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Physical invariants for nonlinear orthotropic solids

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ABSTRACT

Seven invariants, with immediate physical interpretation, are proposed for the strain energy function of nonlinear orthotropic elastic solids. Three of the seven invariants are the principal stretch ratios and the other four are squares of the dot product between the two preferred directions and two principal directions of the right stretch tensor. A strain energy function, expressed in terms of these invariants, has a symmetrical property almost similar to that of an isotropic elastic solid written in terms of principal stretches. Ground state and stress–strain relations are given. Using principal axes techniques, the formulation is applied, with mathematical simplicity, to several types of deformations. In simple shear, a necessary and sufficient condition is given for Poynting relation and two novel deformation-dependent *universal* relations are formulated. Using series expansions and the symmetrical property, the proposed general strain energy function is refined to a particular general form. A type of strain energy function, where the ground state constants are written explicitly, is proposed. Some advantages of this type of function are indicated. An experimental advantage is demonstrated by showing a simple triaxial test can vary a single invariant while keeping the remaining invariants fixed.

1. Introduction

The set of invariants (Spencer, 1984)

$$I_1 = tr\boldsymbol{C}, \quad I_2 = \frac{(tr\boldsymbol{C})^2 - tr\boldsymbol{C}^2}{2}, \quad I_3 = det(\boldsymbol{C}), \quad I_4 = \boldsymbol{a} \bullet \boldsymbol{C}\boldsymbol{a}, \qquad (1.1)$$
$$I_5 = \boldsymbol{a} \bullet \boldsymbol{C}^2\boldsymbol{a}, \quad I_6 = \boldsymbol{b} \bullet \boldsymbol{C}\boldsymbol{b}, \quad I_7 = \boldsymbol{b} \bullet \boldsymbol{C}^2\boldsymbol{b},$$

where $\boldsymbol{C}$ is the right Cauchy–Green deformation tensor, *tr* denotes the trace of a second order tensor, is commonly used to describe the strain energy function of orthotropic materials with orthogonal preferred directions $\boldsymbol{a}$ and $\boldsymbol{b}$. The variables $\sqrt{I_3}$, $\sqrt{I_4}$ and $\sqrt{I_6}$ represent the volume change and the stretches of reference-configuration line elements which were in the directions $\boldsymbol{a}$ and $\boldsymbol{b}$, respectively. However, the remaining invariants do not have immediate physical interpretation. Mechanical responses of a strain energy function, where all of its invariants have immediate physical interpretation are generally easier to analyse than those of strain energy functions with invariants that have some or no immediate physical interpretations. In addition to this, a strain energy function with non-immediate-physical-interpretation is, in general, not experimentally friendly. For example, an isochoric uniaxial stretch in one of the preferred direction will perturbed the invariants I_1, I_2, I_4, I_5, I_6 and I_7, which is not ideal in obtaining a specific form of strain energy function if the specific form is determined by doing tests that vary one invariant and hold the rest of the invariants constant. Note that, the six independent components of $\boldsymbol{C}$ have physical interpretation, but formulating a strain energy function, where the preferred directions (which may depend on position) are not parallel to the coordinate axes, is cumbersome. An example where a strain energy function is formulated using a specific coordinate system, can be found in Criscione (2004), where he developed six independent variables (not invariants) using appropriate bases and decomposing the deformation gradient into three parts. The variables are particularly useful for problems involving tubes. Since they are only related (one-to-one) to $\boldsymbol{C}$ and are independent of the preferred directions, they can be used for isotropic and anisotropic materials. The variables decoupled dilatation and distortion, yield mostly orthogonal response terms and allow the balance equations for straight axisymmetric tubes to be simplified. In spite of all the above mentioned advantages, they are not particularly useful in dealing with non-tubular problems. Similar to the work of Shariff and Parker (2000) and Criscione (2004), in order to separate the dilatation and distortion, Rubin and Jabareen (2008) introduced the modified deformation tensor $\boldsymbol{C}^* = \boldsymbol{C}I_3^{-\frac{1}{3}}$ in their strain energy function. This type of function is particularly useful in the development of numerical methods for nearly incompressible or incompressible solids (Shariff and Parker, 2000). However, only one of their invariants has an immediate physical interpretation. The remaining invariants do not have immediate physical interpretation, although they are physically based (in the sense that they measure distortion that cause deviatoric stress). Nevertheless, their invariants allow the modeling of the distortion in a hydrostatic state of stress independently of the form of the strain energy function. It

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is not apparent that their model is experimentally attractive in obtaining a specific form of strain energy function. There are other types of invariants that can be found in the literature, however, they are mainly not experimentally friendly or not all of the invariants have immediate physical interpretation or they are not design to have physical interpretation (see for example Itskov and Aksel, 2004; Ateshian and Costa, 2009).

