^\alpha_\nu(\mu^{-1})^\beta_\lambda b^\lambda_\beta T^{\nu 3} - \mu b^\alpha_\nu T^{\nu 3} + \mu(\mu^\alpha_\nu T^{\nu 3})_{,3} + \mu^\alpha_\sigma f^\sigma - \rho\ddot{\bar{u}}^\alpha\Big]\delta\bar{u}_\alpha\\ &+\Big[(\mu T^{3\alpha})_{:\alpha} + \mu\mu^\nu_\alpha b_{\nu\beta}T^{\alpha\beta} - \mu(\mu^{-1})^\alpha_\beta b^\beta_\alpha T^{33} + \mu T^{33}_{,3} + f^3 - \rho\ddot{\bar{u}}^3\Big]\delta\bar{u}_3 + \Big[(\mu D^\alpha)_{:\alpha} - \mu(\bar{\mu}^1)^\alpha_\beta b^\beta_\alpha D^3\Big]\delta\phi\\ &+\Big[(\mu B^\alpha)_{:\alpha} - \mu(\bar{\mu}^1)^\alpha_\beta b^\beta_\alpha B^3\Big]\delta\psi\Big\} dz \end{aligned} \tag{5.6}$$

Inserting the expansions of the basic field variables (4.4)–(4.7) into Eq. (5.6) and considering Eq. (5.5), and then integrating with respect to the thickness coordinate and applying the divergence theorem for the shell region, one finally arrives at the divergence equations as follows

$$\delta L_{Sd}\{u^{(n)}_i,\phi_{(n)},\psi_{(n)}\} = \int_T dt \int_A \sum_{n=0}^N \Big\{ (V^i_{(n)} - A^i_{(n)})\delta u^{(n)}_i + \Phi_{(n)}\delta\phi_{(n)} + \Theta_{(n)}\delta\psi_{(n)}\Big\} dA \tag{5.7}$$

In this equation, the quantities of the form

$$V^\alpha_{(n)} = (T^{\beta\alpha}_{(n)} - b^\alpha_\nu T^{\beta\nu}_{(n+1)})_{:\beta} - nT^{3\alpha}_{(n-1)} + (n-1)b^\alpha_\beta T^{3\beta}_{(n)} + (F^\alpha_{(n)} - b^\alpha_\beta F^\beta_{(n)}) + Q^\alpha_{(n)} \tag{5.8a}$$

$$V^3_{(n)} = T^{\alpha 3}_{(n):\alpha} + b_{\alpha\beta}T^{\alpha\beta}_{(n)} - c_{\alpha\beta}T^{\alpha\beta}_{(n+1)} - nT^{33}_{(n-1)} + F^3_{(n)} + Q^3_{(n)},\quad n = 1,2,\ldots,N;\quad \text{on } A\times T \tag{5.8b}$$

$$\Phi_{(n)} = D^\alpha_{(n):\alpha} - nD^3_{(n-1)} + D_{(n)},\quad n = 1,2,\ldots,N;\quad \text{on } A\times T \tag{5.9}$$

$$\Theta_{(n)} = B^\alpha_{(n):\alpha} - nB^3_{(n-1)} + B_{(n)},\quad n = 1,2,\ldots,N;\quad \text{on } A\times T \tag{5.10}$$

in terms of the resultants (4.14) are defined. The effective surface quantities of order (n) are introduced by

$$Q^i_{(n)} = Q^{3i}_{u(n)} - Q^{3i}_{l(n)};\quad \{Q^{3i}_{u(n)}, Q^{3i}_{l(n)}\} = [(T^{3i} - zb^\alpha_\beta T^{3\beta}\delta^i_\alpha)z^n\mu]\quad \text{at } z = \{h,-h\} \tag{5.11}$$

$$D_{(n)} = D^3_{u(n)} - D^3_{l(n)};\quad \{D^3_{u(n)}, D^3_{l(n)}\} = D^3 z^n\mu,\quad n = 1,2,\ldots,N;\quad \text{at } z = \{h,-h\} \tag{5.12}$$

$$B_{(n)} = B^3_{u(n)} - B^3_{l(n)};\quad \{B^3_{u(n)}, B^3_{l(n)}\} = B^3 z^n\mu,\quad n = 1,2,\ldots,N;\quad \text{at } z = \{h,-h\} \tag{5.13}$$

and the acceleration resultants and body force resultants of order (n) by

$$A^i_{(n)} = \sum_{m,p=0}^N \mu_{(n+m+p)}\rho_{(m)}\ddot{u}^i_{(p)},\quad n = 1,2,\ldots,N;\quad \text{on } A\times T \tag{5.14}$$

$$F^i_{(n)} = \sum_{m=0}^N \mu_{(n+m)}(f^i_{(n)} - zb^\alpha_\beta f^\beta_{(n)}\delta^i_\alpha),\quad n = 1,2,\ldots,N;\quad \text{on } A\times T \tag{5.15}$$

Here, the series expansion (4.17) is used for the body force vector components and the mass density for the shell graded functionally across the thickness.

5.2. Boundary conditions

The boundary part (3.27) of the variational principle (3.23) is given by

$$\begin{aligned} \delta L_b\{\Lambda_b\} = &\int_T dt \int_{S_{uf}} (T^i_* - n_3\delta T^{3i})\delta u_i\mu\, dA + \int_T dt \int_{S_{lf}} (T^i_* - n_3\delta T^{3i})\delta u_i\mu\, dA\\ &+ \int_T dt \int_{C_t} dc \int_Z (T^i_* - \nu_\alpha T^{\alpha i})\delta u_i\mu\, dz + \int_T dt \int_{C_u} dc \int_Z (u_i - u^*_i)\nu_\alpha\delta T^{\alpha i}\mu\, dz\\ &+ \int_T dt \int_{S_{uf}} (\phi - \phi^*)n_3\delta D^3\mu\, dA + \int_T dt \int_{S_{lf}} (\phi - \phi^*)n_3\delta D^3\mu\, dA\\ &+ \int_T dt \oint_C dc \int_Z (D_* - \nu_\alpha D^\alpha)\delta\phi\mu\, dz + \int_T dt \int_{S_{uf}} (\psi - \psi^*)n_3\delta B^3\mu\, dA\\ &+ \int_T dt \int_{S_{lf}} (\psi - \psi^*)n_3\delta B^3\mu\, dA + \int_T dt \oint_C dc \int_Z (B_* - \nu_\alpha B^\alpha)\delta\psi\mu\, dz \end{aligned} \tag{5.16}$$

As before, inserting the series exansions (4.4)–(4.7) and evaluating the boundary surface and edge surface integrals in Eq. (5.16), one obtains the boundary conditions of the form

$$\begin{aligned} \delta L_{Sb}\{\Lambda_b\} = &\int_T dt \int_{S_{uf}} \sum_{n=0}^N (T^i_{*u(n)} - Q^{3i}_{u(n)})\delta u^{(n)}_i dA\\ &+ \int_T dt \int_{S_{lf}} \sum_{n=0}^N (T^i_{*l(n)} + Q^{3i}_{l(n)})\delta u^{(n)}_i dA + \int_T dt \int_{C_t} \sum_{n=0}^N (P^i_{*(n)} - \nu_\alpha P^{\alpha i}_{(n)})\delta u^{(n)}_i dc\\ &+ \int_T dt \int_{C_u} \sum_{n=0}^N (u^{(n)}_i - u^{*(n)}_i)\nu_\alpha(\delta T^{\alpha i}_{(n)} - b^\alpha_\beta\delta^i_\alpha\delta T^{3\beta}_{(n+1)})dc + \int_T dt \int_{S_{uf}} \sum_{n=0}^N (\phi_{(n)} - \phi^*_{u(n)})\delta D^3_{u(n)} dA\\ &+ \int_T dt \int_{S_{lf}} \sum_{n=0}^N (\phi^*_{(n)} - \phi^)_{l(n)})\delta D^3_{l(n)} dA + \int_T dt \oint_C \sum_{n=0}^N (D^{(n)}_* - \nu_\alpha D^\alpha_{(n)})\delta\phi_{(n)} dc\\ &+ \int_T dt \oint_C \sum_{n=0}^N (B^{(n)}_* - \nu_\alpha B^\alpha_{(n)})\delta\psi_{(n)} dc\\ &+ \int_T dt \int_{S_{uf}} \sum_{n=0}^N (\psi_{(n)} - \psi^*_{(n)})\delta B^3_{u(n)} dA\\ &+ \int_T dt \int_{S_{lf}} \sum_{n=0}^N (\psi^*_{(n)} - \psi^)_{(n)})\delta B^3_{l(n)} dA \end{aligned} \tag{5.17}$$

where the prescribed electric and magnetic potentials and components of mechanical displacements are expressed as in Eqs. (4.4)–(4.7). In this equation, the resultants of order (n) as

$$(T^i_{*u(n)}, T^i_{*l(n)}) = (T^i_* - z b^\alpha_\beta T^\beta_* \delta^i_\alpha)\mu z^n \quad \text{at } z = (h, -h), \quad n = 1, 2, \ldots, N$$

$$P^i_{*(n)} = \int_Z (T^i_* - z b^\alpha_\beta T^{\lambda\beta}_* \delta^i_\lambda)\mu z^n \, dz, \quad P^{\alpha i}_{(n)} = T^{\alpha i}_{(n)} - b^\alpha_\beta T^{\lambda\beta}_{(n+1)} \delta^i_\lambda,$$
$$n = 1, 2, \ldots, N \tag{5.18}$$

and

$$\{D^{(n)}_*, B^{(n)}_*\} = \int_Z \{D_*, B_*\}\mu z^n \, dz, \quad n = 1, 2, \ldots, N \tag{5.19}$$

are defined.

5.3. Initial conditions

In view of Eqs. (4.4)–(4.7), the initial conditions of higher orders are recorded for the mechanical, electric and magnetic fields as

$$u^{(n)}_i(\theta^\alpha, t_0) - v^{(n)*}_i(\theta^\alpha) = 0, \quad \dot{u}^{(n)}_i(\theta^\alpha, t_o) - w^{(n)*}_i(\theta^\alpha) = 0,$$
$$n = 1, 2, \ldots, N; \quad \text{on } A(t_o) \tag{5.20}$$

$$\phi_{(n)}(\theta^\alpha, t_o) - \alpha^*_{(n)}(\theta^\beta) = 0, \quad \psi_{(n)}(\theta^\alpha, t_o) - \beta^*_{(n)}(\theta^\alpha) = 0,$$
$$n = 1, 2, \ldots, N \quad \text{on } A(t_o) \tag{5.21}$$

where the expansions similar to Eqs. (4.4)–(4.7) are used for the prescribed initial functions.

5.4. Variational shell equations

Thus far, a deterministic system of two-dimensional approximate shear deformation equations is derived for the high frequency motions of a functionally graded electromagnetoelastic shell. A substitution of the gradient equation (4.13), the constitutive relations (4.20), the divergence equations (5.7), and the boundary conditions (5.17) into the variational integral (3.23) leads to the system of shell equations in variational form, namely

$$\delta L_S\{\Lambda_S\} = \delta L_{Sd}\{\Lambda_{Sd}\} + \delta L_{Sg}\{\Lambda_{Sg}\} + \delta L_{Sc}\{\Lambda_{Sc}\} + \delta L_{Sb}\{\Lambda_S\} = 0\} \tag{5.22a}$$

with its admissible state of the form

$$\Lambda_S = \Lambda_{Sd} \cup \Lambda_{Sg} \cup \Lambda_{Sc}$$
$$\Lambda_{Sd} = \{u^{(n)}_i, \phi_{(n)}, \psi_{(n)}\}, \quad \Lambda_{Sg} = \{T^{ij}_{(n)}, D^i_{(n)}, B^i_{(n)}\},$$
$$\Lambda_{Sc} = \{S^{(n)}_{ij}, E^{(n)}_i, M^{(n)}_i\} \tag{5.22b}$$

The system of shell equations has only the initial conditions (5.20) and (5.21), and the symmetry of the stress resultants, which is expressed using Eq. (2.1b) and Eq. (4.14) with the alternating tensor of the reference surface, $\varepsilon_{\alpha\beta}\{= \varepsilon_{\alpha\beta 3}(\theta^\alpha, z = 0)\}$ in the form

$$\varepsilon_{\alpha\beta} T^{3\alpha}_{(n)} + \varepsilon_{\beta\alpha} T^{\alpha 3}_{(n)} = 0, \quad \varepsilon_{\alpha v}(T^{v\alpha}_{(n)} - b^v_\beta T^{\beta\alpha}_{(n+1)}) = 0,$$
$$n = 1, 2, \ldots, N \quad \text{on } A \times T \tag{5.23}$$

as its constraint conditions.

