International Journal of Solids and Structures 48 (2011) 1395–1401

Contents lists available at ScienceDirect

International Journal of Solids and Structures

journal homepage: www.elsevier.com/locate/ijsolstr

Singularities of an interface crack in electrostrictive materials

Cun-Fa Gao [a,b,*], Yiu-Wing Mai [b]

[a] College of Aerospace Engineering, Nanjing University of Aeronautics and Astronautics, Nanjing 210016, China

[b] Centre for Advanced Materials Technology (CAMT), School of Aerospace, Mechanical and Mechatronic Engineering J07, University of Sydney, Sydney, NSW 2006, Australia

ARTICLE INFO

Article history:
Received 17 February 2010
Received in revised form 8 January 2011
Available online 26 January 2011

Keywords:
Electrostrictive materials
Interface crack
Singularities

ABSTRACT

In the present work, the singularities of an interface crack between two dissimilar electrostrictive materials under electric loads are investigated. Within the framework of two-dimensional deformation, the problem is solved using the complex variable method. Three crack models, that is, permeable, impermeable and conducting crack models are considered individually. Complex potentials and intensity factors of total stresses are derived by considering both the Maxwell stresses in the surrounding space at infinity and inside the crack. It is found that, for the above three crack models, the singularities of total stress are the same as those in traditional bi-materials with an interface crack; however, the intensities of the total stress depend on the actual crack model used.

1. Introduction

Multilayer actuators and layered composites which are made of smart materials have many applications in different engineering fields. Cracks may occur at the interface when the materials are subjected to electric and mechanical loads. Suo et al. (1992) studied an impermeable interfacial crack between two piezoelectric materials and showed a new type of singularity around the interface crack tips. Gao and Wang (2000), Beom (2003), and Gao et al. (2004) investigated the interfacial fracture of a permeable crack between two piezoelectric half-planes, and they found that the singularities depend on material properties and applied mechanical loads, but not on applied electric loads. Herrmann et al. (2005) considered the in-plane problem for a moving interface crack with a contact zone in a piezoelectric bi-material and showed increases of the contact zone length and stress intensity factor for the near-critical speed region. Li and Chen (2007) presented the solution for a semi-permeable interface crack between two dissimilar piezoelectric materials and discussed the effect of permittivity of the medium inside the crack on the near-tip singularity. Hausler et al. (2009) studied the fracture behavior of metal-piezoceramic interfaces under both mechanical and electrical loadings by four-point bending using commercial multilayer actuators.

For the problem of interfacial fracture of electrostrictive materials, Ru et al. (1998) studied the electric field induced cracking in multilayer electrostrictive actuators for an interface crack lying between an electrode layer and a ceramic matrix, and an interface crack with one tip at an embedded electrode-edge, respectively. Kim and Beom (2009) presented the numerical analysis of an electrode embedded between dissimilar electrostrictive materials using finite element method. Introducing the Maxwell stresses in the matrix and surrounding space (Kuang, 2008, 2009), Kuang developed the basic equations of Stratton (1941) and Landau and Lifshitz (1960). Gao and Mai (2011) reported the effects of Maxwell stresses on the fracture behavior of a permeable interface crack in an electrostrictive bi-material.

In the present work, we study the singularities of an interfacial crack between two dissimilar electrostrictive materials by using the complex variable method. The crack may be electrically impermeable, permeable and conducting, respectively. Emphasis is placed on two problems: what are the types of singularity for the three crack models, and what factors dominate the intensities of all the singularities? To this end, we first outline the basic equations required in Section 2, and derive the general solutions for the electric potentials and electro-elastic potentials in Sections 3 and 4, respectively. The intensity factors of total stress are given in Section 5 and conclusions presented in Section 6.

2. Basic equations

Consider an isothermal and isotropic electrostrictive material, and neglect the piezo-electricity, the constitutive equations can be expressed as (Jiang and Kuang, 2004):

$$\sigma_{ij} = 2Ge_{ij} + \lambda e_{kk}\delta_{ij} - \frac{1}{2}(a_1 E_i E_j + a_2 E_k E_k \delta_{ij}), \quad (1)$$

$$D_i = (\varepsilon_m \delta_{ij} + a_1 e_{ij} + a_2 e_{kk}\delta_{ij})E_j, \quad (2)$$

where $i, j = 1, 2, 3$. σ_{ij}, e_{ij}, D_i and E_j are stress, strain, electric displacement and electric field intensity, respectively. a_1 and a_2 are

* Corresponding author at: College of Aerospace Engineering, Nanjing University of Aeronautics and Astronautics, Nanjing 210016, China.

E-mail address: cfgao@nuaa.edu.cn (C.-F. Gao).

0020-7683/$ - see front matter © 2011 Elsevier Ltd. All rights reserved.
doi:10.1016/j.ijsolstr.2011.01.026

two independent electrostrictive coefficients in isotropic materials, ε_m is permittivity at zero strain and δ_{ij} is Kronecker delta. λ and G are Lame constants given in terms of Young's modulus E and Poisson ratio ν by: $\lambda = E\nu/[(1+\nu)(1-2\nu)]$ and $G = E/[2(1+\nu)]$.

The equilibrium equations become (Jiang and Kuang, 2004):

$$\frac{\partial \sigma_{ij}}{\partial x_j} + f_i^e = 0, \tag{3}$$

$$\frac{\partial D_i}{\partial x_i} + q = 0, \tag{4}$$

where f_i^e is body force induced by the electric fields, and q is free charge density in the body, the repeated indices represent their summation, and

$$f_i^e = \frac{\partial \sigma_{ij}^M}{\partial x_j}, \quad \sigma_{ij}^M = \varepsilon_m E_i E_j - \frac{1}{2}\varepsilon_m E_k E_k \delta_{ij}, \tag{5}$$

where σ_{ij}^M is the Maxwell stress.

Substituting Eq. (5) into Eq. (3) yields:

$$\frac{\partial \tilde{\sigma}_{ij}}{\partial x_j} = 0, \quad \tilde{\sigma}_{ij} = \sigma_{ij} + \sigma_{ij}^M, \tag{6}$$

where $\tilde{\sigma}_{ij}$ is the total stress.

