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Stochastic homogenization analysis on elastic properties of fiber reinforced composites using the equivalent inclusion method and perturbation method

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ABSTRACT

This paper describes a methodology for evaluation of influence of microscopic uncertainty in material properties and geometry of a microstructure on a homogenized macroscopic elastic property of an inhomogeneous material. For the analysis of the stochastic characteristics of a homogenized elastic property, the first-order perturbation method is used. In order to analyze the influence of microscopic geometrical uncertainty, the perturbation-based equivalent inclusion method is formulated. In this paper, an analytical form of the perturbation term using the equivalent inclusion method is provided.

As a numerical example, macroscopic stochastic characteristics such as an expected value or variance of the homogenized elastic tensor of a unidirectional fiber reinforced plastic, which is caused by microscopic uncertainty in material properties or geometry of a microstructure, are estimated with computing the first order perturbation term of the homogenized elastic tensor. Compared the results of the proposed method with the results of the Monte-Carlo simulation, validity, effectiveness and a limitation of the perturbation-based homogenization method is investigated.

1. Introduction

Inhomogeneous materials such as composite materials have a complex microstructure, and its microstructure have uncertainty in a material property or geometry. This uncertainty will cause a stochastic variation of a homogenized material property, it should be estimated in order to evaluate reliability of a composite structure.

From this point of view, several results of stochastic homogenization analysis have been reported. Kaminski reported the first-order perturbation-based homogenization analysis of two-phase composites (Kami'nski and Kleiber, 2000), the perturbation-based homogenization analysis for thermal conductivity of unidirectional fiber reinforced composites (Kami'nski, 2001). Koishi also reported a result of the first-order perturbation-based homogenization analysis (Koishi et al., 1996). Sakata reported some results of stochastic homogenization analysis using the Monte-Carlo simulation (Sakata et al., 2008a), the three-dimensional result of perturbation analysis for the homogenized elastic tensor and the equivalent elastic properties (Sakata et al., 2008b) or the second-order perturbation-based homogenization method (Sakata et al., 2008c). Kaminski also reported a higher order perturbation-based analysis (Kaminski, 2007). Ostoja (Ostoja-Starzewski (2002)) or Xu (Xu and Brady, 2005) reported other approaches for stochastic homogenization analysis.

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The perturbation based homogenization method will give a good estimation of a stochastic characteristics such as the expectation or variance of the homogenized material property with a relatively small random variation of material properties of component materials, however, its application will be limited to estimate a stochastic characteristic of a homogenized elastic property caused by uncertainty in a material property of a component material. The approximation-based stochastic homogenization approach has been proposed (Sakata et al., 2008d), however, the perturbation-based approach will be still effective for the stochastic analysis with considering many random variables.

In using the finite element method based on the homogenization theory, a geometrical property such as shape or volume fraction of inclusions is not expressed explicitly. Therefore, in perturbation analysis, an analytical form of a perturbation term of a stiffness matrix cannot be obtained. This problem causes difficulty in analyzing a stochastic homogenization problem considering a microscopic geometrical uncertainty.

From these backgrounds, a perturbation-based stochastic homogenization analysis for a unidirectional fiber reinforced composite using the equivalent inclusion method is attempted in this paper. In the equivalent inclusion method, a geometrical parameter such as shape or volume fraction of inclusions is expressed analytically. Therefore, the perturbation-based stochastic analysis procedure can be applied to a problem considering both microscopic uncertainties in geometry and material properties of a microstructure.

In this paper, at first, outline of the equivalent inclusion method is introduced. Next, the perturbation-based stochastic homogenization analysis method using the equivalent inclusion method is proposed. Finally, several numerical results are illustrated for discussion on validity, accuracy and effectiveness of the proposed method.

2. Equivalent inclusion method (EI)

The equivalent inclusion method (EI) is one of the effective methods for estimating a homogenized elastic property of composite materials. For a unidirectional fiber reinforced composite material, an equivalent inclusion method formula based on Mori–Tanaka theory (Mori and Tanaka, 1972; Tohgo, 2004) can be used for the estimation. The homogenized elastic tensor is computed using the following form:

$$\boldsymbol{E}^{\mathrm{EI}} = \boldsymbol{E}_{\mathrm{m}}\{\boldsymbol{E}_{\mathrm{m}} - (1-V_{\mathrm{f}})(\boldsymbol{E}_{\mathrm{m}} - \boldsymbol{E}_{\mathrm{f}})\boldsymbol{S}\}^{-1} \times [\boldsymbol{E}_{\mathrm{m}} - (\boldsymbol{E}_{\mathrm{m}} - \boldsymbol{E}_{\mathrm{f}})\{\boldsymbol{S} - V_{\mathrm{f}}(\boldsymbol{S} - I)\}] = \boldsymbol{E}_{\mathrm{m}}\boldsymbol{X}^{-1}\boldsymbol{Y} \tag{1}$$

$$\boldsymbol{X} = \boldsymbol{E}_{\mathrm{m}} - (1-V_{\mathrm{f}})(\boldsymbol{E}_{\mathrm{m}} - \boldsymbol{E}_{\mathrm{f}})\boldsymbol{S} \tag{2}$$

$$\boldsymbol{Y} = \boldsymbol{E}_{\mathrm{m}} - (\boldsymbol{E}_{\mathrm{m}} - \boldsymbol{E}_{\mathrm{f}})\{\boldsymbol{S} - V_{\mathrm{f}}(\boldsymbol{S} - I)\} \tag{3}$$

where $\boldsymbol{E}_{\mathrm{m}}$ is the elastic tensor of a base material, V_{f} is the volume fraction of the inclusions, $\boldsymbol{E}_{\mathrm{f}}$ is the elastic tensor of the inclusions, $\boldsymbol{S}$ is the Eshelby tensor and I is a unit tensor. For an isotropic material, the elastic tensor can be expressed as

$$\boldsymbol{E} = \frac{E(1-\nu)}{(1+\nu)(1-2\nu)}\begin{bmatrix} 1 & \frac{\nu}{1-\nu} & \frac{\nu}{1-\nu} & 0 & 0 & 0 \\ & 1 & \frac{\nu}{1-\nu} & 0 & 0 & 0 \\ & & 1 & 0 & 0 & 0 \\ & & & \frac{1-2\nu}{2(1-\nu)} & 0 & 0 \\ & \text{Sym.} & & & \frac{1-2\nu}{2(1-\nu)} & 0 \\ & & & & & \frac{1-2\nu}{2(1-\nu)} \end{bmatrix}$$