5.5. Differential shell equations

The variational equation (5.22) generates the differential divergence equation (5.7) by

$$V^\alpha_{(n)} = (T^{\beta\alpha}_{(n)} - b^\alpha_v T^{\beta v}_{(n+1)})_{:\beta} - n T^{3\alpha}_{(n-1)} + (n-1) b^\alpha_\beta T^{3\beta}_{(n)}) + (F^\alpha_{(n)} - b^\alpha_\beta F^\beta_{(n)})$$
$$+ Q^\alpha_{(n)} - A^\alpha_{(n)} = 0$$
$$V^3_{(n)} = T^{\alpha 3}_{(n):\alpha} + b_{\alpha\beta} T^{\alpha\beta}_{(n)} - c_{\alpha\beta} T^{\alpha\beta}_{(n+1)} - n T^{33}_{(n-1)} + F^3_{(n)} + Q^3_{(n)} - A^3_{(n)} = 0,$$
$$n = 1, 2, \ldots, N; \quad \text{on } A \times T \tag{5.24}$$

in terms of Eq. (5.8), and

$$\Phi_{(n)} = D^\alpha_{(n):\alpha} - n D^3_{(n-1)} + D_{(n)} = 0, \quad n = 1, 2, \ldots, N; \quad \text{on } A \times T \tag{5.25}$$

$$\Theta_{(n)} = B^\alpha_{(n):\alpha} - n B^3_{(n-1)} + B_{(n)} = 0, \quad n = 1, 2, \ldots, N; \quad \text{on } A \times T \tag{5.26}$$

in terms of Eqs. (5.9) and (5.10), gradient equations (4.13) by

$$S^{(n)}_{\alpha\beta} = \frac{1}{2}\left[u^{(n)}_{\alpha:\beta} + u^{(n)}_{\beta:\alpha} - 2b_{\alpha\beta}u^{(n)}_3 - \left(b^v_\alpha u^{(n-1)}_{v:\beta} + b^v_\beta \bar{u}^{(n-1)}_{v:\alpha} - 2c_{\alpha\beta}u^{(n-1)}_3\right)\right],$$
$$n = 1, 2, \ldots, N$$
$$S^{(n)}_{\alpha 3} = \frac{1}{2}\left[(n+1)u^{(n+1)}_\alpha + u^{(n)}_{3,\alpha} - (n-1)b^v_\alpha u^{(n)}_v\right],$$
$$S^{(n)}_{33} = (n+1)u^{(n+1)}_3 \quad \text{on } A \times T \tag{5.27}$$

$$E^{(n)}_\alpha = -\phi^{(n)}_{,\alpha}, \quad E^{(n)}_3 = -(n+1)\phi^{(n+1)},$$
$$n = 1, 2, \ldots, N; \quad \text{on } A \times T \tag{5.28}$$

$$M^{(n)}_\alpha = -\psi^{(n)}_{,\alpha}, \quad M^{(n)}_3 = -(n+1)\psi^{(n+1)},$$
$$n = 1, 2, \ldots, N; \quad \text{on } A \times T \tag{5.29}$$

constitutive relations (4.20) by

$$T^{ij}_{(n)} - T^{ij}_{(n)c} = 0, \quad n = 1, 2, \ldots, N; \quad \text{on } A \times T \tag{5.30}$$
$$D^i_{(n)} - D^i_{(n)c} = 0, \quad n = 1, 2, \ldots, N; \quad \text{on } A \times T \tag{5.31}$$
$$B^i_{(n)} - B^i_{(n)c} = 0, \quad n = 1, 2, \ldots, N; \quad \text{on } A \times T \tag{5.32}$$

and natural boundary conditions (5.17) by

$$T^i_{u(n)*} - Q^i_{u(n)} = 0 \quad \text{on } S_{uf} \times T, T^i_{l(n)*} + Q^i_{l(n)} = 0 \quad \text{on } S_{lf} \times T,$$
$$n = 1, 2, \ldots, N; \quad \text{on } A \times T$$
$$P^i_{*(n)} - v_\alpha P^{\alpha\beta}_{(n)} = 0 \quad \text{on } C_t \times T, u^{(n)}_i - u^{*(n)}_i = 0 \quad \text{on } C_u \times T,$$
$$n = 1, 2, \ldots, N; \quad \text{on } A \times T \tag{5.33}$$

and

$$\phi_{(n)} - \phi^*_{u(n)} = 0 \quad \text{on } S_{uf} \times T, \phi_{(n)} - \phi^*_{l(n)} = 0 \quad \text{on } S_{lf} \times T,$$
$$n = 1, 2, \ldots, N; \quad \text{on } A \times T$$
$$D^{(n)}_* - v_\alpha D^\alpha_{(n)} = 0 \quad \text{on } C \times T,$$
$$n = 1, 2, \ldots, N; \quad \text{on } A \times T \tag{5.34}$$

and

$$\psi_{(n)} - \psi^*_{u(n)} = 0 \quad \text{on } S_{uf} \times T, \psi_{(n)} - \psi^*_{l(n)} = 0 \quad \text{on } S_{lf} \times T,$$
$$n = 1, 2, \ldots, N; \quad \text{on } A \times T$$
$$B^{(n)}_* - v_\alpha B^\alpha_{(n)} = 0 \quad \text{on } C \times T,$$
$$n = 1, 2, \ldots, N; \quad \text{on } A \times T \tag{5.35}$$

as the Euler–Lagrange equations.

In the next section, the boundary and initial conditions (5.20), (5.21), and (5.33)–(5.35) are shown to be sufficient for the uniqueness in solutions of the system of two dimensional approximate equations of the shell.

6. Uniqueness of solutions

To investigate the internal consistency of solutions in the lower order equations of structural elements and, in particular, the uniqueness of solutions may be traced back to Kirchhoff (1850). He gave the earliest proof of uniqueness for plates and Byrne (1944) for shells using the classical energy argument. A uniqueness theorem was also given by a number of authors for elastic shells at low frequency vibrations (e.g., Gol'denveizer, 1944; Green and Naghdi, 1971; Naghdi and Trapp, 1972; Weinitschke, 1988) and for piezoelectric, thermoelastic and thermopiezoelectric shells at high frequency vibrations (Dökmeci, 1974a,b; Altay and Dökmeci, 2001; Altay and Dökmeci, 2002a,b). Naghdi (1972) discussed the uniqueness theorems for solutions of the initial (isothermal) boundary-value problems of elastic shells and plates based on the assumption of the nonnegative free energy (or the strain energy) density. Now, a theorem of uniqueness is proved below for the system of two-dimensional linear equations of an electromagnetoelastic homogeneous or non-graded shell derived in the previous section.

To prove the theorem, the existence of two possible sets of admissible state of the shell equations as

$$\Lambda_S^{(\alpha)}\left\{u_i^{(n)},\phi_{(n)},\psi_{(n)};,T^{ij}_{(n)},D^i_{(n)},B^i_{(n)};S^{(n)}_{ij},E^{(n)}_i,M^{(n)}_i\right\}^{(\alpha)},\quad n=1,2,\ldots,N;\quad \text{on } A\times T \tag{6.1}$$

and their difference by

$$\Lambda_S=\Lambda_S^{(2)}\{u_i^{(n)},\ldots,S^{(n)}_{ij},E^{(n)}_i,M^{(n)}_i\}^{(2)}\cap\Lambda_S^{(1)}\{u_i^{(n)},\ldots,S^{(n)}_{ij},E^{(n)}_i,M^{(n)}_i\}^{(1)} \tag{6.2}$$

is considered at the outset. The difference set of admissible solutions (6.2) satisfies the two-dimensional homogeneous system of linear shell equations. The homogeneous shell equations consist of the gradient equations (5.27)–(5.29), the linear constitutive relations (4.26)–(4.28), and the homogeneous divergence equations from Eqs. (5.8)–(5.10) as

$$\begin{aligned}&(T^{\beta\alpha}_{(n)}-b^\alpha_\nu T^{\beta\nu}_{(n)})_{:\beta}-b^\alpha_\beta T^{\beta3}_{(n)}-n(T^{3\alpha}_{(n-1)}-b^\alpha_\beta T^{3\beta}_{(n)})+Q^\alpha_{(n)}-\rho\mathscr{A}^\alpha_{(n)}=0,\\&\quad n=1,2,\ldots,N;\quad \text{on } A\times T\\&T^{\alpha3}_{(n):\alpha}+b_{\alpha\beta}T^{\alpha\beta}_{(n)}-c_{\alpha\beta}T^{\alpha\beta}_{(n+1)}-nT^{33}_{(n-1)}+Q^3_{(n)}-\rho\mathscr{A}^3_{(n)}=0\\&\quad \text{on } A\times T,\quad n=1,2,\ldots,N;\quad \text{on } A\times T\end{aligned} \tag{6.3a}$$

where

$$\mathscr{A}^i_{(n)}=\sum_{m=0}^{N}\mu_{(m+n)}\ddot{u}^i_{(m)},\quad n=1,2,\ldots,N;\quad \text{on } A\times T \tag{6.3b}$$

and

$$D^\alpha_{(n):\alpha}-nD^3_{(n-1)}+D_{(n)}=0,\quad n=1,2,\ldots,N;\quad \text{on } A\times T \tag{6.4}$$

$$B^\alpha_{(n):\alpha}-nB^3_{(n-1)}+B_{(n)}=0,\quad n=1,2,\ldots,N;\quad \text{on } A\times T \tag{6.5}$$

the boundary conditions from Eqs. (5.26)–(5.29) as

$$\begin{aligned}&Q^i_{u(n)}=0\quad \text{on } S_{uf}\times T, Q^i_{l(n)}=0\quad \text{on } S_{lf}\times T,\quad n=1,2,\ldots,N;\quad \text{on } A\times T\\&\nu_\alpha P^{\alpha\beta}_{(n)}=0\quad \text{on } C_t\times T, u^{(n)}_i=0\quad \text{on } C_u\times T,\quad n=1,2,\ldots,N;\quad \text{on } A\times T\end{aligned} \tag{6.6}$$

and

$$\begin{aligned}&\phi_{(n)}=0\quad \text{on } S_{uf}\times T,\phi_{(n)}=0\quad \text{on } S_{lf}\times T,\nu_\alpha D^\alpha_{(n)}=0,\\&\quad n=1,2,\ldots,N\quad \text{on } C\times T\end{aligned} \tag{6.7}$$

$$\begin{aligned}&\psi_{(n)}=0\quad \text{on } S_{uf}\times T,\psi_{(n)}=0\quad \text{on } S_{lf}\times T,\nu_\alpha B^\alpha_{(n)}=0,\\&\quad n=1,2,\ldots,N\quad \text{on} C\times T\end{aligned} \tag{6.8}$$

and the initial conditions by

$$u_i^{(n)}(\theta^\alpha,t_0)=0,\quad \dot{u}_i^{(n)}(\theta^\alpha,t_o)=0,\quad n=1,2,\ldots,N\quad \text{on } A(t_o) \tag{6.9}$$

$$\phi_{(n)}(\theta^\alpha,t_o)=0,\quad n=1,2,\ldots,N\quad \text{on } A(t_o) \tag{6.10}$$

$$\psi_{(n)}(\theta^\alpha,t_o)=0,\quad n=1,2,\ldots,N\quad \text{on } A(t_o) \tag{6.11}$$

in terms of the difference admissible state.

6.1. Energy of the electromagnetoelastic shell

Let K denote the kinetic energy per unit area of the reference surface A and K_S the total kinetic energy of the electromagnetoelastic shell given by

$$K=\int_Z k\mu\,dz,\quad K_S=\int_A K\,dA,\quad k=\frac{1}{2}\rho\dot{u}_i\dot{u}^i \tag{6.12}$$

in terms of the kinetic energy density k from Eq. (2.7b). The density k and its rate are expressed in the form

$$k=\frac{1}{2}\rho\left[\mu^\beta_\alpha\dot{u}_\beta(\bar{\mu}^1)^\alpha_\sigma\dot{u}^\sigma+\dot{u}^3\dot{u}_3\right]=\frac{1}{2}\rho\dot{u}_i\dot{u}^i,\quad \dot{k}=\rho a_i\dot{u}^i \tag{6.13}$$

in terms of the shifted components of the mechanical displacements, (4.6) and (4.7). Inserting Eq. (4.4) with Eq. (4.7) into Eq. (6.12) and integrating with respect to the thickness coordinate, the rate of the areal density K is obtained as

$$\dot{K}=\rho\sum_{n=0}^{N}\mathscr{A}^i_{(n)}\dot{u}^{(n)}_i \tag{6.14}$$

in terms of the acceleration components (6.3b).

