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Determining of three collinear cracks opening displacement using the process zone model

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ABSTRACT

The general failure of structures is often preceded by the growth and coalescence of intermediate cracks. In this work, a model with process zone is used to study the deformed state of an infinite isotropic body with three collinear cracks. A numerical algorithm is presented to obtain crack opening displacements effectively. The problem is analyzed for all possible cases of mutual location of the cracks and different levels of loading. The numerical solution shows negligible influence of two cracks coalescence process on the opening displacement of the third crack. An example of the results implementation to calculate crack initiation duration in a problem for viscoelastic body is given.

1. Introduction

The modern works on fracture mechanics of viscoelastic bodies are mostly devoted to the problems for single isolated crack. So the failure state is reached by the studied body in one of the following cases: when the loading reaches some critical value (Irwin, 1948) or as a result of subcritical crack growth up to its critical length (Schapery, 1986; Knauss, 1993; Kaminsky, 1994; Kaminsky, 1998). However, there is another mechanism that can lead to a failure of the solid body. It is multisite fracture when the failure is a result of coalescence of several subcritical crack into a single big crack (Gutzul and Kaminsky, 1989; Kaminsky, 1990). The growth of this big crack is just a final stage of the fracture process. This mechanism is known to happen in some aircraft structures.

However, it should be noted that the theoretical investigation of the last mechanism is not complete, so there is a need in solutions of the several problems for elastic, elastoplastic and viscoelastic bodies weakened by the systems of cracks taking into account the modern models of fracture mechanics. At the time when this paper was written there are only a few solutions for the problems of this class are given (Vitvitskiy, 1965; Parton and Morozov, 1989; Collins and Cartwright, 1996; Hao, 2001; Nishimura, 2002; Zhou and Wang, 2006; Xu and Wu, 2012; Chang and Kotousov, 2012; Chang and Kotousov, 2012).

Numerous experimental studies, see (Williams, 1984) and surveys (Kaminsky, 2004; Kaminsky and Nizhnik, 1995) show that there are partial fracture (process) zones in the front of crack which move as the crack grows. These zones appear due to the high level of the stress near the crack front. The material in the process zones converts to a semi-fractured state (e.g. the state in craze zones in polymers).

The pattern of the fracture process zone, its structure and size are the crucial factors of adequate description of the fracture process. As it is shown in the modern studies (Parton and Morozov, 1989; Kaminsky and Selivanov, 2001; Kaminsky and Chernoivan, 2004), the most effective techniques to describe crack growth in viscoelastic and elasto-plastic bodies make use models that taking into account process zones. These models are also referred as two-phase models as they are taking into account two phases of fracture instead of one phase in Griffith–Irwin model where solid material failure during the fracture occurs rapidly (without a transitional state). The model of Leonov–Panasyuk that is used to study elastic and viscoelastic bodies, Dugdales model for elasto-plastic bodies and some other models (Knauss, 1993; Schapery, 1986) are two-phase models. The choice of the model to describe the fracture of materials should be made taking into account physical and mechanical properties of the material.

As in many cases the process zone is a thin wedge-shaped defect align the crack it can be modeled using Leonov–Panasyuk–Dugdale model as a split with self-balanced stresses σ applied along of this split (σ is a tensile strength of the material for the Leonov–Panasyuk model (Panasyuk, 1969) and a yield stress for the Dugdale model (Dugdale, 1960)).

Modern fracture mechanics uses energy, stress and deformation criteria to describe the process of fracture for materials of different types. The deformation criteria can be effective for the elasto-plastic materials with considerable plastic zones near the crack front. The stress criterion based on SIF can give an inappropriate

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precision when used for such materials. Furthermore, deformation criterion, namely the COD-criterion, is widely used to study the subcritical growth of the cracks in viscoelastic materials. This criterion allows to obtain kinetic equations of slow crack growth in viscoelastic media (see surveys (Schapery, 1986; Kaminsky, 2004)).

To use the deformation criteria in mode I problems it is necessary to determine opening displacement $\delta_I(l,p) = 2v(l,p)$ at the crack tip (where p is the loading intensity and v is a displacement component which is normal to the crack). It is worth noting that the corresponding calculations have not been given in the above cited works. Then the fracture criterion is as follows (Wells, 1961; Panasyuk, 1969)

$$\delta_I(l,p_*) = \delta_{Ic} \tag{1}$$

where p_* is a critical intensity of the external loading that makes the crack grow; δ_{Ic} is the COD. Using Eq. (1) one can determine the critical loading p_* by the known COD.

For a crack in viscoelatic body under the subcritical level of external loading, the COD criterion can be written as (Savin and Kaminsky, 1967)

$$\delta_I(t,l,p) = \delta_{Ic}. \tag{2}$$

Eq. (2) can be solved to determine the duration of mode I crack initiation for a known value of p.

Thus, to investigate long-term fracture of viscoelastic bodies it is needed to determine the displacement of the crack faces using the model which takes into account the process zone at the crack front. The aim of this work is to establish the relations to obtain the crack opening in the system of collinear cracks.

This work deals with a stressed state of an infinite plate with a system of three collinear cracks. To determine the length of process zones and the opening displacement a solution for the plane with rectilinear slits given in Muskhelishvili (1953) is used. An approach that is used in this work is based on a polynomial representation of the general solution of one-dimensional problem of linear conjunction on the segments where the loading is applied. This representation allows us to determine principal values of the integrals in the solution of the linear conjunction problem. The obtained equations were used to solve a problem of initial period of crack development in a viscoelastic plate using Leonov–Panasyuk–Dugdale model and COD criterion. A comparison of the obtained results with the results which are known from literature is given (see Appendix A), as well as a discussion on possibility to expand the solution on the determination of service life for viscoelastic bodies with crack sets.

2. Problem statement

Consider a system of three collinear cracks of arbitrary length in an infinite isotropic elastic body. The body is assumed to be under uniform normal to the crack line tension p applied at infinity. To formulate boundary conditions of the problem Cartesian coordinate system with Ox axis along the crack line is used.

According to Leonov–Panasyuk–Dugdale model a zone of nonlinear behavior at the crack tip can be substituted by a slit with compressing stresses of intensity σ applied on its faces. These stresses are shown as a uniformly distributed loading along the corresponding segments in Fig. 1. Thus the problem of elasticity theory for the upper half-plane has the following boundary conditions for stresses along Ox:

$$\tau_{xy}(t) = 0, \quad t \in L; \quad \sigma_y(t) = \begin{cases} 0, & t \in L' \\ \sigma, & t \in L'' \end{cases},$$

where

$$L = \begin{cases} \bigcup_{k=1}^{3} L_k, & L_k = (c_k, d_k), & \text{problem } a \\ L_1 \cup L_2, & L_1 = (c_1, d_2), L_2 = (c_3, d_3), & \text{problem } b \\ L_1 \cup L_2, & L_1 = (c_1, d_1), L_2 = (c_2, d_3), & \text{problem } c \\ L_1, & L_1 = (c_1, d_3), & \text{problem } d \end{cases},$$

$$L' = \bigcup_{k=1}^{3} (a_k, b_k), \quad L'' = L - L'.$$

The positions of c_k and d_k should be determined from the condition of stress finiteness at these points. If the cracks (segments of L' on Fig. 1) are close enough to each other adjacent zones of non-linear behavior (process zones) can form a continuous united zone. A condition of forming this united zone is the condition of the problem a being unsolvable. All possible cases of the mutual cracks positions are shown on Fig. 1.

Solutions of the problems a and b are given below. It can be seen that the solution of problem c can be obtained from the solution of the problem b by the inverse of x-axis direction. The problem d has a simple analytic solution that is also given below.

3. Solution for the separate process zones (problem *a*)

3.1. General solution of the problem

The general solution of a problem for a plane with collinear slits under the normal to the slits tension applied at infinity can be expressed using two complex functions $\Phi(z)$ and $\Omega(z)$ (see (Muskhelishvili, 1953)):

$$\Phi(z) = \Phi_0(z) + \frac{P_n(z)}{X(z)} - \frac{p}{4}, \quad \Omega(z) = \Phi(z) + \frac{p}{2}, \tag{3}$$

where

$$\Phi_0(z) = \sigma\frac{F(z)}{X(z)}, \quad F(z) = \frac{1}{2\pi i}\int_{L''}\frac{X^+(t)}{t-z}\mathrm{d}t;$$

$$X(z) = \prod_{k=1}^{n}(z-c_k)^{1/2}(z-d_k)^{1/2}, \tag{4}$$

$$P_n(z) = \sum_{k=0}^{n} C_k z^{n-k}, \quad C_0 = \frac{p}{2},$$

n is the number of slits, c_k and d_k are the ends of kth slit $(k = 1,2,\ldots,n)$.

Polynomial coefficients C_k can be obtained from the condition of displacement uniqueness:

$$2\int_{L_k}\frac{P_n(t)}{X^+(t)}\mathrm{d}t + \int_{L_k}[\Phi_0^+(t) - \Phi_0^-(t)]\mathrm{d}t = 0, \quad k = 1,2,\ldots,n. \tag{5}$$

Then the displacement can be obtained from the following equation

$$2\mu(u+iv) = \varkappa\varphi(z) - \omega(\bar{z}) - (z-\bar{z})\overline{\Phi(z)}, \tag{6}$$

where $\varkappa = (3\lambda+\mu)/(\lambda+\mu)$ for the plain stress; λ and μ are Lamé constants,

$$\omega(z) = \int\Omega(z)\mathrm{d}z, \quad \varphi(z) = \int\Phi(z)\mathrm{d}z.$$

3.2. Numerical solution of the problem

The function in (4) for $n = 3$ can be written on the upper faces of slits as

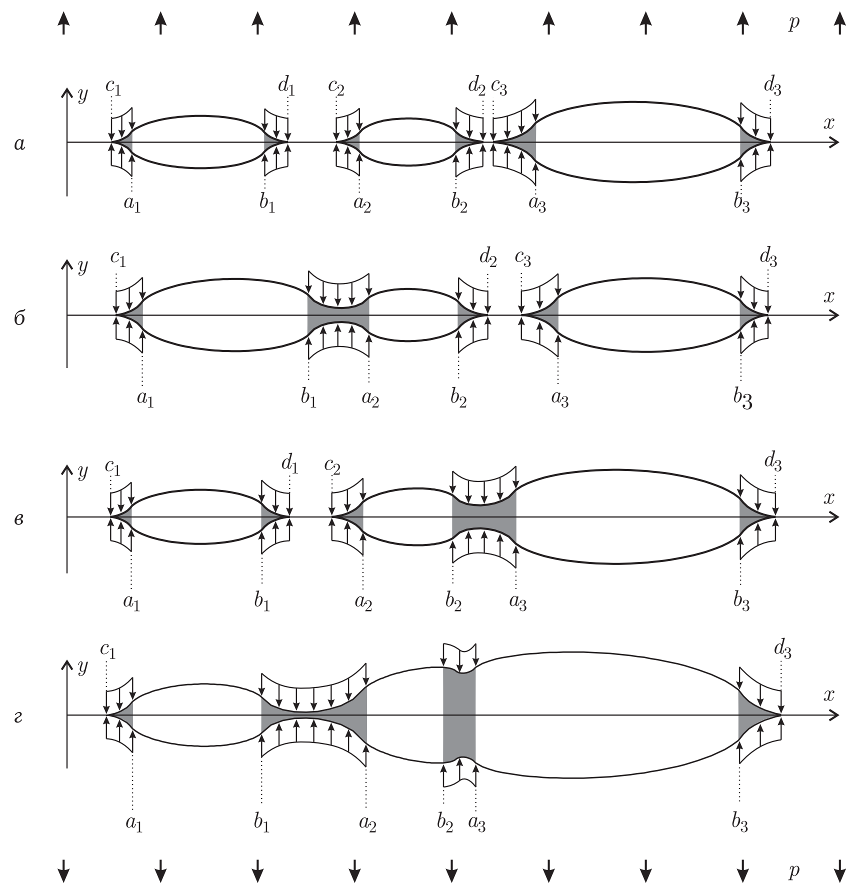

Fig. 1. Illustration of possible boundary condition dependency on crack disposition.