The Eshelby tensor depends on the shape of inclusion. Each component of the Eshelby tensor in case of continuous long unidirectional fiber can be expressed as

$$S_{1111} = \frac{1}{2(1-\nu_{\mathrm{m}})}\left[\frac{a_2^2 + 2a_1a_2}{\{a_1+a_2\}^2} + (1-2\nu_{\mathrm{m}})\frac{a_2}{a_1+a_2}\right]$$

$$S_{1122} = \frac{1}{2(1-\nu_{\mathrm{m}})}\left\{\frac{a_2^2}{(a_1+a_2)^2} - (1-2\nu_{\mathrm{m}})\frac{a_2}{a_1+a_2}\right\}$$

$$S_{1133} = \frac{1}{2(1-\nu_{\mathrm{m}})}\frac{2\nu_{\mathrm{m}}a_2}{a_1+a_2}$$

$$S_{2211} = \frac{1}{2(1-\nu_{\mathrm{m}})}\left\{\frac{a_1^2}{(a_1+a_2)^2} - (1-2\nu_{\mathrm{m}})\frac{a_1}{a_1+a_2}\right\}$$

$$S_{2222} = \frac{1}{2(1-\nu_{\mathrm{m}})}\left[\frac{a_1^2 + 2a_1a_2}{(a_1+a_2)^2} + (1-2\nu_{\mathrm{f}})\frac{a_1}{a_1+a_2}\right]$$

$$S_{2233} = \frac{1}{2(1-\nu_{\mathrm{m}})}\frac{2\nu_{\mathrm{m}}a_1}{a_1+a_2}$$

$$S_{1212} = \frac{1}{2(1-\nu_{\mathrm{m}})}\left[\frac{a_1^2+a_2^2}{2(a_1^2+2a_1a_2+a_2^2)} + \frac{1-2\nu_{\mathrm{m}}}{2}\right]$$
$$S_{2323} = \frac{a_1}{2(a_1+a_2)}$$
$$S_{3131} = \frac{a_2}{2(a_1+a_2)}$$

and other components are zero. a_i is a size of an inclusion as shown in Fig. 1. In this paper, $a_3 \gg a_1, a_2$ is assumed.

3. Perturbation analysis for equivalent inclusion method

In order to estimate stochastic characteristics such as expectation or variance of a homogenized elastic property of a composite material, the perturbation method is used in this paper. The perturbation based approach for stochastic homogenization analysis has been reported (Kami'nski, 2004), and the basic concept is applied to the equivalent inclusion method-based homogenization analysis in this paper. Based on the first-order approximation second moment method (FOSM), the expectation and variance of the equivalent elastic tensor can be computed as

$$\left.\begin{aligned} &E[\boldsymbol{E}^{\mathrm{EI}^*}] = [\boldsymbol{E}^{\mathrm{EI0}}] \\ &\mathrm{cov}[\boldsymbol{E}^{\mathrm{EI}^*}] = \sum_i \sum_j [\boldsymbol{E}^{\mathrm{EI1}}]_i [\boldsymbol{E}^{\mathrm{EI1}}]_j \mathrm{cov}[\alpha_i, \alpha_j] \end{aligned}\right\} \tag{4}$$

where $E[\boldsymbol{E}^{\mathrm{EI}^*}] = [\boldsymbol{E}^{\mathrm{EI0}}]$ is the expectation of the homogenized elastic tensor, $\mathrm{cov}[\boldsymbol{E}^{\mathrm{EI}^*}]$ is the covariance of the homogenized elastic tensor and $\mathrm{cov}[\alpha_i, \alpha_j]$ is the covariance of two random variables. $[\boldsymbol{E}^{\mathrm{EI0}}]$ is the 0th order perturbation term, $[\boldsymbol{E}^{\mathrm{EI1}}]_i$ is the first order perturbation term of the homogenized elastic tensor with respect to the random variable α_i.

Also, the CoV (coefficient of variance) can be estimated using the following equation.

$$\mathrm{CoV}[\boldsymbol{E}^{EI^*}] = \frac{\sqrt{|\,\mathrm{Var}[\boldsymbol{E}^{EI^*}]\,|}}{E[\boldsymbol{E}^{EI^*}]} \tag{5}$$

In this case, stochastic variations in material properties, volume fraction and cross-sectional shape of fiber is taken into account. Detailed formulations for computing the perturbation term of the equivalent elastic tensor against the each stochastic variations are introduced in the following sections.

3.1. Stochastic variation in material properties

In case of considering stochastic variation in material properties, for example, variation in Young's modulus of fiber, an observed value of the Young's modulus can be expressed as

$$E_{\mathrm{f}}^* = E_{\mathrm{f}}^0(1+\alpha) \tag{6}$$

where superscript $*$ indicates an observed value, superscript 0 indicates an expected value and α is a random variable. In this case, the first order perturbation term for E_{f} (Young's modulus of inclusion) variation can be expressed as

$$\boldsymbol{E}^{EI1}|_{E_{\mathrm{f}}} = \boldsymbol{E}_{\mathrm{m}} \frac{\partial(\boldsymbol{X}^{-1}\boldsymbol{Y})}{\partial\alpha} = \boldsymbol{E}_{\mathrm{m}}[-\boldsymbol{X}^{-1}(1-V_{\mathrm{f}})\boldsymbol{E}_{\mathrm{f}}\boldsymbol{S}\boldsymbol{X}^{-1}\boldsymbol{Y} + \boldsymbol{X}^{-1}\boldsymbol{E}_{\mathrm{f}}\{\boldsymbol{S} - V_{\mathrm{f}}(\boldsymbol{S}-\boldsymbol{I})\}] \tag{7}$$

Also, the first order perturbation terms for E_{m} (Young's modulus of matrix), ν_{f}(Poisson's ratio of inclusion) and ν_{m} (Poisson's ratio of matrix) variation can be formulated as

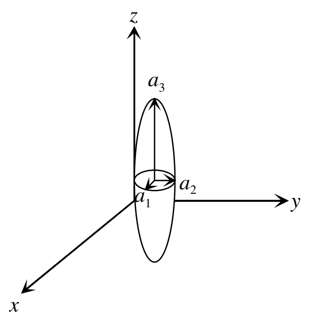

Fig. 1. Schematic view of an inclusion.

