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Lattice with long-range interaction of power-law type for fractional non-local elasticity

Vasily E. Tarasov *

Skobeltsyn Institute of Nuclear Physics, Lomonosov Moscow State University, Moscow 119991, Russia

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ABSTRACT

Lattice models with long-range interactions of power-law type are suggested as a new type of microscopic model for fractional non-local elasticity. Using the transform operation, we map the lattice equations into continuum equation with Riesz derivatives of non-integer orders. The continuum equations that are obtained from the lattice model describe fractional generalization of non-local elasticity models. Particular solutions and correspondent asymptotic of the fractional differential equations for displacement fields are suggested for the static case.

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1. Introduction

Lattice with long-range interaction is a subject of investigations in different areas of mechanics and physics (see for example Kröner, 1967; Eringen and Kim, 1977; Ostoja-Starzewski, 2002; Luo and Afraimovich, 2010; Tarasov, 2011; Dyson, 1971; Frohlich et al., 1978; Nakano and Takahashi, 1995; Campa et al., 2009). The long-range interactions have been studied in discrete systems as well as in their continuous analogs. As it was shown in Tarasov (2006b,a) (see also Tarasov and Zaslavsky, 2006; Tarasov, 2011), the continuum equations with derivatives of non-integer orders can be directly connected to lattice models with long-range interactions of power law type.

The theory of derivatives and integrals of non-integer orders (Samko et al., 1993; Kilbas et al., 2006) allow us to investigate the behavior of materials and media that are characterized by non-locality of power-law type. Fractional calculus has a wide application in mechanics and physics (for example see Carpinteri and Mainardi, 1997; Hilfer, 2000; Sabatier et al., 2007; Mainardi, 2010; Luo and Afraimovich, 2010; Tarasov, 2011, 2013a; Klafter et al., 2011). The fractional calculus allows us to formulate a fractional generalization of non-local elasticity models in two forms: the fractional gradient elasticity models (weak power-law non-locality) and the fractional integral non-local models (strong power-law non-locality). Fractional models of non-local elasticity and some microscopic models are considered in different articles (see for example Lazopoulos, 2006; Cottone et al., 2009; Carpinteri et al., 2009a,b, 2011; Di Paola and Zingales, 2008, 2009, 2011; Di Paola et al., 2010, 2014; Tarasov, 2014, 2013, 2014a). Elastic waves in nonlocal continua modeled by a fractional calculus approach are considered in Cottone et al. (2009), Atanackovic and Stankovic (2009), Zingales (2011), Sapora et al. (2013) and Challamel et al. (2013). In Tarasov (2014) and Tarasov (2013) a general approach to describe lattice model with power-law spatial dispersion for fractional elasticity has been proposed. This approach can be used for different type of interaction of lattice particles. Therefore explicit forms of the long-range interactions are not considered in Tarasov (2014, 2013). In Tarasov (2014a) a model of lattice with long-range interaction of Grünwald–Letnikov–Riesz type has been suggested to describe fractional gradient and integral elasticity of continuum. In this paper we focus on the lattice models with long-range interaction of power-law type as new type of microscopic models for fractional generalization of elasticity theory. We suggest lattice models with power-law long-range interaction as microscopic model of fractional non-local continuum. The equations for displacement field are directly derived from the suggested lattice models by the methods of Tarasov (2006b,a). The suggested generalization of the elasticity equations contains the fractional Laplacian in the Riesz's form (Kilbas et al., 2006). We demonstrate a connection between the dynamics of lattice system of particles with long-range interactions and the fractional continuum equations by using the transform operation suggested in Tarasov (2006b,a). We show how the continuous limit for the lattice with long-range interactions of power-law type gives the continuum equation of the fractional elasticity. We get particular

* Tel.: +7 495 939 03 97.
E-mail address: tarasov@theory.sinp.msu.ru

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solutions of the fractional differential elasticity equations for some special cases.

2. Equations of lattice model

As a microscopic model, we use unbounded homogeneous lattices, such that all particles are displaced from its equilibrium positions in one direction, and the displacement of particle is described by a scalar field. We consider one-dimensional lattice system of interacting particles. The equations of motion for particles are

$$M\frac{d^2u_n(t)}{dt^2} = g_2 \sum_{\substack{m=-\infty \\ m \neq n}}^{+\infty} K_2(n,m)\,(u_n - u_m) + g_\alpha \sum_{\substack{m=-\infty \\ m \neq n}}^{+\infty} K_\alpha(n,m)\,(u_n - u_m) + F(n), \quad (1)$$

where $u_n(t)$ are displacements from the equilibrium, g_2 and g_α are the coupling constants of particle interactions, and the terms $F(n)$ characterize an interaction of the particles with the external on-site force. For simplicity, we assume that all particles have the same mass M. The function $K_2(n,m)$ describes the nearest-neighbor interaction with coupling constant $g_2 = K$, which is the spring stiffness. The function $K_\alpha(n,m)$ describes the long-range interaction with a coupling constant g_α. For a simple case each particle can be considered an inversion center and

$$K_\alpha(n,m) = K_\alpha(|n-m|).$$

Equations of motion (1) have the invariance with respect to its displacement of lattice as a whole in case of absence of external forces. It should be noted that the non-invariant terms lead to the divergences in the continuous limit (Tarasov, 2011).

Using the approach suggested in Tarasov (2006a,b, 2011), we can consider a set operations that transforms the lattice equations for $u_n(t)$ into continuum equation for displacement field $u(x,t)$. We assume that $u_n(t)$ are Fourier coefficients of the field $\hat{u}(k,t)$ on $[-k_0/2, k_0/2]$ that is described by the equations

$$u_n(t) = \frac{1}{k_0}\int_{-k_0/2}^{+k_0/2} dk\ \hat{u}(k,t)\, e^{ikx_n} = \mathcal{F}_\Delta^{-1}\{\hat{u}(k,t)\}, \quad (2)$$

$$\hat{u}(k,t) = \sum_{n=-\infty}^{+\infty} u_n(t)\, e^{-ikx_n} = \mathcal{F}_\Delta\{u_n(t)\}, \quad (3)$$

where $x_n = nd$ and $d = 2\pi/k_0$ is distance between equilibrium positions of the lattice particles. Eqs. (3) and (2) are the basis for the Fourier series transform $\mathcal{F}_\Delta$ and the inverse Fourier series transform $\mathcal{F}_\Delta^{-1}$.