$$X^{+}(t) = (-1)^{3-k} i\hat{X}(t), \quad t \in L_k, \quad k = 1, 2, 3, \tag{7}$$

$$\hat{X}(t) = \sqrt{\left|\prod_{k=1}^{3}(t - c_k)(t - d_k)\right|}.$$

Using Sokhotskiy–Plemel formulas one can obtain

$$\Phi_0^{\pm}(t) = \frac{\sigma}{2}g(t) + \frac{\sigma_1 G(t)}{X^{\pm}(t)},$$

where

$$g(t) = \begin{cases} 1, & t \in L'' \\ 0, & t \in L' \end{cases}, \quad \sigma_1 = \frac{\sigma}{2\pi};$$
$$G(t) = \int_{L''}\frac{\hat{X}(\tau)}{\tau - t}d\tau = \sum_{k=1}^{3}(-1)^{3-k}[F_{ck}(t) + F_{dk}(t)] \tag{8}$$

and

$$F_{ck}(t) = \int_{c_k}^{a_k}\frac{\hat{X}(\tau)}{\tau - t}d\tau, \quad F_{dk}(t) = \int_{b_k}^{d_k}\frac{\hat{X}(\tau)}{\tau - t}d\tau. \tag{9}$$

Then

$$\Phi_0^{+}(t) - \Phi_0^{-}(t) = 2\frac{\sigma_1 G(t)}{X^{+}(t)};$$

$$\int_{L_k}[\Phi_0^{+}(t) - \Phi_0^{-}(t)]dt = \frac{2\sigma_1 J_k}{(-1)^{3-k}i}, \; J_k = \int_{L_k}\frac{G(t)}{\hat{X}(t)}dt, \; k = 1, 2, 3. \tag{10}$$

The function in the numerator of (9) can be presented as

$$\hat{X}(y) = \sqrt{y\sum_{i=0}^{5}q_{(ck)i}y^i}, \quad y = t - c_k;$$
$$\hat{X}(y) = \sqrt{y\sum_{i=0}^{5}q_{(dk)i}y^i}, \quad y = d_k - t, \tag{11}$$

near the ends of the slits c_k, d_k. The analytical integration of F_{ck} and F_{dk} leads to automorphic functions. To simplify the calculations function $\hat{X}$ can be presented as polynomial on each of the segments $(0, y'_{ck}), (0, y'_{dk})$, where

$$y_{ck} = a_k - c_k, \quad y'_{ck} = \kappa y_{ck} = a'_k - c_k;$$
$$y_{dk} = d_k - b_k, \quad y'_{dk} = \kappa y_{dk} = d_k - b'_k$$

(see Fig. 2), and κ is lightly larger than 1.

One can obtain for the arguments from $(0, y'_{ck})$ and $(0, y'_{dk})$

$$\hat{X}(y) \approx \sqrt{y}\sum_{m=0}^{n_{ck}}g_{(ck)m}(y/y'_{ck})^m, \quad \hat{X}(y) \approx \sqrt{y}\sum_{m=0}^{n_{dk}}g_{(dk)m}(y/y'_{dk})^m, \tag{12}$$

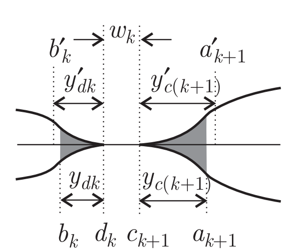

Fig. 2. The notation for geometric parameters used for crack tips.

where n is determined by the accuracy of the representation.

Substitutions $\tau = c_k + y_{ck}x^2$ and $\tau = d_k - y_{dk}x^2$ in (9) give us

$$F_{ck}(t) \approx \int_0^1 \frac{\sqrt{y_{ck}}\, x \sum g_{(ck)m}(y_{ck}x^2/y'_{ck})^m}{y_{ck}x^2 - (t - c_k)} 2y_{ck}x\mathrm{d}x;$$
$$F_{dk}(t) \approx \int_0^1 \frac{\sqrt{y_{dk}}\, x \sum g_{(dk)m}(y_{dk}x^2/y'_{dk})^m}{d_k - t - y_{dk}x^2} 2y_{dk}x\mathrm{d}x \quad (13)$$

or

$$F_{ck}(\delta_{ck}) \approx 2\sqrt{y_{ck}} \int_0^1 \sum_{m=0}^{n_{ck}} h_{(ck)m} \frac{x^{2m+2}\mathrm{d}x}{x^2 - \delta_{ck}};$$
$$F_{dk}(\delta_{dk}) \approx -2\sqrt{y_{dk}} \int_0^1 \sum_{m=0}^{n_{dk}} h_{(dk)m} \frac{x^{2m+2}\mathrm{d}x}{x^2 - \delta_{dk}}, \quad (14)$$

where

$$h_{(ck)m} = g_{(ck)m}\left(\frac{y_{ck}}{y'_{ck}}\right)^m = \frac{g_{(ck)m}}{\kappa^m}, \quad \delta_{ck} = \delta_{ck}(t) = \frac{t - c_k}{y_{ck}};$$
$$h_{(dk)m} = g_{(dk)m}\left(\frac{y_{dk}}{y'_{dk}}\right)^m = \frac{g_{(dk)m}}{\kappa^m}, \quad \delta_{dk} = \delta_{dk}(t) = \frac{d_k - t}{y_{dk}}. \quad (15)$$

Then integration in (14) yields

$$F_{ck}(\delta_{ck}) \approx 2\sqrt{y_{ck}} \sum_{m=0}^{n} h_{(ck)m}\zeta_m(\delta_{ck});$$
$$F_{dk}(\delta_{dk}) \approx -2\sqrt{y_{dk}} \sum_{m=0}^{n} h_{(dk)m}\zeta_m(\delta_{dk}), \quad (16)$$

where

$$\zeta_m(\delta) = \int_0^1 \frac{x^{2m+2}}{x^2 - \delta}\mathrm{d}x.$$

If $|\delta| > 1$

$$\zeta_m(\delta) = -\delta^{-1} \sum_{j=0}^{\infty} \frac{\delta^{-j}}{2(m + j + 1) + 1};$$

if $|\delta| < 1$ it is possible to use recurrent formula

$$\zeta_m(\delta) = \frac{1}{2m + 1} + \delta\zeta_{m-1}(\delta), \quad m = 0, 1, \ldots, n,$$

where

$$\delta\zeta_{-1}(\delta) = \delta \int_0^1 \frac{\mathrm{d}x}{x^2 - \delta} = \sqrt{|\delta|} \cdot \begin{cases} \frac{1}{2}\ln\frac{1-\sqrt{\delta}}{1+\sqrt{\delta}}, & \delta \geqslant 0 \\ -\tan^{-1}\frac{1}{\sqrt{-\delta}}, & \delta < 0 \end{cases}.$$

Further,

$$\zeta_m(\delta) = \sum_{j=0}^{m} \frac{\delta^j}{2(m - j) + 1} + \delta^{m+1}\zeta_{-1}(\delta)$$

and

$$\sum_{m=0}^{n} h_m\zeta_m(\delta) = V(\delta) + \delta\zeta_{-1}(\delta)H(\delta),$$

where

$$V(\delta) = \sum_{m=0}^{n} \upsilon_m\delta^m, \quad H(\delta) = \sum_{m=0}^{n} h_m\delta^m, \quad \upsilon_m = \sum_{j=m}^{n} \frac{h_j}{2(j - m) + 1}.$$

Whence, for the given problem parameters c_k, a_k, b_k and d_k $(k = 1, 2)$ one can obtain the following representations of $F_{ck}(t)$ and $F_{dk}(t)$ for $c_k \leqslant t \leqslant a'_k$ and $b'_k \leqslant t \leqslant d_k$ accordingly

$$F_{ck}(\delta) \approx 2\sqrt{y_{ck}}[V_{ck}(\delta) + \delta\zeta_{-1}(\delta)H_{ck}(\delta)], \quad \delta = \delta_{ck};$$
$$F_{dk}(\delta) \approx -2\sqrt{y_{dk}}[V_{dk}(\delta) + \delta\zeta_{-1}(\delta)H_{dk}(\delta)], \quad \delta = \delta_{dk}. \quad (17)$$

Each representation is defined by its own set of quotients $h_{(ck)m}, \upsilon_{(ck)m}$ $(m = 0, 1, \ldots, n_{ck})$ for polynomials H_{ck} and V_{ck}, and $h_{(dk)m}, \upsilon_{(dk)m}$ $(m = 0, 1, \ldots, n_{dk})$ for polynomials H_{dk} and V_{dk}.