When the strain is very small, the coupling effects between strain and electric field may be neglected. Hence, the electric field can be obtained directly from the theory of electrostatics. In this case, Eq. (2) is reduced to:

$$D_i = \varepsilon_m E_i, \tag{7}$$

and furthermore, we have:

$$E_i = -\frac{\partial \varphi}{\partial x_i}, \tag{8}$$

$$\frac{\partial D_i}{\partial x_i} = 0 \quad (i = 1, 2). \tag{9}$$

Substituting Eqs. (7) and (8) in (9) yields:

$$\frac{\partial^2 \varphi}{\partial x_1^2} + \frac{\partial^2 \varphi}{\partial x_2^2} = 0. \tag{10}$$

The general solution of Eq. (10) is:

$$\varphi = Re[w(z)], \quad z = x_1 + ix_2, \tag{11}$$

where $w(z)$ is the unknown potential function.

Inserting Eq. (11) in (8) gives:

$$E_1 = -\mathrm{Re}[w'(z)], \quad E_2 = \mathrm{Im}[w'(z)]. \tag{12}$$

From Eqs. (12) and (7), we obtain:

$$E_1 - iE_2 = -w'(z), \tag{13}$$

$$D_1 - iD_2 = -\varepsilon_m w'(z). \tag{14}$$

After $w(z)$ is determined from given electric boundary conditions, all stress fields can be obtained and the final results are outlined below (Jiang and Kuang, 2004):

Total stresses $\tilde{\sigma}_{ij}$

$$\tilde{\sigma}_{22} + \tilde{\sigma}_{11} = \kappa w'(z)\overline{w'(z)} + 2\left[\phi'(z) + \overline{\phi'(z)}\right], \tag{15}$$

$$\tilde{\sigma}_{22} - \tilde{\sigma}_{11} + 2i\tilde{\sigma}_{12} = \kappa w''(z)\overline{w(z)} + 2[\bar{z}\phi''(z) + \psi'(z)], \tag{16}$$

where $\kappa = -(1-2\nu)(a_1 + 2a_2)/[4(1-\nu)]$, and $\phi(z)$ and $\psi(z)$ are two complex functions to be determined.

Maxwell stresses σ_{ij}^M

$$\sigma_{22}^M + \sigma_{11}^M = 0, \tag{17}$$

$$\sigma_{22}^M - \sigma_{11}^M + 2i\sigma_{12}^M = -\varepsilon_m \Omega'(z), \tag{18}$$

where

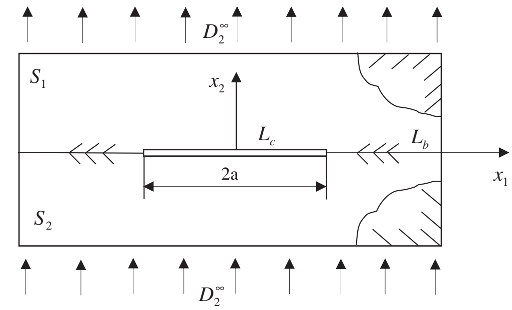

Fig. 1. Permeable or impermeable interface crack in electrostrictive materials.

$$\Omega'(z) = [w'(z)]^2. \tag{19}$$

Similarly, the displacement field is:

$$2G(u_1 + iu_2) = K\phi(z) - z\overline{\phi'(z)} - \overline{\psi(z)} + \chi\overline{\Omega(z)} - \frac{\kappa}{2}w(z)\overline{w'(z)}, \tag{20}$$

where $K = 3 - 4\nu$ and $\chi = (a_1 - 2\varepsilon_m)/4$.

From Eqs. (15) and (16) we have:

$$\tilde{\sigma}_{22} - i\tilde{\sigma}_{12} = \frac{1}{2}\kappa\left[w'(z)\overline{w'(z)} + w(z)\overline{w''(z)}\right] + \phi'(z) + \overline{\phi'(z)} + z\overline{\phi''(z)} + \overline{\psi'(z)}. \tag{21}$$

Introduce a new function, $W(z)$, which is:

$$W(z) = \bar{\phi}'(z) + z\bar{\phi}''(z) + \bar{\psi}'(z). \tag{22}$$

Then, we have:

$$W(\bar{z}) = \overline{\phi'(z)} + \bar{z}\overline{\phi''(z)} + \overline{\psi'(z)}. \tag{23}$$

Eq. (23) gives:

$$\overline{\phi'(z)} + \overline{\psi'(z)} = W(\bar{z}) - \bar{z}\overline{\phi''(z)}. \tag{24}$$

Substituting Eq. (24) into (21) yields:

$$\tilde{\sigma}_{22} - i\tilde{\sigma}_{12} = \sigma_w(z,\bar{z}) + \Phi(z) + W(\bar{z}) + (z - \bar{z})\overline{\Phi'(z)}, \tag{25}$$

where $\Phi(z) = \phi'(z)$, and

$$\sigma_w(z,\bar{z}) = \frac{1}{2}\kappa\left[w'(z)\overline{w'(z)} + w(z)\overline{w''(z)}\right], \tag{26}$$

which is not an analytical function of z.

Similarly, we have from Eq. (20) that:

$$2G(u_{1,1} + iu_{2,1}) = K\Phi(z) - W(\bar{z}) - (z - \bar{z})\overline{\Phi'(z)} + \chi\overline{\Omega'(z)} - \sigma_w(z,\bar{z}), \tag{27}$$

where $u_{i,1} = \partial u_i/\partial x_1$.

3. Solutions for electric potential function $w(z)$

Consider an interface crack between two dissimilar electrostrictive half-planes S_1 and S_2 , as shown in Fig. 1. Assume that the upper and lower planes are surrounded by free space with a dielectric constant ε_{env} and the electric loading at infinity is D_2^∞.