From Eq. (2.11) the rate of the total internal energy of the shell is given by

$$\dot{U}_S=\dot{U}_a+\dot{U}_e+\dot{U}_m \tag{6.15}$$

where

$$\begin{aligned}&\dot{U}_a=\int_A dA\int_Z T^{ij}\dot{S}_{ij}\mu\,dz,\quad \dot{U}_e=\int_A dA\int_Z \dot{D}^iE_i\mu\,dz\\&\dot{U}_m=\int_A dA\int_Z \dot{B}^iM_i\mu\,dz\end{aligned} \tag{6.16}$$

After inserting Eqs. (5.23)–(5.29) into Eq. (6.16), and then integrating it with respect to the thickness coordinate, one obtains the rate of the internal energies of the form

$$\begin{aligned}\dot{U}_a=\sum_{n=0}^{N}\int_A dA\Big\{&\frac{1}{2}T^{\alpha\beta}_{(n)}\Big[\dot{u}^{(n)}_{\alpha:\beta}+\dot{u}^{(n)}_{\beta:\alpha}-2b_{\alpha\beta}\dot{u}^{(n)}_3-(b^\nu_\alpha\dot{u}^{(n-1)}_{\nu:\beta}\\&+b^\nu_\beta\dot{u}^{(n-1)}_{\nu:\alpha}-2c_{\alpha\beta}\dot{u}^{(n-1)}_3)\Big]+T^{\alpha3}_{(n)}\Big[(n+1)\dot{u}^{(n+1)}_\alpha+\dot{u}^{(n)}_{3,\alpha}\\&-(n-1)b^\nu_\alpha\dot{u}^{(n)}_\nu\Big]+(n+1)T^{33}_{(n)}\dot{u}^{(n+1)}_3\Big\}\end{aligned} \tag{6.17a}$$

Considering the symmetry of the stress resultants (5.23), the rate of the mechanical energy takes the form

$$\begin{aligned}\dot{U}_a=\sum_{n=0}^{N}\int_A dA\Big\{&T^{\alpha\beta}_{(n)}\Big[\dot{u}^{(n)}_{\alpha:\beta}-b^\nu_\alpha\dot{u}^{(n-1)}_{\nu:\beta}-b_{\alpha\beta}\dot{u}^{(n)}_3+c_{\alpha\beta}\dot{u}^{(n-1)}_3\Big]\\&+T^{\alpha3}_{(n)}\Big[(n+1)\dot{u}^{(n+1)}_\alpha+\dot{u}^{(n)}_{3,\alpha}-(n-1)b^\nu_\alpha\dot{u}^{(n)}_\nu\Big]+(n+1)T^{33}_{(n)}\dot{u}^{(n+1)}_3\Big\}\end{aligned} \tag{6.17b}$$

and the rate of the electromagnetic energy as

$$\dot{U}_e=-\sum_{n=0}^{N}\int_A dA\Big\{\dot{D}^\alpha_{(n)}\phi^{(n)}_{,\alpha}+(n+1)\dot{D}^3_{(n+1)}\phi^{(n+1)}\Big\} \tag{6.18}$$

$$\dot{U}_m=-\sum_{n=0}^{N}\int_A dA\Big\{\dot{B}^\alpha_{(n)}\psi^{(n)}_{,\alpha}+(n+1)\dot{B}^3_{(n+1)}\psi^{(n+1)}\Big\} \tag{6.19}$$

is obtained. Applying the divergence theorem and collecting the surface and line integrals, Eqs. (6.17)–(6.19) are written as

$$\dot{U}_a = \int_A dA \sum_{n=0}^{N} \Big\{ -(T^{\beta\alpha}_{(n)} - b^\alpha_\nu T^{\beta\nu}_{(n+1)})_{:\alpha} \dot{u}^{(n)}_\beta - T^{\alpha 3}_{(n),\alpha} \dot{u}^{(n)}_3$$
$$-(b_{\alpha\beta} T^{\alpha\beta}_{(n)} - c_{\alpha\beta}) T^{\alpha\beta}_{(n-1)} \dot{u}^{(n)}_3 + T^{\alpha 3}_{(n)} \left[(n+1)\dot{u}^{(n+1)}_\alpha - (n-1) b^\nu_\alpha \dot{u}^{(n)}_\nu \right]$$
$$+(n+1) T^{33}_{(n)} \dot{u}^{(n+1)}_3 \Big\} + \Gamma_a \tag{6.20}$$

$$\dot{U}_e = \sum_{n=0}^{N} \int_A dA \{ \dot{D}^\alpha_{(n):\alpha} \phi^{(n)} - (n+1) \dot{D}^3_{(n+1)} \phi^{(n+1)} \} + \Gamma_e \tag{6.21}$$

$$\dot{U}_m = \sum_{n=0}^{N} \Big\{ \dot{B}^\alpha_{(n):\alpha} \psi^{(n)} - (n+1) \dot{B}^3_{(n+1)} \psi^{(n+1)} \Big\} + \Gamma_c \tag{6.22}$$

Here, the denotations of the form

$$\Gamma_a = \sum_{n=0}^{N} \oint_C \nu_\alpha \Big[\Big(T^{\beta\alpha}_{(n)} - b^\alpha_\nu T^{\beta\nu}_{(n+1)} \Big) \dot{u}^{(n)}_\beta + T^{\alpha 3}_{(n)} \dot{u}^{(n)}_3 \Big] dc \tag{6.23}$$

$$\Gamma_e = -\sum_{n=0}^{N} \oint_C \dot{D}^\alpha_{(n)} \phi \, dc \tag{6.24}$$

$$\Gamma_m = -\sum_{n=0}^{N} \oint_C \dot{B}^\alpha_{(n)} \psi \, dc \tag{6.25}$$

are defined.

6.2. Sufficient conditions

Now, as a last step toward proving the theorem, using Eqs. (6.3) and (6.4) one writes the integral of the form as

$$\int_T dt \int_A \sum_{n=0}^{N} \Big\{ \Big[(T^{\beta\alpha}_{(n)} - b^\alpha_\nu T^{\beta\nu}_{(n)})_{;\beta} - b^\alpha_\beta T^{\beta 3}_{(n)} - n(T^{3\alpha}_{(n-1)} - b^\alpha_\beta T^{3\beta}_{(n)}) + Q^\alpha_{(n)}$$
$$-\rho \mathscr{A}^\alpha_{(n)} \Big] \dot{u}^{(n)}_\alpha + \Big[T^{\alpha 3}_{(n):\alpha} + b_{\alpha\beta} T^{\alpha\beta}_{(n)} - c_{\alpha\beta} T^{\alpha\beta}_{(n+1)} - n T^{33}_{(n-1)} + Q^3_{(n)}$$
$$-\rho \mathscr{A}^3_{(n)} \Big] \dot{u}^{(n)}_3 - (\dot{D}^\alpha_{(n):\alpha} - n \dot{D}^3_{(n-1)} + \dot{D}_{(n)}) \phi_{(n)}$$
$$-(\dot{B}^\alpha_{(n):\alpha} - n \dot{B}^3_{(n-1)} + \dot{B}_{(n)}) \psi_{(n)} \Big\} dA = 0 \tag{6.26}$$

This equation is readily expressed in the form

$$-\int_T dt \int_A \Big\{ \dot{U}_a + \dot{U}_e + \dot{U}_m + \dot{K} \Big\} dA + \int_T dt \int_A \sum_{n=0}^{N} \Big\{ Q^i_{(n)} \dot{u}^{(n)}_i$$
$$-\dot{D}_{(n)} \phi_{(n)} - \dot{B}_{(n)} \psi_{(n)} \Big\} dA \quad + \int_T dt \{ \Gamma_a + \Gamma_e + \Gamma_m \} = 0 \tag{6.27}$$

with the aid of Eq. (6.14) and Eqs. (6.20)–(6.24). Integrating equation (6.27) with respect to time, one finally has

$$U_S(t_2) + K_S(t_2) - U_S(t_1) - K_S(t_1)$$
$$= \int_T dt \int_A \sum_{n=0}^{N} \Big\{ Q^i_{(n)} \dot{u}^{(n)}_i - \dot{D}_{(n)} \phi_{(n)} - \dot{B}_{(n)} \psi_{(n)} \Big\} dA$$
$$+ \int_T dt \{ \Gamma_a + \Gamma_e + \Gamma_m \} \tag{6.28}$$

The integrands in the right hand side of Eq. (6.28) vanish due to the boundary and initial conditions (6.6)–(6.11). Moreover, the individual internal and kinetic energy densities are positive definite, by definition, and initially zero. Hence, the total energies calculated by integration from them in terms of the difference system have the same properties. Guided by the usual arguments based on the positive definiteness of the energies, one finally arrives at the condition of the form

$$U_S(t_2) = U_S(t_1) = K_S(t_2) = K_S(t_1) = 0 \tag{6.29}$$

This implies a trivial solution for the difference admissible state $\Lambda_S (= \Lambda^{(2)}_S - \Lambda^{(1)}_S) = 0$, that is, the two solutions are identical. Thus, the following theorem of uniqueness is concluded.

6.3. Theorem of uniqueness

In a θ^i- system of geodesic normal coordinates in the Euclidean space Ξ, consider a regular electromagnetoelastic shell region V+S with its piecewise smooth boundary surface $S(= S_{uf} \cup S_{lf} \cup S_e)$, closure $\overline{V}(V \cup S)$ and middle surface A at the time interval $T = [t_0, t_1]$, under a prescribed initial data. The region is set in a motion by application of assigned surface traction and electric and magnetic potentials and prescribed mechanical and electric displacements and magnetic induction on the edge boundary surface. Let

$$\Lambda_S = \{ u_i \in C_{12}, \phi_{(n)} \in C_{10}, \psi_{(n)} \in C_{10}; T^{ij}_{(n)} \in C_{10}, D^i_{(n)} \in C_{10}, B^i_{(n)}$$
$$\in C_{10}; S^{(n)}_{ij} \in C_{00}, E^{(n)}_i \in C_{00}, M^{(n)}_i \in C_{00} \} \text{ on } A \times T$$

be an admissible state of single-valued continuous functions which satisfies the hierarchic linear system of two-dimensional divergence equations , 6.4 and 6.5, gradient equations (5.27)–(5.29), linear constitutive relations (4.26)–(4.28), boundary conditions (6.6)–(6.8), initial conditions (6.9)–(6.11) and the symmetry of the stress resultants (5.23). Also, let the mass density ρ is positive everywhere on A and the symmetry relations of material elasticity's (2.24) with Eq. (2.25) hold. Then, there exist at most one admissible state Λ_S, which satisfies the hierarchic linear system of electromagnetoelastic shell equations.

The theorem of uniqueness is proved on the basis of the energy argument, and the conditions are enumerated, which are sufficient to the uniqueness in solutions of the system of two dimensional shell equations. It is a generalization of the theorems reported for the high frequency vibrations of plates and shells made of thermoelastic, piezoelectric and thermopiezoelectric shells (Dökmeci, 1974a,b; Altay and Dökmeci, 2001; Altay and Dökmeci, 2002a). The complementary existence theorem which states the conditions for which there exists at least one solution to the initial-mixed boundary value problems defined by the system of two dimensional shell equations has yet to be investigated even in the case of high frequency vibrations of elastic shells.

7. Electromagnetoelastic laminae equations

This section is an extension of the results reported in Section 5 to the motion of a functionally graded electromagnetoelastic curved thin laminae within the discrete layer modeling of composite structures. A system of two dimensional approximate equations is derived in variational and differential forms for the laminae at the low frequency motions (i.e., long-wave approximation). All the shear and transverse normal strains effects as well as the interface conditions are taken into account.

7.1. Geometry of laminae

Consider a regular, finite and bounded region of thin laminae $\Omega_l + \partial\Omega_l$ called the laminae space with its entire boundary surface $\partial\Omega_l$, closure $\overline{\Omega}_l (= \Omega_l \cup \partial\Omega_l)$ and overall thickness $2H$ in the Euclidean space Ξ. The laminae region (see Fig. 3) contains a number of regular sub-regions (or layers) $\Omega_{(n)} + \partial\Omega_{(n)}$ with its bounding surface $\partial\Omega_{(n)} (= S^{(n)}_e \cup S_{n,(n-1)} \cup S_{n,(n+1)})$, each of which may possess distinct but uniform thickness $2h_n$, curvature, and electromagnetoelastic properties. The layers are sequentially numbered starting with (1) at the lowermost (or bottom) layer and terminating with (N) at the uppermost (or top) layer. The laminae is referred to by a fixed, right-handed system of geodesic normal coordinates θ^i situated on the reference surface $A_{(1)} (\equiv A)$ of the bottom layer. The θ^α-coordinate curves form a system of curvilinear coordinates on the reference surface, and $\theta^3 (\equiv z)$-axis is chosen positively upward so that

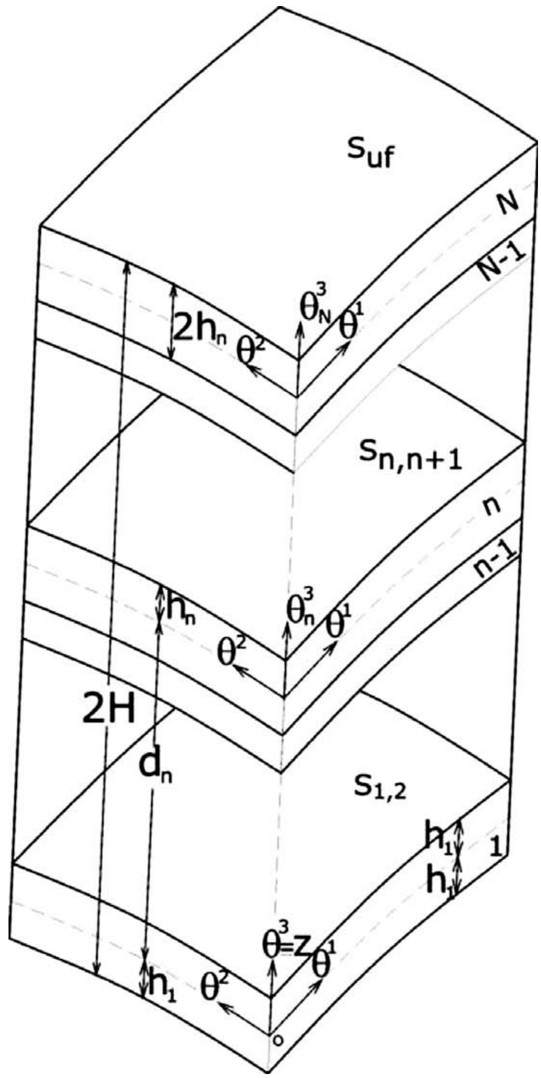

Fig. 3. An element of electromagnetoelastic laminae.