The Fourier transform can be derived from (3) and (2) in the limit as $d \to 0$ ($k_0 \to \infty$). In this limit the sum is transformed into an integral, and Eqs. (2) and (3) become

$$\tilde{u}(k,t) = \int_{-\infty}^{+\infty} dx\ e^{-ikx} u(x,t) = \mathcal{F}\{u(x,t)\}, \quad (4)$$

$$u(x,t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} dk\ e^{ikx}\tilde{u}(k,t) = \mathcal{F}^{-1}\{\tilde{u}(k,t)\}. \quad (5)$$

Here we use the lattice function

$$u_n(t) = \frac{2\pi}{k_0} u(x_n, t)$$

with continuous function $u(x,t)$, where $x_n = nd = (2\pi n)/k_0 \to x$. We assume that $\tilde{u}(k,t) = \mathcal{L}\hat{u}(k,t)$, where $\mathcal{L}$ denotes the passage to the limit $d \to 0$ ($k_0 \to \infty$), i.e. the function $\tilde{u}(k,t)$ can be derived from $\hat{u}(k,t)$ in the limit $d \to 0$. Note that $\tilde{u}(k,t)$ is a Fourier transform of the field $u(x,t)$. The function $\hat{u}(k,t)$ is a Fourier series transform of $u_n(t)$, where we can use $u_n(t) = (2\pi/k_0)u(nd,t)$.

We can state that a lattice model transforms into continuum model by the combination $\mathcal{F}^{-1}\mathcal{L}\,\mathcal{F}_\Delta$ of the following operation (Tarasov, 2006a,b):

The Fourier series transform:

$$\mathcal{F}_\Delta : \quad u_n(t) \to \mathcal{F}_\Delta\{u_n(t)\} = \hat{u}(k,t). \quad (6)$$

The passage to the limit $d \to 0$:

$$\mathcal{L} = \lim_{d\to 0} : \quad \hat{u}(k,t) \to \mathcal{L}\{\hat{u}(k,t)\} = \tilde{u}(k,t). \quad (7)$$

The inverse Fourier transform:

$$\mathcal{F}^{-1} : \quad \tilde{u}(k,t) \to \mathcal{F}^{-1}\{\tilde{u}(k,t)\} = u(x,t). \quad (8)$$

These operations allow us to get continuum equations from the lattice equations (Tarasov, 2006a,b, 2011).

3. Lattice with nearest-neighbor interaction

Let us consider the lattice with nearest-neighbor interaction that is described by (1), where $K_\alpha(n-m) = 0$, and

$$\sum_{\substack{m=-\infty \\ m \neq n}}^{+\infty} K_2(n,m)\, u_m(t) = u_{n+1}(t) - 2u_n(t) + u_{n-1}(t), \quad (9)$$

where the term $K_2(n,m)$ describes the nearest-neighbor interaction. Let us derive the usual elastic equation from the lattice model with the nearest-neighbor interaction with coupling constant $g_2 = K$, which is the spring stiffness. The following statement (Tarasov, 2006a,b, 2011) gives for this lattice model with the nearest-neighbor interaction the corresponding continuum equation in the limit $d \to 0$.

Proposition 1. *In the continuous limit ($d \to 0$) the lattice equations of motion*

$$M\frac{d^2u_n(t)}{dt^2} = K\cdot(u_{n+1}(t) - 2u_n(t) + u_{n-1}(t)) + F(n) \quad (10)$$

are transformed by the combination $\mathcal{F}^{-1}\mathcal{L}\,\mathcal{F}_\Delta$ of the operations (6)–(8) into the continuum equation:

$$\frac{\partial^2 u(x,t)}{\partial t^2} = C_e^2\,\Delta u(x,t) + \frac{1}{\rho} f(x), \quad (11)$$

where

$$\rho = \frac{M}{A\,d}, \quad C_e^2 = \frac{E}{\rho} = \frac{K\,d^2}{M}, \quad E = \frac{K\,d}{A} \quad (12)$$

and C_e^2 is a finite parameter, A is the cross-section area of the medium, E is the Youngs modulus, and $f(x) = F(x)/(Ad)$ is the force density.

A detailed proof of Proposition 1 is given in Appendix A.

As a result, we prove that lattice Eq. (10) in the limit $d \to 0$ give the continuum equation with the Laplacian (see also Tarasov, 2014b). Note that this result can be derived by methods described in Section 8 of Maslov (1976), where the relation

$$\exp i\left(-i\,d\,\frac{\partial}{\partial x}\right) u(x,t) = u(x+d,t)$$

and the representation of (10) by pseudo-differential equation are used.

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4. Lattice model with long-range interaction

Let us derive a continuum equation for the lattice with long-range interaction that is described by (1), where $K_\alpha(n-m)$ satisfies the conditions

$$K_\alpha(n-m) = K_\alpha(|n-m|), \quad \sum_{n=1}^{\infty} |K_\alpha(n)|^2 < \infty. \tag{13}$$

To have fractional gradient elasticity models, we assume that the function

$$\hat{K}_\alpha(k) = \sum_{\substack{n=-\infty \\ n \neq 0}}^{+\infty} e^{-ikn} K_\alpha(n) = 2\sum_{n=1}^{\infty} K_\alpha(n)\cos(kn), \tag{14}$$

satisfies the condition

$$\lim_{k\to 0} \frac{\hat{K}_\alpha(k) - \hat{K}_\alpha(0)}{|k|^\alpha} = A_\alpha, \tag{15}$$

where A_α has a finite value. Condition (15) means that

$$\hat{K}_\alpha(k) - \hat{K}_\alpha(0) = A_\alpha |k|^\alpha + R_\alpha(k), \tag{16}$$

for $k \to 0$, where

$$\lim_{k\to 0} R_\alpha(k)/|k|^\alpha = 0. \tag{17}$$

The interaction terms $K_\alpha(|n-m|)$, which give the continuum equations of gradient elasticity models, can be defined as

$$K_\alpha(n) = \frac{(-1)^n}{\Gamma(\alpha/2+1+n)\Gamma(\alpha/2+1-n)}. \tag{18}$$

Using the series (Ref. Prudnikov et al., 1986, Section 5.4.8.12)

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{\Gamma(\nu+1+n)\Gamma(\nu+1-n)} \cos(nk) = \frac{2^{2\nu-1}}{\Gamma(2\nu+1)} \sin^{2\nu}\left(\frac{k}{2}\right) - \frac{1}{2\Gamma^2(\nu+1)}, \tag{19}$$

where $\nu > -1/2$ and $0 < k < 2\pi$, we get

$$\hat{K}_\alpha(k) - \hat{K}_\alpha(0) = \frac{2^{2\alpha-1}}{\Gamma(\alpha+1)} \sin^\alpha\left(\frac{k}{2}\right) = \frac{1}{2\Gamma(\alpha+1)} k^\alpha + O(k^{\alpha+2}). \tag{20}$$

Here we use $\nu = \alpha/2$ and $\sin(k/2) = k/2 + O(k^3)$. The limit $k \to 0$ gives

$$\lim_{k\to 0} \frac{\hat{K}_\alpha(k) - \hat{K}_\alpha(0)}{|k|^\alpha} = \frac{1}{2\Gamma(\alpha+1)} \tag{21}$$

and we have $A_\alpha = 1/(2\Gamma(\alpha+1))$. To consider a fractional generalization of the elastic theory, the variables x and $d = \Delta x$ are dimensionless. Note that the interaction (18) for integer values of α is discussed in Tarasov (2014b).