Similar to solving the problems of thermal stresses, $w(z)$ can be obtained from the theory of electrostatics. For the three crack models, detailed derivations for $w(z)$ are given in Appendix A; and final results (from Eqs. (A21), (A34) and (A45)) are summarized below:

$$w_k(z) = \begin{cases} i\frac{D_2^\infty}{\varepsilon_k} z, & \textit{permeable crack}, \\ i\frac{D_2^\infty}{\varepsilon_k}\sqrt{z^2-a^2}, & \textit{impermeable crack}, \quad z \in S_k \quad (k=1,2). \\ -E_1^\infty\sqrt{z^2-a^2}, & \textit{conducting crack}, \end{cases} \tag{28}$$

Substituting Eq. (28) into (26) we have:

$$\sigma_w(z,\bar{z})|_{z\to\infty} = \sigma_w^\infty = \begin{cases} \frac{1}{2}\kappa_k\left(\frac{D_2^\infty}{\varepsilon_k}\right)^2, \ \textit{permeable crack}, \\ \frac{1}{2}\kappa_k\left(\frac{D_2^\infty}{\varepsilon_k}\right)^2, \ \textit{impermeable crack}, \ z\in S_k \ (k=1,2). \\ \frac{1}{2}\kappa_k(E_1^\infty)^2, \ \textit{conducting crack}, \end{cases} \tag{29}$$

Specifically, along the crack-faces where $z = |x_1| < a$, we find:

$$\sigma_w(z,\bar{z})|_{z=|x_1|<a} = \sigma_w^c = \begin{cases} \frac{1}{2}\kappa_k\left(\frac{D_2^\infty}{\varepsilon_k}\right)^2, \quad \textit{permeable crack}, \\ -\frac{1}{2}\kappa_k\left(\frac{D_2^\infty}{\varepsilon_k}\right)^2, \ \textit{impermeable crack}, \ z\in S_k \ (k=1,2). \\ -\frac{1}{2}\kappa_k(E_1^\infty)^2, \ \textit{conducting crack}, \end{cases} \tag{30}$$

Similarly, ahead of the crack tip where $z = |x_1| > a$, we obtain:

$$\sigma_w(z,\bar{z})|_{z=|x_1|>a} = \sigma_w^l = \begin{cases} \frac{1}{2}\kappa_k\left(\frac{D_2^\infty}{\varepsilon_k}\right)^2, \ \textit{permeable crack}, \\ \frac{1}{2}\kappa_k\left(\frac{D_2^\infty}{\varepsilon_k}\right)^2, \ \textit{impermeable crack}, \ z\in S_k \ (k=1,2). \\ \frac{1}{2}\kappa_k(E_1^\infty)^2, \ \textit{conducting crack}, \end{cases} \tag{31}$$

4. Solutions for electro-elastic potentials

Mathematically, the upper and lower half-spaces come together at infinity; thus, the following continuous conditions apply (Gao and Wang, 2000):

$$[\tilde{\sigma}_{22} - i\tilde{\sigma}_{12}]_1^\infty = [\tilde{\sigma}_{22} - i\tilde{\sigma}_{12}]_2^\infty, \tag{32}$$

$$[u_{1,1} + iu_{2,1}]_1^\infty = [u_{1,1} + iu_{2,1}]_2^\infty. \tag{33}$$

Taking the limit $z \to \infty$ in Eqs. (25) and (27) and inserting those two equations into Eqs. (32) and (33) above gives:

$$\sigma_{w1}^\infty + \Phi_1(\infty) + W_1(\infty) = \sigma_{w2}^\infty + \Phi_2(\infty) + W_2(\infty) = [\tilde{\sigma}_{22} - i\tilde{\sigma}_{12}]_1^\infty, \tag{34}$$

$$\frac{1}{2G_1}\left[K_1\Phi_1(\infty) - W_1(\infty) + \chi_1\overline{\Omega_1'(\infty)} - \sigma_{w1}^\infty\right] = \frac{1}{2G_2}\left[K_2\Phi_2(\infty) - W_2(\infty) + \chi_2\overline{\Omega_2'(\infty)} - \sigma_{w2}^\infty\right]. \tag{35}$$

In addition, it is assumed that $K_1/\varepsilon_1^2 = K_2/\varepsilon_2^2$ for impermeable crack and $K_1 = K_2$ for conducting crack. Conversely, along the x_1-axis, we have:

$$[\tilde{\sigma}_{22} - i\tilde{\sigma}_{12}]_1 = [\tilde{\sigma}_{22} - i\tilde{\sigma}_{12}]_2, \quad -\infty < x_1 < +\infty. \tag{36}$$

From Eq. (25) we can obtain:

$$[\tilde{\sigma}_{22} - i\tilde{\sigma}_{12}]_1 = \sigma_{w1}^c + \Phi_1^+(x_1) + W_1^-(x_1), \tag{37}$$

$$[\tilde{\sigma}_{22} - i\tilde{\sigma}_{12}]_2 = \sigma_{w2}^c + \Phi_2^-(x_1) + W_2^+(x_1). \tag{38}$$

Substituting Eqs. (37) and (38) into Eq. (36) yields:

$$\left[\Phi_1^+(x_1) - W_2^+(x_1) + \sigma_{w1}^c\right] - \left[\Phi_2^-(x_1) - W_1^-(x_1) + \sigma_{w2}^c\right] = 0, \quad -\infty < x_1 < +\infty,$$

which gives (Muskhelisvili, 1975):

$$\Phi_1(z) - W_2(z) = \Phi_1(\infty) - W_2(\infty), \quad z \in S_1, \tag{39}$$

$$\Phi_2(z) - W_1(z) = \Phi_2(\infty) - W_1(\infty), \quad z \in S_2. \tag{40}$$

Now, define a jump function as:

$$\Delta u_{,1}(x_1) = [u_{1,1}(x_1) + iu_{2,1}(x_1)]_1 - [u_{1,1}(x_1) + iu_{2,1}(x_1)]_2. \tag{41}$$

Then, on the crack-faces, we have from Eq. (27) that:

$$[u_{1,1} + iu_{2,1}]_1 = \frac{1}{2G_1}\left[K_1\Phi_1^+(x_1) - W_1^-(x_1) + \chi_1\overline{\Omega_1'(x_1)} - \sigma_{w1}^c\right], \tag{42}$$

$$[u_{1,1} + iu_{2,1}]_2 = \frac{1}{2G_2}\left[K_2\Phi_2^-(x_1) - W_2^+(x_1) + \chi_2\overline{\Omega_2'(x_1)} - \sigma_{w2}^c\right]. \tag{43}$$