$$z=-h, \quad z=h, \quad f(\theta^1,\theta^2)=0 \tag{7.1}$$

define the lower and upper faces and the edge boundary surface of the bottom layer, respectively. An (n)-th layer is bounded by the edge or lateral boundary surfaces $S_e^{(n)}$, its lower face $S_{n,(n-1)}$ and upper face $S_{n,n+1}$, and edge surface $S_e^{(n)}$, which is a right cylindrical surface with generators perpendicular to the middle (reference) surface $A_{(n)}(\cong A)$ of the layer. The edge boundary surface intersects the reference surface along a smooth and non-intersecting Jordan curve C_n, an outward unit vector normal to the edge surface is denoted by $v_i^{(n)}$ and to the faces by n_i. The bottom layer is bounded by its lower face S_{lf}, the interface $S_{1,2}$ and the edge boundary surface $S_e^{(1)}$, and the top layer by its upper face S_{uf}, the interface $S_{N-1,N}$ and the edge boundary surface $S_e^{(N)}$. Furthermore, a set of local geodesic normal coordinates θ_n^i is introduced on the midsurface $A_{(n)}$ of the (n)th layer, and hence, the midsurface A_n and the interface $S_{n,n+1}$ are defined by the parametric equations of the form

$$\theta_n^\alpha \equiv \theta^\alpha, \quad \theta_n^3(\equiv z_n)=\theta^3-d_n=0; \quad n=1,2,\ldots,N \tag{7.2}$$

and

$$\theta_n^3-h_n=0, \quad \theta_{n+1}^3+h_{n+1}=0 \tag{7.3}$$

$$\theta^3-(d_n+h_n)=0, \quad \theta^3-(d_{n+1}-h_{n+1})=0 \tag{7.4}$$

where the distance d_n is given by

$$d_n=\sum_{r=1}^{n}(2-\delta_{1r}-\delta_{nr})h_{(r)} \tag{7.5}$$

between the middle surfaces of the first and (n)th layers.

7.2. Basic field variables

The laminae region under the assumptions of thin shells, similar to those recorded in Eqs. (4.1)–(4.3) and the alike, is mathematically treated as a two dimensional continuum. All the field variables together with their derivatives are assumed to be exist, single-valued and continuous functions of the space coordinates and time in the laminae region with the absence of any kind of singularities, and also, they are assumed, as usual, not varied widely across the thickness. Lur'e and Shumova (1996) examined the kinematic models used for construction of energetically consistent equations of laminated structural elements, as did Vasiliev and Lurie (1992), Reddy and Robbins (1994), and Reddy (2004). As before, a kinematic type of models is chosen, and the mechanical displacements, and the electric and magnetic potentials are taken as a basis for the derivation of laminae equations. The basic field variables are represented by a truncated version of the series expansions (4.4)–(4.7) of order $N = 1$, namely

$$\bar{u}_i^n(\theta^\alpha,\theta^3\equiv z;t)=\alpha_i^n(\theta^\alpha,t)+z\beta_i^n(\theta^\alpha,t) \tag{7.6}$$

$$\phi_n(\theta^\alpha,\theta^3,t)=\alpha_n(\theta^\alpha,t)+z\beta_n(\theta^\alpha,t) \tag{7.7}$$

$$\psi_n(\theta^\alpha,\theta^3,t)=\xi_n(\theta^\alpha,t)+z\zeta_n(\theta^\alpha,t) \tag{7.8}$$

with the denotations as

$$\begin{aligned}&u_i^{n(0)}=\alpha_i^n(\theta^\alpha,t), \quad u_i^{n(1)}(\theta^\alpha,t)=\beta_i^n(\theta^\alpha,t)\\&\phi_n^{(0)}=\alpha_n(\theta^\alpha,t), \quad \phi_n^{(1)}=\beta_n(\theta^\alpha,t), \quad \psi_n^{(0)}=\xi_n(\theta^\alpha,t), \quad \psi_n^{(1)}=\zeta_n(\theta^\alpha,t)\end{aligned} \tag{7.9}$$

Alternatively, these equations may be expressed as

$$\begin{aligned}&\bar{u}_i^{(n)}(\theta_n^\alpha\equiv\theta^\alpha,\theta_n^3\equiv z_n;t)=\left[\alpha_i^{(n)}(\theta^\alpha,t)+d_n\beta_i^{(n)}(\theta^\alpha,t)\right]+z_n\beta_i^{(n)}(\theta^\alpha,t)\\&\phi_{(n)}(\theta^\alpha,\theta_n^3,t)=\left[\alpha_n(\theta^\alpha,t)+d_{(n)}\beta_n(\theta^\alpha,t)\right]+z_n\beta_n(\theta^\alpha,t)\\&\psi_{(n)}(\theta^\alpha,\theta_n^3,t)=\left[\xi_n(\theta^\alpha,t)+d_{(n)}\zeta_n(\theta^\alpha,t)\right]+z_n\zeta_n(\theta^\alpha,t)\end{aligned} \tag{7.10}$$

in the system of local coordinates. In Eq. (7.6), α_n^σ stand for the extensional motions (or the stretching), α_n^3 and β_σ^3 for the flexural motions (or the bending), and β_n^3 for the thickness stretching of the nth layer. A linear approximation is introduced for the mechanical displacement components in Eqs. (7.6) or (7.9) as a generalization of the Kirchhoff–Love hypothesis of classical shells, namely

$$\beta_n^3=0, \quad \beta_\sigma^n=-(\alpha_{3,\sigma}^n+b_\sigma^v\alpha_v^n) \tag{7.11}$$

where only three of the displacement components (i.e., α_n^i) are selected independently in lieu of six in the linear approximation (7.9), which revokes the contradictions of Eq. (7.11). Thus, the effects of transverse shear and normal strains, the rotatory inertia and the coupling of adherent layers are all taken into account with the aid of Eq. (7.6).

7.3. Continuity conditions. Independent basic variables

The layers of the laminae are well attached one another, and the relative deformations are prevented. Accordingly, the continuity of the mechanical displacements as well as the electric and magnetic potential is imposed at the interfaces of layers. The continuity conditions are expressed from Eq. (3.28) by

$$\alpha_i^{(n)}(\theta^\alpha,t)+(d_n+h_n)\beta_i^{(n)}(\theta^\alpha,t)=\alpha_i^{(n+1)}(\theta^\alpha,t)+(d_{n+1}-h_{n+1})\beta_i^{(n+1)}(\theta^\alpha,t)$$
$$\alpha_{(n)}(\theta^\alpha,t)+(d_n+h_n)\beta_{(n)}(\theta^\alpha,t)=\alpha_{(n+1)}(\theta^\alpha,t)+(d_{n+1}-h_{n+1})\beta_{(n+1)}(\theta^\alpha,t)$$
$$\xi_{(n)}(\theta^\alpha,t)+(d_n+h_n)\zeta_{(n)}(\theta^\alpha,t)=\xi_{(n)}(\theta^\alpha,t)+(d_{n+1}-h_{n+1})\zeta_{(n+1)}(\theta^\alpha,t)\quad \text{on } A_{n,n+1}\times T,\quad n=1,2,\ldots,(N-1) \tag{7.12}$$

for the basic field variables. In addition, the continuity conditions are recorded by

$$n_3(T^{3j}_{(n+1)}-T^{3j}_{(n)})=0$$
$$n_3(D^3_{(n+1)}-D^3_{(n)})=0,\quad n_3(B^3_{(n+1)}-B^3_{(n)})=0\quad \text{on } A_{n,n+1}\times T,\quad n=1,2,\ldots,(N-1) \tag{7.13}$$

for the components of traction, surface charge and magnetic induction at the interfaces between the adjacent layers. Eq. (7.12) represents $5(N-1)$ constraint conditions for the basic field variables, and then, the number of independent functions ($10N$) in Eqs. (7.6)–(7.8) is reduced to $5(N+1)$. The independent functions are chosen as

$$\alpha_i^1\in C_{12}\quad \text{and}\quad \beta_i^n\in C_{12}, \alpha_1\in C_{12}\quad\text{and}\quad \beta_n\in C_{12},$$
$$\xi_1\in C_{12}\quad\text{and}\quad \zeta_n\in C_{12}\quad n=1,2,\ldots,N \tag{7.14}$$

Eq. (7.12) are solved for the rest of the dependent functions as

$$\alpha_i^n=\alpha_i^1+\sum_{r=1}^{n}\kappa_{nr}\beta_i^{(r)},\quad \alpha_n=\alpha_1+\sum_{r=1}^{n}\kappa_{nr}\beta_{(r)},$$
$$\xi_n=\xi_1+\sum_{r=1}^{n}\kappa_{nr}\zeta_{(r)};\quad n=2,\ldots,N \tag{7.15a}$$

where

$$\kappa_{nr}(\neq\kappa_{rn})=(2-\delta_{1r}-\delta_{nr})h_{(r)}-\delta_{nr}d_{(n)},\quad n=1,2,\ldots N$$
$$d_1=0,\quad d_N=2H-(h_1+h_N) \tag{7.15b}$$

in terms of the independent functions (7.14).

7.4. Gradient equations

The approximations (7.6)–(7.8) for the basic field variables imply similar distributions for the strain tensor, and the electric and magnetic field vectors in the form

$$(S_{ij},E_i,M_i)=\sum_{r=o}^{R=2}(S_{ij}^{(r)},E_i^{(r)},M_i^{(r)})z^r \tag{7.16}$$

The distributions are obtained by inserting Eqs. (7.6)–(7.8) into the variational integral (3.23), and then integrating with respect to the thickness coordinate with the result

$$\begin{aligned}\delta L_{Lg}=\int_T dt\sum_{n=1}^{N}\int_A\{&(S_{\alpha\beta}^{(o)}-e_{\alpha\beta})\delta N^{\alpha\beta}+(S_{\alpha\beta}^{(1)}-\varepsilon_{\alpha\beta})\delta M^{\alpha\beta}\\&+(S_{\alpha\beta}^{(2)}-\gamma_{\alpha\beta})\delta K^{\alpha\beta}+(S_{\alpha3}^{(0)}-e_{\alpha3})\delta Q^\alpha+(S_{\alpha3}^{(1)}-\varepsilon_{\alpha3})\delta R^\alpha\\&+(S_{33}^{(0)}-e_{33})\delta N^{33}+(E_\alpha^{(0)}-e_\alpha)\delta\mathscr{D}^\alpha+((E_\alpha^{(1)}-\varepsilon_\alpha)\delta\mathscr{E}^\alpha\\&+(E_3^{(0)}-e_3)\delta\mathscr{D}^3+(\mathscr{M}_\alpha^{(0)}-\gamma_\alpha)\delta\mathscr{M}^\alpha+((M_\alpha^{(1)}\\&-\upsilon_\alpha)\delta\mathscr{N}^\alpha+(\mathscr{M}_3^{(0)}-\gamma_3)\delta\mathscr{M}^3\}^{(n)}\end{aligned} \tag{7.17}$$

Here, the quantities of the form

$$e_{\sigma\nu}=\frac{1}{2}(\alpha_{\sigma:\nu}+\alpha_{\nu:\sigma}-2b_{\nu\beta}\alpha_3),\quad e_{\sigma3}=\frac{1}{2}(\alpha_{3,\alpha}+b_\alpha^\sigma\alpha_\sigma+\beta_\alpha),\quad e_{33}=\beta_3$$
$$\varepsilon_{\sigma\nu}=\frac{1}{2}(-b_\sigma^\lambda u_{\lambda:\nu}-b_\nu^\lambda\alpha_{\lambda:\sigma}+2c_{\sigma\nu}\alpha_3+\beta_{\sigma:\nu}+\beta_{\nu:\sigma}-2b_{\sigma\nu}\beta_3),\quad \varepsilon_{\alpha3}=\frac{1}{2}\beta_{3,\alpha}$$
$$\gamma_{\sigma\nu}=\frac{1}{2}(-b_\sigma^\lambda\beta_{\lambda:\nu}-b_\nu^\lambda\beta_{\lambda:\sigma}+2c_{\sigma\nu}\beta_3),\quad \varepsilon_{33}=\gamma_{i3}=0\quad \text{on } A\times T \tag{7.18}$$

and

$$e_\sigma=-\alpha_{,\sigma},\quad \varepsilon_\sigma=-\beta_{,\sigma},\quad e_3=-\beta\quad \text{on } A\times T \tag{7.19}$$
$$\gamma_\alpha=-\xi_{,\alpha},\quad \upsilon_\alpha=-\zeta_{,\alpha},\quad \gamma_3=-\zeta\quad \text{on } A\times T \tag{7.20}$$

are defined. In the equations above, the stress resultants by

$$(N^{\alpha\beta},M^{\alpha\beta},K^{\alpha\beta},Q^\alpha=Q^{\alpha3},R^\alpha=R^{3\alpha},N^{33})=\int_Z(T^{\alpha\beta},zT^{\alpha\beta},z^2T^{\alpha\beta},T^{\alpha3},zT^{\alpha3},T^{33})\mu\,dz \tag{7.21}$$

are introduced. Also, it is pertinent to note here that the conventional stress resultants are defined in the form

$$\mathscr{N}^{\alpha\beta}=\int_Z T^{\alpha\nu}\mu_\nu^\beta\mu\,dz=N^{\alpha\beta}-b_\nu^\beta M^{\alpha\nu},$$
$$\mathscr{M}^{\alpha\beta}=\int_Z T^{\alpha\nu}\mu_\nu^\beta\mu z\,dz=M^{\alpha\beta}-b_\nu^\beta K^{\alpha\nu} \tag{7.22}$$

in terms of Eq. (7.21). Likewise, the gross electric displacement and magnetic induction by

$$(\mathscr{D}^i,\mathscr{E}^\alpha;\mathscr{M}^i,\mathscr{N}^\alpha)=\int_Z(D^i,zD^\alpha;B^i,zB^\alpha)\mu\,dz \tag{7.23}$$

are considered. The resultant quantities are measured per unit length of coordinate curves on the reference surface A.