Proposition 2. *The lattice equations*

$$M\frac{d^2 u_n}{dt^2} = g_2 \sum_{\substack{m=-\infty \\ m \neq n}}^{+\infty} K_2(n,m)\,(u_n - u_m) + g_\alpha \sum_{\substack{m=-\infty \\ m \neq n}}^{+\infty} K_\alpha(n-m)\,(u_n - u_m) + F(n), \tag{22}$$

where g_2 and g_α are coupling constants, $K_2(n,m)$ is defined by (9) and $K_\alpha(|n-m|)$ is defined by (18), are transformed by the combination $\mathcal{F}^{-1}\mathcal{L}\,\mathcal{F}_\Delta$ of the operations (6)–(8) into the continuum equation:

$$\frac{\partial^2 u(x,t)}{\partial t^2} = C_2\,\Delta u(x,t) - C_\alpha\,(-\Delta)^{\alpha/2} u(x,t) + \frac{1}{\rho} f(x), \tag{23}$$

where $(-\Delta)^{\alpha/2}$ is the fractional Laplacian in the Riesz's form (Kilbas et al., 2006; Samko et al., 1993), and

$$C_2 = \frac{g_2 d^2}{4M}, \quad C_\alpha = \frac{g_\alpha d^\alpha}{2\Gamma(\alpha+1)M} \tag{24}$$

are finite parameters.

A detailed proof of Proposition 2 is given in Appendix B.

In the Proposition 2, we use the Riesz fractional derivative $(-\Delta)^{\alpha/2}$. It can be defined as non-integer power of the Laplace operator in terms of the Fourier transform $\mathcal{F}$ by

$$((-\Delta)^{\alpha/2} f)(x) = \mathcal{F}^{-1}\left(|k|^\alpha (\mathcal{F}f)(k)\right). \tag{25}$$

This fractional Laplacian can be also defined in the form of the hypersingular integral (Samko et al., 1993; Kilbas et al., 2006) by

$$((-\Delta)^{\alpha/2} f)(x) = \frac{1}{d_n(m,\alpha)} \int_{\mathbb{R}^n} \frac{1}{|z|^{\alpha+n}} (\Delta_z^m f)(z)\,dz,$$

where $m > \alpha > 0$, and $(\Delta_z^m f)(z)$ is the finite difference of order m of a function $f(x)$ with a vector step $z \in \mathbb{R}^n$ and centered at the point $x \in \mathbb{R}^n$:

$$(\Delta_z^m f)(z) = \sum_{k=0}^{m} (-1)^k \frac{m!}{k!(m-k)!} f(x - kz),$$

where the constant $d_n(m,\alpha)$ is defined by

$$d_n(m,\alpha) = \frac{\pi^{1+n/2} A_m(\alpha)}{2^\alpha \Gamma(1+\alpha/2)\Gamma(n/2+\alpha/2)\sin(\pi\alpha/2)},$$

and

$$A_m(\alpha) = \sum_{j=0}^{m} (-1)^{j-1} \frac{m!}{j!(m-j)!} j^\alpha.$$

This hypersingular integral does not depend (Samko et al., 1993; Kilbas et al., 2006) on the choice of $m > \alpha > 0$.

We can note another possibility to set the interactions described by $K_2(n,m)$. The term $K_2(n,m)$ that describes the nearest-neighbor interaction can be represented in forms that differ from (9). In general $K_2(n,m)$ describes a special form of the long-range interaction. Let us give some example of these forms.

Example 1. Instead of the nearest-neighbor interaction function (9) we can use the long-range interaction with

$$K_2(n,m) = \frac{(-1)^{|n-m|}}{|n-m|^2}. \tag{26}$$

Using (Ref. Prudnikov et al., 1986, Section 5.4.2.12)

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos(nk) = \frac{1}{4}\left(k^2 - \frac{\pi^2}{3}\right), \quad |k| \leqslant \pi,$$

we obtain

$$\hat{K}_2(k) = 2\sum_{n=1}^{+\infty} \frac{(-1)^n}{n^2} \cos(kn) = \frac{1}{2}k^2 - \frac{\pi^2}{6}, \quad |k| \leqslant \pi.$$

Then we have

$$\hat{K}_2(k) - \hat{K}_2(0) = \frac{1}{2}k^2. \tag{27}$$

Example 2. If we consider the long-range interaction in the form

$$K_2(n) = \frac{(-1)^{n+1}}{n^2 - a^2}, \tag{28}$$

then Eq. (14) gives

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$$\hat{K}(k) = \frac{\pi}{a\sin(\pi a)}\cos(ak) - \frac{1}{a^2}.$$

For $k \to 0$, we obtain

$$\hat{K}_2(k) - \hat{K}_2(0) = \frac{\pi a}{2\sin(\pi a)}k^2 + O(k^4). \tag{29}$$

Example 3. The long-range interaction

$$K_2(n,m) = \frac{1}{|n-m|^{\alpha}} \tag{30}$$

with the non-integer parameter $\alpha > 3$, gives (see Theorem 8.7 in Tarasov, 2011) the relation

$$\hat{K}_2(k) - \hat{K}_2(0) = -\zeta(\alpha-2)k^2 + \dots,$$

where $\zeta(z)$ is the Riemann zeta-function.

Proposition 2 allows to demonstrate the close relation between the lattice structure and the fractional gradient non-local continuum. Let us describe the well-known special cases.

Lattice Eq. (22) have two parameters g_2 and g_{α}. The corresponding continuum equation (23) have two finite parameters C_2 and C_{α}. If we use

$$g_2 = 4K, \quad g_{\alpha} = 0.$$

then

$$C_2 = C_e^2 = Kd^2/M, \quad C_{\alpha} = 0$$

and Eq. (23) gives Eq. (11). If we assume that

$$g_2 = g_{\alpha} = 4K,$$

then

$$C_2 = C_e^2 = \frac{Kd^2}{M}, \quad C_{\alpha} = \frac{2C_e^2 d^{\alpha-2}}{\Gamma(\alpha+1)} \tag{31}$$

and we get relation

$$\frac{\partial^2 u(x,t)}{\partial t^2} = C_e^2 \Delta u(x,t) - \frac{2d^{\alpha-2}C_e^2}{\Gamma(\alpha+1)}(-\Delta)^{\alpha/2}u(x,t) + \frac{1}{\rho}f(x), \tag{32}$$

where $C_e = \sqrt{E/\rho}$ is the elastic bar velocity. Let us give a remark about the scale parameter $l_s(\alpha)$. Eq. (31) can lead to incorrect conclusion about the behavior of the scale parameter

$$l_s^2(\alpha) = \frac{C_{\alpha}\rho}{E} = \frac{C_{\alpha}}{C_e^2} \tag{33}$$

for $d \to 0$ in the case $0 < \alpha < 2$. Using $C_e^2 = Kd^2/M$, the parameter (33) can be written as

$$l_s^2(\alpha) = \frac{2Kd^{\alpha}}{\Gamma(\alpha+1)C_e^2 M}. \tag{34}$$

Using that the value of C_e^2 is finite, then behavior of the parameter $l_s^2(\alpha)$ for $d \to 0$ has the same form for $\alpha > 2$ and $0 < \alpha < 2$, such that $l_s^2(\alpha)$ is proportion to d^{α}. Therefore we assume that the range of validity of alpha parameter is arbitrary real positive α.