Inserting Eqs. (41) and (43) into (41), and using Eqs. (39) and (40), we have:

$$2\Delta u_{,1}(x_1) = \lambda_1\Phi_1^+(x_1) - \lambda_2\Phi_2^-(x_1) + \frac{\chi_1}{G_1}\overline{\Omega_1'(x_1)} - \frac{\chi_2}{G_2}\overline{\Omega_2'(x_1)} + C_1^\infty - C_2^\infty, \tag{44}$$

where

$$\lambda_1 = \frac{K_1}{G_1} + \frac{1}{G_2}, \quad \lambda_2 = \frac{K_2}{G_2} + \frac{1}{G_1}, \tag{45}$$

$$C_1^\infty = \frac{1}{G_1}\left[\Phi_2(\infty) - W_1(\infty) - \sigma_{w1}^c\right], \tag{46}$$

$$C_2^\infty = \frac{1}{G_2}\left[\Phi_1(\infty) - W_2(\infty) - \sigma_{w2}^c\right]. \tag{47}$$

Introduce a new function, $F(z)$, whereby:

$$F(z) = \begin{cases} \lambda_1\Phi_1(z) - \frac{\chi_2}{G_2}\overline{\Omega'}_2(z) + C_1^\infty, & z \in S_1 \\ \lambda_2\Phi_2(z) - \frac{\chi_1}{G_1}\overline{\Omega'}_1(z) + C_2^\infty, & z \in S_2. \end{cases} \tag{48}$$

Then, Eq. (44) becomes:

$$2\Delta u_{,1} = F^+(x_1) - F^-(x_1). \tag{49}$$

Except on the crack, we have $\Delta u_{,1} = 0$, that is,

$$F^+(x_1) = F^-(x_1) = F(x_1), \quad |x_1| > a, \tag{50}$$

which means that $F(z)$ is an analytic function in the whole plane except the crack.

Using Eq. (40) we obtain from Eq. (37) that:

$$[\tilde{\sigma}_{22} - i\tilde{\sigma}_{12}]_1 = \Phi_1^+(x_1) + \Phi_2^-(x_1) + \sigma_{w1}^c - \Phi_2(\infty) + W_1(\infty). \tag{51}$$

From Eq. (34) we also obtain:

$$W_1(\infty) = [\tilde{\sigma}_{22} - i\tilde{\sigma}_{12}]_1^\infty - \sigma_{w1}^\infty - \Phi_1(\infty). \tag{52}$$

Inserting Eq. (52) into (51) gives:

$$[\tilde{\sigma}_{22} - i\tilde{\sigma}_{12}]_1 = [\tilde{\sigma}_{22} - i\tilde{\sigma}_{12}]_1^\infty + \left(\sigma_{w1}^c - \sigma_{w1}^\infty\right) + \Phi_1^+(x_1) + \Phi_2^-(x_1) - [\Phi_2(\infty) + \Phi_1(\infty)]. \tag{53}$$

Since the crack is traction-free on the crack-faces, we have:

$$[\tilde{\sigma}_{22} - i\tilde{\sigma}_{12}]_1 = \left[\sigma_{22}^M - i\sigma_{12}^M\right]_c, \tag{54}$$

where $\left[\sigma_{22}^M - i\sigma_{12}^M\right]_c$ are the Maxwell stresses inside the crack.

Inserting Eq. (54) in Eq. (53) gives:

$$\Phi_1^+(x_1) + \Phi_2^-(x_1) = \sigma_0 + [\Phi_2(\infty) + \Phi_1(\infty)], \tag{55}$$

where

$$\sigma_0 = \left[\sigma_{22}^M - i\sigma_{12}^M\right]_c - \left\langle[\tilde{\sigma}_{22} - i\tilde{\sigma}_{12}]_1^\infty + \left(\sigma_{w1}^c - \sigma_{w1}^\infty\right)\right\rangle. \tag{56}$$

Conversely, using the following identities:

$$\Phi_1^+(x_1) = \frac{1}{\lambda_1}F^+(x_1) + \frac{1}{\lambda_1}\left(\frac{\chi_2}{G_2}\overline{\Omega_2'(x_1)} - C_1^\infty\right), \tag{57}$$

$$\Phi_2^-(x_1) = \frac{1}{\lambda_2}F^-(x_1) + \frac{1}{\lambda_2}\left(\frac{\chi_1}{G_1}\overline{\Omega_1'(x_1)} - C_2^\infty\right), \tag{58}$$

Eq. (55) can be reduced to:

$$F^+(x_1) + \frac{\lambda_1}{\lambda_2}F^-(x_1) = \lambda_1\sigma_0 + \lambda_1[\Phi_2(\infty) + \Phi_1(\infty)] - \left(\frac{\chi_2}{G_2}\overline{\Omega_2'(x_1)} - C_1^\infty\right) - \frac{\lambda_1}{\lambda_2}\left(\frac{\chi_1}{G_1}\overline{\Omega_1'(x_1)} - C_2^\infty\right). \tag{59}$$

Using Eqs. (34), (35), (46) and (47), Eq. (59) becomes:

$$F^+(x_1) - gF^-(x_1) = f(x_1), \tag{60}$$

where

$$g = -\lambda_1/\lambda_2 < 0, \tag{61}$$

$$f(x_1) = \lambda_1\sigma_0 - \frac{\chi_2}{G_2}\overline{\Omega_2'(x_1)} - \frac{\lambda_1}{\lambda_2}\frac{\chi_1}{G_1}\overline{\Omega_1'(x_1)}. \tag{62}$$

The general solution of Eq. (60) is (Muskhelisvili, 1975):

$$F(z) = \frac{X(z)}{2\pi i}\int_{L_c}\frac{f(x_1)dx_1}{X^+(x_1)(x_1 - z)} + X(z)(c_1 z + c_0), \tag{63}$$

where c_1 and c_0 are constants to be determined, and

$$X(z) = (z+a)^{-\gamma}(z-a)^{\gamma-1}, \tag{64}$$

$$\gamma = \frac{1}{2} - i\varepsilon, \quad \varepsilon = \frac{1}{2\pi}\ln|g|. \tag{65}$$

From Eqs. (64) and (65), we have:

$$X(z) = \frac{1}{\sqrt{z^2 - a^2}}\left(\frac{z+a}{z-a}\right)^{i\varepsilon}. \tag{66}$$

Following Muskhelisvili (1975) we can determine from Eq. (63) that:

$$F(z) = \frac{1}{1-g}[f(z) - zX(z)\lambda_1\sigma_0] + X(z)(c_1 z + c_0). \tag{67}$$

In this case, the single-valued condition of displacement can be expressed as:

$$\oint_\Lambda F(z)dz = 0, \tag{68}$$

where Λ is a clockwise contour enclosing the crack.