7.5. Constitutive relations

The material of electromagnetoelastic thin laminae is functionally graded along its thickness, and all the material coefficients of each layer in Eqs. (2.21)–(2.23) are taken to be continuously dependent on the thickness coordinate z in a linear form

$$\{c^{ijkl}(z),e^{ijk}(z),\varepsilon^{ijk}(z),e^{ij}(z),\varepsilon^{ij}(z),c^{ij}(z)\}=\{(c_0^{ijkl}+zc_1^{ijkl}),\ldots,(c_0^{ij}+zc_1^{ij})\} \tag{7.24}$$

in consistent with Eqs. (7.6)–(7.8). To obtain the constitutive relations of laminae, the variational integral (3.26) with Eq. (4.17) is written for all the layers, and then the distribution (7.16) is invoked and the integration is performed with respect to the thickness coordinate with the result

$$\begin{aligned}\delta L_{Lc}=\int_T dt\int_A\sum_{r=1}^{N}\Big\{&(N^{\alpha\beta}-N_c^{\alpha\beta})\delta e_{\alpha\beta}+(M^{\alpha\beta}-M_c^{\alpha\beta})\delta\varepsilon_{\alpha\beta}\\&+(K^{\alpha\beta}-K_c^{\alpha\beta})\delta\gamma_{\alpha\beta}+(Q^\alpha-Q_c^\alpha)\delta e_{\alpha3}+(R^\alpha-R_c^\alpha)\delta\varepsilon_{\alpha3}\\&+(N^{33}-N_c^{33})\delta e_{33}+(\mathscr{D}^\alpha-\mathscr{O}_c^\alpha)\delta e_\alpha+(\mathscr{E}^\alpha-\mathscr{E}_c^\alpha)\delta\varepsilon_\alpha\\&+(\mathscr{D}^3-\mathscr{D}_c^3)\delta e_3+(\mathscr{M}^\alpha-\mathscr{M}_c^\alpha)\delta\gamma_\alpha+(\mathscr{N}^\alpha-\mathscr{N}_c^\alpha)\delta\upsilon_\alpha\\&+\big(\mathscr{M}^3-\mathscr{M}_c^3\big)\delta\gamma_3\Big\}^r dA\end{aligned} \tag{7.25}$$

Here, the constitutive denotations of the form

$$
\begin{aligned}
N_c^{ij} &= (c_0^{ijkl}\mu_0 + c_1^{ijkl}\mu_1)e_{kl} + (c_0^{ijkl}\mu_1 + c_1^{ijkl}\mu_2)\varepsilon_{kl} \\
&\quad + (c_0^{ijkl}\mu_2 + c_1^{ijkl}\mu_3)\gamma_{kl} - (e_0^{ijk}\mu_0 + e_1^{ijk}\mu_1)_k \\
&\quad - (e_0^{ijk}\mu_1 + e_1^{ijk}\mu_2)\varepsilon_k - (\varepsilon_0^{ijk}\mu_0 + \varepsilon_1^{ijk}\mu_1)\gamma_k - (\varepsilon_0^{ijk}\mu_1 + \varepsilon_1^{ijk}\mu_2)\upsilon_k \\
M_c^{\alpha i} &= (c_0^{\alpha ikl}\mu_1 + c_1^{\alpha ikl}\mu_2)e_{kl} + (c_0^{\alpha ikl}\mu_2 + c_1^{\alpha ikl}\mu_3)\varepsilon_{kl} \\
&\quad + (c_0^{\alpha ikl}\mu_3 + c_1^{\alpha ikl}\mu_4)\gamma_{kl} - (e_0^{\alpha ik}\mu_1 + e_1^{\alpha ik}\mu_2)e_k - (e_0^{\alpha ik}\mu_2 + e_1^{\alpha ik}\mu_3)\varepsilon_k \\
&\quad - (\varepsilon_0^{\alpha ik}\mu_1 + \varepsilon_1^{\alpha ik}\mu_2)\gamma_k - (\varepsilon_0^{\alpha ik}\mu_2 + \varepsilon_1^{\alpha ik}\mu_3)\upsilon_k \\
K_c^{\alpha\beta} &= (c_0^{\alpha\beta kl}\mu_2 + c_1^{\alpha\beta kl}\mu_3)e_{kl} + (c_0^{\alpha\beta kl}\mu_3 + c_1^{\alpha\beta kl}\mu_4)\varepsilon_{kl} \\
&\quad + (c_0^{\alpha\beta kl}\mu_4 + c_1^{\alpha\beta kl}\mu_5)\gamma_{kl} - (e_0^{\alpha\beta l}\mu_2 + e_1^{\alpha\beta k}\mu_3)e_k \\
&\quad - (e_0^{\alpha\beta l}\mu_3 + e_1^{\alpha\beta k}\mu_4)\varepsilon_k - (\varepsilon_0^{\alpha\beta l}\mu_2 + \varepsilon_1^{\alpha\beta k}\mu_3)\gamma_k - (\varepsilon_0^{\alpha\beta l}\mu_3 + \varepsilon_1^{\alpha\beta k}\mu_4)\upsilon_k \\
Q^\alpha &= N^{\alpha 3}, \quad R^\alpha = M^{\alpha 3}
\end{aligned} \tag{7.26}
$$

with

$$
\mathscr{N}_c^{\alpha\beta} = N_c^{\alpha\beta} - b_\nu^\beta M_c^{\nu\alpha}, \mathscr{M}_c^{\alpha\beta} = M_c^{\alpha\beta} - b_\nu^\beta K_c^{\alpha\nu} \tag{7.27}
$$

and

$$
\begin{aligned}
\mathscr{D}_c^i &= (e_0^{ikl}\mu_0 + e_1^{ikl}\mu_1)e_{kl} + (e_0^{ikl}\mu_1 + e_1^{ikl}\mu_2)\varepsilon_{kl} \\
&\quad + (e_0^{ikl}\mu_2 + e_1^{ikl}\mu_3)\gamma_{kl} + (e_0^{ik}\mu_0 + e_1^{ik}\mu_1)e_k \\
&\quad + (e_0^{ik}\mu_1 + e_1^{ik}\mu_2)\varepsilon_k + (\varepsilon_0^{ik}\mu_0 + \varepsilon_1^{ik}\mu_1)\gamma_k \\
&\quad + (\varepsilon_0^{ik}\mu_1 + \varepsilon_1^{ik}\mu_2)\upsilon_k \\
\mathscr{E}_c^\alpha &= (e_0^{\alpha kl}\mu_1 + e_1^{\alpha kl}\mu_2)e_{kl} + (e_0^{\alpha kl}\mu_2 + e_1^{\alpha kl}\mu_3)\varepsilon_{kl} \\
&\quad + (e_0^{\alpha kl}\mu_3 + e_1^{\alpha kl}\mu_4)\gamma_{kl} + (e_0^{\alpha l}\mu_1 + e_1^{\alpha k}\mu_2)e_k \\
&\quad + (e_0^{\alpha l}\mu_2 + e_1^{\alpha k}\mu_3)\varepsilon_k + (\varepsilon_0^{\alpha l}\mu_1 + \varepsilon_1^{\alpha k}\mu_2)\gamma_k \\
&\quad + (\varepsilon_0^{\alpha l}\mu_2 + \varepsilon_1^{\alpha k}\mu_3)\upsilon_k
\end{aligned} \tag{7.28}
$$

and

$$
\begin{aligned}
\mathscr{M}_c^i &= (\varepsilon_0^{ikl}\mu_0 + \varepsilon_1^{ikl}\mu_1)e_{kl} + (\varepsilon_0^{ikl}\mu_1 + \varepsilon_1^{ikl}\mu_2)\varepsilon_{kl} \\
&\quad + (\varepsilon_0^{ikl}\mu_2 + \varepsilon_1^{ikl}\mu_3)\gamma_{kl} + (c_0^{ik}\mu_0 + c_1^{ik}\mu_1)\gamma_k \\
&\quad + (c_0^{ik}\mu_1 + c_1^{ik}\mu_2)\upsilon_k + (\varepsilon_0^{ik}\mu_0 + \varepsilon_1^{ik}\mu_1)e_k + (\varepsilon_0^{ik}\mu_1 \\
&\quad + \varepsilon_1^{ik}\mu_2)\varepsilon_k \\
\mathscr{N}_c^\alpha &= (\varepsilon_1^{\alpha kl}\mu_1 + \varepsilon_2^{\alpha kl}\mu_2)e_{kl} + (\varepsilon_0^{\alpha kl}\mu_2 + \varepsilon_1^{\alpha kl}\mu_3)\varepsilon_{kl} \\
&\quad + (\varepsilon_0^{\alpha kl}\mu_3 + \varepsilon_1^{\alpha kl}\mu_4)\gamma_{kl} + (c_0^{\alpha l}\mu_1 + e_1^{\alpha k}\mu_2)\gamma_k \\
&\quad + (c_0^{\alpha l}\mu_2 + c_1^{\alpha k}\mu_3)\upsilon_k + (\varepsilon_0^{ik}\mu_1 + \varepsilon_1^{ik}\mu_2)e_k \\
&\quad + (\varepsilon_0^{\alpha l}\mu_2 + \varepsilon_1^{\alpha k}\mu_3)\varepsilon_k
\end{aligned} \tag{7.29}
$$

are defined in terms of Eqs. (4.18) and (7.18)–(7.20).

7.6. Divergence equations

Within the concept of a discrete layer modeling, the divergence equations are derived for a thin electromagnetoelastic laminae paralleling to the formulation of shell equations in Section 5. The derivation is essentially based on the selection of the basic field variables (7.6)–(7.8) as a linear function in the thickness coordinate, and using a variational averaging procedure for smoothly varying field quantities across the thickness of laminae. To begin with, the mass density and the body force vector are expressed as

$$
\rho(z) = \rho_0 + \rho_1 z, \quad f(z) = f_0 + f_1 z \tag{7.30}
$$

for a functionally graded material in consistent with the material coefficients of the laminae given in Eq. (7.24). Also, the resultants of traction, surface charge and quantities of the form

$$
\begin{aligned}
\{P_u^i, P_l^i\}^{(n)} &= (T^{3i} - z b_\beta^\alpha T^{3\beta}\delta_\alpha^i)\mu \quad \text{at } z = \{h, -h\}_{(n)} \\
\{R_u^i, R_l^i\}^{(n)} &= (T^{3i} - z b_\beta^\alpha T^{3\beta}\delta_\alpha^i)z\mu \quad \text{at } z = \{h, -h\}_{(n)}
\end{aligned} \tag{7.31}
$$

and

$$
\{d_u^3, d_l^3\}^{(n)} = D^3\mu, \quad \{e_u^3, e_l^3\} = D^3 z\mu \quad \text{at } z = \{h, -h\}_{(n)} \tag{7.32}
$$

$$
\{m_u^3, m_l^3\}^{(n)} = B^3\mu; \quad \{n_u^3, n_l^3\} = B^3 z\mu \quad \text{at } z = \{h, -h\}_{(n)} \tag{7.33}
$$

and the displacements and the body force resultants by

$$
\begin{aligned}
A^i &= (\rho_0\mu_0 + \rho_1\mu_1)\alpha^i + (\rho_0\mu_1 + \rho_1\mu_2)\beta^i, \\
B^i &= (\rho_0\mu_1 + \rho_1\mu_2)\alpha^i + (\rho_0\mu_2 + \rho_1\mu_3)\beta^i
\end{aligned} \tag{7.34}
$$

and

$$
\begin{aligned}
F^i &= (f_0^i\mu_0 + f_1^i\mu_1) - b_\nu^\sigma \delta_\sigma^i (f_0^\nu\mu_1 + f_1^\nu\mu_2), \\
G^i &= (f_0^i\mu_1 + f_1^i\mu_2) - b_\nu^\sigma \delta_\sigma^i (f_0^\nu\mu_2 + f_1^\nu\mu_3)
\end{aligned} \tag{7.35}
$$

are defined. Now, Eqs. (7.6)–(7.8) are inserted into the variational integral (3.24) with Eq. (5.2), and then, it is evaluated for an individual layer as in the derivation of Eq. (5.7) and then for the laminae with the following result

$$
\begin{aligned}
\delta L_{Ld}\{\Lambda\} = \int_T dt \int_A \sum_{r=1}^N \{ &(V^i + P_u^i - P_l^i)\delta\alpha_i + (W^i + R_u^i - R_l^i)\delta\beta_i \\
&+ (\mathscr{D}_{:\alpha}^\alpha + d_u - d_l)\delta\alpha + (\mathscr{E}_{:\alpha}^\alpha + e_u - e_l)\delta\beta \\
&+ (\mathscr{M}_{:\alpha}^\alpha + m_u - m_l)\delta\xi + (\mathscr{N}_{:\alpha}^\alpha + n_u - n_l)\delta\zeta\}^{(r)} = 0
\end{aligned} \tag{7.36}
$$

where

$$
V^\alpha = \mathscr{N}_{:\beta}^{\beta\alpha} - b_\nu^\alpha Q^\nu + F^\alpha - A^\alpha, \quad V^3 = Q_{:\alpha}^\alpha + b_{\alpha\beta}\mathscr{N}^{\alpha\beta} + F^3 - A^3 \tag{7.37}
$$

$$
W^\alpha = \mathscr{M}_{:\beta}^{\beta\alpha} - Q^\alpha + F^\alpha - B^\alpha, \quad W^3 = R_{:\alpha}^\alpha - N^{33} + b_{\alpha\beta}\mathscr{M}^{\alpha\beta} + F^3 - B^3 \tag{7.38}
$$