Consider the computational procedure to obtain J_k $(k = 1, 2, 3)$ in (10). To proceed, we first divide the interval of integration into three parts; $(c_k, d_k) = (c_k, a'_k) \cup (a'_k, b'_k) \cup (b'_k, d_k)$ then rewrite J_k as a sum of three integrals over these three parts

$$J_k = \int_{c_k}^{d_k} \frac{G(t)}{\hat{X}(t)}\mathrm{d}t = I_{ck} + I_{abk} + I_{dk}. \quad (18)$$

Among the functions F_{ck}, F_{dk} $(k = 1, 2, 3)$, function F_{cl} has nonintegrable singularity on (c_l, a'_l) and F_{dl} has nonintegrable singularity on (b'_l, d_l). Consider the computation of I_{cl} principal value. Similar formulas that can be obtained for I_{dl}, I_{abk} have no singularities. Using (8) for $G(t)$ one can obtain

$$I_{cl} = \int_{c_l}^{a'_l} \frac{(-1)^{3-l}F_{cl}(\delta_{cl})}{\hat{X}(t)}\mathrm{d}t + \int_{c_l}^{a'_l} \frac{B_{cl}(t)}{\hat{X}(t)}\mathrm{d}t, \quad (19)$$

where

$$B_{cl}(t) = \sum_{\substack{k=1 \\ k \neq l}}^{3} (-1)^{3-k}F_{ck}(\delta_{ck}) + \sum_{k=1}^{3}(-1)^{3-k}F_{dk}(\delta_{dk}), \quad (20)$$

δ_{ck} and δ_{dk} are the functions of t, they are defined by (15).Substituting $t = c_l + y_{cl}x^2$ into (19) and taking into account (12) and (15) it can be shown that

$$\hat{X}(t) = \sqrt{y_{cl}}\, xH_{cl}(x^2). \quad (21)$$

Then using (17) and (21)

$$I_{cl} = 2\sqrt{y_{cl}} \int_0^{\sqrt{\kappa}} \frac{B_{cl}(c_l + y_{cl}x^2)}{H_{cl}(x^2)}\,\mathrm{d}x + (-1)^{3-l}$$
$$\cdot\, 4y_{cl} \int_0^{\sqrt{\kappa}} \frac{V_{cl}(x^2) + x^2\zeta_{-1}(x^2)H_{cl}(x^2)}{H_{cl}(x^2)}\,\mathrm{d}x.$$

The first integral has no singularities. The second integral can be found as

$$(-1)^{3-l} \cdot 4\int_0^{\sqrt{\kappa}} \left[\frac{V_{cl}(x^2)}{H_{cl}(x^2)} + \frac{x}{2}\ln\frac{1 - x}{1 + x}\right]\mathrm{d}x = (-1)^{3-l}\left[4\int_0^{\sqrt{\kappa}} \frac{V_{cl}(x^2)}{H_{cl}(x^2)}\mathrm{d}x + \kappa_0\right],$$

where

$$\kappa_0 = (\kappa - 1)\ln\frac{\sqrt{\kappa} - 1}{\sqrt{\kappa} + 1} - 2\sqrt{\kappa}.$$

To calculate displacement at the crack tips we need to put $\kappa = 1$. This gives us $\kappa_0 = -2$.

Now it can be shown that

$$I_{cl} = 2y_{cl}\left[\int_0^{\sqrt{\kappa}} \frac{(-1)^{3-l} \cdot 2\sqrt{y_{cl}}V_{cl}(x^2) + B_{cl}(c_l + y_{cl}x^2)}{\sqrt{y_{cl}}\,H_{cl}(x^2)}\,\mathrm{d}x + (-1)^{3-l} \cdot \frac{\kappa_0}{2}\right] \quad (22)$$

has no singularities.

Similarly,

$$I_{dl} = -2y_{dl}\left[\int_0^{\sqrt{\kappa}} \frac{(-1)^{3-l+1} \cdot 2\sqrt{y_{dl}}V_{cl}(x^2) + B_{dl}(d_l - y_{dl}x^2)}{\sqrt{y_{dl}}\,H_{dl}(x^2)}\,\mathrm{d}x - (-1)^{3-l} \cdot \frac{\kappa_0}{2}\right], \quad (23)$$

where

$$B_{dl}(t) = \sum_{k=1}^{3}(-1)^{3-k}F_{ck}(\delta_{ck}) + \sum_{\substack{k=1 \\ k \neq l}}^{3} (-1)^{3-k}F_{dk}(\delta_{dk}).$$

3.3. Adjacent cracks

Now we calculate the quotients J and I for the case adjacent slit ends d_k and c_{k+1} Fig. 2. Decreasing of $w_k = c_{k+1} - d_k$ values lead to increasing of $q_{(dk)1}$ and $q_{(c,k+1)1}$ in (11) and increasing of n_{dk} and $n_{c,k+1}$ in (12).

Eqs (11) can be rewritten as

$$\hat{X}(y) = \sqrt{y(y+w_k)\sum_{i=0}^{4} q'_{(c,k+1)i} y^i};$$
$$\hat{X}(y) = \sqrt{y(w_k - y)\sum_{i=0}^{4} q'_{(dk)i} y^i}, \tag{24}$$

The quotients with primes below denote the case where the representation (24) is used for $\hat{X}$. The quotients without primes are for representation (11).Then it can be written for $\hat{X}(t)$ near d_k and c_{k+1} using (12) that

$$\hat{X}(y) \approx \sqrt{y(y+w_k)}\sum_{m=0}^{n_{c,k+1}} g'_{(c,k+1)m}(y/y'_{c,k+1})^m;$$
$$\hat{X}(y) \approx \sqrt{y(y+w_k)}\sum_{m=0}^{n_{dk}} g'_{(dk)m}(y/y'_{dk})^m. \tag{25}$$

Further,

$$F_{c,k+1}(t) \approx \int_0^1 \frac{\sqrt{y_{c,k+1}}\, x\sqrt{y_{c,k+1}x^2 + w_k}\sum g'_{(c,k+1)m}(y_{c,k+1}x^2/y'_{c,k+1})^m}{y_{c,k+1}x^2 - (t - c_{k+1})} 2y_{c,k+1} x\mathrm{d}x;$$
$$F_{dk}(t) \approx \int_0^1 \frac{\sqrt{y_{dk}}\, x\sqrt{y_{dk}x^2 + w_k}\sum g'_{(dk)m}(y_{dk}x^2/y'_{dk})^m}{d_k - t - y_{dk}x^2} 2y_{dk} x\mathrm{d}x.$$

The expressions that correspond to (14) for the distant cracks, are as follows

$$F_{c,k+1}(\delta,\varepsilon) \approx 2y_{c,k+1}\int_0^1 \sum_{m=0}^{n_{c,k+1}} h'_{(c,k+1)m}\frac{\sqrt{x^2+\varepsilon}}{x^2-\delta}x^{2m+2}\mathrm{d}x, \quad \delta = \delta_{c,k+1}, \quad \varepsilon = \varepsilon_{c,k+1};$$
$$F_{dk}(\delta,\varepsilon) \approx -2y_{dk}\int_0^1 \sum_{m=0}^{n_{dk}} h'_{(dk)m}\frac{\sqrt{x^2+\varepsilon}}{x^2-\delta}x^{2m+2}\mathrm{d}x, \quad \delta = \delta_{dk}, \quad \varepsilon = \varepsilon_{dk}, \tag{26}$$

where $h'_{(c,k+1)m}$ and $h'_{(dk)m}$ can be determined similarly to (15) and

$$\varepsilon_{c,k+1} = \frac{w_k}{y_{c,k+1}}, \quad \varepsilon_{dk} = \frac{w_k}{y_{dk}}. \tag{27}$$

After consecutive integration in (26) one can obtain

$$F_{c,k+1}(\delta,\varepsilon) \approx 2y_{c,k+1}\sum_{m=0}^{n_{c,k+1}} h'_{(c,k+1)m}\zeta_m(\delta,\varepsilon), \quad \delta = \delta_{c,k+1}, \quad \varepsilon = \varepsilon_{c,k+1};$$
$$F_{dk}(\delta,\varepsilon) \approx -2y_{dk}\sum_{m=0}^{n_{dk}} h'_{(dk)m}\zeta_m(\delta,\varepsilon), \quad \delta = \delta_{dk}, \quad \varepsilon = \varepsilon_{dk}, \tag{28}$$

where

$$\zeta_m(\delta,\varepsilon) = \int_0^1 \frac{\sqrt{x^2+\varepsilon}}{x^2-\delta}x^{2m+2}\mathrm{d}x.$$

If $|\delta| > 1$,

$$\zeta_m(\delta,\varepsilon) = -\delta^{-1}\sum_{j=0}^{\infty}\eta_{m+j+1}(\varepsilon)\,\delta^{-j},$$

$$\eta_n(\varepsilon) = \int_0^1 \sqrt{x^2+\varepsilon}\,x^{2n}\mathrm{d}x, \quad n = 0,1,2,\ldots \tag{29}$$

If $\varepsilon \leqslant 1$ these quotients can be computed using recurrent formula

$$\eta_n(\varepsilon) = \frac{1}{2(n+1)}\left(\varepsilon Y_{n+1} + \sqrt{1+\varepsilon}\right);$$

$$Y_1 = \ln\frac{\sqrt{1+\varepsilon}+1}{\sqrt{\varepsilon}};$$

$$Y_{n+1} = \frac{1}{2n}\left[-(2n-1)\varepsilon Y_n + \sqrt{1+\varepsilon}\right], \quad n = 1,2,\ldots$$

For arbitrary ε,

$$\eta_n(\varepsilon) = \frac{\sqrt{\varepsilon}}{2n+1}F(-1/2,\ n+1/2;\ n+3/2;\ -1/\varepsilon),$$

where F is the hypergeometric function.

To determine the quotients $\zeta_m(\delta,\varepsilon)$ for $|\delta| < 1$ one can use recurrent formula

$$\zeta_m(\delta,\varepsilon) = \eta_m(\varepsilon) + \delta\zeta_{m-1}(\delta,\varepsilon), \quad m = 0,1,\ldots,n,$$

where

$$\zeta_{-1}(\delta,\varepsilon) = \begin{cases} Y_1 + \frac{\gamma}{2}\ln\frac{\sqrt{1+\varepsilon}-\gamma}{\sqrt{1+\varepsilon}+\gamma}, & \gamma^2 = 1 + \frac{\varepsilon}{\delta} > 0 \\ Y_1 + \gamma\tan^{-1}\frac{\gamma}{\sqrt{1+\varepsilon}}, & \gamma^2 = -1 - \frac{\varepsilon}{\delta} > 0 \end{cases}.$$

Then

$$\zeta_m(\delta,\varepsilon) = \sum_{j=0}^{m}\eta_{m-j}(\varepsilon)\delta^j + \delta^{m+1}\zeta_{-1}(\delta,\varepsilon)$$

and

$$\sum_{m=0}^{n} h'_m\zeta_m(\delta,\varepsilon) = V'(\delta) + \delta\zeta_{-1}(\delta,\varepsilon)H'(\delta),$$

where

$$V'(\delta) = \sum_{m=0}^{n} v_m\delta^m, \quad H'(\delta) = \sum_{m=0}^{n} h'_m\delta^m, \quad v'_m = \sum_{j=m}^{n} h_j\eta_{j-m}.$$

Thus, the expressions for the functions $F_{ck}(t)$ and $F_{dk}(t)$ are obtained for $c_k \leqslant t \leqslant a'_k$ and $b'_k \leqslant t \leqslant d_k$. These expressions are similar to (17) for the distant cracks:

$$F_{ck}(\delta) \approx 2y_{ck}[V'_{ck}(\delta) + \delta\zeta_{-1}(\delta)H'_{ck}(\delta)], \quad \delta = \delta_{c,k+1};$$
$$F_{dk}(\delta) \approx -2y_{dk}[V'_{dk}(\delta) + \delta\zeta_{-1}(\delta)H'_{dk}(\delta)], \quad \delta = \delta_{dk}, \tag{30}$$

Every expression has its own set of quotients $h'_{(ck)m}, v'_{(ck)m}$ $(m = 0,1,\ldots,n_{ck})$ for polynomials H'_{ck}, V'_{ck} and $h'_{(dk)m}, v'_{(dk)m}$ $(m = 0,1,\ldots,n_{dk})$ for polynomials H'_{dk}, V'_{dk}.