For $\alpha = 4$ Eq. (32) is the usual equations of the gradient elasticity models

$$\frac{\partial^2 u(x,t)}{\partial t^2} = C_e^2 \Delta u(x,t) - \frac{d^2 C_e^2}{12}\Delta^2 u(x,t) + \frac{1}{\rho}f(x). \tag{35}$$

The correspondent stress–strain relation for linear one-dimensional elasticity has the form

$$\sigma(x,t) = E\left(1 - l_s^2\Delta\right)\varepsilon(x,t),$$

where $\sigma(x,t)$ is the stress, $\varepsilon(x,t)$ is the strain, and l_s is the scale parameter.

In general, the coupling constants g_2 and g_{α} are independent. Therefore the sign of the coupling constant g_{α} (including the case $\alpha = 4$) may differ from the sign of the constant $g_2 = 4K$. For $\alpha = 4$ the second-gradient parameter is defined by the relation

$$l_s^2 = \frac{|g_4|d^2}{48K}, \tag{36}$$

where the sign in front of the factor l_s^2 is determined by the sign of the coupling constant g_4. If the constant g_4 is positive then we get the gradient elasticity model with negative sign (Askes and Aifantis, 2011).

As a result the second-gradient model with negative and positive sign can be derived from a microstructure of lattice particles by suggested approach. The suggested approach as shown above uniquely leads to second-order strain gradient terms that are preceded by the positive and negative signs.

5. Stationary solution for fractional gradient elasticity

We can consider more general model of lattice with long-range interaction, where all particles are displaced from its equilibrium in one direction, and the displacement of particles is described by a scalar field $u(\mathbf{r},t)$, where $\mathbf{r} \in \mathbb{R}^n$ $(n = 1,2,3)$. The correspondent continuum equation of the fractional elasticity model is

$$\frac{\partial^2 u(\mathbf{r},t)}{\partial t^2} = C_2\Delta u(\mathbf{r},t) - C_{\alpha}(-\Delta)^{\alpha/2}u(\mathbf{r},t) + \frac{1}{\rho}f(\mathbf{r}), \tag{37}$$

where $\mathbf{r}$ and $r = |\mathbf{r}|$ are dimensionless variables.

Let us consider the static case ($\partial u(\mathbf{r},t)/\partial t = 0$, i.e. $u(\mathbf{r},t) = u(\mathbf{r})$) in this fractional elasticity model. Then Eq. (37) has the form

$$C_2\Delta u(\mathbf{r}) - C_{\alpha}(-\Delta)^{\alpha/2}u(\mathbf{r}) + \frac{1}{\rho}f(\mathbf{r}) = 0. \tag{38}$$

We can use the Fourier method to solve fractional differential Eq. (38), which is based on the relations

$$\mathcal{F}[(-\Delta)^{\alpha/2}u(\mathbf{r})](\mathbf{k}) = |\mathbf{k}|^{\alpha}\hat{u}(\mathbf{k}), \quad \mathcal{F}[\Delta u(\mathbf{r})](\mathbf{k}) = -\mathbf{k}^2\hat{u}(\mathbf{k}). \tag{39}$$

Applying the Fourier transform $\mathcal{F}$ to both sides of (38) and using (39), we have

$$(\mathcal{F}u)(\mathbf{k}) = \frac{1}{\rho}\left(C_2|\mathbf{k}|^2 + C_{\alpha}|\mathbf{k}|^{\alpha_j}\right)^{-1}(\mathcal{F}f)(\mathbf{k}). \tag{40}$$

Eq. (38) (see, for example, Section 5.5.1. in Kilbas et al., 2006) has a particular solution that can be represented in the form of the convolution of the functions $G_{\alpha}^n(|\mathbf{r}|)$ and $f(|\mathbf{r}|)$ as follow

$$u(\mathbf{r}) = \frac{1}{\rho}\int_{\mathbb{R}^n} G_{\alpha}^n(\mathbf{r}-\mathbf{r}')f(\mathbf{r}')d^n\mathbf{r}', \tag{41}$$

where $G_{\alpha}^n(\mathbf{r})$ is the Green function (see Section 5.5.1. in Kilbas et al., 2006) of the form

$$G_{\alpha}^n(\mathbf{r}) = \mathcal{F}^{-1}\left[\left(C_2|\mathbf{k}|^2 + C_{\alpha}|\mathbf{k}|^{\alpha}\right)^{-1}\right](\mathbf{r}) = \int_{\mathbb{R}^n}\left(C_2|\mathbf{k}|^2 + C_{\alpha}|\mathbf{k}|^{\alpha}\right)^{-1}e^{+i(\mathbf{k},\mathbf{r})}d^n\mathbf{k}. \tag{42}$$

We can use the relation

$$\int_{\mathbb{R}^n} e^{i(\mathbf{k},\mathbf{r})}f(|\mathbf{k}|)d^n\mathbf{k} = \frac{(2\pi)^{n/2}}{|\mathbf{r}|^{(n-2)/2}}\int_0^{\infty} f(\lambda)\lambda^{n/2}J_{n/2-1}(\lambda|\mathbf{r}|)d\lambda \tag{43}$$

that holds (see Lemma 25.1 of Samko et al., 1993) for any suitable function f such that the integral in the right-hand side of (43) is convergent. Here J_{ν} is the Bessel function of the first kind.

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Using relation (43), the Green function (42) can be represented (see Theorem 5.22 in Kilbas et al., 2006) in the form of the integral with respect to one parameter λ as

$$G^n_\alpha(\mathbf{r}) = \frac{|\mathbf{r}|^{(2-n)/2}}{(2\pi)^{n/2}} \int_0^\infty \frac{\lambda^{n/2} J_{(n-2)/2}(\lambda|\mathbf{r}|)\, d\lambda}{C_2\,\lambda^2 + C_\alpha\,\lambda^\alpha}, \tag{44}$$

where $n = 1, 2, 3$, and $J_{(n-2)/2}$ is the Bessel function of the first kind.