The variational integral (7.37) is expressed by

$$
\delta L_{Ld}\{\Lambda_{ld}\} = \delta L_{Lda}\{\Lambda_{la}\} + \delta L_{Lde}\{\Lambda_{le}\} + \delta L_{Ldm}\{\Lambda_{lm}\} = 0 \tag{7.39}
$$

where

$$
\begin{aligned}
\delta L_{Lda}\{\Lambda_{la}\} = \int_T dt \int_A \Bigg\{ &\left(\sum_{r=1}^N V_{(r)}^i + P_{u(N)}^i - P_{l(1)}^i \right)\delta\alpha_i^{(1)} \\
&+ \left[W_{(1)}^i + h_1 \sum_{r=2}^N V_{(r)}^i + h_1(P_{(l)}^i + P_{u(N)}^i) \right]\delta\beta_i^{(1)} \\
&+ \sum_{r=n}^{N-1} \left(W_{(n)}^i + \sum_{r=n}^N \kappa_{nr} V_{(r)}^i + 2h_n P_u^{i(N)} \right)\delta\beta_i^{(r)} \\
&+ \left[W_{(N)}^i - (d_N - h_N)V_{(N)}^i + 2h_N P_u^{i(N)} \right]\delta\beta_i^{(N)} \Bigg\} dA
\end{aligned} \tag{7.40}
$$

for the mechanical field,

$$
\begin{aligned}
\delta L_{Lde}\{\Lambda_{le}\} = \int_T dt \int_A \Bigg\{ &\left(\sum_{r=1}^N \mathscr{D}_{(r):\alpha}^\alpha + d_{u(N)} - d_{l(1)} \right)\delta\alpha_{(1)} \\
&+ \left[\mathscr{E}_{:\alpha}^\alpha + h_1 \sum_{r=2}^N \mathscr{D}_{(r):\alpha}^\alpha + h_1(d_{u(N)} + d_{l(1)}) \right]\delta\beta_{(1)} \\
&+ \sum_{n=1}^{N-1} (\mathscr{E}_{(n):\alpha}^\alpha + \sum_{r=n}^N \kappa_{nr}\mathscr{D}_{(r):\alpha}^\alpha + 2h_n d_u^{(N)})\delta\beta_{(r)}) \\
&+ \left[\mathscr{E}_{(n):\alpha}^\alpha - (d_N - h_N)\mathscr{D}_{(N):\alpha}^\alpha + 2h_N d_u^{(N)} \right]\delta\beta_{(N)} \Bigg\} dA
\end{aligned} \tag{7.41}
$$

for the electrical field, and

$$
\begin{aligned}
\delta L_{Ldm}\{\Lambda_{lm}\} = \int_T dt \int_A \Bigg\{ &\left(\sum_{r=1}^N \mathscr{M}_{(r):\alpha}^\alpha + m_{u(N)} - m_{l(1)} \right)\delta\xi_{(1)} \\
&+ \left[\mathscr{N}_{:\alpha}^\alpha + h_1 \sum_{r=2}^N \mathscr{M}_{(r):\alpha}^\alpha + h_1(m_{u(N)} + m_{l(1)}) \right]\delta\zeta_{(1)} \\
&+ \sum_{n=1}^{N-1} (\mathscr{N}_{(n):\alpha}^\alpha + \sum_{r=n}^N \kappa_{nr}\mathscr{M}_{(r):\alpha}^\alpha + 2h_n m_u^{(N)})\delta\zeta_{(r)} \\
&+ \left[\mathscr{N}_{(n):\alpha}^\alpha - (d_N - h_N)\mathscr{M}_{(N):\alpha}^\alpha + 2h_N m_u^{(N)} \right]\delta\zeta_{(N)} \Bigg\}
\end{aligned} \tag{7.42}
$$

for the magnetic field. Their admissible states by

$$\Lambda_{la} = \{\alpha_i^{(1)}, \beta_i^{(n)}\}, \quad \Lambda_{le} = \{\alpha_{(1)}, \beta_{(n)}\}, \quad \Lambda_{lm} = \{\xi_{(1)}, \zeta_{(n)}\}; \quad n = 1, 2, \ldots, N \tag{7.43}$$

are given. Here, the variations are expressed with respect to the independent basic functions (7.15), and the resultants of mechanical, electrical and magnetic quantities are used. Also,the continuity of traction, electric displacements and magnetic inductions across interfaces of layers (7.13) are all taken all into account.

7.7. Boundary conditions

In the laminae region, the mechanical displacements are taken to be prescribed on some part of the lower face S_{lfu} and of the edge surface S_{leu} of the lowermost layer. The tractions are given on the upper face S_{uf}, and some part of the lower face $S_{lft}(= S_{lf} \cap S_{lfu})$ and the edge surface $S_{et}(= S_e \cap S_{leu})$ of laminae. The electric and magnetic potentials are given on the faces and the surface charge and the magnetic charge on the edge surface. Accordingly, the variational surface integral (3.27) is expressed for the laminae as follows:

$$\delta L_{lb}\{\Lambda_{lb}\} = \delta L_{lba}\{\Lambda_{ba}\} + \delta L_{lbe}\{\Lambda_{be}\} + \delta L_{lbm}\{\Lambda_{bm}\} \tag{7.44a}$$

with the denotations by

$$\delta L_{lb}\{\Lambda_{lb}\} = \delta L_{af} + \delta L_{ae} + \delta L_{au}, \quad \delta L_{le} = \delta L_{ef} + \delta L_{ee}, \quad \delta L_{lm} = \delta L_{mf} + \delta L_{me} \tag{7.44b}$$

where

$$\begin{aligned}
\delta L_{af} &= \int_T dt \int_{S_{uf} \cup S_{lft}} \left[(T^i_* - n_3 T^{3i}) - z(T^\sigma_* - n_3 T^{3\sigma}) b^\nu_\sigma \delta^i_\nu\right](\delta\alpha_i + z\delta\beta_i)\mu\, dA \\
\delta L_{ae} &= \int_T dt \int_{C_t} dc \sum_{r=1}^N \int_Z \left\{[(T^i_* - \nu_\alpha T^{\alpha i}) - z(T^\sigma_* - \nu_\lambda T^{\lambda\sigma}) b^\nu_\sigma \delta^i_\nu]\delta\alpha_i + z\delta\beta_i)\mu\, dz\right\}^{(r)} \\
\delta L_{au} &= \int_T dt \int_{C_u} dc \left\{\int_Z [(\alpha_i + z\beta_i) - u^*_i]\nu_\lambda(\delta T^{\lambda i} - z b^\eta_\sigma \delta^i_\eta \delta T^{\lambda\sigma})\mu\, dz\right\}^{(1)} \\
&\quad + \int_T dt \int_{S_{lfu}} \left\{\left[(\alpha_i + z\beta_i) - u^*_i n_3(\delta T^{3i} - z b^\eta_\sigma \delta^i_\eta \delta T^{\lambda\sigma})\right]\mu\, dA\right\}^{(1)}
\end{aligned} \tag{7.45}$$

and

$$\begin{aligned}
\delta L_{ef} &= \int_T dt \int_{S_f} [(\alpha + z\beta) - \phi_*] n_3 \delta D^3\, dA \\
\delta L_{ee} &= \int_T dt \oint_C dc \sum_{r=1}^N \left\{\int_Z (D_* - \nu_\sigma D^\sigma)(\delta\alpha + z\delta\beta)\right\}^{(r)} \mu\, dz
\end{aligned} \tag{7.46}$$

and

$$\begin{aligned}
\delta L_{mf} &= \int_T dt \int_{S_f} [(\xi + z\zeta) - \psi_*] n_3 \delta B^3\, dA \\
\delta L_{me} &= \int_T dt \int_C dc \sum_{r=1}^N \left\{\int_Z (B_* - \nu_\sigma B^\sigma)(\delta\xi + z\delta\zeta)\right\}^{(r)} \mu\, dz
\end{aligned} \tag{7.47}$$

As before, taking the integration with respect to the thickness coordinate in Eqs. (7.45)–(7.47), using the resultants (7.21)–(7.23), (7.27) and (7.31)–(7.33), and considering the independent functions (7.15), the boundary conditions are expressed by

$$\begin{aligned}
\delta L_{af} &= \int_T dt \int_{S_{lft}} \left[(\mathscr{P}^i_{l^*} + P^i_l)(\delta\alpha_i + h_1\delta\beta_i)\right]^{(1)} dA \\
&\quad + \int_T dt \int_{S_{uf}} \left\{(\mathscr{P}^i_{u^*} - P^i_u)^{(N)}\left[\delta\alpha^{(1)}_i + \sum_{n=1}^N (2 - \delta_{1n}) h_n \delta\beta^{(n)}_i\right]\right\} dA \\
\delta L_{ae} &= \int_T dt \int_{C_t} \delta\alpha^{(1)}_i \sum_{n=1}^N (\mathscr{N}^i_* - \nu_\alpha \mathscr{N}^{\alpha i})^{(n)} \\
&\quad + \sum_{n=1}^N \delta\beta^{(n)}_i \left[\sum_{r=n}^N \kappa_{nr}(\mathscr{N}^i_* - \nu_\alpha \mathscr{N}^{\alpha i})^{(r)} + (\mathscr{M}^i_* - \nu_\alpha \mathscr{M}^{\alpha i})^{(n)}\right]\Bigg\} dc \\
\delta L_{au} &= \int_T dt \int_{C_u} \nu_\sigma\{(\alpha_\lambda - \alpha_{*\lambda})\delta\mathscr{N}^{\sigma\lambda} + (\beta_\lambda - \beta_{*\lambda})\delta\mathscr{M}^{\sigma\lambda} \\
&\quad + (\beta_3 - \beta_{*3})\delta R^\sigma\}^{(1)} dc + \int_T dt \int_{S_{lfu}} \left\{[(\alpha_i - \alpha^*_i) - h_1(\beta_i - \beta^*_i)]\delta P^i_l\right\}^{(1)} dA
\end{aligned} \tag{7.48}$$

for the mechanical field. Here, the prescribed mechanical displacement is approximated as in Eq. (7.6), and

$$\{\mathscr{P}^i_{*u}, \mathscr{P}^i_{*l}\}^{(n)} = (T^i_* - z b^\alpha_\beta T^\beta_* \delta^i_\alpha)\mu \quad \text{at } z = \{h, -h\}_{(n)} \tag{7.49}$$

$$\{\mathscr{N}^i_*, \mathscr{M}^i_*\}^{(n)} = \int_Z (T^i_* - z b^\alpha_\beta T^\beta_* \delta^i_\alpha)\{1, z\}\mu\, dz \tag{7.50}$$

are defined. Likewise, the boundary conditions are obtained as

$$\begin{aligned}
\delta L_{ef} &= \int_T dt \int_{S_{lf}} \{[-(\alpha - h_1\beta) + \phi_*]\delta d^3_l\}^{(1)} dA \\
&\quad + \int_T dt \int_{S_{uf}} \left\{\left[-\phi_* + \sum_{n=1}^N (2 - \delta_{1n}) h_n \beta_{(n)}\right]\delta d^3_u\right\}^{(N)} dA \\
\delta L_{ee} &= \int_T dt \oint_C dc \Bigg\{\delta\alpha_1 \sum_{n=1}^N (\mathscr{D}_* - \nu_\alpha \mathscr{D}^\alpha)^{(n)} \\
&\quad + \sum_{n=1}^N \delta\beta^{(n)} \left[\sum_{r=n}^N \kappa_{nr}(\mathscr{D}_* - \nu_\alpha \mathscr{D}^\alpha)^{(r)} + (\mathscr{E}_* - \nu_\alpha \mathscr{E}^\alpha)^{(n)}\right]\Bigg\} dc
\end{aligned} \tag{7.51}$$

and

$$\begin{aligned}
\delta L_{mf} &= \int_T dt \int_{S_{lf}} \{[-(\xi_- h_1\zeta) + \psi_*]\delta m^3_l\}^{(1)} dA \\
&\quad + \int_T dt \int_{S_{uf}} \left\{\left[-\psi_* + \sum_{n=1}^N (2 - \delta_{1n}) h_n \zeta_{(n)}\right]\delta m^3_u\right\}^{(N)} dA \\
\delta L_{me} &= \int_T dt \oint_C dc \Bigg\{\delta\zeta_1 \sum_{n=1}^N (\mathscr{M}_* - \nu_\alpha \mathscr{M}^\alpha)^{(n)} \\
&\quad + \sum_{n=1}^N \delta\zeta^{(n)} \left[\sum_{r=n}^N \kappa_{nr}(\mathscr{M}_* - \nu_\alpha \mathscr{M}^\alpha)^{(r)} + (\mathscr{N}_* - \nu_\alpha \mathscr{N}^\alpha)^{(n)}\right]\Bigg\} dc
\end{aligned} \tag{7.52}$$

for the electromagnetic field.