Now we can determine J_k $(k = 1,2,3)$ using (18). As it was mentioned above, among the other F_{ck}, F_{dk} $(k = 1,2,3)$ functions, F_{cl} has nonintegrable singularity on (c_l, a'_l) and F_{dl} has nonintegrable singularity on (b'_l, d_l). Consider the computation of I_{cl} principal value for both values of l, 2 and 3. Substituting $t = c_l + y_{cl}x^2$ into (19) and using (25) and (30) yields

$$\hat{X}(t) = \sqrt{y_{cl}}\,x\sqrt{y_{cl}x^2 + w_l}\,H'_{cl}(x^2). \tag{31}$$

Taking into account (30) one can obtain

$$I_{cl} = (-1)^{3-l}\cdot 4y_{cl}\int_0^{\sqrt{\kappa}}\frac{V'_{cl}(x^2) + x^2\zeta_{-1}(x^2)H'_{cl}(x^2)}{\sqrt{x^2+\varepsilon_{cl}}H'_{cl}(x^2)}\,\mathrm{d}x$$
$$+ 2\int_0^{\sqrt{\kappa}}\frac{B_{cl}(c_l + y_{cl}x^2)}{H'_{cl}(x^2)}\,\mathrm{d}x,$$

where $B_{cl}(t)$ is defined by (20) and

$$\begin{aligned}\kappa_0(\kappa,\varepsilon) &= 4\int_0^{\sqrt{\kappa}} \frac{x^2\zeta_{-1}(x^2)}{\sqrt{x^2+\varepsilon}}\mathrm{d}x \\ &= 2\left(Y_1\sqrt{\kappa(\kappa+\varepsilon)} - \left(\varepsilon Y_1 + \sqrt{1+\varepsilon}\right)\ln\frac{\sqrt{\kappa+\varepsilon}+\sqrt{\kappa}}{\sqrt{\varepsilon}}\right) \\ &\quad + \kappa\ln\frac{\sqrt{\kappa(1+\varepsilon)}-\sqrt{\kappa+\varepsilon}}{\sqrt{\kappa(1+\varepsilon)}+\sqrt{\kappa+\varepsilon}}.\end{aligned}$$

To determine the displacements at the crack tips we need the last integral for $\kappa = 1$. Then $\kappa_0(1,\varepsilon) = -2Y_1^2\varepsilon$.

Further,

$$\begin{aligned}I_{cl} &= 2\int_0^{\sqrt{\kappa}} \frac{(-1)^{3-l}\cdot 2y_{cl}V'_{cl}(x^2) + B_{cl}\left(c_l + y_{cl}x^2\right)}{\sqrt{x^2+\varepsilon_{cl}}H'_{cl}(x^2)}\,\mathrm{d}x \\ &\quad + (-1)^{3-l}\cdot y_{cl}\kappa_0(\kappa,\varepsilon_{cl}),\end{aligned} \tag{32}$$

where there is no singularities.

Using the similar approach one can obtain

$$\begin{aligned}I_{dl} &= -2\int_0^{\sqrt{\kappa}} \frac{(-1)^{3-l+1}\cdot 2y_{dl}V'_{dl}(x^2) + B_{dl}\left(d_l - y_{dl}x^2\right)}{\sqrt{x^2+\varepsilon_{dl}}H'_{dl}(x^2)}\,\mathrm{d}x \\ &\quad + (-1)^{3-l}\cdot y_{dl}\kappa_0(\kappa,\varepsilon_{dl}).\end{aligned} \tag{33}$$

Now, it suffices to define the conditions of (11) or (24) usage in expression for $\hat{X}$: for $k = 1$ or $\varepsilon_{ck} \geqslant 1$ we use the first of Eqs. (11), for $k > 1$ and $\varepsilon_{ck} < 1$ the first of Eqs. (24); for $k = n$ or $\varepsilon_{dk} \geqslant 1$ the second of Eqs. (11), for $k < n$ and $\varepsilon_{dk} < 1$ the second of Eqs. (24). Values of ε_{ck} and ε_{dk} can be determined using (27).

3.4. Lengths of process zones and vertical displacement of crack faces

Denote

$$I_{k,m} = \int_{c_k}^{d_k} \frac{t^m\,\mathrm{d}t}{\hat{X}(t)}. \tag{34}$$

The corresponding integrals can be determined using the scheme explained above

$$\begin{aligned}I_{k,m} &= \int_{c_k}^{a_k}\frac{t^m\,\mathrm{d}t}{\hat{X}(t)} + \int_{a_k}^{b_k}\frac{t^m\,\mathrm{d}t}{\hat{X}(t)} + \int_{b_k}^{d_k}\frac{t^m\,\mathrm{d}t}{\hat{X}(t)} \\ &= 2\int_0^1 \frac{(c_k+y_{ck}x^2)^m}{D_{ck}(x^2)}\mathrm{d}x + \int_{a_k}^{b_k}\frac{t^m\,\mathrm{d}t}{\hat{X}(t)} + 2\int_0^1\frac{(d_k - y_{dk}x^2)^m}{D_{dk}(x^2)}\mathrm{d}x,\end{aligned}$$

where

$$D_{ck}(\xi) = \begin{cases} H_{ck}(\xi)/\sqrt{y_{ck}}, & k = 1 \text{ or } \varepsilon_{ck} > 1 \\ \sqrt{\xi+\varepsilon_{ck}}H'_{ck}(\xi), & k > 1 \text{ and } \varepsilon_{ck} < 1\end{cases};$$

$$D_{dk}(\xi) = \begin{cases} H_{dk}(\xi)/\sqrt{y_{dk}}, & k = 3 \text{ or } \varepsilon_{dk} > 1 \\ \sqrt{\xi+\varepsilon_{dk}}H'_{dk}(\xi), & k < 3 \text{ and } \varepsilon_{dk} < 1\end{cases}.$$

The condition of displacement uniqueness (5) can be written as a system of linear algebraic equations

$$\sum_{n=1}^{3} I_{k,3-n}\tilde{C}_n = -J_1 - I_{k,3}\tilde{C}_0, \quad \tilde{C}_k = \frac{C_k}{\sigma_1}, \quad k = 0,\ldots,3,$$

This system allows us to determine C_1, C_2 and C_3.

$\Phi(z)$ then is as follows

$$\Phi(z) = \sigma\frac{F(z)}{X(z)} + \frac{\tilde{P}_3(z)}{X(z)} - \frac{p}{4}, \quad \tilde{P}_3(z) = \sum_{n=0}^{3}\tilde{C}_n z^{3-n}.$$

To find the crack opening displacement we have to determine the values of this function on the crack faces. Using Sokhotskiy–Plemel formula we have

$$\Phi^{\pm}(x) = \frac{\sigma}{2}g(x) + \sigma_1\frac{G(x)+\tilde{P}_3(x)}{X^{\pm}(x)} - \frac{p}{4}, \quad \Omega^{\pm}(x) = \Phi^{\pm}(t) + \frac{p}{2}, \tag{35}$$

where functions $g(x)$ and $G(x)$ can be determined using (8) as functions of geometry parameters.

Parameters $c_k, d_k, k = 1,2,3$ can be found using the condition of Φ finiteness:

$$\begin{cases} G(c_k) + \tilde{P}_3(c_k) = 0 \\ G(d_k) + \tilde{P}_3(d_k) = 0\end{cases}, \quad k = 1,2,3.$$

This gives us a system of 6 non-linear equations. Quotients of $P_3(x)$ and values of function G at $z = c_k, \tau = d_k$ in this system should be treated as functions of c_k, d_k.

Consider now the displacements on crack faces. According to (6) the displacement for the upper face of kth slit is as follows

$$[u(x) + iv(x)]^{\pm} = \frac{1}{2\mu}\int\left[\varkappa\Phi^{\pm}(x) - \Omega^{\mp}(x)\right]\mathrm{d}x.$$

It can be shown using (35) that

$$[u(x)+iv(x)]^{\pm} = \Lambda_1\int\left[\sigma_1\frac{G(x)+\tilde{P}_3(x)}{X^{\pm}(x)} - \frac{p}{4}\right]\mathrm{d}x + \Lambda_2\frac{\sigma}{2}\int g(x)\mathrm{d}x,$$

$$\Lambda_1 = \frac{\varkappa+1}{2\mu}, \quad \Lambda_2 = \frac{\varkappa-1}{2\mu}.$$

For the plane stress this can be written as

$$\Lambda_1 = \frac{4}{E}, \quad \Lambda_2 = \frac{2(1-\nu)}{E}.$$

Taking into account (7) one can obtain the displacement for the upper face of kth slit ($k = 1,2,3$) as

$$u^{\pm}(x) = -\Lambda_1\frac{p}{4}x + \Lambda_2\frac{\sigma}{2}\int_0^x g(\tau)\mathrm{d}\tau;$$

$$v^{\pm}(x) = \pm(-1)^{3-k+1}\Lambda_1\sigma_1 v_0(x), \tag{36}$$

where

$$v_0(x) = \int_{c_k}^{x}\frac{G(\tau)+\tilde{P}_3(\tau)}{\hat{X}(\tau)}\mathrm{d}\tau. \tag{37}$$

Using (22), (23), (32), (33) it can be shown that at the crack tips

$$\begin{aligned}v_0(a_k) &= (-1)^{3-k+1}y_{ck}\Big\{(-1)^{3-k+1} \\ &\quad + \int_0^1\frac{(-1)^{3-k}\cdot 2\sqrt{y_{ck}}V_{ck}(x^2) + B_{ck}(c_k+y_{ck}x^2) + \tilde{P}_3(c_k+y_{ck}x^2)}{\sqrt{y_{ck}}\,H_{ck}(x^2)}\mathrm{d}x\Big\},\end{aligned}$$

when $k = 1$ or $\varepsilon_{ck} \geqslant 1$ and

$$\begin{aligned}v_0(a_k) &= (-1)^{3-k+1}\Big\{(-1)^{3-k+1}\cdot y_{ck}Y_1^2\varepsilon_{ck} \\ &\quad + \int_0^1\frac{(-1)^{3-k}\cdot 2y_{ck}V'_{ck}(x^2) + B_{ck}(c_k+y_{ck}x^2) + \tilde{P}_3(c_k+y_{ck}x^2)}{\sqrt{x^2+\varepsilon_{ck}}\,H'_{ck}(x^2)}\mathrm{d}x\Big\},\end{aligned}$$

when $k > 1$ and $\varepsilon_{ck} < 1$;

$$\begin{aligned}v_0(b_k) &= (-1)^{3-k}y_{dk}\Big\{(-1)^{3-k} \\ &\quad + \int_0^1\frac{(-1)^{3-k+1}\cdot 2\sqrt{y_{dk}}V_{dk}(x^2) + B_{dk}(d_k-y_{dk}x^2) + \tilde{P}_3(d_k-y_{dk}x^2)}{\sqrt{y_{dk}}\,H_{dk}(x^2)}\mathrm{d}x\Big\},\end{aligned}$$

when $k = 3$ or $\varepsilon_{dk} \geqslant 1$ and

$$\begin{aligned}v_0(b_k) &= (-1)^{3-k}\Big\{(-1)^{3-k}\cdot y_{dk}Y_1^2\varepsilon_{dk} \\ &\quad + \int_0^1\frac{(-1)^{3-k+1}\cdot 2y_{dk}V'_{dk}(x^2) + B_{dk}(d_k-y_{dk}x^2) + \tilde{P}_3(d_k-y_{dk}x^2)}{\sqrt{x^2+\varepsilon_{dk}}\,H'_{dk}(x^2)}\mathrm{d}x\Big\},\end{aligned}$$

when $k < 3$ and $\varepsilon_{dk} < 1$.