For the 3-dimensional case ($n = 3$), we can use

$$J_{1/2}(z) = \sqrt{\frac{2}{\pi z}} \sin(z) \tag{45}$$

and we have

$$G^3_\alpha(\mathbf{r}) = \frac{1}{2\pi^2|\mathbf{r}|} \int_0^\infty \frac{\lambda \sin(\lambda|\mathbf{r}|)\, d\lambda}{C_2\,\lambda^2 + C_\alpha\,\lambda^\alpha}. \tag{46}$$

For the 1-dimensional case ($n = 1$), we can use

$$J_{-1/2}(z) = \sqrt{\frac{2}{\pi z}} \cos(z) \tag{47}$$

and we have (see Theorem 5.24 in Kilbas et al., 2006) the function

$$G^1_\alpha(\mathbf{r}) = \frac{1}{\pi} \int_0^\infty \frac{\cos(\lambda|\mathbf{r}|)\, d\lambda}{C_2\,\lambda^2 + C_\alpha\,\lambda^\alpha}. \tag{48}$$

Let us determine the deformation of an infinite elastic continuum, when a force is applied to a small region of the medium. This is the well-known Thomson's problem (Landau and Lifshitz, 1986). We solve Thomson's problem in the framework of the fractional elasticity model. If we consider the deformation for $|\mathbf{r}|$, which are larger compare with the size of the region, we can suppose that the force is applied at a point. In this case, we have

$$f(\mathbf{r}) = f_0\,\delta(\mathbf{r}) = f_0\,\delta(x)\delta(y)\delta(z). \tag{49}$$

Then the displacement field $u(\mathbf{r})$ of fractional elasticity has a simple form of the particular solution that is proportional to the Green function

$$u(\mathbf{r}) = \frac{f_0}{\rho}\, G^n_\alpha(\mathbf{r}). \tag{50}$$

Therefore, the displacement field for the case (49) has the form

$$u(\mathbf{r}) = \frac{1}{2\pi^2} \frac{f_0}{\rho\,|\mathbf{r}|} \int_0^\infty \frac{\lambda \sin(\lambda|\mathbf{r}|)}{C_2\,\lambda^2 + C_\alpha\,\lambda^\alpha}\, d\lambda. \tag{51}$$

The asymptotic behavior $|\mathbf{r}| \to \infty$ of the displacement field $u(\mathbf{r})$ in the model described by (51) with (49), is given by

$$u(\mathbf{r}) \;\approx\; \frac{f_0\,\Gamma(2-\alpha)\,\sin(\pi\alpha/2)}{\pi^3\, C_\alpha\, \rho} \cdot \frac{1}{|\mathbf{r}|^{3-\alpha}} \quad (\alpha < 2), \tag{52}$$

$$u(\mathbf{r}) \;\approx\; \frac{1}{2\pi^2} \frac{f_0}{\rho\,|\mathbf{r}|}, \quad (\alpha > 2). \tag{53}$$

Note that the asymptotic behavior $|\mathbf{r}| \to \infty$ does not depend on the parameter α for $\alpha > 2$. In the case $\alpha < 2$ the displacement field on the long distances is determined only by term with the fractional Laplacian of the order α.

The asymptotic behavior $|\mathbf{r}| \to 0$ of the displacement field $u(\mathbf{r})$ that is described by Eq. (51), where the force $f(\mathbf{r})$ is applied at a point (49), is given by

$$u(\mathbf{r}) \;\approx\; \frac{1}{2\pi^2} \frac{f_0}{\rho\,|\mathbf{r}|}, \quad (\alpha < 2), \tag{54}$$

$$u(\mathbf{r}) \;\approx\; \frac{f_0\,\Gamma((3-\alpha)/2)}{2^\alpha\,\pi^2\sqrt{\pi}\,\rho\, C_\alpha\,\Gamma(\alpha/2)} \cdot \frac{1}{|\mathbf{r}|^{3-\alpha}}, \quad (2 < \alpha < 3), \tag{55}$$

$$u(\mathbf{r}) \;\approx\; \frac{f_0}{2\pi\,\alpha\,\rho\, c_\beta^{1-3/\alpha}\, c_\alpha^{3/\alpha}\,\sin(3\pi/\alpha)}, \quad (\alpha > 3). \tag{56}$$

Here the Euler's reflection formula for Gamma function is used. Note that the asymptotic behavior $|\mathbf{r}| \to 0$ does not depend on the parameter α for $\alpha < 2$. In the case $\alpha > 2$, the displacement field on the short distances is determined only by term with the fractional Laplacian of the order α.

The functions

$$u(x) = \frac{1}{x} \int_0^\infty \frac{\lambda \sin(\lambda x)}{C_2\,\lambda^2 + C_\alpha\,\lambda^\alpha}\, d\lambda$$

for the different orders of $1 < \alpha < 6$ and with $C_2 = C_\alpha = 1$ are present on Figs. 1–4, where $x = |\mathbf{r}|$. Figs. 1,2,4 allows us to see that the field $u(x)$ tends to a constant value ($u(x) \to const$) at $x \to 0$ for the parameters $\alpha > 3$ ($\alpha = 3.6, \alpha = 4.1, \alpha = 5.2$ and $\alpha = 5.6$). Figs. 2 and 3 demonstrate that the asymptotic behavior of the type $u(x) \approx 1/x^{3-\alpha}$ for the field $u(x)$ at $x \to 0$ for the parameters $2 < \alpha < 3$ ($\alpha = 2.6, \alpha = 2.7$). Figs. 3 and 4 show that the asymptotic behavior of the type $u(x) \approx 1/x$ for the field $u(x)$ at $x \to 0$ for the parameters $0 < \alpha < 2$ ($\alpha = 1.4, \alpha = 1.9$).

We can determine the deformation of an infinite non-local elastic continuum, when a pair of forces with equal in magnitude and oppositely directed is applied to a small region of the medium. We assume that these forces are separated by small distance. If we consider the deformation for $|\mathbf{r}|$, which are larger compare with the size of the region, we can suppose that two force are applied at two points such that

$$f(\mathbf{r}) = f_0\,\delta(\mathbf{r}+\mathbf{a}) - f_0\,\delta(\mathbf{r}-\mathbf{a}). \tag{57}$$

This problem is analogous to a dipole system as a pair of electric charges of equal magnitude but opposite sign, separated by small distance. Then the displacement field $u(\mathbf{r})$ of fractional elasticity has a form of the particular solution

$$u(\mathbf{r}) = \frac{f_0}{\rho}\left(G^n_\alpha(\mathbf{r}+\mathbf{a}) - G^n_\alpha(\mathbf{r}-\mathbf{a})\right). \tag{58}$$

Therefore, the displacement field for the case (57) has the form

$$u(\mathbf{r}) = \frac{1}{2\pi^2}\frac{f_0}{\rho} \int_0^\infty \left(\frac{\sin(\lambda|\mathbf{r}+\mathbf{a}|)}{|\mathbf{r}+\mathbf{a}|} \frac{\sin(\lambda|\mathbf{r}-\mathbf{a}|)}{|\mathbf{r}-\mathbf{a}|}\right) \frac{\lambda}{C_2\,\lambda^2 + C_\alpha\,\lambda^\alpha}\, d\lambda. \tag{59}$$