$$\begin{aligned}\boldsymbol{E}^{EI1}|_{E_\mathrm{m}} &= \tfrac{\partial(\boldsymbol{E}_\mathrm{m}\boldsymbol{X}^{-1}\boldsymbol{Y})}{\partial\alpha}\\ &= \boldsymbol{E}_\mathrm{m}\boldsymbol{X}^{-1}\boldsymbol{Y} + \boldsymbol{E}_\mathrm{m}[-\boldsymbol{X}^{-1}\{\boldsymbol{E}_\mathrm{m} - (1-V_\mathrm{f})\boldsymbol{E}_\mathrm{f}\boldsymbol{S}\}\boldsymbol{X}^{-1}\boldsymbol{Y} + \boldsymbol{X}^{-1}[\boldsymbol{E}_\mathrm{m} - \boldsymbol{E}_\mathrm{m}\{\boldsymbol{S} - V_\mathrm{f}(\boldsymbol{S}-\boldsymbol{I})\}]]\end{aligned} \tag{8}$$

$$\boldsymbol{E}^{EI1}|_{v_\mathrm{f}} = \boldsymbol{E}_\mathrm{m}\frac{\partial(\boldsymbol{X}^{-1}\boldsymbol{Y})}{\partial\alpha} = \boldsymbol{E}_\mathrm{m}[-\boldsymbol{X}^{-1}(1-V_\mathrm{f})\overline{\boldsymbol{E}_\mathrm{f}}\boldsymbol{S}\boldsymbol{X}^{-1}\boldsymbol{Y} + \boldsymbol{X}^{-1}\overline{\boldsymbol{E}_\mathrm{f}}\{\boldsymbol{S} - V_\mathrm{f}(\boldsymbol{S}-\boldsymbol{I})\}] \tag{9}$$

$$\begin{aligned}\boldsymbol{E}^{EI1}|_{E_\mathrm{m}} &= \tfrac{\partial(\boldsymbol{E}_\mathrm{m}\boldsymbol{X}^{-1}\boldsymbol{Y})}{\partial\alpha}\\ &= \boldsymbol{E}_\mathrm{m}\boldsymbol{X}^{-1}\boldsymbol{Y} + \boldsymbol{E}_\mathrm{m}[-\boldsymbol{X}^{-1}\{\boldsymbol{E}_\mathrm{m} - (1-V_\mathrm{f})\boldsymbol{E}_\mathrm{f}\boldsymbol{S}\}\boldsymbol{X}^{-1}\boldsymbol{Y} + \boldsymbol{X}^{-1}[\boldsymbol{E}_\mathrm{m} - \boldsymbol{E}_\mathrm{m}\{\boldsymbol{S} - V_\mathrm{f}(\boldsymbol{S}-\boldsymbol{I})\}]]\end{aligned} \tag{10}$$

$$\begin{aligned}\boldsymbol{E}^{EI1}|_{v_\mathrm{m}} &= \tfrac{\partial(\boldsymbol{E}_\mathrm{m}\boldsymbol{X}^{-1}\boldsymbol{Y})}{\partial\alpha}\\ &= \overline{\boldsymbol{E}_\mathrm{m}}\boldsymbol{X}^{-1}\boldsymbol{Y} - \boldsymbol{E}_\mathrm{m}\boldsymbol{X}^{-1}\{\overline{\boldsymbol{E}_\mathrm{m}} - (1-V_\mathrm{f})\overline{\boldsymbol{E}_\mathrm{m}}\boldsymbol{S} - (1-V_\mathrm{f})(\boldsymbol{E}_\mathrm{m} - \boldsymbol{E}_\mathrm{f})\overline{\boldsymbol{S}}\}\boldsymbol{X}^{-\boldsymbol{1}}\boldsymbol{Y}\\ &\quad + \boldsymbol{E}_m\boldsymbol{X}^{-\boldsymbol{1}}[\overline{\boldsymbol{E}_m} - \overline{\boldsymbol{E}_m}\{\boldsymbol{S} - \boldsymbol{V}_f(\boldsymbol{S}-\boldsymbol{I})\} - (\boldsymbol{E}_m - \boldsymbol{E}_f)(\overline{\boldsymbol{S}} - \boldsymbol{V}_f\overline{\boldsymbol{S}})]\end{aligned} \tag{11}$$

where

$$\overline{\boldsymbol{E}_\mathrm{f}} = \begin{bmatrix} \eta_\mathrm{f} & \beta_\mathrm{f} & \beta_\mathrm{f} & 0 & 0 & 0\\ & \eta_\mathrm{f} & \beta_\mathrm{f} & 0 & 0 & 0\\ & & \eta_\mathrm{f} & 0 & 0 & 0\\ & & & \gamma_\mathrm{f} & 0 & 0\\ & \mathrm{Sym.} & & & \gamma_\mathrm{f} & 0\\ & & & & & \gamma_\mathrm{f}\end{bmatrix}$$

$$\eta_\mathrm{f} = \frac{-2v_\mathrm{f}^3 + 4v_\mathrm{f}^2}{(v_\mathrm{f} + 2v_\mathrm{f}^2 - 1})^2 E_\mathrm{f}, \quad \beta_\mathrm{f} = \frac{E_\mathrm{f} v_\mathrm{f}(1+2v_\mathrm{f}^2)}{(1 - v_\mathrm{f} - 2v_\mathrm{f}^2)^2}, \quad \gamma_\mathrm{f} = \frac{-E_\mathrm{f} v_\mathrm{f}}{2(1 + v_\mathrm{f} + 2v_\mathrm{f}^2)}$$

$$\overline{\boldsymbol{E}_\mathrm{m}} = \begin{bmatrix} \eta_\mathrm{m} & \beta_\mathrm{m} & \beta_\mathrm{m} & 0 & 0 & 0\\ & \eta_\mathrm{m} & \beta_\mathrm{m} & 0 & 0 & 0\\ & & \eta_\mathrm{m} & 0 & 0 & 0\\ & & & \gamma_\mathrm{m} & 0 & 0\\ & \mathrm{Sym.} & & & \gamma_\mathrm{m} & 0\\ & & & & & \gamma_\mathrm{m}\end{bmatrix}$$

$$\eta_\mathrm{m} = \frac{-2v_\mathrm{m}^3 + 4v_\mathrm{m}^2}{(v_\mathrm{m} + 2v_\mathrm{m}^2 - 1)^2} E_\mathrm{m}, \quad \beta_\mathrm{m} = \frac{E_\mathrm{m} v_\mathrm{m}(1+2v_\mathrm{m}^2)}{(1 - v_\mathrm{m} - 2v_\mathrm{m}^2)^2}, \quad \gamma_\mathrm{m} = \frac{-E_\mathrm{f} v_\mathrm{m}}{2(1 + v_\mathrm{m} + 2v_\mathrm{m}^2)}$$

$$\overline{S_{1111}} = \frac{v_\mathrm{m}}{2(1-v_\mathrm{m})^2}\left\{\frac{a_2^2 + 2a_1a_2}{(a_1+a_2)^2} + (1-2v_\mathrm{m})\frac{a_2}{a_1+a_2}\right\} + \frac{1}{2(1-v_\mathrm{m})}\left(-2v_\mathrm{m}\frac{a_2}{a_1+a_2}\right)$$

$$\overline{S_{1122}} = \frac{v_\mathrm{m}}{2(1-v_\mathrm{m})^2}\left\{\frac{a_2^2}{(a_1+a_2)^2} - (1-2v_\mathrm{m})\frac{a_2}{a_1+a_2}\right\} + \frac{1}{2(1-v_\mathrm{m})}\left(2v_\mathrm{m}\frac{a_2}{a_1+a_2}\right)$$