4. Coalescence of inner process zones (problem *b*)

In this problem there are two slits and five process zones.

The function from (4) on the upper faces of cracks can be rewritten for $n = 2$ as

$$X^+(t) = (-1)^{2-k} i\hat{X}(t), \quad t \in L_k, \quad k = 1, 2,$$

$$\hat{X}(t) = \sqrt{\left|\prod_{\substack{k=1\\k\neq 2}}^{3}(t - c_k)\prod_{k=2}^{3}(t - d_k)\right|}.$$

Then the second integral in the condition of displacement uniqueness (5) is as follows

$$\int_{L_k}[\Phi_0^+(t) - \Phi_0^-(t)]\mathrm{d}t = \frac{2\sigma_1 J_k}{(-1)^{2-k} i}, \quad J_k = \int_{L_k}\frac{G(t)}{\hat{X}(t)}\mathrm{d}t, \quad k = 1, 2, \tag{38}$$

$$G(t) = -[F_{c1}(t) + F_{e1}(t) + F_{d2}(t)] + F_{c3}(t) + F_{d3}(t),$$

where F_{ck} and F_{dk} can be determined from (9) and

$$F_{e1}(t) = \int_{b_1}^{a_2}\frac{\hat{X}(\tau)}{\tau - t}\mathrm{d}\tau, \tag{39}$$

$$\hat{X}(y) = \sqrt{\sum_{i=0}^{4} q_{(ck)i} y^i}, \quad y = t - e_1, \quad e_1 = \frac{b_1 + a_2}{2}. \tag{40}$$

Denote

$$y_{e1} = a_2 - b_1, \quad y'_{e1} = \kappa y_{e1} = a'_2 - b'_1, \quad a'_2 = a_2 + \frac{\kappa - 1}{2} y_{e1}, \quad b'_1 = b_1 - \frac{\kappa - 1}{2} y_{e1}.$$

An approximation of $\hat{X}(y)$ for $(-y'_{e1}/2, y'_{e1}/2)$ can be built as a polynomial

$$\hat{X}(y) \approx \sum_{m=0}^{n_{e1}} g_{(e1)m}(y/y'_{e1})^m, \tag{41}$$

where n_{e1} is determined by the accuracy of approximation.

Substituting $\tau = e_1 + y_{e1}x$ into (39) one can obtain

$$F_{e1}(t) \approx \int_{-1/2}^{1/2}\sum\frac{g_{(e1)m}(y_{e1}x/y'_{e1})^m}{y_{e1}x - (t - e_1)} y_{e1}\mathrm{d}x \tag{42}$$

or

$$F_{e1}(\delta_{e1}) \approx \int_{-1/2}^{1/2}\sum_{m=0}^{n_{e1}} h_{(e1)m}\frac{x^m \mathrm{d}x}{x - \delta_{e1}}, \tag{43}$$

where

$$h_{(e1)m} = g_{(e1)m}\left(\frac{y_{e1}}{y'_{e1}}\right)^m = \frac{g_{(e1)m}}{\kappa^m}, \quad \delta_{e1} = \delta_{e1}(t) = \frac{t - e_1}{y_{e1}}. \tag{44}$$

The sequential integration in (43) yields

$$F_{e1}(\delta_{e1}) \approx \sum_{m=0}^{n} h_{(e1)m}\zeta_m(\delta_{e1}), \tag{45}$$

where

$$\zeta_m(\delta) = \int_{-1/2}^{1/2}\frac{x^m}{x - \delta}\mathrm{d}x.$$

When $|\delta| > 1$

$$\zeta_m(\delta) = -\delta^{-1}\sum_{j=0}^{\infty}\eta_{m+j+1}\delta^{-j}, \quad \eta_n = \begin{cases} 0, & \text{when } n \text{ is even} \\ 1/(n2^{n-1}), & \text{when } n \text{ is odd} \end{cases}$$

For $|\delta| < 1$ a recurrent formula can be used:

$$\zeta_m(\delta) = \eta_m + \delta\zeta_{m-1}(\delta), \quad m = 1, 2, \ldots, n,$$

where

$$\zeta_0(\delta) = \int_{-1/2}^{1/2}\frac{\mathrm{d}x}{x - \delta} = \ln\left|\frac{1/2 - \delta}{1/2 + \delta}\right|.$$

Then

$$\zeta_m(\delta) = \sum_{j=0}^{m-1}\eta_{m-j}\delta^j + \delta^m\zeta_0(\delta)$$

and

$$\sum_{m=0}^{n} h_m\zeta_m(\delta) = V(\delta) + \zeta_0(\delta)H(\delta),$$

where

$$V(\delta) = \sum_{m=0}^{n-1} v_m\delta^m, \quad H(\delta) = \sum_{m=0}^{n} h_m\delta^m, \quad v_m = \sum_{j=m+1}^{n} h_j\eta_{j-m}.$$

Thus, for the given parameters of the model c_k, a_k, b_k and d_k $(k = 1, 2, 3)$ we have the following value of $F_{e1}(t)$ for $b'_1 \leqslant t \leqslant a'_2$

$$F_{e1}(\delta) \approx V_{e1}(\delta) + \zeta_0(\delta)H_{e1}(\delta), \quad \delta = \delta_{e1}. \tag{46}$$

To determine J_1 from (38) the interval of integration can be split into five subintervals $(c_1, d_2) = (c_1, a'_1) \cup (a'_1, b'_1) \cup (b'_1, a'_2) \cup (a'_2, b'_2) \cup (b'_2, d_2)$. We denote the corresponding integrals as follows

$$J_1 = \int_{c_1}^{d_2}\frac{G(t)}{\hat{X}(t)}\mathrm{d}t = I_{c1} + I_{ce1} + I_{ab1} + I_{de1} + I_{d2}. \tag{47}$$

Among the functions $F_{c1}, F_{e1}, F_{d2}, F_{c3}, F_{d3}$, function F_{cl} has nonintegrable singularity on (c_l, a'_l) $(l = 1$ or $l = 3)$, F_{dl} has nonintegrable singularity on (b'_l, d_l) $(l = 2$ or $l = 3)$ and F_{e1} has nonintegrable singularity on (b'_1, a'_2).

The value of I_{c1} can be calculated using (22). The value of I_{d2} can be calculated using (23) when $\varepsilon_{d2} \geqslant 1$ and using (33) when $\varepsilon_{d2} < 1$. It can be seen that in the above-mentioned equations $(-1)^{3-k}$ should be replaced with $(-1)^{2-1} = -1$ and $B_{c1}(t)$ should be defined as

$$B_{c1}(t) = -[F_{e1}(\delta_{e1}) + F_{d2}(\delta_{d2})] + F_{c3}(\delta_{c3}) + F_{d3}(\delta_{d3}).$$

5. Coalescence of all inner process zones (problem *c*)

In this case a vertical displacement of upper face of a slit with two process zones can be obtained as

$$v(x) = \Lambda_1\sigma_1\sum_{k=1}^{3}\{K(x, b_k) - K(x, a_k)\}, \tag{48}$$

where

$$K(x, \xi) = (x - \xi)\ln\left|\frac{\delta(x) - \delta(\xi)}{\delta(x) + \delta(\xi)}\right|, \quad \delta(x) = \sqrt{\frac{x - c_1}{d_3 - x}},$$

Coordinates of process zones ends can be found using a system of equations

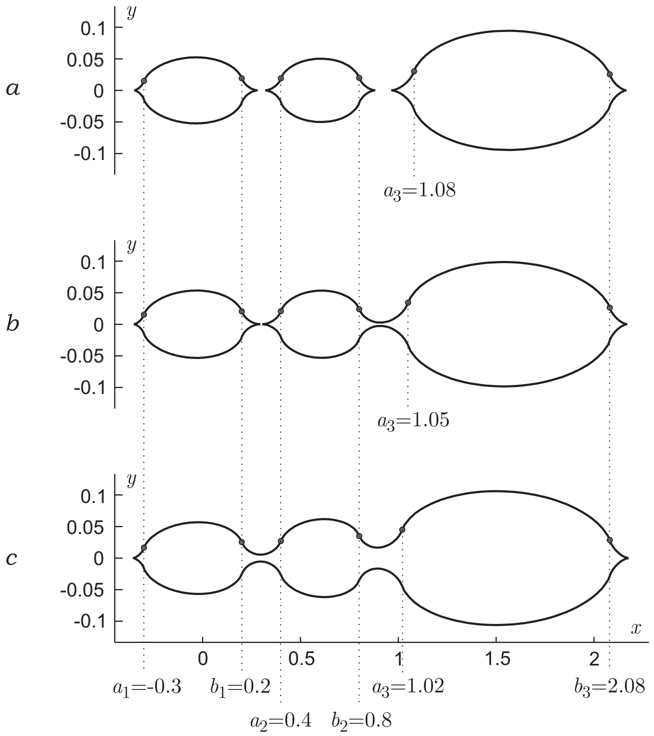

Fig. 3. An example of the crack opening displacement plots calculated for three different positions of the longest crack.

$$\begin{cases} \sum_{k=1}^{3}\left\{\tan^{-1}\delta(b_k) - \tan^{-1}\delta(a_k)\right\} = \frac{\pi}{2}\left(1-\frac{p}{\sigma}\right) \\ \sum_{k=1}^{3}\left\{\hat{X}(b_k) - \hat{X}(a_k)\right\} = 0 \end{cases}, \tag{49}$$

$$\hat{X}(x) = \sqrt{(x-c_1)(d_3-x)}.$$

For a single crack one can obtain from (48) that

$$v(x) = \Lambda_1\sigma_1\{K(x,b) - K(x,-b)\}. \tag{50}$$

In this case, the first condition from (5) gives the following dependence

$$2\arccos\frac{b}{d} = \pi\frac{p}{\sigma}.$$

6. Numerical results and discussion

Fig. 3 shows crack opening displacement for three cracks of different lengths $(0.5, 0.4, 1)$ and three positions of the longest crack. This opening displacement was calculated using (36) and shown up to a factor $\Lambda_1\sigma$, which is a function of elastic constants.