Eq. (68) combined with the result obtained by taking limit $z \to \infty$ in Eq. (67) leads to $c_1 = c_0 = 0$. Thus, the final solution for $F(z)$ can be simplified as:

$$F(z) = \frac{1}{1-g}[f(z) - zX(z)\lambda_1\sigma_0]. \tag{69}$$

With Eqs. (69) and (48), the unknown functions $\Phi_1(z)$ and $\Phi_2(z)$ can be determined, and thence, $W_1(z)$ and $W_2(z)$ are readily obtained from Eqs. (39) and (40).

5. Solutions for intensity factors of total stress

We define the intensity factor of total stress as:

$$k_1 - ik_2 = \lim_{r\to 0} r^{i\varepsilon}\sqrt{2\pi r}[\tilde{\sigma}_{22} - i\tilde{\sigma}_{12}]_1. \tag{70}$$

Ahead of the crack tip, from Eq. (25) and using Eq. (31), we have:

$$[\tilde{\sigma}_{22} - i\tilde{\sigma}_{12}]_1 = \sigma_{w1}^l + \Phi_1^+(x_1) + W_1^-(x_1), \tag{71}$$

Similarly, Eq. (71) can be reduced to:

$$[\tilde{\sigma}_{22} - i\tilde{\sigma}_{12}]_1 = \lambda_1^{-1}\left[(1-g)F(x_1) - \frac{\chi_1}{G_1}\overline{\Omega_{10}'(x_1)} - \frac{\lambda_1}{\lambda_2}\frac{\chi_2}{G_2}\overline{\Omega_{20}'(x_1)}\right], \tag{72}$$

where a constant without any effect on stress singularities is neglected.

Inserting Eq. (69) into (72) leads to:

$$[\tilde{\sigma}_{22} - i\tilde{\sigma}_{12}]_1 = \lambda_1^{-1}\left[f(x_1) - x_1X(x_1)\lambda_1\sigma_0 - \frac{\chi_1}{G_1}\overline{\Omega_{10}'(x_1)} - \frac{\lambda_1}{\lambda_2}\frac{\chi_2}{G_2}\overline{\Omega_{20}'(x_1)}\right]. \tag{73}$$

Substituting Eq. (62) into (73) yields:

$$[\tilde{\sigma}_{22} - i\tilde{\sigma}_{12}]_1 = [\sigma_0 - x_1X(x_1)\sigma_0]. \tag{74}$$

Inserting Eq. (74) into (70) we finally obtain:

$$k_1 - ik_2 = -(2a)^{i\varepsilon}\sqrt{\pi a}\sigma_0. \tag{75}$$

Substituting Eq. (56) into (75) now gives:

$$k_1 - ik_2 = (2a)^{i\varepsilon}\sqrt{\pi a}\langle[\tilde{\sigma}_{22} - i\tilde{\sigma}_{21}]_1^\infty + [\sigma_{w1}^c - \sigma_{w1}^\infty] - [\sigma_{22}^M - i\sigma_{21}^M]_c\rangle. \tag{76}$$

For traditional materials loaded by mechanical stresses σ_{22}^∞ and σ_{21}^∞ at infinity, we have:

$$[\tilde{\sigma}_{22} - i\tilde{\sigma}_{21}]_1^\infty = \sigma_{22}^\infty - i\sigma_{21}^\infty, \quad [\sigma_{w1}^c - \sigma_{w1}^\infty] = 0, \quad [\sigma_{22}^M - i\sigma_{21}^M]_c = 0.$$

In this case, Eq. (76) degenerates to:

$$k_1 - ik_2 = (2a)^{i\varepsilon}\sqrt{\pi a}(\sigma_{22}^\infty - i\sigma_{21}^\infty), \tag{77}$$

which is expected for a traditional bi-material.

If the mechanical stresses are zero at infinity, then

$$[\tilde{\sigma}_{22} - i\tilde{\sigma}_{21}]^\infty = [\sigma_{22}^M - i\sigma_{21}^M]_\infty, \tag{78}$$

where $[\sigma_{22}^M - i\sigma_{21}^M]_\infty$ are the Maxwell stresses in the surrounding space at infinity.

In this case, Eq. (76) becomes:

$$k_1 - ik_2 = (2a)^{i\varepsilon}\sqrt{\pi a}\left\langle[\sigma_{22}^M - i\sigma_{21}^M]_\infty + [\sigma_{w1}^c - \sigma_{w1}^\infty] - [\sigma_{22}^M - i\sigma_{21}^M]_c\right\rangle. \tag{79}$$

Eq. (79) shows that the singularity of total stress depends on the Maxwell stresses in the surrounding space at infinity and on the crack-faces. Since the Maxwell stresses on the crack-faces relate to the electric boundary conditions there, we will discuss below three specific cases individually.