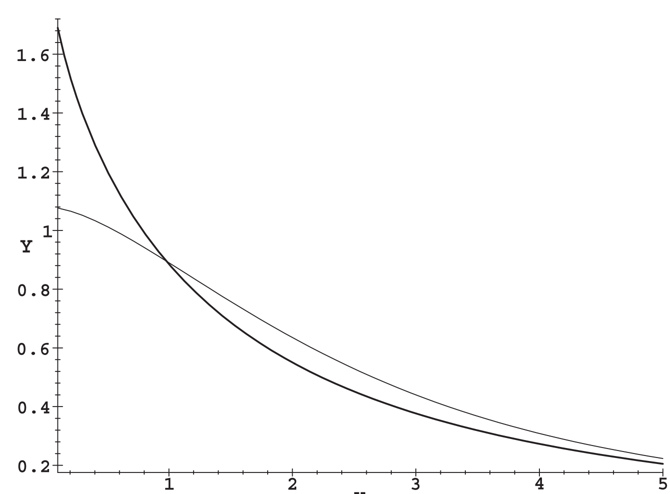

Fig. 1. Plots of function $u(x)$ for the orders $\alpha = 3.6$ and $\alpha = 5.2$, where $x = |\mathbf{r}|$ and $C_2 = C_\alpha = 1$.

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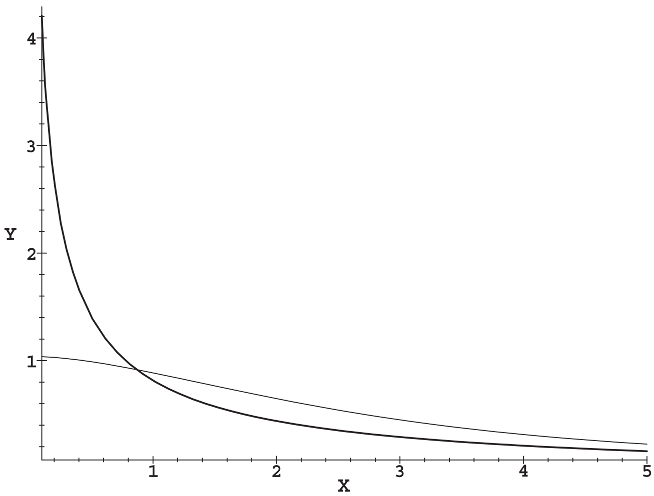

Fig. 2. Plots of function $u(x)$ for the orders $\alpha = 2.6$ and $\alpha = 5.6$, where $x = |\mathbf{r}|$ and $C_2 = C_\alpha = 1$.

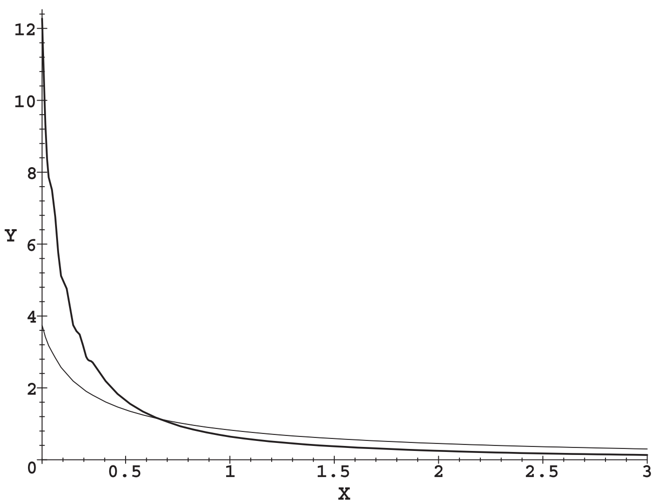

Fig. 3. Plots of function $u(x)$ for the orders $\alpha = 1.4$ and $\alpha = 2.7$, where $x = |\mathbf{r}|$ and $C_2 = C_\alpha = 1$.

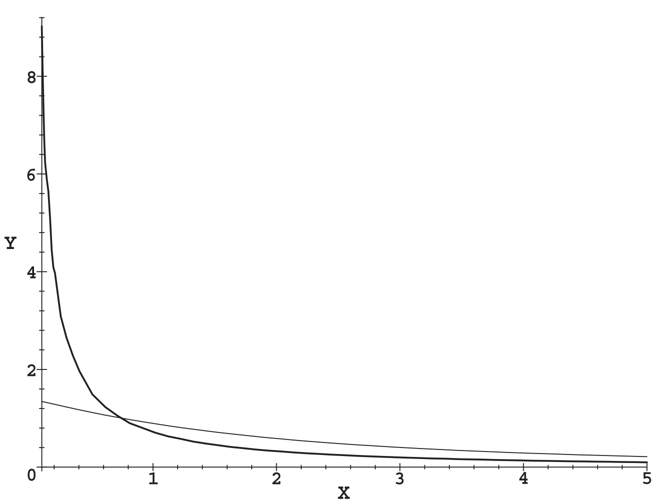

Fig. 4. Plots of function $u(x)$ for the orders $\alpha = 1.9$ and $\alpha = 4.1$, where $x = |\mathbf{r}|$ and $C_2 = C_\alpha = 1$.

For $|\mathbf{r}| \gg |\mathbf{a}|$, we can use $|\mathbf{r}-\mathbf{a}| - |\mathbf{r}+\mathbf{a}| \approx 2\,|\mathbf{a}|\cos\theta$, where θ is the angle between $\mathbf{r}$ and $\mathbf{a}$, and $|\mathbf{r}-\mathbf{a}|\,|\mathbf{r}+\mathbf{a}| \approx |\mathbf{r}|^2$. Then the displacement field can be represented in the form

$$u(\mathbf{r}) = \frac{f_0\,|\mathbf{a}|\cos\theta}{\pi^2\,\rho\,|\mathbf{r}|^2}\int_0^\infty \frac{\lambda\,\sin(\lambda|\mathbf{r}|)}{C_2\,\lambda^2 + C_\alpha\,\lambda^\alpha}\,d\lambda + \frac{f_0}{\pi^2\,\rho\,|\mathbf{r}|}\int_0^\infty \frac{\lambda\,\sin(\lambda\,|\mathbf{a}|\cos\theta)\,\cos(\lambda|\mathbf{r}|)}{C_2\,\lambda^2 + C_\alpha\,\lambda^\alpha}\,d\lambda \qquad (60)$$

For the suggested lattice equation and correspondent continuum limit, we can use all positive values of alpha parameter. There is no reason to limit of the range alpha values in the used fractional differential equations for the suggested form of fractional gradient and integral elasticity. It allows us to state that the range of validity of alpha parameter is arbitrary real positive values. As a result, we can distinguish two following particular cases in the fractional elasticity model described by (38): (1) fractional integral elasticity ($\alpha < 2$); (2) fractional gradient elasticity ($\alpha > 2$). Note that for the first case the order of the fractional Laplacian in Eq. (37) is less than the order of the term related to the usual Hooke's law. In the second case the order of the fractional Laplacian is greater of the order of the term related to the Hooke's law.