Fig. 3 *a* corresponds to the problem *a* when all process zones are separated. Fig. 3 *b* is for problem *c*. For these crack positions the problem *a* has no solution, process zones between second and third cracks collate and the distance w_1 between the first and the second slits is small. Fig. 3 *c* corresponds to the problem *d* when all inner process zones collate.

Fig. 4 shows a transition from different kinds of the problem. So we have problem *a* for $a_3 = 1.08, 1.07, 1.06$, problem *c* for $a_3 = 1.05$ and problem *d* for $a_3 = 1.04, 1.03, 1.02$.

The crack opening displacement near the tips is shown separately in Fig. 5. We use dots to indicate the displacements of the upper face of the cracks at the ends of process zones, v_{ak}, v_{bk} $(k = 1, 2, 3)$.

Consider the influence of the mutual positions of the cracks on a deformation field. To do so we can compare the process zones length and the opening displacement at the crack tips with the corresponding values for the single crack. For a single crack of half-length l the length of process zones and the displacement at the crack tips can be found as follows

$$y_{l1} = l\left(\cos^{-1}\frac{\pi p}{2\sigma} - 1\right), \qquad v_{l1} = \Lambda_1\sigma_1 2l\ln\cos^{-1}\frac{\pi p}{2\sigma},$$

where displacements are given up to the factor, which depends on the elastic constants only.

Fig. 6 *a* shows the dependence of $y'_{ck} = y_{ck}/y_{l1}$ (curves *ck*) and $y'_{dk} = y_{dk}/y_{l1}$ (curves $dk, k = 1, 2, 3$) on crack mutual positions according to notation given in Fig. 2. Fig. 6 *b* shows the similar dependencies of $v'_{ak} = v_{ak}/v_{l1}$ and $v'_{bk} = v_{bk}/v_{l1}$ on the position of the left tip of the third crack. As it was beforehand, we lock the positions of two cracks, $a_1 = -0.3, b_1 = 0.2, a_2 = 0.4, b_2 = 0.8$, and the position of the right tip of the third crack $b_3 = 2.08$. The position of the left tip of the third crack, a_3, has the values from 1.02 to 1.08. It can be seen that $y'_{ck}, y'_{dk}, v'_{ak}$ and v'_{dk} are growing as the left tip of the third crack approaches the other cracks. The influence the third crack left tip position on the length of outer zones of the other cracks (curves 1*a* and 3*b*) is negligible.

Fig. 7 shows combined plots of crack opening displacement for the following crack tips positions $a_1 = -1, b_1 = -0.5, a_3$

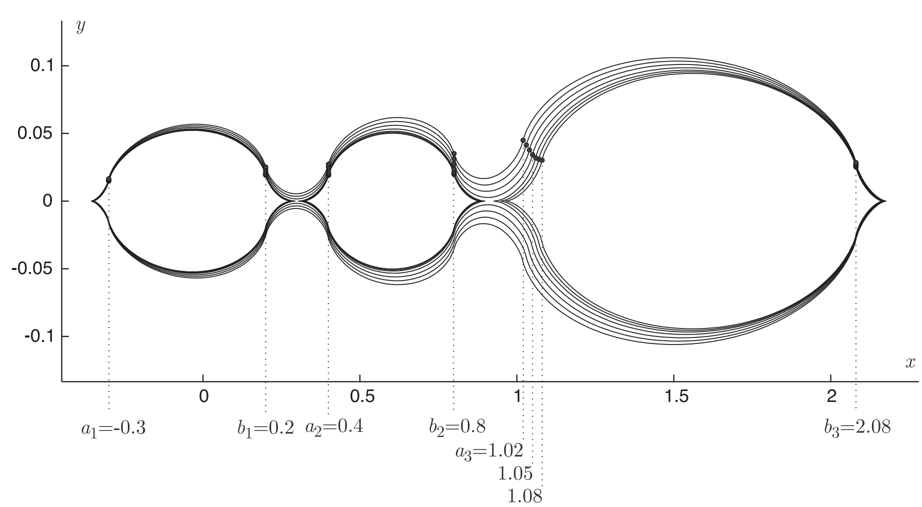

Fig. 4. Combined representation of crack opening displacement from Fig. 3 with intermediate crack positions ($a_3 = 1.03, 1.04, 1.06, 1.07$).

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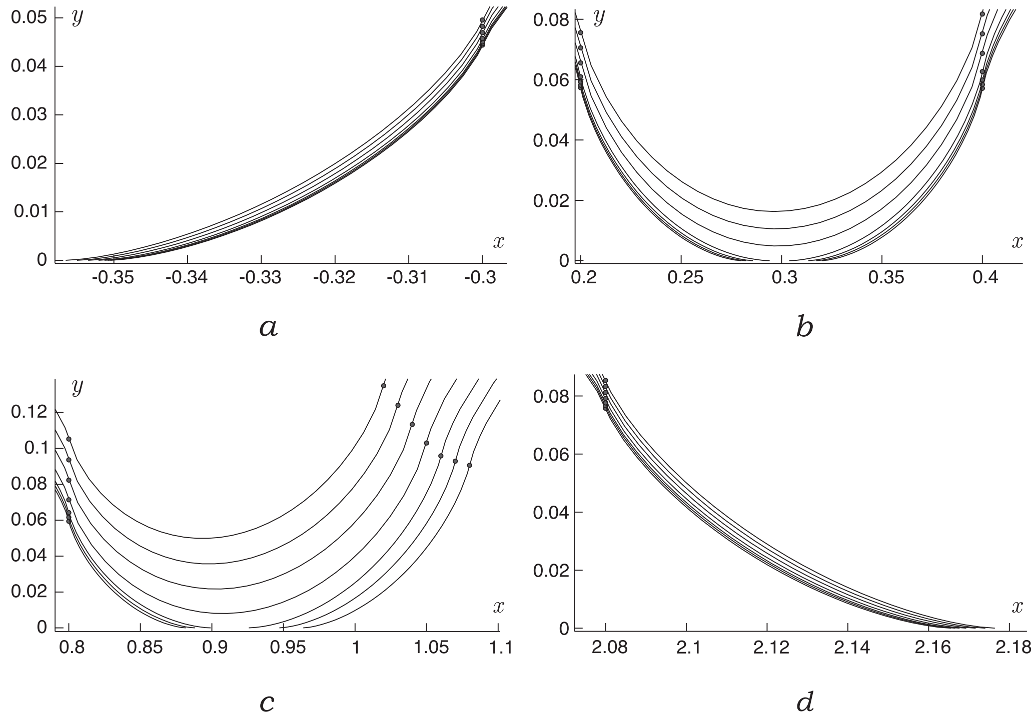

Fig. 5. Crack opening displacement near the tips for Fig. 4.

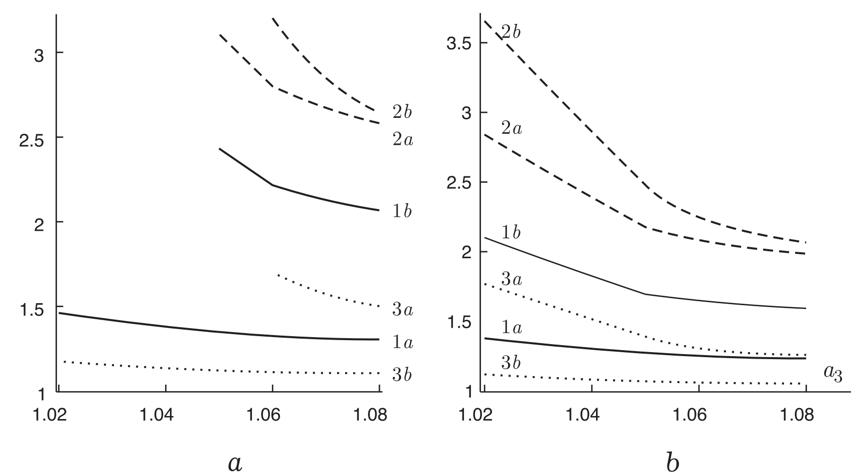

Fig. 6. The dimensionless process zone parameters of collating cracks for various position of the third crack.

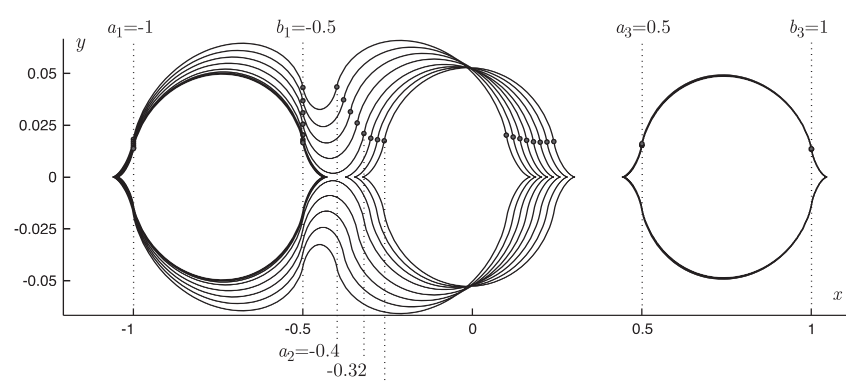

Fig. 7. The crack opening displacement for two collating cracks and the distant third crack.

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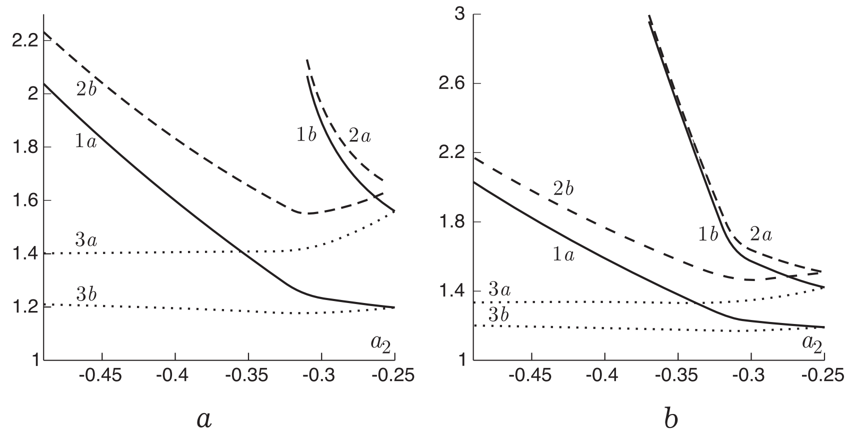

Fig. 8. The dimensionless process zone parameters dependence on the third crack position.