$$\overline{S_{1133}} = \frac{v_\mathrm{m}}{2(1-v_\mathrm{m})^2}\frac{2v_\mathrm{m}a_2}{a_1+a_2} + \frac{1}{2(1-v_\mathrm{m})}\frac{2v_\mathrm{m}a_2}{a_1+a_2}$$

$$\overline{S_{2211}} = \frac{v_\mathrm{m}}{2(1-v_\mathrm{m})^2}\left\{\frac{a_1^2}{(a_1+a_2)^2} - (1-2v_\mathrm{m})\frac{a_1}{a_1+a_2}\right\} + \frac{1}{2(1-v_\mathrm{m})}\left(2v_\mathrm{m}\frac{a_1}{a_1+a_2}\right)$$

$$\overline{S_{2222}} = \frac{v_\mathrm{m}}{2(1-v_\mathrm{m})^2}\left\{\frac{a_1^2 + 2a_1a_2}{(a_1+a_2)^2} + (1-2v_\mathrm{m})\frac{a_1}{a_1+a_2}\right\} + \frac{1}{2(1-v_\mathrm{m})}\left(-2v_\mathrm{m}\frac{a_1}{a_1+a_2}\right)$$

$$\overline{S_{2233}} = \frac{v_\mathrm{m}}{2(1-v_\mathrm{m})^2}\frac{2v_\mathrm{m}a_1}{a_1+a_2} + \frac{1}{2(1-v_\mathrm{m})}\frac{2v_\mathrm{m}a_1}{a_1+a_2}$$

$$\overline{S_{1212}} = \frac{v_\mathrm{m}}{2(1-v_\mathrm{m})^2}\left\{\frac{a_1^2 + a_2^2}{2(a_1+a_2)^2} + \frac{1-2v_\mathrm{m}}{2}\right\} + \frac{1}{2(1-v_\mathrm{m})}(-v_\mathrm{m})$$

$$\overline{S_{2323}} = \overline{S_{3131}} = 0$$

3.2. Stochastic variation in geometry of fiber

In a composite material, geometry of a reinforcement material has also uncertainty. For instance, volume fraction, cross-sectional shape or shape along longitudinal axis may have a random variation. From this point of view, perturbation terms for geometrical random variation are also derived. In this paper, the perturbation terms for a random variation in volume fraction and cross-sectional fiber of fiber are introduced.

For a random variation in volume fraction of fiber, we assume that an observed value of the volume fraction can be obtained as

$$V_{\mathrm{f}}^{*} = V_{\mathrm{f}}^{0}(1+\alpha) \tag{13}$$

Then, the first order perturbation term of the equivalent elastic tensor can be formulated as

$$\boldsymbol{E}^{H1}|_{V_{\mathrm{f}}} = \boldsymbol{E}_{\mathrm{m}}\frac{\partial(\boldsymbol{X}^{-1}\boldsymbol{Y})}{\partial\alpha} = \boldsymbol{E}_{\mathrm{m}}[-\boldsymbol{X}^{-1}V_{\mathrm{f}}(\boldsymbol{E}_{\mathrm{m}}-\boldsymbol{E}_{\mathrm{f}})\boldsymbol{S}\boldsymbol{X}^{-1}\boldsymbol{Y} + \boldsymbol{X}^{-1}(\boldsymbol{E}_{\mathrm{m}}-\boldsymbol{E}_{\mathrm{f}})V_{\mathrm{f}}(\boldsymbol{S}-\boldsymbol{I})] \tag{14}$$

Also, if it is assumed that the cross-sectional shape of fiber is ellipsoidal, we can consider that an observed value of the size of cross section, which is illustrated in Fig. 1, is obtained as

$$a_i^{*} = a_i^{0}(1+\alpha) \tag{15}$$

For instance, the first-order perturbation term in case of a1 variation can be expressed as

$$\boldsymbol{E}^{H1}|_{a_1} = \boldsymbol{E}_{\mathrm{m}}\frac{\partial(\boldsymbol{X}^{-1}\boldsymbol{Y})}{\partial\alpha} = \boldsymbol{E}_{\mathrm{m}}[-\boldsymbol{X}^{-1}(1-V_{\mathrm{f}})(\boldsymbol{E}_{\mathrm{m}}-\boldsymbol{E}_{\mathrm{f}})\boldsymbol{S}'\boldsymbol{X}^{-1}\boldsymbol{Y} - \boldsymbol{X}^{-1}(\boldsymbol{E}_{\mathrm{m}}-\boldsymbol{E}_{\mathrm{f}})(\boldsymbol{S}'-V_{\mathrm{f}}\boldsymbol{S}')] \tag{16}$$

where

$$S'_{1111} = \frac{1}{2(1-\nu_{\mathrm{m}})}\left[\frac{-2a_1^3a_2 - 2a_1^2a_2^2}{(a_1+a_2)^4} - (1-2\nu_{\mathrm{m}})\frac{a_1a_2}{(a_1+a_2)^2}\right]$$
$$S'_{1122} = \frac{1}{2(1-\nu_{\mathrm{m}})}\left[\frac{-2a_1^2a_2^2 - 2a_1a_2^3}{(a_1+a_2)^4} + (1-2\nu_{\mathrm{m}})\frac{a_1a_2}{(a_1+a_2)^2}\right]$$
$$S'_{1133} = \frac{1}{2(1-\nu_{\mathrm{m}})}\frac{-2\nu_{\mathrm{m}}a_1a_2}{(a_1+a_2)^2}$$
$$S'_{2211} = \frac{1}{2(1-\nu_{\mathrm{m}})}\left[\frac{2a_1^2a_2^2 + 2a_1a_2^3}{(a_1+a_2)^4} - (1-2\nu_{\mathrm{m}})\frac{a_1a_2}{(a_1+a_2)^2}\right]$$
$$S'_{2222} = \frac{1}{2(1-\nu_{\mathrm{m}})}\left[\frac{2a_1^2a_2^2 + 2a_1a_2^3}{(a_1+a_2)^4} + (1-2\nu_{\mathrm{m}})\frac{a_1a_2}{(a_1+a_2)^2}\right]$$
$$S'_{2233} = \frac{1}{2(1-\nu_{\mathrm{m}})}\frac{2\nu_{\mathrm{m}}a_1a_2}{(a_1+a_2)^2}$$
$$S'_{1212} = \frac{1}{2(1-\nu_{\mathrm{m}})}\left[\frac{a_1^3a_2^{-}a_1a_2^3}{(a_1+a_2)^4}\right]$$
$$S'_{2323} = \frac{a_1a_2}{2(a_1+a_2)^2}$$
$$S'_{3131} = \frac{-a_1a_2}{2(a_1+a_2)^2}$$

4. Numerical results

4.1. Uncertainty in material properties

In this paper, stochastic characteristics such as expectation, variance and coefficient of variance (CoV) of the equivalent elastic tensor of a unidirectional fiber reinforced plastic are estimated using Eqs. (4) and (5).