5.1. For a permeable crack

In this case, from Eqs. (29) and (30), we have:

$$\sigma_{w1}^c = \sigma_{w1}^\infty = \frac{1}{2}\kappa_1\left(\frac{D_2^\infty}{\varepsilon_1}\right)^2. \tag{80}$$

Also, Eq. (28) implies that the electric displacement is uniform everywhere, meaning that the electric displacement inside the crack equals that applied. Moreover, assume the interior of the crack and the surrounding space at infinity are filled with different gases, and $(\varepsilon_{env}^\infty, \varepsilon_m, \varepsilon_c)$ represent the dielectric constants of the crack interior, matrix and surrounding space at infinity, respectively. If we assume $\varepsilon_{env}^\infty \ll \varepsilon_m$ and $\varepsilon_c \ll \varepsilon_m$, the Maxwell stresses become:

$$\sigma_{22\infty}^M = \frac{(D_2^\infty)^2}{2\varepsilon_{env}}, \quad \sigma_{12\infty}^M = 0, \tag{81}$$

$$\sigma_{22c}^M = \frac{(D_2^\infty)^2}{2\varepsilon_c}, \quad \sigma_{12c}^M = 0. \tag{82}$$

Inserting Eqs. (81) and (82) into Eq. (79) we obtain:

$$k_1 - ik_2 = (2a)^{i\varepsilon}\sqrt{\pi a}\left[\frac{1}{2\varepsilon_{env}} - \frac{1}{2\varepsilon_c}\right](D_2^\infty)^2. \tag{83}$$

5.2. For an impermeable crack

In this case, Eqs. (29) and (30) together give:

$$\sigma_{w1}^{\infty} = \frac{1}{2}\kappa_1\left(\frac{D_2^{\infty}}{\varepsilon_1}\right)^2, \quad \sigma_{w1}^{c} = -\frac{1}{2}\kappa_1\left(\frac{D_2^{\infty}}{\varepsilon_1}\right)^2. \tag{84}$$

The Maxwell stresses inside the crack are zero, that is,

$$\sigma_{22c}^{M} = 0, \quad \sigma_{12c}^{M} = 0. \tag{85}$$

Inserting Eqs. (81), (84) and (85) into (79) yields:

$$k_1 - ik_2 = (2a)^{i\varepsilon}\sqrt{\pi a}\left[\frac{1}{2\varepsilon_{env}} - \frac{\kappa_1}{\varepsilon_1^2}\right](D_2^{\infty})^2. \tag{86}$$

5.3. For a conducting crack

In this case, we obtain from Eqs. (29) and (30) that:

$$\sigma_{w1}^{\infty} = \frac{1}{2}\kappa_k(E_1^{\infty})^2, \quad \sigma_{w1}^{c} = -\frac{1}{2}\kappa_k(E_1^{\infty})^2, \tag{87}$$

and the Maxwell stresses at infinity are given by:

$$\sigma_{22\infty}^{M} = -\frac{\varepsilon_{env}}{2}(E_1^{\infty})^2, \quad \sigma_{12\infty}^{M} = 0, \tag{88}$$

Thus, substituting Eqs. (87), (88) and (85) into (79) yields:

$$k_1 - ik_2 = -(2a)^{i\varepsilon}\sqrt{\pi a}\left[\frac{\varepsilon_{env}}{2} + \kappa_1\right](E_1^{\infty})^2. \tag{89}$$

When the interface crack degenerates to a crack in a *homogeneous* material, it can be shown that the above results for a permeable crack agree with those obtained by Gao et al. (2010a,b) who derived the solutions for a single crack by using an elliptical-hole model and the solutions for collinear cracks by solving the Riemann–Hilbert boundary-value problem, respectively. However, for an impermeable crack or a conducting crack, previous results (Gao et al., 2010a,b) have missed a constant in the solutions for intensity factors of total stresses (i.e., second term within the brackets in Eqs. (86) and (89), respectively).

6. Conclusions

We have studied the 2D problem of an interface crack between two dissimilar electrostrictive solids by using the complex variable method. The general solutions for complex potentials and intensity factors of total stress are obtained for a permeable interface crack, an impermeable interface crack and a conducting interface crack. It is found that for these three crack models, the singular nature of total stress is the same as that in traditional bi-materials with an interface crack, but the intensities of the total stress depend on the adopted crack models because the Maxwell stresses on the crack-faces are directly related to these crack models. In general, the applied electric field may enhance or retard crack growth depending on the electric boundary conditions on the crack-faces, the medium inside the crack and the surrounding space at infinity.

Acknowledgements

C.-F.G. thanks the financial support from the National Natural Science Foundation of China (10972103 and 10672076) and the Australian Research Council (ARC) as Visiting Professor to the CAMT at Sydney University. Y.-W.M. also thanks the ARC for supporting this research project (DP0665856).

Appendix A. Solutions of $w(z)$ for the three crack models

From Eqs. (13) and (14) we have:

$$D_2 = \frac{\varepsilon}{2i}\left[w'(z) - \overline{w'(z)}\right], \tag{A1}$$

$$E_1 = -\frac{\varepsilon}{2}\left[w'(z) + \overline{w'(z)}\right]. \tag{A2}$$

In general, $w'(z)$ can be expressed as:

$$w'(z) = \Gamma_0 + w_0'(z), \tag{A3}$$

where Γ_0 is a constant, and $w_0'(z)$ is analytic at infinity, i.e., $w_0'(\infty) = 0$.

Inserting Eq. (A3) in (A1) and (A2) and taking the limit $z \to \infty$ gives:

$$\Gamma_0 - \overline{\Gamma}_0 = 2iD_2^{\infty}/\varepsilon, \tag{A4}$$

$$\Gamma_0 + \overline{\Gamma}_0 = -2E_1^{\infty}, \tag{A5}$$

which leads to:

$$\Gamma_0 = \frac{iD_2^{\infty}}{\varepsilon} - E_1^{\infty}. \tag{A6}$$

Finally, substituting Eq. (A3) into (A1) and (A3) produces:

$$D_2 = D_2^{\infty} + \frac{\varepsilon}{2i}\left[w_0'(z) - \overline{w_0'(z)}\right], \tag{A7}$$

$$E_1 = E_1^{\infty} - \frac{\varepsilon}{2}\left[w_0'(z) + \overline{w_0'(z)}\right]. \tag{A8}$$

A.1. Case 1: Permeable crack model

For the case shown in Fig. 1 with a permeable interface crack, we can use the following conditions:

$$D_2^{+}(x_1) = D_2^{-}(x_1), E_1^{+}(x_1) = E_1^{-}(x_1), \text{ along the crack}, \tag{A9}$$

$$D_2 = D_2^{\infty}, \quad E_1 = E_1^{\infty} = 0, \quad z \to \infty, \tag{A10}$$

$$\oint_{\Lambda} D_2\, dz = 0, \tag{A11}$$

where Λ is a clockwise contour enclosing the crack.