7.8. Initial conditions

A set of initial conditions based on Eqs. (2.29)–(2.31) is recorded as

$$\begin{aligned}
&\alpha^{(1)}_i(\theta^\alpha, t_0) - v^{*(1)}_i(\theta^\alpha) = 0, \quad \beta^{(n)}_i(\theta^\alpha, t_0) - w^{*(n)}_i(\theta^\alpha) = 0, \\
&\quad n = 1, 2, \ldots, N \quad \text{on } A(t_0) \\
&\dot{\alpha}^{(1)}_i(\theta^\alpha, t_0) - \eta^{*(1)}_i(\theta^\alpha) = 0, \quad \dot{\beta}^{(n)}_i(\theta^\alpha, t_0) - \lambda^{*(n)}_i(\theta^\alpha) = 0, \\
&\quad n = 1, 2, \ldots, N; \quad \text{on } A(t_0)
\end{aligned} \tag{7.53}$$

and

$$\alpha_{(1)}(\theta^\alpha, t_0) - \eta^*(\theta^\alpha) = 0, \quad \beta_{(n)}(\theta^\alpha, t_0) - \lambda^*_{(n)}(\theta^\alpha) = 0, \quad n = 1, 2, \ldots, N \quad \text{on } A(t_0) \tag{7.54}$$

$$\xi_{(1)}(\theta^\alpha, t_0) - \gamma^*(\theta^\alpha) = 0, \quad \zeta_{(n)}(\theta^\alpha, t_0) - \varphi^*_{(n)}(\theta^\alpha) = 0, \quad n = 1, 2, \ldots, N \quad \text{on } A(t_0) \tag{7.55}$$

for the mechanical and electromagnetic fields, respectively.

8. Laminae equations in differential form. Some special cases

Thus far, the unified variational principles are formulated for certain electromagnetoelastic regions in Section 3, and the approximate two dimensional equations of a functionally graded thin shell are derived in both variational and differential form in Sections 4–6, and those of laminae in variational form in Section 7. The resulting equations in invariant form are capable of tackling a large variety of similar results for piezoelectric, electromagnetic and magnetoelastic regions and structural elements in a system of coordinates most appropriate to their geometry. To begin with, the approximate equations of laminae are given in differential form, and then, some cases involving with special geometry, material and type of vibrations are indicated.

8.1. Differential laminae equations

The laminae equations in variational form consist of the gradient equations (7.17), the constitutive relations (7.25), the divergence equation (7.40) with Eqs. (7.41)–(7.43), and the boundary conditions (7.48), (7.51) and (7.52), and the initial conditions (7.53)–(7.55). These equations readily yield, as their Euler–Lagrange equations, to the laminae equations in differential form. First, setting the variational integral (7.17) equal to zero and considering Eq. (7.14) with Eq. (7.15), the gradient equations in differential form are expressed by

$$
\begin{aligned}
&e^{(1)}_{\sigma\nu} = \frac{1}{2}(\alpha_{\sigma:\nu} + \alpha_{\nu:\sigma} - 2b_{\nu\sigma}\alpha_3)^{(1)},\\
&e^{(n)}_{\sigma\nu} = e^{(1)}_{\sigma\nu} + \frac{1}{2}\sum_{r=1}^{n}\kappa_{nr}(\beta_{\sigma:\nu} + \beta_{\nu:\sigma} - 2b_{\sigma\nu}\beta_3)^{(r)}\\
&e^{(1)}_{\sigma3} = \frac{1}{2}(\alpha_{3,\sigma} + b^{\nu}_{\sigma}\alpha^{+}_{\nu}\beta_{\sigma})^{(1)},\\
&e^{(n)}_{\alpha3} = e^{(1)}_{\alpha3} + \beta^{(n)}_{\alpha} + \frac{1}{2}\sum_{r=1}^{n}\kappa_{nr}(\beta_{3,\alpha} + b^{\sigma}_{\alpha}\beta_{\sigma})^{(r)} \quad \text{on } A\times T\\
&\varepsilon^{(1)}_{\sigma\nu} = \frac{1}{2}(-b^{\lambda}_{\sigma}\alpha_{\lambda:\nu} - b^{\lambda}_{\nu}\alpha_{\lambda:\sigma} + 2c_{\sigma\nu}\alpha_3 + \beta_{\sigma:\nu} + \beta_{\nu:\sigma} - 2b_{\sigma\nu}\beta_3)^{(1)};\\
&e^{(1)}_{33} = \beta^{(1)}_{3}, \quad e^{(n)}_{33} = \beta^{(n)}_{3}\\
&\varepsilon^{(n)}_{\sigma\nu} = \varepsilon^{(1)}_{\sigma\nu} + \frac{1}{2}\sum_{r=1}^{n}\kappa_{nr}(-b^{\lambda}_{\sigma}\beta_{\lambda:\nu} - b^{\lambda}_{\nu}\beta_{\lambda:\sigma} + 2c_{\sigma\nu}\beta_3)^{(r)}\\
&\qquad + (\beta_{\sigma:\nu} + \beta_{\nu:\sigma} - 2b_{\sigma\nu}\beta_3)^{(n)}\\
&\gamma^{(1)}_{\sigma\nu} = \frac{1}{2}(-b^{\lambda}_{\sigma}\beta_{\lambda:\nu} - b^{\lambda}_{\nu}\beta_{\lambda:\sigma} + 2c_{\sigma\nu}\beta_3)^{(1)},\\
&\gamma^{(n)}_{\sigma\nu} = \frac{1}{2}(-b^{\lambda}_{\sigma}\beta_{\lambda:\nu} - b^{\lambda}_{\nu}\beta_{\lambda:\sigma} + 2c_{\sigma\nu}\beta_3)^{(n)},\\
&\varepsilon^{(1)}_{\alpha3} = \frac{1}{2}\beta^{(1)}_{3,\alpha}, \quad \varepsilon^{(n)}_{\alpha3} = \frac{1}{2}\beta^{(n)}_{3,\alpha}; (\varepsilon^{(1)}_{33} = \gamma^{(1)}_{i3} = 0, \varepsilon^{(n)}_{33} = \gamma^{(n)}_{i3} = 0,\\
&\quad n = 2,3,\ldots,(N)
\end{aligned} \tag{8.1}
$$

and

$$
\begin{aligned}
&e^{(1)}_{\sigma} = -\alpha^{(1)}_{,\alpha}, \quad e^{(n)}_{\sigma} = -\alpha^{(1)}_{,\sigma} - \sum_{r=1}^{n}\kappa_{nr}\beta^{(r)}_{,\sigma} \quad \text{on } A\times T\\
&\varepsilon^{(1)}_{\sigma} = -\beta^{(1)}_{,\sigma}, \quad \varepsilon^{(n)}_{\sigma} = -\beta^{(n)}_{,\sigma}; \quad e^{(1)}_{3} = -\beta^{(1)}, \quad e^{(n)}_{3} = -\beta^{(n)}, \quad \varepsilon^{(1)}_{3} = 0;\\
&\quad n = 2,3,\ldots,(N)
\end{aligned} \tag{8.2}
$$

and

$$
\begin{aligned}
&\gamma^{(1)}_{\sigma} = -\xi^{(1)}_{,\alpha}, \quad \gamma^{(n)}_{\sigma} = -\xi^{(1)}_{,\sigma} - \sum_{r=1}^{n}\kappa_{nr}\zeta^{(r)}_{,\sigma}; \quad n = 2,3,\ldots,(N) \quad \text{on } A\times T\\
&\upsilon^{(1)}_{\sigma} = -\zeta^{(1)}_{,\sigma}, \quad \upsilon^{(n)}_{\sigma} = -\zeta^{(n)}_{,\sigma}; \quad \gamma^{(1)}_{3} = -\zeta^{(1)}, \quad \gamma^{(n)}_{3} = -\zeta^{(n)},\\
&\upsilon^{(1)}_{3} = 0, \quad \upsilon^{(n)}_{3} = 0
\end{aligned} \tag{8.3}
$$

in terms of the independent functions. Likewise, the constitutive relations in differential form follow from the variational Eq. (7.25) as

$$
\begin{aligned}
&(N^{\alpha\beta} - N^{\alpha\beta}_{c})^{(n)} = 0, \quad (M^{\alpha\beta} - M^{\alpha\beta}_{c})^{(n)} = 0, \quad (K^{\alpha\beta} - K^{\alpha\beta}_{c})^{(n)} = 0,\\
&n = 1,2,\ldots,N; \quad \text{on } A\times T\\
&(Q^{3\alpha} - Q^{3\alpha}_{c})^{(n)} = 0, (R^{3\alpha} - R^{3\alpha}_{c})^{(n)} = 0, (N^{33} - N^{33}_{c})^{(n)} = 0,\\
&n = 1,2,\ldots,N; \quad \text{on } A\times T
\end{aligned} \tag{8.4}
$$

and

$$
\begin{aligned}
&(\mathscr{D}^{\alpha} - \mathscr{D}^{\alpha}_{c})^{(n)} = 0, \quad (\mathscr{E}^{\alpha} - \mathscr{E}^{\alpha}_{c})^{(n)} = 0, \quad (\mathscr{D}^{3} - \mathscr{D}^{3}_{c})^{(n)} = 0,\\
&\quad n = 1,2,\ldots,N \quad \text{on } A\times T
\end{aligned} \tag{8.5}
$$

$$
\begin{aligned}
&(\mathscr{M}^{\alpha} - \mathscr{M}^{\alpha}_{c})^{(n)} = 0, \quad (\mathscr{N}^{\alpha} - \mathscr{N}^{\alpha}_{c})^{(n)} = 0, \quad (\mathscr{M}^{3} - \mathscr{M}^{3}_{c})^{(n)} = 0,\\
&\quad n = 1,2,\ldots,N \quad \text{on } A\times T
\end{aligned} \tag{8.6}
$$

in terms of Eqs. (7.26), (7.28) and (7.29), and the divergence equations in differential form by

$$
\begin{aligned}
&\sum_{r=1}^{N} V^{i}_{(r)} + (P^{i}_{u(N)} - P^{i}_{l(1)}) = 0 \quad \text{on } A\times T\\
&W^{i}_{(1)} + h_1\sum_{r=2}^{N} V^{i}_{(r)} + h_1(P^{i}_{(l)} + P^{i}_{(N)}) = 0 \quad \text{on } A\times T\\
&W^{i}_{(n)} + \sum_{r=n}^{N-1}K_{nr}V^{i}_{(r)} + 2h_nP^{(N)}_{u} = 0, \quad n = 2,3,\ldots,(N-1) \quad \text{on } A\times T\\
&W^{i}_{(N)} - (d_N - h_N)V^{i}_{(N)} + 2h_{(N)}P^{(N)}_{u} = 0 \quad \text{on } A\times T
\end{aligned} \tag{8.7}
$$

together with the symmetric stress resultants which readily follows from Eq. (5.23) as

$$
\begin{aligned}
&\varepsilon_{\alpha\beta}Q^{3\alpha}_{(n)} + \varepsilon_{\beta\alpha}Q^{\alpha3}_{(n)} = 0, \varepsilon_{\alpha\beta}R^{3\alpha}_{(n)} + \varepsilon_{\beta\alpha}R^{\alpha3}_{(n)} = 0 \quad n = 1,2,3,..,N; \quad \text{on } A\times T\\
&\varepsilon_{\alpha\nu}(N^{\nu\alpha}_{(n)} - b^{\nu}_{\beta}M^{\beta\alpha})_{(n)} = 0, \varepsilon_{\alpha\nu}(\mathscr{N}^{\nu\alpha} - b^{\nu}_{\beta}\mathscr{M}^{\beta\alpha})_{(n)} = 0\\
&n = 1,2,3,..,N; \quad \text{on } A\times T
\end{aligned} \tag{8.8}
$$