The properties of fiber and matrix are employed correspond to E-glass and Epoxy resin. Volume fraction of fiber (V_{f}) is 0.25 in this example. Elastic properties of the component materials are listed in Table 1.

For this material, uncertainty in Young's modulus and Poisson's ratio of fiber and resin are taken into account. In this case, as stochastic characteristics of the component materials, $E[\alpha] = 0$ and $\sqrt{\mathrm{Var}[\alpha]} = 0.055$ are assumed.

Figs. 2–5 shows the estimated results of expected value for each equivalent elastic tensor considering Fig. 2 shows the result for E_{f} variation, Fig. 3 shows the result of E_{m} variation, Fig. 4 shows the result of ν_{f} variation and Fig. 5 shows the result of ν_{m} variation.

Table 1
Expected values of elastic properties for fiber and matrix

	Fiber (E-glass)	Matrix (Epoxy)
Young's modulus (GPa)	73.0	4.5
Poisson's ratio	0.2156	0.39

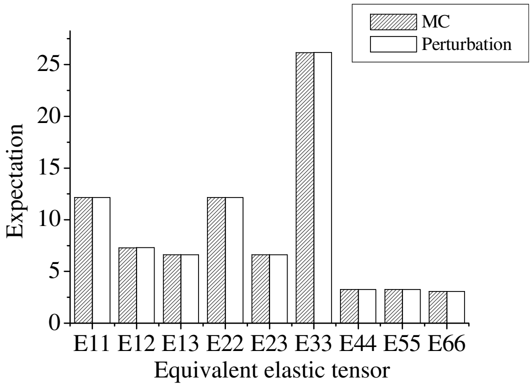

Fig. 2. Estimated expectation of equivalent elastic tensor for E_f variation.

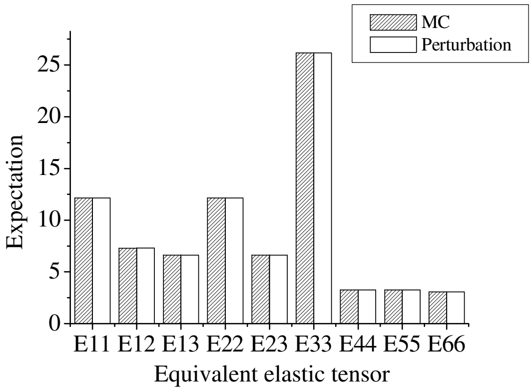

Fig. 3. Estimated expectation of equivalent elastic tensor for E_m variation.

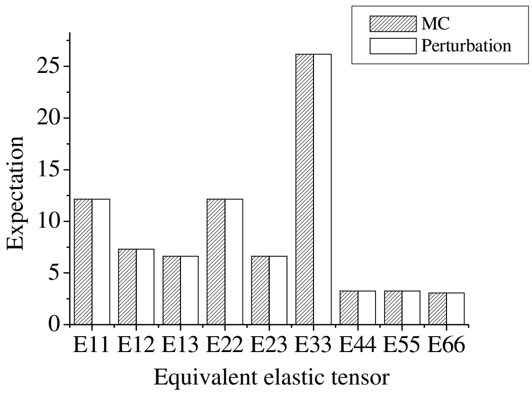

Fig. 4. Estimated expectation of equivalent elastic tensor for ν_f variation.

For comparison, the estimated results obtained with using the Monte-Carlo simulation are also illustrated. MC in the figures and tables shows the results of the Monte-Carlo simulation. In this study, 10,000 trials are carried out for estimating the stochastic characteristics. The expected value and variances are computed from the result of each trial as

$$\left.\begin{aligned} &E_{\mathrm{MC}}[E^{\mathrm{EI}^*}] = \int_{-\infty}^{\infty} E^{\mathrm{EI}}(\alpha)f(\alpha)\,\mathrm{d}\alpha = \sum E^{\mathrm{EI}}(\alpha) \\ &\mathrm{Var}_{\mathrm{MC}}[E^{\mathrm{EI}^*}] = \int_{-\infty}^{\infty} (E^{\mathrm{EI}}(\alpha) - E_{\mathrm{MC}}[E^{\mathrm{EI}^*}])^2 f(\alpha)\,\mathrm{d}\alpha = \sum (E^{\mathrm{EI}}(\alpha) - \mathrm{E}_{\mathrm{MC}}[E^{\mathrm{EI}}])^2 \end{aligned}\right\} \tag{17}$$

where the subscript "MC" means the result of the Monte-Carlo simulation. In order to generate a set of normal distributed random number, the following Box–Mullar randomization formula (Press et al., 1993) is used.

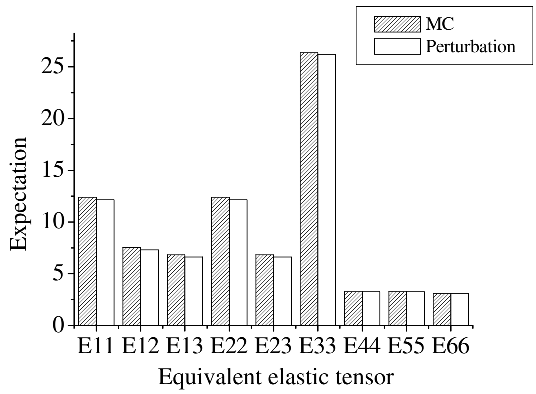

Fig. 5. Estimated expectation of equivalent elastic tensor for ν_m variation.

$$\alpha = \sqrt{-2\sigma^2 \log U_1} \times \sin 2\pi U_2 \tag{18}$$

where U_1 and U_2 are observed values of a uniform random number, σ^2 is variance of α.

"MC" in the figure also indicates the result of the Monte-Carlo simulation and "Perturbation" indicates the result of the proposed perturbation-based analysis.

From Figs. 2–5, it can be recognized that each expectation of the homogenized elastic tensor can be well estimated with using the proposed method. For more detailed discussion, the relative estimation errors, which can be computed using Eq. (19), are listed in Table 2. E_{ij} in the figures and table shows the elements of the homogenized elastic tensor.

$$R_E = \frac{|\, E_{\mathrm{MC}}[E^{\mathrm{EI}^*}] - E[E^{\mathrm{EI}^*}]\,|}{E_{\mathrm{MC}}[E^{\mathrm{EI}^*}]} \times 100(\%) \tag{19}$$

From Table 2, it can be found that the relative estimation errors are less than 0.1% in case of E_{f}, E_{m} and ν_{f} variation. For ν_{m} variation, the estimation error is larger than the others, but it is less than about 3%. From these results, it can be considered that the proposed method gives accurate estimation of the expectation in these cases.