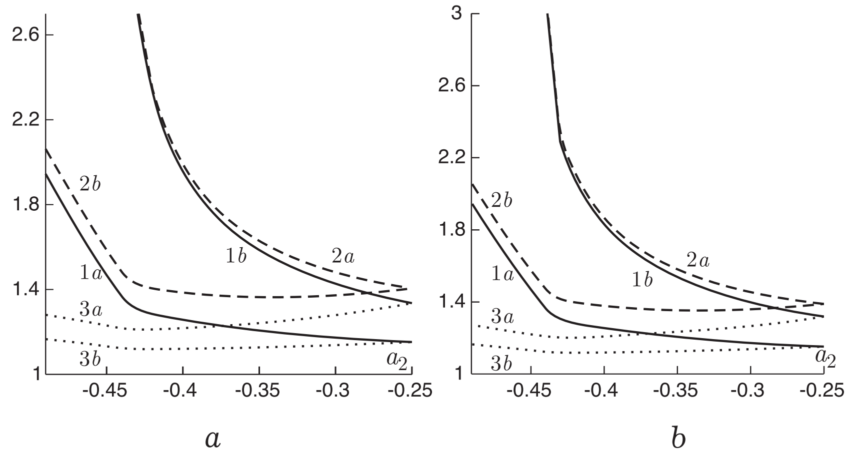

Fig. 9. The dimensionless process zone parameters for $\sigma/p = 6$.

$= 0.5, b_3 = 1$. We use eight equidistant values for a_2, from $a_2 = -0.4$ to $a_2 = -0.26$. The problem should be solved as problem b for $a_2 = -0.4, \ldots, -0.32$ and problem a for $a_2 = -0.3, -0.28, -0.26$. As it can be seen the influence of the position of a_2 on the opening displacement of the third crack is negligible.

Fig. 8 a shows dependencies of $y'_{ck} = y_{ck}/v_{l1}$ (curves ka) and $y'_{dk} = y_{dk}/v_{l1}$ (curves $dk, k = 1, 2, 3$) on a_2. Fig. 8 b shows the similar dependencies for $v'_{ak} = v_{ak}/v_{l1}, v'_{dk} = v_{dk}/v_{l1}$. The bending points on the curves indicate the position when the process zones collate.

The numerical results are given for $\sigma/p = 3$. The dependencies for the values depicted on Fig. 8 and the lower levels of loading are shown in Fig. 9 ($\sigma/p = 6$) and Fig. 10 ($\sigma/p = 9$). As it can be seen the curves become steeper for the lower levels of loading (as the coalescence occurs for the closer cracks).

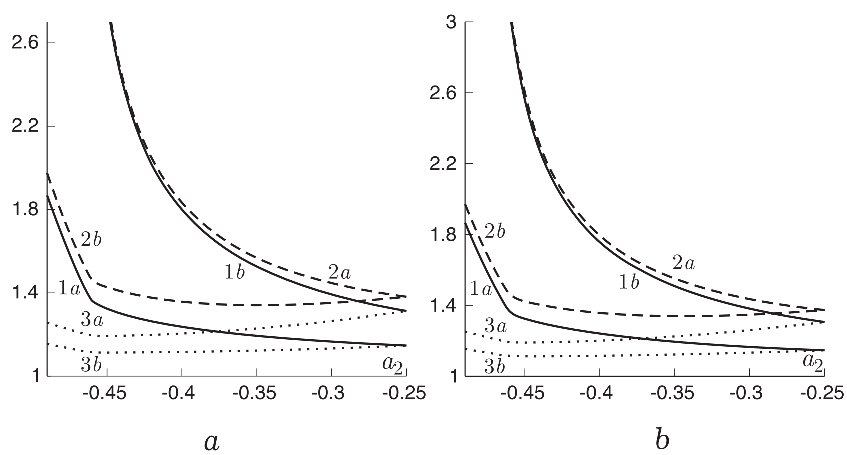

Fig. 10. The dimensionless process zone parameters for $\sigma/p = 9$.

Table 1
Parameters of PMMA creep function.

k	λ_k, days^{-1}	β_k, days^{-1}
1	$3.60{\cdot}10^5$	$3.95{\cdot}10^6$
2	$6.12{\cdot}10^4$	$3.74{\cdot}10^5$
3	$1.52{\cdot}10^4$	$3.02{\cdot}10^4$
4	$3.26{\cdot}10^3$	$2.27{\cdot}10^3$
5	$7.44{\cdot}10^2$	$1.68{\cdot}10^2$
6	$2.06{\cdot}10^2$	13.2
7	69.5	1.44
8	21.7	$1.47{\cdot}10^{-1}$
9	6.11	$2.03{\cdot}10^{-2}$
10	$5.44{\cdot}10^{-1}$	$3.39{\cdot}10^{-3}$
11	$9.51{\cdot}10^{-2}$	$2.98{\cdot}10^{-4}$

7. Mode I crack initiation in a viscoelastic plate

The solution that is given above can be used to determine parameters of initial period of mode I crack development in a viscoelastic body under subcritical loadings.

According to [Christensen, 2003], the stress–strain relation for isotropic non-aging linear viscoelastic material for uniaxial tension and isothermal conditions can be written as a Boltzmann's integral

$$\varepsilon(t) = \int_0^t D(t-\tau)\mathrm{d}\sigma(\tau),$$

where D is a material creep function. This function can be described with a satisfactory precision using the following well-established model (Rabotnov, 1980)

$$D(t) = \frac{1}{E}\left[1 + \sum_k \lambda_k \int_0^t \exp(-\beta_k\tau)\mathrm{d}\tau\right] \tag{51}$$

Parameters of this function for PMMA ($E = 2240$ MPa) are given in Table 1 according to (Kaminsky and Selivanov, 2005).

Equations for the stressed state of a linear viscoelastic plate with cracks can be obtained from the equations of the previous sections using the correspondence principle (Christensen, 2003). So the time dependence of the transverse crack opening displacement can also be written as Boltzmann's integral. The moment when the crack starts to grow can be determined as a moment when the opening displacement at the tip of some crack reaches its critical value. During the studied period of crack initiation a_k and b_k should be constant but the opening displacement should grow.

According to Eqs. (36) and (51), the displacement of crack faces as a function of time can be written as

$$v(x,t) = \frac{2\sigma}{\pi}D(t)v_0(x), \tag{52}$$

where t is time and x is the coordinate and $v_0(x)$ can be determined using Eq. (37).

Whence the equation to determine the duration of the crack initiation period t_0 is

$$\frac{4\sigma}{\pi}D(t_0)v_{0\max} = \delta_c, \quad v_{0\max} = \max_{\substack{x=a_k\\ x=b_k\\ k=1,2,3}} v(x)$$

and the condition of subcritical crack state is

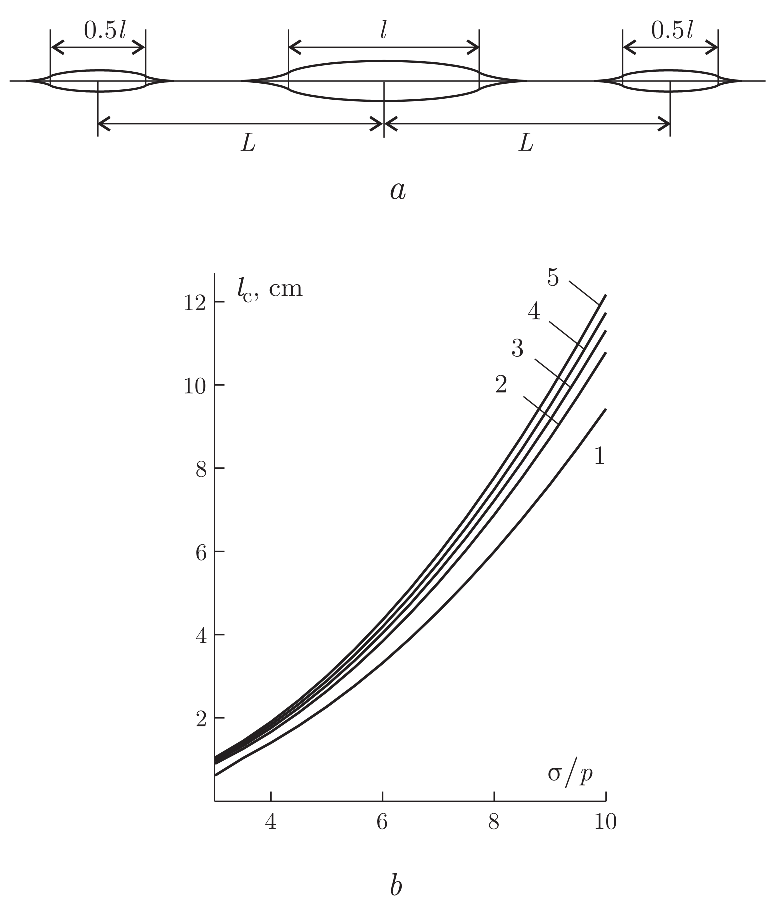

Fig. 11. The critical length vs. dimensionless parameter of external loading.

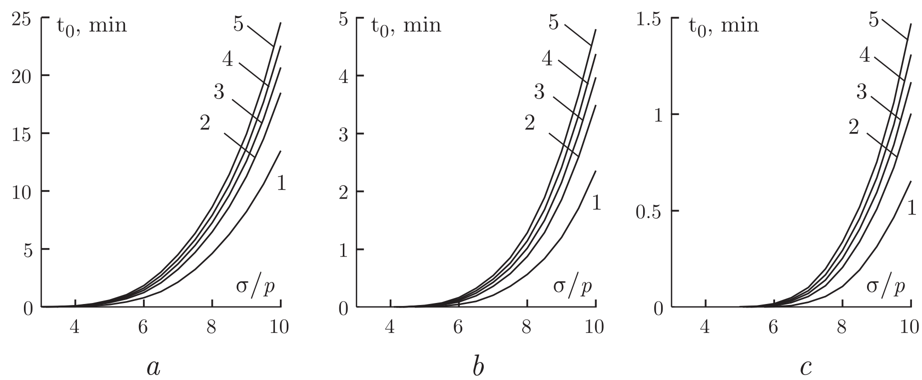

Fig. 12. The initiation period duration t_0 vs. dimensionless parameter of external loading.

$$\frac{4\sigma}{\pi E} v_{0\,\max} < \delta_c.$$

For further numerical computations we use $\delta_c = 30 \cdot 10^{-6}$ m, $\sigma = 35$ MPa (Gain et al., 2011).

Fig. 11 *b* shows a dependence of the critical length of the largest crack in the system of three cracks where one of them is twice as big as two others Fig. 11 *a* on the dimensionless external loading parameter. The curves were plotted for the following values of L: 1 for $L = l$, 2 for $L = 1.25l$, 3 for $L = 1.5l$, 4 for $L = 2l$, 5 for $L = \infty$ (the last case corresponds to the critical length of a single crack). The critical length is determined as

$$l_c = \frac{\pi E \delta_c}{4\sigma v_{0\,\max}},$$

where $v_{0\,\max}$ is calculated for $a_2 = -0.5, b_2 = 0.5, a_{1,3} = \mp L - 0.25, b_{1,3} = \mp L + 0.25$.