Substituting Eqs. (A7) and (A8) into (A9) gives:

$$\varepsilon_1\left[w_{10}'(x_1^{+}) - \bar{w}_{10}'(x_1^{-})\right] = \varepsilon_2\left[w_{20}'(x_1^{-}) - \bar{w}_{20}'(x_1^{+})\right], \quad -\infty < x_1 < +\infty, \tag{A12}$$

$$\left[w_{10}'(x_1^{+}) + \bar{w}_{10}'(x_1^{-})\right] = \left[w_{20}'(x_1^{-}) + \bar{w}_{20}'(x_1^{+})\right], \quad -\infty < x_1 < +\infty. \tag{A13}$$

That is,

$$\left[\varepsilon_1 w_{10}'(x_1) + \varepsilon_2\bar{w}_{20}'(x_1)\right]^{+} - \left[\varepsilon_2 w_{20}'(x_1) + \varepsilon_1\bar{w}_{10}'(x_1)\right]^{-} = 0, \quad -\infty < x_1 < +\infty, \tag{A14}$$

$$\left[w_{10}'(x_1) - \bar{w}_{20}'(x_1)\right]^{+} - \left[w_{20}'(x_1) - \bar{w}_{10}'(x_1)\right]^{-} = 0, \quad -\infty < x_1 < +\infty. \tag{A15}$$

The solutions of Eqs. (A14) and (A15) are given by:

$$\varepsilon_1 w_{10}'(z) + \varepsilon_2\bar{w}_{20}'(z) = 0, \quad z \in S_1, \tag{A16}$$

$$\varepsilon_2 w_{20}'(z) + \varepsilon_1\bar{w}_{10}'(z) = 0, \quad z \in S_2, \tag{A17}$$

and

$$w_{10}'(z) - \bar{w}_{20}'(z) = 0, \quad z \in S_1, \tag{A18}$$

$$w_{20}'(z) - \bar{w}_{10}'(z) = 0, \quad z \in S_2. \tag{A19}$$

From Eqs. (A16)–(A19) we have:

$$w_{10}'(z) = 0, \quad w_{20}'(z) = 0. \tag{A20}$$

Thus, from Eq. (A3), (A6) and (A20), we obtain:

$$w_k(z) = \frac{iD_2^\infty}{\varepsilon_k} z, \quad z \in S_k \ (k = 1, 2). \tag{A21}$$

A.2. Case 2: Impermeable crack model

For the case shown in Fig. 1 with an impermeable interface crack, the following conditions can be used:

$$D_2^+(x_1) = D_2^-(x_1) = 0, \text{ along the crack,} \tag{A22}$$

$$D_2 = D_2^\infty, \quad E_1 = E_1^\infty = 0, \quad z \to \infty, \tag{A23}$$

$$\oint_\Lambda D_2 dz = 0. \tag{A24}$$

Here, $D_2^+(x_1) = D_2^-(x_1)$ is true along the whole x_1-axis so that Eqs. (A16) and (A17) are still valid.

But the condition $D_2^+(x_1) = 0$ on the upper face of the crack results in:

$$\varepsilon_1 w'_{10}(x_1^+) - \varepsilon_1 \bar{w}'_{10}(x_1^-) = -2iD_2^\infty, \quad -a < x_1 < +a, \tag{A25}$$

Similarly, the condition $D_2^-(x_1) = 0$ leads to:

$$\varepsilon_2 w'_{20}(x_1^-) - \varepsilon_2 \bar{w}'_{20}(x_1^+) = -2iD_2^\infty, \quad -a < x_1 < +a. \tag{A26}$$

Eqs. (A25) and (A26) give:

$$\left[\varepsilon_1 w'_{10}(x_1) - \varepsilon_2 \bar{w}'_{20}(x_1)\right]^+ + \left[\varepsilon_2 w'_{20}(x_1) - \varepsilon_1 \bar{w}'_{10}(x_1)\right]^- = -4iD_2^\infty, \quad -a < x_1 < +a. \tag{A27}$$

Introduce a new analytic function $\eta_1(z)$ in the whole plane as:

$$\eta_1(z) = \begin{cases} \varepsilon_1 w'_{10}(z) - \varepsilon_2 \bar{w}'_{20}(z), & z \in S_1, \\ \varepsilon_2 w'_{20}(z) - \varepsilon_1 \bar{w}'_{10}(z), & z \in S_2. \end{cases} \tag{A28}$$

Then,

$$\eta_1(x_1^+) + \eta_1(x_1^-) = -4iD_2^\infty, \quad -a < x_1 < +a. \tag{A29}$$

The solution of Eq. (A29) is:

$$\eta_1(z) = -2iD_2^\infty \left[1 - \frac{z}{\sqrt{z^2 - a^2}}\right] + \frac{\hat{c}_1 z + \hat{c}_0}{\sqrt{z^2 - a^2}}. \tag{A30}$$

Using the condition $\eta_1(\infty) = 0$ and Eq. (A24), we obtain $\hat{c}_1 = \hat{c}_0 = 0$ and finally:

$$\eta_1(z) = -2iD_2^\infty \left[1 - \frac{z}{\sqrt{z^2 - a^2}}\right]. \tag{A31}$$

Solving Eqs. (A16) and (A28), we have:

$$w'_{k0}(z) = -i\frac{D_2^\infty}{\varepsilon_k}\left[1 - \frac{z}{\sqrt{z^2 - a^2}}\right], \quad z \in S_k \ (k = 1, 2) \tag{A32}$$

Eq. (A32), combined with Eqs. (A3) and (A6), yields:

$$w'_k(z) = i\frac{D_2^\infty}{\varepsilon_k}\frac{z}{\sqrt{z^2 - a^2}}, \quad z \in S_k \ (k = 1, 2) \tag{A33}$$

That is,

$$w_k(z) = i\frac{D_2^\infty}{\varepsilon_k}\sqrt{z^2 - a^2}, \quad z \in S_k \ (k = 1, 2). \tag{A34}$$

A.3. Case 3: Conducting crack model

For the case shown in Fig. 2 with a conducting interface crack, we can now use the following boundary conditions:

$$E_1^+(x_1) = E_1^-(x_1) = 0, \text{ along the crack,} \tag{A35}$$

$$E_1 = E_1^\infty, \quad D_2 = D_2^\infty = 0, \ z \to \infty, \tag{A36}$$

$$\oint_\Lambda E_1 dz = 0. \tag{A37}$$

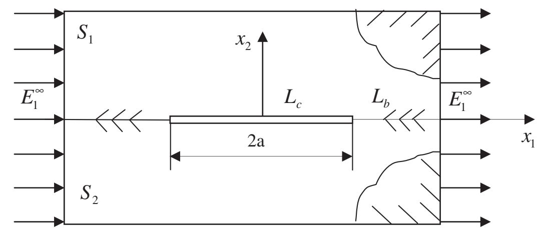

Fig. 2. Conducting interface crack in electrostrictive materials.