Using the proposed method and the Monte-Carlo simulation, CoV of each equivalent elastic tensor is also estimated. Figs. 6–9 show the estimated results using those methods, and Table 3 shows the relative estimation error between the results of the proposed method and the Monte-Carlo simulation.

Table 2
Relative error between expectations estimated by the proposed method and MC in case of uncertainty in material properties

	E11	E12	E13	E22	E23	E33	E44	E55	E66
E_f	0.018	0.021	0.018	0.018	0.018	0.065	0.037	0.037	0.031
E_m	0.097	0.099	0.097	0.097	0.097	0.028	0.104	0.104	0.102
ν_f	0.001	0.001	0.015	0.001	0.015	0.003	0.002	0.002	0.002
ν_m	1.983	3.224	3.094	1.983	3.094	0.701	0.037	0.037	0.059

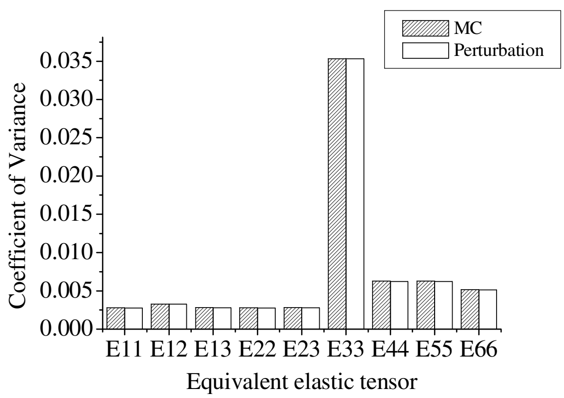

Fig. 6. Estimated CoV of equivalent elastic tensor for E_f variation.

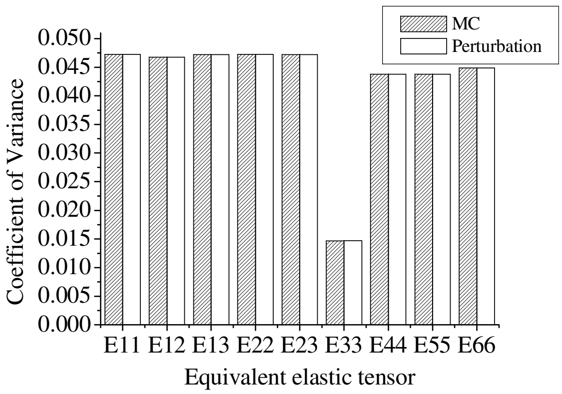

Fig. 7. Estimated CoV of equivalent elastic tensor for E_m variation.

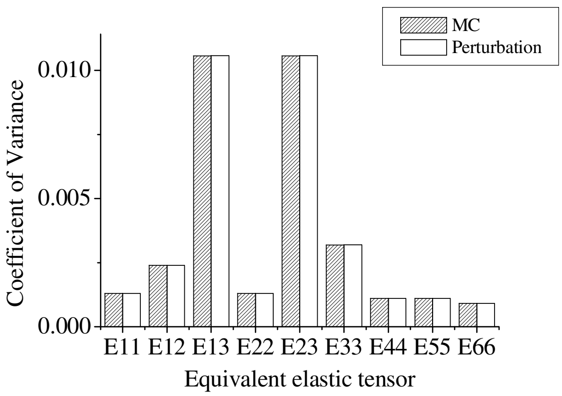

Fig. 8. Estimated CoV of equivalent elastic tensor for v_f variation.

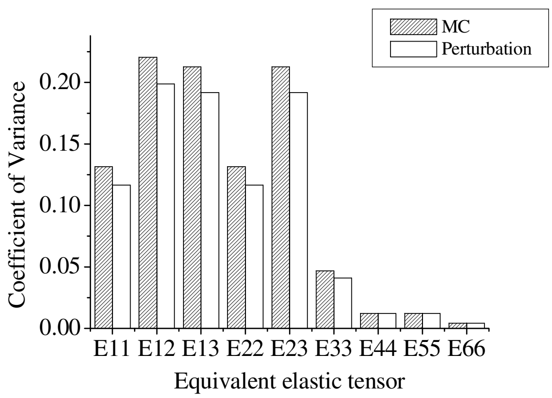

Fig. 9. Estimated CoV of equivalent elastic tensor for v_m variation.

Table 3
Relative estimation error between CoV by the proposed method and MC in case of material properties

	E11	E12	E13	E22	E23	E33	E44	E55	E66
E_f	0.957	0.949	0.952	0.957	0.952	0.028	0.758	0.758	0.809
E_m	0.007	0.012	0.008	0.007	0.008	0.068	0.042	0.042	0.030
v_f	0.119	0.120	0.094	0.119	0.094	0.116	0.099	0.099	0.098
v_m	11.341	9.799	9.816	11.341	9.816	12.406	0.030	0.030	0.637

From these figures and Table 3, it can be also considered that CoV of each homogenized elastic constant are well estimated by the proposed method. Especially, the relative estimation errors are less than 1% in case of E_f, E_m and ν_f variation, those of CoV can be accurately estimated. On the other hand, the estimation error for ν_m variation is larger than the others, and more than 10% of the estimation error is sometimes included. Since it is confirmed from Fig. 5 and Table 2 that the expectation can be well estimated in case of ν_m variation, it can be considered that this error is caused by variance estimation for ν_m variation with using the first-order perturbation-based approach. This result shows that the proposed method would better to avoid to be used for estimation of CoV in case of large ν_m variation.

4.2. Uncertainty in geometrical parameters

Next, influence of uncertainty in microscopic geometrical parameter on an equivalent elastic tensor is investigated with using the proposed method. In this case, the volume fraction of fiber V_f or size of cross-sectional shape of fiber, which is illustrated in Fig. 1, is considered as a random variable. It is assumed that V_f and the size a_1 are random variables distributed according to the normal distribution. Similar to the previous example, $E[\alpha] = 0$ and $\sqrt{\mathrm{Var}[\alpha]} = 0.055$ are assumed.

Figs. 10 and 11 show the estimated result of expectation and CoV of each equivalent elastic tensor using the proposed method and the Monte-Carlo simulation for V_f variation. From these figures, it can be recognized that the proposed method gives good estimations in case of V_f. Figs. 12 and 13 show the estimated result of expectation and CoV of each equivalent elastic tensor for a_1 variation. These figures show that the proposed method also give good estimations of those stochastic characteristic in case of a_1 variation.

For more detailed evaluation, the relative errors in estimating CoV between the result of the proposed method and the Monte-Carlo simulation are listed in Table 4. From this result, it can be recognized that the maximum estimation error is less than 1% in case of these geometrical variation, and it is considered that the proposed method will be usable for estimation of the stochastic characteristics of the equivalent elastic tensor in case of microscopic uncertainty in geometry of a microstructure such as the volume fraction or size of fiber.