from Eq. (7.41)–(7.43) for the mechanical field, and

$$
\begin{aligned}
&\sum_{r=1}^{N}\mathscr{D}^{\alpha}_{(r):\alpha} + d_{u(N)} - d_{l(1)} = 0\\
&\mathscr{E}^{\alpha}_{(1):\alpha} + h_1\sum_{r=2}^{N}\mathscr{D}^{\alpha}_{(r):\alpha} + h_1(d_{u(N)} + d_{l(1)}) = 0\\
&\mathscr{E}^{\alpha}_{(r):\alpha} + \sum_{r=n}^{N}\kappa_{nr}\mathscr{D}^{\alpha}_{(r):\alpha} + 2h_ud^{(N)}_{u} = 0, \quad n = 2,3,\ldots(N-1) \quad \text{on } A\times T\\
&\mathscr{E}^{\alpha}_{(N):\alpha} - (d_N - h_N)\mathscr{D}^{\alpha}_{(N):\alpha} + 2h_Nd^{(N)}_{u}) = 0
\end{aligned} \tag{8.9}
$$

for the electrical field, and

$$
\begin{aligned}
&\sum_{r=1}^{N}\mathscr{M}^{\alpha}_{(r):\alpha} + m_{u(N)} - m_{l(1)} = 0\\
&\mathscr{N}^{\alpha}_{(1):\alpha} + h_1\sum_{r=2}^{N}\mathscr{M}^{\alpha}_{(r):\alpha} + h_1(m_{u(N)} + m_{l(1)}) = 0\\
&\mathscr{N}^{\alpha}_{(n):\alpha} + \sum_{r=n}^{N}\kappa_{nr}\mathscr{M}^{\alpha}_{(r):\alpha} + 2h_nm^{(N)}_{u} = 0, \quad n = 2,3,\ldots,(N-1)\\
&\text{on } A\times T\\
&\mathscr{N}^{\alpha}_{(N):\alpha} - (d_N - h_N)\mathscr{M}^{\alpha}_{(N):\alpha} + 2h_Nm^{(N)}_{u} = 0
\end{aligned} \tag{8.10}
$$

for the magnetic field, with the admissible states (7.43). Lastly, the boundary conditions in differential form are written as

$$
\begin{aligned}
&P^{(N)i}_{*u} - P^{(N)i}_{u} = 0 \ \text{ on } S_{uf}\times T, \ P^{(1)i}_{*l} + P^{(1)i}_{l} = 0 \ \text{ on } S_{lft}\times T\\
&\sum_{n=1}^{N}(\mathscr{N}^{i}_{*} - \nu_{\alpha}\mathscr{N}^{\alpha i})^{(n)} = 0, \ \text{ on } A\times T
\end{aligned} \tag{8.11a}
$$

$$
\sum_{r=n}^{N}\kappa_{nr}(\mathscr{N}^{i}_{*} - \nu_{\alpha}\mathscr{N}^{\alpha i})^{(r)} + (\mathscr{M}^{i}_{*} - \nu_{\alpha}\mathscr{M}^{\alpha i})^{(n)} = 0, \text{ along } C_t\times T \tag{8.11b}
$$

and

$$
\alpha^{(1)}_{i} - \alpha^{(1)}_{*i} = 0, \quad \beta^{(1)}_{i} - \beta^{(1)}_{*i} = 0, \quad \text{on } S_{lfu}\times T \quad \text{and} \quad C_u\times T \tag{8.11c}
$$

for the mechanical field, and

$$\alpha_1 - h_1\beta_1 - \phi_{*l} = 0, \quad \text{on } S_{lf} \times T$$
$$-\phi_{*u} + \sum_{n=1}^{N}(2-\delta_{1n})h_n\beta_n = 0, \quad \text{on } S_{uf} \times T \tag{8.12a}$$

and

$$\sum_{n=1}^{N}(\mathscr{D}_* - \nu_\alpha \mathscr{D}^\alpha)^{(n)} = 0, \ \text{along } C \times T$$
$$\sum_{r=n}^{N}\kappa_{nr}(\mathscr{D}_* - \nu_\alpha \mathscr{D}^\alpha)^{(r)} + (\mathscr{E}_* - \nu_\alpha \mathscr{E}^\alpha)^{(n)} = 0 \quad \text{along } C \times T \tag{8.12b}$$

and

$$\xi_1 - h_1\zeta_1 - \psi_{*l} = 0, \quad \text{on } S_{lf} \times T$$
$$-\psi_{*u} + \sum_{n=1}^{N}(2-\delta_{1n})h_n\zeta_n = 0, \quad \text{on } S_{uf} \times T \tag{8.13a}$$

and

$$\sum_{n=1}^{N}(\mathscr{M}_* - \nu_\alpha \mathscr{M}^\alpha)^{(n)} = 0, \ \text{along } C \times T$$
$$\sum_{r=n}^{N}\kappa_{nr}(\mathscr{M}_* - \nu_\alpha \mathscr{M}^\alpha)^{(r)} + (\mathscr{N}_* - \nu_\alpha \mathscr{N}^\alpha)^{(n)} = 0, \ \text{along } C \times T \tag{8.13b}$$

for the electromagnetic field in terms of the prescribed quantities.

The initial conditions (7.53)–(7.55) and the boundary conditions (8.11)–(8.15) are sufficient to ensure the uniqueness in solutions of the system of two dimensional equations of electromagnetoelastic laminae. The proof of the uniqueness is similar to that in Section 6 for the thin shell (cf., Dökmeci, 1978).

8.2. Electromagnetoelastic plates and layered plates

In the absence of curvature effects, that is, $b^\alpha_\beta = 0$ and the shell tensor $\mu^\alpha_\beta = \delta^\alpha_\beta$, the results of the thin shell and laminae are reduced to a system of two dimensional equations governing the motions of a functionally graded electromagnetoelastic plates at high frequencies, and also, plane laminae or layered plates. In addition, omitting the dependency on one of the aerial coordinates, one recovers the system of one-dimensional equations for a functionally graded beam and laminated beam. Further, discarding one of the effects, the results are easily seen to reduce to those of elastic, piezoelectric or magnetoelastic beam, plate and shell, and laminated elements as well (e.g., Dökmeci, 1974b, and the Altay and Dökmeci, 2003b).

8.3. Special variational principles

The variational principles are quite general and contain some of earlier ones as special cases. The variational principle (3.55) operating on all the field variables are readily reduced to the variational principles of piezoelectric and magnetoelastic stratified regions by simply dropping out the magnetic and electric terms, respectively. Also, it is noteworthy to mention a variational principle of the form

$$\delta L_{La}\left\{\Lambda_a = (u_i \in C_{12}, S_{ij} \in C_{00}, T^{ij})^{(n)}\right\}$$
$$= \sum_{n=1}^{N}(\delta L_a\{\Lambda_a\})^{(n)} + \delta L_{Sa} = 0 \tag{8.16a}$$

with

$$\delta L_{Sa} = \int_T dt \sum_{n=1}^{N-1}\left(\int_S \nu_i\{[\![T^{ij}]\!] < \delta u_j > - < \delta T^{ij} > [\![u_j]\!]\}\, dS\right)^{(n)} \tag{8.16b}$$

and its reciprocal variational principle as

$$\delta L_{Lem}\left\{\Lambda_{em} = (\phi, E_i, D^i; \psi, M_i, B^i)^{(n)}\right\}$$
$$= \sum_{n=1}^{N}(\delta L_e\{\Lambda_e\} + \delta L_m\{\Lambda_m\})^{(n)} + \delta L_{Sem} = 0, \tag{8.17a}$$

with

$$\delta L_{Sem} = \int_T dt \sum_{n=1}^{N-1}\Bigg(\int_S \nu_i\{[\![D^i]\!] < \delta\phi >$$
$$- < \delta D^i > [\![\phi]\!] + [\![B^i]\!] < \delta\psi > - < \delta B^i > [\![\psi]\!]\}\, dS\Bigg)^{(n)} \tag{8.17b}$$

from Eqs. (3.53) together with Eqs. (3.20)–(3.22).

9. Some concluding remarks

Presented in the first part of this study are the fundamental equations of an electromagnetoelastic material, including some results for their internal consistency, which is of special importance from the mathematical as well as physical point of view in modeling the response of a continuum. Certain unified variational principles operating on all the field variables with their well-known features are derived so as to express the fundamental equations of the regular region and stratified regular region of an electromagnetoelastic material, as the appropriate Euler–Lagrange equations. The principles are deduced from a general principle of physics by relaxing its constraint conditions through an involutory transformation, and thus, allowing for simultaneous approximation upon all the field variables. They generate all the divergence and gradient equations, the constitutive relations, and the boundary conditions of the regions. Their admissible states are subjected to Cauchy's second law of motion (i.e., the symmetry of the stress tensor), which always remains as a constraint conditions for a non-polar continuum (cf., a polar continuum, Altay and Dökmeci, 2006) and the initial conditions, which can be either directly included or implicitly considered through Laplace transformation (e.g., Dökmeci, 1992; Altay and Dökmeci, 2000). The weight functions of the variational principles are not purely formal as in the weighted residual methods but found in terms of the original field variables, which provide their physical interpretation. The variational principles are the counterpart of the Hu–Washizu variational principle in elasticity, and they are compatible with and contain some of earlier variational principles in elasticity, piezoelectricity, magnetoelasticity, electromagnetism and electromagnetoelasticity, as special cases. However, the unified variational principle (3.55), which generates all the interface conditions for a stratified electromagnetoelastic region as well, has no counterpart for the coupled fields of continua. The reader may be referred to a review of variational principles with an extensive list of references for a regular region of materials subjected to coupled mechanical, electrical, thermal, magnetic and moisture fields (e.g., Altay and Dökmeci, 2007, and references cited therein).

In the second part, a system of two dimensional shear deformation equations of a electromagnetoelastic thin shell graded functionally across its thickness is systematically derived with the use of Mindlin's method of reduction at the short wave approximation. The mechanical displacements and the electric and magnetic potentials are chosen as the basic field variables in terms of the power series expansions in the thickness coordinate. The variational principles are used to derive consistently a system of two dimensional approximate equations of shell at high frequency motions in invariant and fully variational and differential forms, including all the mechanical, electric and magnetic effects of higher orders. The invariant form is useful in expressing the system of equations in any particular system of coordinates, which is most appropriate to the geometry of shell, and the variational form is desirable in making simultaneous approximation upon all the field variables by freely choosing the shape functions in a direct approximate solution of the shell equations. The initial-mixed boundary value problem defined by the shell equations is shown to be well-posed and deterministic. The uniqueness in solutions of the problem is investigated and the boundary and initial conditions are shown to be sufficient for the uniqueness, and then a theorem of uniqueness is stated. The resulting equations are quite general and agree with some of earlier ones, and they can be readily reduced to a system of two dimensional shear deformation equations of a functionally graded plates by simply neglecting the curvature effects, that is, replacing the shell tensor μ^{α}_{β} by the Kronecker delta δ^{α}_{β}. The shell equations, which are geometrically linear but physically nonlinear recover the high frequency equations of shells, piezoelectric crystal surfaces, and even bars and piezoelectric bars by simply eliminating the dependency on one of the aerial coordinates, that is, $u_i^{(n)} = u_i^{(n)}(\theta^1, t),\ \phi_{(n)} = \phi(\theta^1, t)$ and $\psi(\theta^1, t)$, as well as the curvature effects (Dökmeci, 1972, 1974b). Moreover, discarding the electric and curvature effects, the shell equations govern the motions of a magnetoelastic plate and shell at the short wave approximation, and also, retaining only the linear terms in Eqs. (4.4)–(4.7) with $N = 1$ at long-wave approximation.

In the third part, the system of two dimensional shear deformation equations is developed for the motions of a functionally graded electromagnetoelastic laminae under the long-wave approximation within a discrete layer modeling. The laminae may comprise any number of bonded layers, each with a distinct but uniform thickness, curvature and physical properties. The system of laminae equations is derived for the case when the mechanical displacements and the electric and magnetic potentials vary linearly across the thickness of each layer. In the derivation, the Kirchoff–Love hypothese of shells and its contradictions are eliminated, and hence, the dynamic interactions among the constituents are fully taken into account. Also, the usual continuity of field variables are fully maintained in accordance with perfectly bonded constituents. Paralleling the uniqueness theorem for the shell case in Section 6, the boundary conditions (8.10)–(8.15) and the initial conditions (7.53)–(7.55) can be shown to be sufficient to assure the uniqueness in solutions of the laminae equations (cf., Dökmeci, 1978). As before, the resulting equations in invariant form are applicable to arbitrary shaped laminae in a desirable system of coordinates. Besides, they can be reduced to those of piezoelectric and magnetoelastic layered beams, plates and shells, which are unavailable in the open literature. In closing, what remain to complete this work are applications to structural elements with special geometry, material and motion, including numerical results which can provide an evaluation of the formulation, some experimental results and technological applications as well.

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