In the light of the remarks made in the preceding paragraph, based on the work of Shariff (2006, 2008, 2009), we formulate seven simple invariants that have immediate physical interpretation. Three of the invariants are the principal extension ratios λ_i ($i = 1,2,3$) and the other four are $1 \geqslant \zeta_i = (\boldsymbol{a} \bullet \boldsymbol{e}_i)^2 \geqslant 0$ and $1 \geqslant \xi_i = (\boldsymbol{b} \bullet \boldsymbol{e}_i)^2 \geqslant 0$ ($i = 1,2$), where $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$ are any two principal directions of the right stretch tensor $\boldsymbol{U}$. The physical meaning of λ_i is obvious and it is clear that ζ_i and ξ_i are the square of the cosine of the angle between the principal direction $\boldsymbol{e}_i$ and the preferred directions $\boldsymbol{a}$ and $\boldsymbol{b}$, respectively. To fully characterize three-dimensional mechanical properties of a material, we need an experimental test that can vary a single invariant while keeping the remaining invariants fixed; in Section 6, we show that this can be done via a triaxial test using the proposed invariants. In Section 2, we show that the proposed strain energy function has a symmetry which facilitates the formulation of a general functional form given in Section 5, and the construction of a specific form via a triaxial test described in Section 6. In Section 5, we also proposed a type of strain energy function, where the ground state constants are expressed explicitly in the function and advantages of such a function are explored using simple shear problems.

Principal axes techniques can be useful in solving some boundary value problems. This is particularly evident in relation to the calculation of *instantaneous* moduli of isotropic elasticity, as pointed out by Hill (1970). Ogden (1972) stated that "Principal techniques obviate the need for any special choice of invariants and, moreover, by use of such techniques, the basic elegance and simplicity of isotropic elasticity is underlined". These techniques are extended to orthotropic elasticity and their elegance and simplicity are expressed in Section 4. In Section 4.3, where simple shear is discussed, a necessary and sufficient condition is given for Poynting relation and two novel deformation-dependent *universal* relations are formulated.

To the author's present knowledge, a strain energy function where *all* of its invariants have immediate physical interpretation does not exist in the literature.

2. Physical invariants and strain energy function

We first recall some essential kinematics of finite deformation of an orthotropic elastic solid. Consider a body occupying the region B_0 in some reference configuration. Let $\boldsymbol{F}$ be the deformation tensor and $\boldsymbol{X}$ a position vector of a point in B_0. Under this deformation the point moves to a new position $\boldsymbol{x}(\boldsymbol{X}) \in B$, where B is the current configuration of the deformed body. The principal stretch λ_i ($i = 1,2,3$) is given by

$$\lambda_i = \boldsymbol{e}_i \bullet \boldsymbol{U}\boldsymbol{e}_i, \tag{2.1}$$

where $\boldsymbol{U}^2 = \boldsymbol{F}^T\boldsymbol{F}$. In this communication all subscripts i and j take the values 1, 2 and 3, unless stated otherwise.

In this paper we only consider an orthotropic material with preferred orthogonal directions $\boldsymbol{a}$ and $\boldsymbol{b}$. Following the work of Spencer (1984), the mechanical behavior of an orthotropic solid can be characterized by a strain energy function

$$W_e = \widehat{W}(\boldsymbol{U}, \boldsymbol{A}, \boldsymbol{B}), \tag{2.2}$$

where the tensor $\boldsymbol{A} = \boldsymbol{a} \otimes \boldsymbol{a}$ ($\otimes$ denotes the dyadic product) and the tensor $\boldsymbol{B} = \boldsymbol{b} \otimes \boldsymbol{b}$.

Since

$$\boldsymbol{U} = \lambda_1\boldsymbol{E}_1 + \lambda_2\boldsymbol{E}_2 + \lambda_3\boldsymbol{E}_3, \tag{2.3}$$

where $\boldsymbol{E}_i = \boldsymbol{e}_i \otimes \boldsymbol{e}_i$, we can express

$$\widehat{W}(\boldsymbol{U}, \boldsymbol{A}, \boldsymbol{B}) = \overline{W}(\lambda_1, \lambda_2, \lambda_3, \boldsymbol{E}_1, \boldsymbol{E}_2, \boldsymbol{E}_3, \boldsymbol{A}, \boldsymbol{B}). \tag{2.4}$$

$\overline{W}$ is an isotropic invariant of $\boldsymbol{E}_1$, $\boldsymbol{E}_2$, $\boldsymbol{E}_3$, $\boldsymbol{A}$ and $\boldsymbol{B}$ i.e.,

$$\begin{aligned}\overline{W}(\lambda_1, \lambda_2, \lambda_3, \boldsymbol{E}_1, \boldsymbol{E}_2, \boldsymbol{E}_3, \boldsymbol{A}, \boldsymbol{B}) = \overline{W}(\lambda_1, \lambda_2, \lambda_3, &\boldsymbol{Q}\boldsymbol{E}_1\boldsymbol{Q}^T, \boldsymbol{Q}\boldsymbol{E}_2\boldsymbol{Q}^T, \\ &\boldsymbol{Q}\boldsymbol{E}_3\boldsymbol{Q}^T, \boldsymbol{Q}\boldsymbol{A}\boldsymbol{Q}^T, \boldsymbol{Q}\boldsymbol{B}\boldsymbol{Q}^T)\end{aligned} \tag{2.5}$$