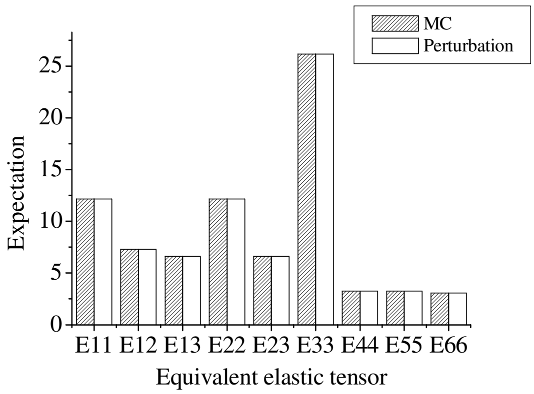

Fig. 10. Estimated expectation of equivalent elastic tensor for V_f variation.

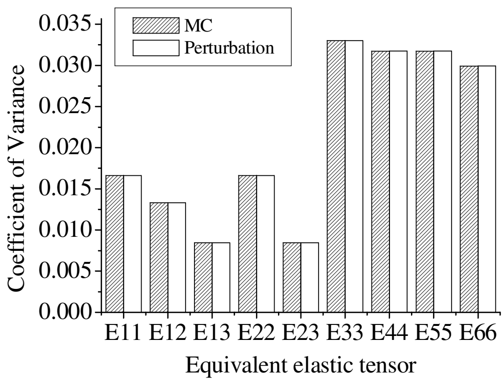

Fig. 11. Estimated CoV of equivalent elastic tensor for V_f variation.

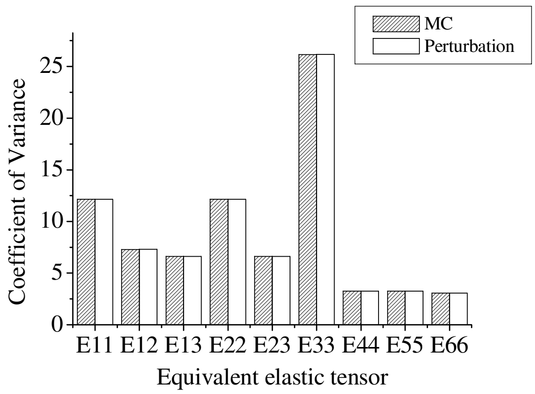

Fig. 12. Estimated expectation of equivalent elastic tensor for a_1 variation.

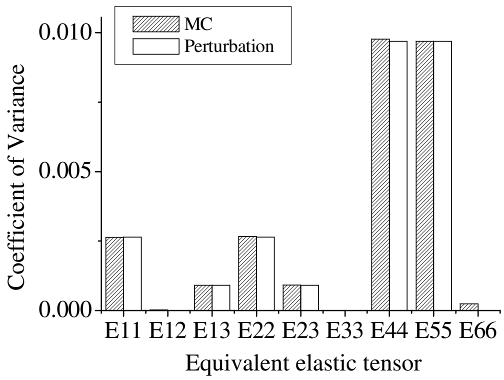

Fig. 13. Estimated CoV of equivalent elastic tensor for a_1 variation.

Table 4
Relative estimation error between CoV by the proposed method and MC in case of variation in microscopic geometry

	E11	E12	E13	E22	E23	E33	E44	E55	E66
V_f	0.076	0.085	0.082	0.076	0.082	0.046	0.072	0.072	0.072
a_1	0.092	0.000	0.031	1.000	0.922	0.000	0.802	0.045	0.000

4.3. Uncertainty in material property and geometry

As a more complex problem, the case that all material properties, volume fraction and shape of fiber have uncertainty is taken into account. It is assumed that the six random variables change independent to each other, and those distribute according to the normal distribution. The expectation and variance of the random variable α is same as the previous examples.

The stochastic characteristics such as expectation and CoV of each equivalent elastic constant are estimated using the proposed method and the Monte-Carlo simulation. For the Monte-Carlo simulation, 10,000 samples are used for trial.

The estimated results using the proposed method and the Monte-Carlo simulation are illustrated in Figs. 14 and 15. Fig. 14 shows the estimated results of expectations of the equivalent elastic tensor, and Fig. 15 shows the estimated results of CoV. From these figures, it can be considered that both of the stochastic characteristics can be well estimated using the proposed method.

For more detailed evaluation, the relative estimation errors in expectation and CoV are listed in Table 5. From this table, it can be found that the maximum error in estimated expectation of the equivalent elastic tensor is about 3%, and the maximum error in the CoV estimation is about 9%.

The estimation error in the expectation is same degree of the result of v_m variation, and the estimation error in the CoV is also same degree or slightly less than the result of v_m variation. It can be considered from this result that the v_m variation is dominant to accuracy of the estimation.

As a numerical investigation, the stochastic homogenization analysis is performed under the following condition.

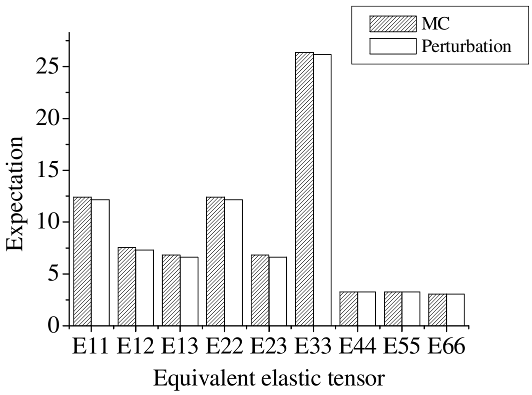

Fig. 14. Estimated expected value of equivalent elastic tensor for the six random variables.

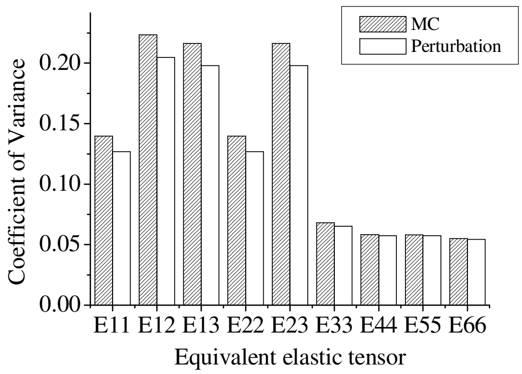

Fig. 15. Estimated CoV of equivalent elastic tensor for the six random variables.