6. Conclusion

We suggest lattice models with long-range interaction of power-law type as microscopic model of fractional non-local elastic continuum. The continuum equations with derivatives of non-integer orders are directly derived from the suggested lattice models. We prove that the fractional gradient and fractional integral models can be derived from lattice models with long-range particle interactions. The suggested approach uniquely leads to second-order and fractional-order strain gradient terms that are preceded by the positive and negative signs. Fractional calculus allows us to obtain exact analytical solutions of the fractional differential equations for continuum models of a wide class of material with fractional gradient and fractional integral non-locality. A characteristic feature of the behavior of a fractional non-local continuum is the spatial power-tails of non-integer orders in the asymptotic behavior. The fractional elasticity models, which are suggested in this paper to describe complex materials with fractional non-locality, can be characterized by a common or universal spatial behavior of elastic materials by analogy with the universal temporal behavior of low-loss dielectrics (Jonscher, 1977, 1996, 1999; Tarasov, 2008). The asymptotic behavior (52) and (54) allows us to state that fractional integral elasticity effects are important on the macroscopic scales. The asymptotic behavior (53), (55) and (56) allows us to state that fractional gradient elasticity effects are very important for the mesoscopic and nano scales. As a results the fractional gradient elasticity models can be very important for nanomechanics (Cleland, 2003; Chuang et al., 2006; Liu et al., 2006; Li and Gao, 2013; Gopalakrishnan and Narendar, 2013) of nonlocal materials with long-range particle interactions.

Appendix A. Proof of Proposition 1

To derive the equation for the field $\hat{u}(k,t)$, we multiply Eq. (10) by $\exp(-iknd)$, and summing over n from $-\infty$ to $+\infty$. Then

$$\sum_{n=-\infty}^{+\infty} e^{-iknd} M \frac{d^2 u_n}{dt^2} = K \cdot \sum_{n=-\infty}^{+\infty} e^{-iknd}(u_{n+1} - 2u_n + u_{n-1}) + \sum_{n=-\infty}^{+\infty} e^{-iknd} F(n). \qquad (61)$$

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The first term on the right-hand side of (61) is

$$K \cdot \sum_{n=-\infty}^{+\infty} e^{-iknd} K_2(n,m) u_n = K \cdot \sum_{n=-\infty}^{+\infty} e^{-iknd} (u_{n+1} - 2u_n + u_{n-1})$$
$$= K \cdot \sum_{n=-\infty}^{+\infty} e^{-iknd} u_{n+1} - 2K \cdot \sum_{n=-\infty}^{+\infty} e^{-iknd} u_n + K \cdot \sum_{n=-\infty}^{+\infty} e^{-iknd} u_{n-1}$$
$$= e^{ikd} K \cdot \sum_{m=-\infty}^{+\infty} e^{-ikmd} u_m - 2K \cdot \sum_{n=-\infty}^{+\infty} e^{-iknd} u_n + e^{-ikd} K \cdot \sum_{j=-\infty}^{+\infty} e^{-ikjd} u_j.$$

Using the definition of $\hat{u}(k,t)$, we obtain

$$K \cdot \sum_{n=-\infty}^{+\infty} e^{-iknd} K_2(n,m) u_n = K \cdot \left(e^{ikd} \hat{u}(k,t) - 2\hat{u}(k,t) + e^{-ikd} \hat{u}(k,t)\right)$$
$$= K \cdot \left(e^{ikd} + e^{-ikd} - 2\right) \hat{u}(k,t)$$
$$= 2K \cdot (\cos(kd) - 1)\, \hat{u}(k,t).$$

As a result, we have

$$K \cdot \sum_{n=-\infty}^{+\infty} e^{-iknd} K_2(n,m) u_n = -4K \cdot \sin^2\left(\frac{kd}{2}\right) \hat{u}(k,t). \tag{62}$$

Substitution of (62) into (61) gives

$$M \frac{\partial^2 \hat{u}(k,t)}{\partial t^2} = -4K \cdot \sin^2\left(\frac{kd}{2}\right) \hat{u}(k,t) + \mathcal{F}_{\Delta}\{F(u_n(t))\}. \tag{63}$$

For $d \to 0$, the asymptotic behavior of the sine is $\sin(kd/2) = kd/2 + O((kd)^3)$. Then

$$-4 \sin^2\left(\frac{kd}{2}\right) = -(kd)^2 + O((kd)^4).$$

Using the finite parameter $C_e^2 = K d^2/M$, the transition to the limit $d \to 0$ in Eq. (63) gives

$$\frac{\partial^2 \tilde{u}(k,t)}{\partial t^2} = -C_e^2 k^2 \tilde{u}(k,t) + \frac{1}{M} \mathcal{F}\{F(x)\}, \tag{64}$$

where C_e^2 is defined by (12). The inverse Fourier transform $\mathcal{F}^{-1}$ of (64) has the form

$$\frac{\partial^2 \mathcal{F}^{-1}\{\tilde{u}(k,t)\}}{\partial t^2} = -C_e^2 \mathcal{F}^{-1}\{k^2 \tilde{u}(k,t)\} + \frac{1}{\rho} f(x), \tag{65}$$

where $f(x) = F(x)/(Ad)$ is the force density, and $\rho = M/(Ad)$ is the mass density. Then using

$$\mathcal{F}^{-1}\{\tilde{u}(k,t)\} = u(x,t), \quad \mathcal{F}^{-1}\{k^2 \tilde{u}(k,t)\} = -\Delta u(x,t),$$

we obtain the continuum Eq. (11). This ends the proof.

Appendix B. Proof of Proposition 2

To derive the equation for the field $\hat{u}(k,t)$, we multiply Eq. (22) by $\exp(-iknd)$, and summing over n from $-\infty$ to $+\infty$. Then

$$\sum_{n=-\infty}^{+\infty} e^{-iknd} M \frac{d^2}{dt^2} u_n(t) = g_2 \sum_{n=-\infty}^{+\infty} \sum_{\substack{m=-\infty \\ m \neq n}}^{+\infty} e^{-iknd} K_2(n,m)\, (u_n - u_m)$$
$$+ g_\alpha \sum_{n=-\infty}^{+\infty} \sum_{\substack{m=-\infty \\ m \neq n}}^{+\infty} e^{-iknd} K_\alpha(n-m)\, (u_n - u_m) + \sum_{n=-\infty}^{+\infty} e^{-iknd} F(n). \tag{66}$$

The left-hand side of (66) gives

$$\sum_{n=-\infty}^{+\infty} e^{-iknd} \frac{\partial^2 u_n(t)}{\partial t^2} = \frac{\partial^2}{\partial t^2} \sum_{n=-\infty}^{+\infty} e^{-iknd} u_n(t) = \frac{\partial^2 \hat{u}(k,t)}{\partial t^2}, \tag{67}$$

where $\hat{u}(k,t)$ is defined by (3). The second term of the right-hand side of (66) is

$$\sum_{n=-\infty}^{+\infty} e^{-iknd} F(n) = \mathcal{F}_{\Delta}\{F(n)\}. \tag{68}$$

The limit for the first term on the right-hand side of (66) is described in Proposition 1.