In this case, $E_1^+(x_1) = E_1^-(x_1)$ holds along the whole x_1-axis so that Eqs. (A18) and (A19) are still true.

However, the condition $E_1^\pm(x_1) = 0$ on the crack-faces gives:

$$w'_{10}(x_1^+) + \bar{w}'_{10}(x_1^-) = 2E_1^\infty, \quad -a < x_1 < +a, \tag{A38}$$

$$w'_{20}(x_1^-) + \bar{w}'_{20}(x_1^+) = 2E_1^\infty, \quad -a < x_1 < +a. \tag{A39}$$

Similarly, introduce another new function as:

$$\eta_2(z) = \begin{cases} w'_{10}(z) + \bar{w}'_{20}(z), & z \in S_1, \\ w'_{20}(z) + \bar{w}'_{10}(z), & z \in S_2. \end{cases} \tag{A40}$$

Then, Eqs. (A38) and (A39) become:

$$\eta_2(x_1^+) + \eta_2(x_1^-) = 4E_1^\infty, \quad -a < x_1 < +a. \tag{A41}$$

Similarly, the final solution of Eq. (A41) becomes:

$$\eta_2(z) = 2E_1^\infty \left[1 - \frac{z}{\sqrt{z^2 - a^2}}\right]. \tag{A42}$$

Solving Eqs. (40), (A18) and (A19) we have:

$$w'_{k0}(z) = E_1^\infty \left[1 - \frac{z}{\sqrt{z^2 - a^2}}\right], \quad z \in S_k \ (k = 1, 2). \tag{A43}$$

Eq. (A43), combined with Eqs. (A3) and (A6), leads to:

$$w'_k(z) = -E_1^\infty \frac{z}{\sqrt{z^2 - a^2}}, \quad z \in S_k \ (k = 1, 2). \tag{A44}$$

Hence, we have:

$$w_k(z) = -E_1^\infty \sqrt{z^2 - a^2}, \quad z \in S_k \ (k = 1, 2). \tag{A45}$$

References

Beom, H.G., 2003. Permeable cracks between two dissimilar piezoelectric materials. International Journal of Solids and Structures 40 (24), 6669–6679.

Gao, C.F., Wang, M.Z., 2000. Collinear permeable cracks between dissimilar piezoelectric materials. International Journal of Solids and Structures 37 (36), 4969–4986.

Gao, C.F., Mai, Y.-W., 2011. Permeable interfacial crack in electrostrictive materials. In: Kuna, M., Ricoeur, A. (Eds.), The Proceedings of the IUTAM Symposium On Multiscale Modelling of Fatigue, Damage and Fracture in Smart Materials Systems. Springer, Dordrecht, pp. 133–139.

Gao, C.F., Hausler, C., Balke, H., 2004. Periodic permeable interface cracks in piezoelectric materials. International Journal of Solids and Structures 41 (2), 323–335.

Gao, C.F., Mai, Y.-W., Zhang, N., 2010a. Solution of a crack in an electrostrictive solid. International Journal of Solids and Structures 47 (3–4), 444–453.

Gao, C.F., Mai, Y.-W., Zhang, N., 2010b. Solution of collinear cracks in an electrostrictive solid. Philosophical Magazine 90 (10), 1245–1262.

Hausler, C., Jelitto, H., Neumeister, P., Balke, H., Schneider, G.A., 2009. Interfacial fracture of piezoelectric multilayer actuators under mechanical and electrical loading. International Journal of Fracture 160 (1), 43–54.

Herrmann, K.P., Komarov, A.V., Loboda, V.V., 2005. On a moving interface crack with a contact zone in a piezoelectric biomaterial. International Journal of Solids and Structures 42 (16–17), 4555–4573.

Jiang, Q., Kuang, Z.B., 2004. Stress analysis in two dimensional electrostrictive material with an elliptic rigid conductor. European Journal of Mechanics A – Solids 23 (6), 945–956.

Kim, Y.H., Beom, H.G., 2009. Numerical analysis of an electrode embedded between dissimilar electrostrictive materials. International Journal of Solids and Structures 46 (18–19), 3441–3450.

Kuang, Z.B., 2008. Some variational principles in elastic dielectric and elastic magnetic materials. European Journal of Mechanics A – Solids 27 (3), 504–514.

Kuang, Z.B., 2009. Internal energy variational principles and governing equations in electroelastic analysis. International Journal of Solids and Structures 46 (3–4), 902–911.

Landau, L.D., Lifshitz, E.M., 1960. Electrodynamics of Continous Media. Pergamon Press, Oxford.

Li, Q., Chen, Y.H., 2007. Solution for a semi-permeable interface crack between two dissimilar piezoelectric materials. Journal of Applied Mechanics 2007 (74), 833–844.

Muskhelisvili, N.I., 1975. Some Basic Problems of Mathematical Theory of Elasticity. Noordhoff, Gronigen.

Ru, C.Q., Mao, X., Epstein, M., 1998. Electric-field induced interfacial cracking in multilayer electrostrictive actuators. Journal of the Mechanics and Physics of Solids 46 (8), 1301–1318.

Stratton, J.A., 1941. Electromagnetic Theory. McGraw-Hill, New York.

Suo, Z., Kuo, C.M., Barnett, D.M., Willis, J.R., 1992. Fracture mechanics for piezoelectric ceramics. Journal of the Mechanics and Physics of Solids 40 (4), 739–765.