Table 5
Relative estimation error between CoV by the proposed method and MC in case of the six random variables

	E11	E12	E13	E22	E23	E33	E44	E55	E66
Exp	2.048	3.334	3.198	2.056	3.201	0.722	0.029	0.002	0.021
CoV	9.099	8.461	8.450	9.142	8.456	4.295	1.285	0.942	1.178

$$E[\alpha] = 0$$
$$\sqrt{\mathrm{Var}[\alpha]} = \begin{cases} 0.055 & : \text{for } E_{\mathrm{f}}, E_{\mathrm{m}}, \nu_{\mathrm{f}}, V_{\mathrm{f}}, a_1 \\ 0.0055 & : \text{for } \nu_{\mathrm{m}} \end{cases} \tag{20}$$

Namely, it is assumed that the variance of ν_m is less than the other random variables. Fig. 16 shows the result of CoV estimation, and Table 6 shows the relative estimation error between the proposed method and the Monte-Carlo simulation.

From this figure and table, it can be recognized that the maximum estimation error is less than 0.1% in case of the expectation estimation, and that of the CoV estimation is less than about 1%. Compared this result with the Table 5, the estimation in case of less ν_{m} can be accurately performed. Therefore, it can be concluded that the proposed method will be useful for estimation of the stochastic characteristics of the equivalent elastic tensor for microscopic uncertainty in geometry and material properties of a microstructure. In case that the stochastic variation in ν_{m} is observed, however, the proposed method should be used carefully for the stochastic homogenization analysis.

4.4. Accuracy of the perturbation analysis for large stochastic variation

In order to evaluate applicability of the proposed method, influence of degree of a stochastic variation on accuracy of estimation is investigated. In this case, the estimated results using the proposed method and the MC simulation in case of $0 < \mathrm{CoV}[\alpha] \leqslant 0.067$ are investigated. The relative estimation errors between the result of the proposed method and the Monte-Carlo simulation are illustrated in Fig. 17.

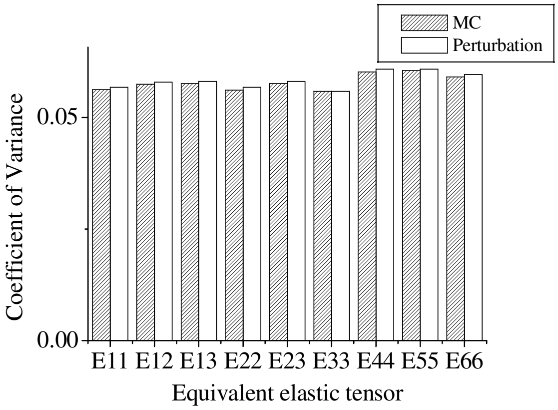

Fig. 16. Estimated CoV equivalent elastic tensor for the six random variables in case of less variance of v_m.

Table 6
Relative estimation error between CoV by the proposed method and MC in case of the six random variables in case of less variance of v_m

	E11	E12	E13	E22	E23	E33	E44	E55	E66
Exp	0.022	0.002	0.003	0.004	0.024	0.024	0.003	0.094	0.080
CoV	0.879	0.809	0.870	1.058	0.924	0.040	1.049	0.555	0.885

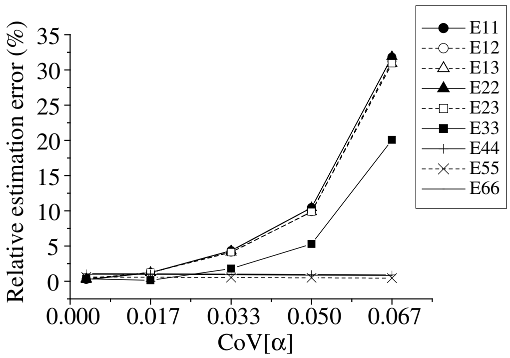

Fig. 17. Relationship between the estimation error and degree of CoV of α.

From Fig. 17, it can be recognized that the error in the estimation of CoV for E44, E55 E66 does not increase when CoV[α] becomes large. This fact shows that the proposed method is effective for estimation of CoV for E44, E55, E66 even in case of a large stochastic variation of component materials. On the other hand, the estimation error increases nonlinearly, and is rapidly increases when CoV[α] becomes large. It should be noticed that the estimated result of CoV using the proposed method may include over 10% of estimation error when CoV[α] is large, and this result shows a limitation on application of the proposed method to the stochastic homogenization analysis considering microscopic uncertainty in material properties and geometry of a microstructure. From Fig. 17, it may be recommended that the proposed method is used for CoV estimation of E11, E12, E13, E22, E23, E33 when CoV[α] is less than 0.05 in this case. From these results, it can be concluded that the lower order perturbation technique as the proposed first-order perturbation based method should be applied to a problem with a very small coefficient of variation for a random input in case that a nonlinear relationship between a response and a random variable is observed.

5. Conclusion

In this paper, the perturbation analysis is applied to the equivalent inclusion method, which is widely used for estimating the equivalent elastic property of composite materials. The perturbation-based equivalent inclusion method is applied to a stochastic homogenization problem of a unidirectional fiber reinforced composite material.

At first, the analytical form of the first order perturbation term of the equivalent elastic tensor with respect to several kinds of random variables such as material properties of component materials, volume fraction and size of an inclusion.

Using the perturbation terms, the stochastic characteristics such as the expectation, variance and coefficient of variance can be estimated based on the first order approximation.

In order to investigate validity and effectiveness of the proposed method, the stochastic homogenization analysis for several kinds of microscopic stochastic variations are performed. Compared to the result of the Monte-Carlo simulation, validity and accuracy of the proposed method are confirmed.

For more discussion, the stochastic homogenization analysis considering six random variables such as Young's modulus and Poisson's ratio of fiber and matrix, volume fraction and size of the inclusion is performed. From the numerical result, it can be confirmed that the proposed method is usable for estimating the stochastic characteristics in this case, but it is found that the v_m variation has a large influence on accuracy of the estimation.

Also the proposed method is applied to the stochastic homogenization problem considering several degrees of stochastic variation. From the numerical result, it can be recognized that the relative estimation error is less than 10% in case that CoV of a random variable is less than 0.05 even if all of the microscopic uncertainty in geometry and material properties of a microstructure under the given condition.

From the numerical results, it can be concluded that the proposed method will be useful for estimation of the stochastic characteristics of the equivalent elastic tensor for microscopic uncertainty in geometry and material properties of a microstructure with taking the normal distributed random variables into account, however, it should be used carefully for stochastic homogenization analysis especially when a large stochastic variation in v_m is observed. Also, it should be noticed that the proposed first order perturbation based technique should be applied to a problem with a very small coefficient of variation for a random input in case that a nonlinear response is observed,

In addition, it should be mentioned that this paper focuses on the case that a material property of a composite can be regarded as a "homogenized macroscopic property" of an inhomogeneous media and a uniform random variability can be observed over a certain region including a very large number of inclusions. In this case, namely, the scale of the inclusion is assumed to be sufficiently (or infinitely) small. If we consider a case of a more general problem such as a non-uniform random variability or finite size of inclusions, an influence of the length scale (Ostoja-Starzewski (2008)) should be taken into account. This problem may not able to be solved using the proposed method, and a new approach will be studied as a next step of this study.

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