The second term on the right-hand side of (66) with a multiplier g_α is

$$\sum_{n=-\infty}^{+\infty} \sum_{\substack{m=-\infty \\ m \neq n}}^{+\infty} e^{-iknd} K_\alpha(n-m)(u_n - u_m)$$
$$= \sum_{n=-\infty}^{+\infty} \sum_{\substack{m=-\infty \\ m \neq n}}^{+\infty} e^{-iknd} K_\alpha(n-m) u_n$$
$$- \sum_{n=-\infty}^{+\infty} \sum_{\substack{m=-\infty \\ m \neq n}}^{+\infty} e^{-iknd} K_\alpha(n-m) u_m. \tag{69}$$

Using (3), the first term in the right-hand side of (69) gives

$$\sum_{n=-\infty}^{+\infty} \sum_{\substack{m=-\infty \\ m \neq n}}^{+\infty} e^{-iknd} K_\alpha(n-m) u_n = \sum_{n=-\infty}^{+\infty} e^{-iknd} u_n \sum_{\substack{m'=-\infty \\ m' \neq 0}}^{+\infty} K_\alpha(m')$$
$$= \hat{K}_\alpha(0)\, \hat{u}(k,t), \tag{70}$$

where we use (13) and $K_\alpha(m' + n - n) = K_\alpha(m')$, and

$$\hat{K}_\alpha(kd) = \sum_{\substack{n=-\infty \\ n \neq 0}}^{+\infty} e^{-iknd} K_\alpha(n) = \mathcal{F}_{\Delta}\{K_\alpha(n)\}. \tag{71}$$

Note that

$$\sum_{n=-\infty}^{+\infty} \sum_{\substack{m=-\infty \\ m \neq n}}^{+\infty} e^{-iknd} K_\alpha(n-m) u_m = \sum_{\substack{n=-\infty \\ n \neq m}}^{+\infty} e^{-iknd} K_\alpha(n-m) \sum_{m=-\infty}^{+\infty} u_m$$
$$= \sum_{\substack{n'=-\infty \\ n' \neq 0}}^{+\infty} e^{-ikn'd} K_\alpha(n') \sum_{m=-\infty}^{+\infty} u_m e^{-ikmd} = \hat{K}_\alpha(kd)\ \hat{u}(k,t), \tag{72}$$

where $K_\alpha(m - (n' + m)) = K_\alpha(n')$ is used.

As a result, Eq. (66) has the form

$$M \frac{\partial^2 \hat{u}(k,t)}{\partial t^2} = g_2 \left(\hat{K}_2(0) - \hat{K}_2(kd)\right) \hat{u}(k,t)$$
$$+ g_\alpha \left(\hat{K}_\alpha(0) - \hat{K}_\alpha(kd)\right) \hat{u}(k,t) + \mathcal{F}_{\Delta}\{F(n)\}, \tag{73}$$

where $\mathcal{F}_{\Delta}\{F(n)\}$ is an operator notation for the Fourier series transform of $F(n)$.

The Fourier series transform $\mathcal{F}_{\Delta}$ of (22) gives (73). We will consider the limit $d \to 0$. Using (16), Eq. (73) can be written as

$$\frac{\partial^2}{\partial t^2} \hat{u}(k,t) = \frac{g_2 d^2}{M} \hat{\mathcal{T}}_{2,\Delta}(k)\, \hat{u}(k,t) + \frac{g_\alpha d^\alpha}{M} \hat{\mathcal{T}}_{\alpha,\Delta}(k)\, \hat{u}(k,t) + \frac{1}{M} \mathcal{F}_{\Delta}\{F(n)\}, \tag{74}$$

where we use (20), the Proposition 1 for $K_2(n,m)$, and the following notations

$$\hat{\mathcal{T}}_{\alpha,\Delta}(k) = -\frac{1}{2\Gamma(\alpha+1)} |k|^\alpha + d^2\, O(|k|^{\alpha+2}), \tag{75}$$

$$\hat{\mathcal{T}}_{2,\Delta}(k) = -k^2 + d^2\, O(k^4). \tag{76}$$

In the limit $d \to 0$, we get

$$\hat{\mathcal{T}}_\alpha(k) = \mathcal{L}\hat{\mathcal{T}}_{\alpha,\Delta}(k) = -\frac{1}{2\Gamma(\alpha+1)} |k|^\alpha, \tag{77}$$

$$\hat{\mathcal{T}}_2(k) = \mathcal{L}\hat{\mathcal{T}}_{2,\Delta}(k) = -k^2. \tag{78}$$

As a result, Eq. (74) in the limit $d \to 0$ gives

$$\frac{\partial^2}{\partial t^2} \tilde{u}(k,t) = \frac{g_2 d^2}{M} \hat{\mathcal{T}}_2(k)\, \tilde{u}(k,t) + \frac{g_\alpha d^\alpha}{M} \hat{\mathcal{T}}_\alpha(k)\, \tilde{u}(k,t) + \frac{1}{M} \mathcal{F}\{F(x)\}, \tag{79}$$

where $\tilde{u}(k,t) = \mathcal{L}\hat{u}(k,t)$ and $F(k) = \mathcal{F}\{F(x)\} = \mathcal{L}\mathcal{F}_\Delta\{F(n)\}$. The inverse Fourier transform of (79) is

$$\frac{\partial^2}{\partial t^2} u(x,t) = \frac{g_2 d^2}{M}\, \mathcal{T}_2(x)\, u(x,t) + \frac{g_\alpha d^\alpha}{M}\, \mathcal{T}_\alpha(x)\, u(x,t) + \frac{1}{\rho} f(x), \quad (80)$$

where $f(x) = F(x)/(Ad)$ is the force density, the operators $\mathcal{T}_2(x)$ and $\mathcal{T}_\alpha(x)$ are defined by

$$\mathcal{T}_2(x) = \mathcal{F}^{-1}\{\hat{\mathcal{T}}_2(k)\} = \Delta, \quad \mathcal{T}_\alpha(x) = \mathcal{F}^{-1}\{\hat{\mathcal{T}}_\alpha(k)\} = -\frac{1}{2\Gamma(\alpha+1)} (-\Delta)^{\alpha/2}. \quad (81)$$

Here, we use the connection between the Riesz's fractional Laplacian and its Fourier transform (Kilbas et al., 2006) in the form

$$\mathcal{F}[(-\Delta)^{\alpha/2} u(x)](k) = |k|^\alpha\, \hat{u}(k). \quad (82)$$

Using the finite parameters C_2 and C_α, which are defined by (24), the substitution of (81) into (80) gives continuum Eq. (23). This ends the proof.

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