Fig. 12 shows a dependence of the initiation period duration on the value of the external loading for the positions of the cracks from Fig. 11 (plot *a* is for $l = 1$ *cm*, plot *b* is for $l = 2$ *cm* and plot *c* is for $l = 3$ *cm*).

As it can be seen from figures, the bigger values of t_0 correspond to the bigger values of L and σ/p.

8. Conclusions

As it can be seen, the approach used in this work allows us to obtain and analyze solutions for the problem of mutual influence of positions of three collinear cracks in isotropic elastic body on their opening displacements. The numerical analysis shows that the opening displacement of the third cracks dependence on the process of coalescence of other two cracks is negligible even for the cases of very close cracks. Thus the conclusion is that the coalescence process can be treated as almost independent from the process of the third crack development.

The scheme that is used to solve the problem herein can also be used to solve the problem for arbitrary number of cracks. It can also be used to determine the service life duration of viscoelastic isotropic and anisotropic bodies with crack sets, expanding the results from Kaminsky (1990), Kaminsky (1998) and Selivanov and Chernoivan (2007).

Appendix A. Comparison of the results of this work with the results from literature

Herein, a comparison is given between the results for the opening displacement in a system of three collinear cracks obtained in this paper and the results for a periodical system of collinear cracks (Parton and Morozov, 1989). The potential functions for the periodical system of cracks is

$$\Phi(z) = \sigma_1 i C[\zeta(z)] + \frac{\sigma}{2} - \frac{p}{4}, \quad \Omega(z) = \Phi(z) + \frac{p}{2}, \quad \sigma_1 = \frac{\sigma}{2\pi},$$

where

$$C(\zeta) = \ln \frac{\sqrt{a^2 - a_1^2}\zeta - l\sqrt{a^2 - z^2}}{\sqrt{a^2 - a_1^2}\zeta + l\sqrt{a^2 - z^2}}; \tag{A.1}$$

$$\zeta(z) = \sin\frac{\pi z}{L}, \quad a = \sin\frac{\pi(l+d)}{L}, \quad a_1 = \sin\frac{\pi l}{L},$$

where L is the distance between two adjacent cracks.

The displacement for the upper face of the crack is

$$\begin{aligned} v(x) &= \Lambda_1 \sigma_1 \int_{-(l+d)}^{x} C_x[\zeta(\tau)]\mathrm{d}\tau \\ &= \Lambda_1 \sigma_1 \frac{L}{\pi} \int_{-a}^{\zeta(x)} \frac{C_x(\xi)\mathrm{d}\xi}{\sqrt{1-\xi^2}}, \quad C_x(x) = \mathrm{Re}\, C^+(x). \end{aligned} \tag{A.2}$$

To calculate this displacement for $-(l+d) \leqslant x \leqslant 0$ we present $C_x(x)$ as

$$C_x(x) = \tilde{C}_x(x) - \ln|a_1 + x|,$$

$$\tilde{C}_x(x) = \ln\left|(a_1 - x)\frac{\sqrt{a^2 - a_1^2}\sqrt{a^2 - x^2} + a^2 - a_1 x}{\sqrt{a^2 - a_1^2}\sqrt{a^2 - x^2} + a^2 + a_1 x}\right|$$

$$\begin{aligned} K_x(x) &= \int_{-(l+d)}^{x} C_x(\tau)\mathrm{d}\tau = \int_{-a}^{\zeta(x)} \frac{\tilde{C}_x(\tau)}{\sqrt{1-\tau^2}}\mathrm{d}\tau - I(x), \quad I(x) \\ &= \int_{-a}^{\zeta(x)} \frac{\ln|a_1 + \tau|}{\sqrt{1-\tau^2}}\mathrm{d}\tau. \end{aligned}$$

The principal value of second integral in the above expression can be found as follows.

1. When $|\zeta(x) + a_1| < \varepsilon$

$$I(x) = \int_{-a}^{-a_1-\varepsilon} \frac{\ln|a_1+\tau|}{\sqrt{1-\tau^2}}\mathrm{d}\tau + F[\zeta(x)] - F[-a_1 - \varepsilon]; \tag{A.3}$$

2. When $-a_1 + \varepsilon < \zeta(x) \leqslant 0$

$$\begin{aligned} I(x) = &\int_{-a}^{-a_1-\varepsilon} \frac{\ln|a_1+\tau|}{\sqrt{1-\tau^2}}\mathrm{d}\tau + F[-a_1+\varepsilon] - F[-a_1-\varepsilon] \\ &+ \int_{-a_1+\varepsilon}^{\zeta(x)} \frac{\ln|a_1+\tau|}{\sqrt{1-\tau^2}}\mathrm{d}\tau. \end{aligned} \tag{A.4}$$

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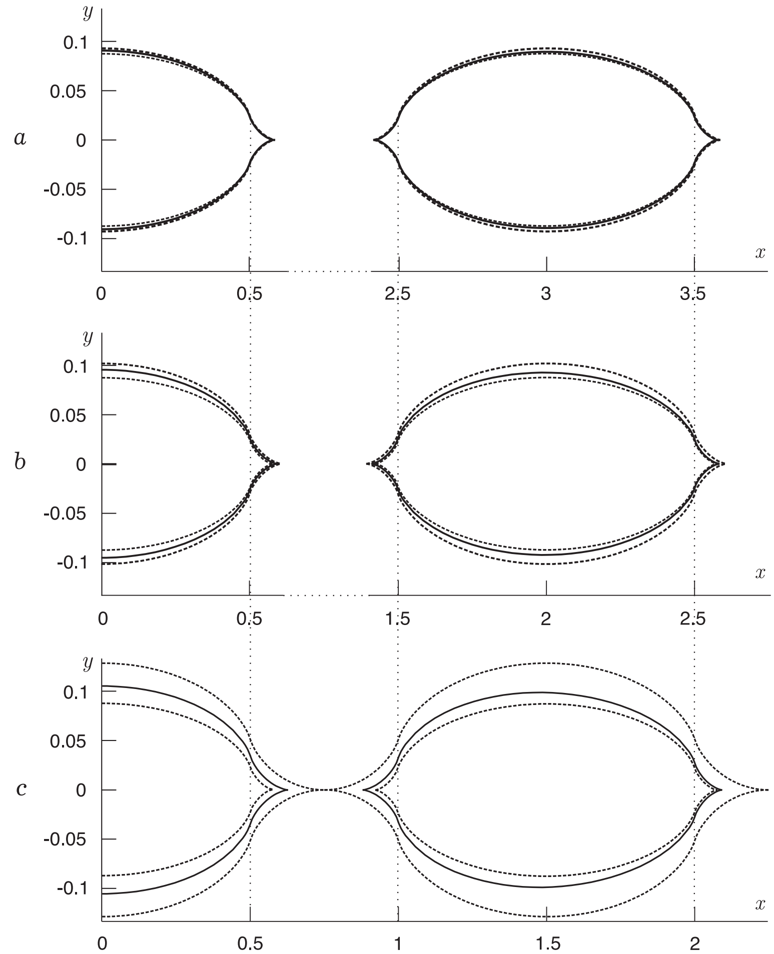

Fig. A.1. A comparison of normalized opening displacement in the problem of three cracks with two limiting cases of its solution known from literature.

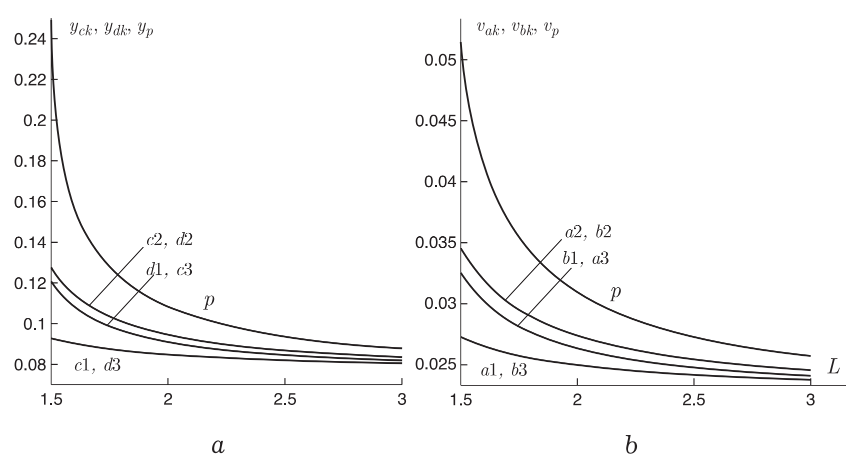

Fig. A.2. A comparison of the process zone lengths in the problem of three cracks with two limiting cases of its solution known from literature.

Functions under integrals in (A.3) and (A.4) are regular so

$$F(x) = \int \ln|a_1 + x| \sum_{k=0}^{\infty} b_k (a_1 + x)^k \mathrm{d}x$$
$$= \sum_{k=0}^{\infty} b_k (a_1 + x)^{k+1} \left[\frac{\ln|a_1 + x|}{k+1} - \frac{1}{(k+1)^2} \right],$$

$$b_k = \frac{y^{(k)}(-a_1)}{k!}, \quad y(x) = \frac{1}{\sqrt{1 - x^2}};$$

$$y^{(2n)}(-a_1) = \frac{\sum_{i=0}^{n} A_{(2n)i} \cdot (a_1^2)^{n-i}}{(1 - a_1^2)^{2n+1/2}}, \quad y^{(2n+1)}(-a_1)$$
$$= \frac{-a_1 \sum_{i=0}^{n} A_{(2n+1)i} \cdot (a_1^2)^{n-i}}{(1 - a_1^2)^{2n+3/2}}.$$

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Below is some rows from the top of matrix A:

1					
2	1				
6	9				
24	72	9			
120	600	225			
720	5400	4050	225		
5040	52920	66150	11025		
40320	564480	1058400	352800	11025	
362880	6531840	17146080	9525600	893025	
3628800	81648000	285768000	238140000	44651250	893025

Fig. A.1 shows the normalized opening displacement $(v(x)/(\Lambda_1\sigma), \sigma/p = 3)$ in the system of three collinear cracks (solid lines) and in the periodical system of cracks (outer dotted lines). For a comparison, the normalized opening displacement for a single crack is shown according to Eq. (50) (inner dotted lines). The difference between the results is bigger for the smaller L.

Using Fig. A.2*a* one can compare lengths of process zones (y_{ck}, y_{dk}) in the system of three equidistant collinear cracks of equal lengths with a length of process zones in the periodical system of cracks (y_p). The corresponding opening displacements at the crack tips $(v_{ak}, v_{bk}$ and $v_p)$ can be compared using Fig. A.2*b*. The difference between the results diminishes for the large values of L. This approves reliability of the results presented in this paper.

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