for all proper orthogonal tensors $\boldsymbol{Q}$. Taking note that $tr\boldsymbol{E}_i = tr\boldsymbol{A} = tr\boldsymbol{B} = 1$, $\boldsymbol{E}_i = \boldsymbol{E}_i^2 = \boldsymbol{E}_i^3 = \cdots$, $\boldsymbol{A} = \boldsymbol{A}^2 = \boldsymbol{A}^3 = \cdots$, $\boldsymbol{B} = \boldsymbol{B}^2 = \boldsymbol{B}^3 = \cdots$ and $\boldsymbol{E}_i\boldsymbol{E}_j = \boldsymbol{0}$, $i \neq j$, and using the results of Spencer (1971) for five matrices, it follows that W_e can be expressed as

$$W_e = W_f(\lambda_1, \lambda_2, \lambda_3, \zeta_1, \zeta_2, \zeta_3, \xi_1, \xi_2, \xi_3), \tag{2.6}$$

where the invariants $\zeta_i = tr(\boldsymbol{E}_i\boldsymbol{A})$ and $\xi_i = tr(\boldsymbol{E}_i\boldsymbol{B})$. We call ζ_i and ξ_i "invariants" because they are invariants of the tensors involving $\boldsymbol{E}_i$, $\boldsymbol{A}$ and $\boldsymbol{B}$, although some of them do not have unique values if two or three eigenvalues of $\boldsymbol{U}$ have the same value. However,

$$\zeta_3 = 1 - \zeta_1 - \zeta_2 \quad \text{and} \quad \xi_3 = 1 - \xi_1 - \xi_2. \tag{2.7}$$

Hence, we can omit ζ_3 and ξ_3 in the arguments given in (2.6) and we have,

$$\begin{aligned}W_e &= \widetilde{W}(\lambda_1, \lambda_2, \lambda_3, \zeta_1, \zeta_2, \xi_1, \xi_2) \\ &= W_f(\lambda_1, \lambda_2, \lambda_3, \zeta_1, \zeta_2, 1 - \zeta_1 - \zeta_2, \xi_1, \xi_2, 1 - \xi_1 - \xi_2).\end{aligned} \tag{2.8}$$

The invariant set $\{\lambda_1, \lambda_2, \lambda_3, \zeta_1, \zeta_2, \xi_1, \xi_2\}$ is a minimal integrity basis with a syzygy (Spencer, 1971) (see Appendix A).

The function W_f enjoys the symmetrical property

$$\begin{aligned}&W_f(\lambda_1, \lambda_2, \lambda_3, \zeta_1, \zeta_2, \zeta_3, \xi_1, \xi_2, \xi_3) \\ &= W_f(\lambda_1, \lambda_3, \lambda_2, \zeta_1, \zeta_3, \zeta_2, \xi_1, \xi_3, \xi_2) \\ &= W_f(\lambda_3, \lambda_1, \lambda_2, \zeta_3, \zeta_1, \zeta_2, \xi_3, \xi_1, \xi_2) = \text{etc.}\end{aligned} \tag{2.9}$$

To prove the above symmetry we consider an arbitrary proper orthogonal tensor $\boldsymbol{Q}$ written in the form $\boldsymbol{Q} = \widehat{\boldsymbol{Q}}\boldsymbol{Q}_0$, where $\boldsymbol{Q}_0$ is a proper orthogonal rotation tensor (rotation of $\frac{\pi}{2}$ about $\boldsymbol{e}_3$) having the properties $\boldsymbol{Q}_0\boldsymbol{e}_1 = \boldsymbol{e}_2$, $\boldsymbol{Q}_0\boldsymbol{e}_2 = -\boldsymbol{e}_1$, $\boldsymbol{Q}_0\boldsymbol{e}_3 = \boldsymbol{e}_3$ and $\widehat{\boldsymbol{Q}}$ is an arbitrary proper orthogonal tensor. The function $\widehat{W}$ have the property

$$\begin{aligned}\widehat{W}(\boldsymbol{U}, \boldsymbol{A}, \boldsymbol{B}) &= \widehat{W}\left(\boldsymbol{Q}\boldsymbol{U}\boldsymbol{Q}^T, \boldsymbol{Q}\boldsymbol{A}\boldsymbol{Q}^T, \boldsymbol{Q}\boldsymbol{B}\boldsymbol{Q}^T\right) = \widehat{W}\Big(\lambda_2\widehat{\boldsymbol{Q}}\boldsymbol{E}_1\widehat{\boldsymbol{Q}}^T \\ &\quad + \lambda_1\widehat{\boldsymbol{Q}}\boldsymbol{E}_2\widehat{\boldsymbol{Q}}^T + \lambda_3\widehat{\boldsymbol{Q}}\boldsymbol{E}_3\widehat{\boldsymbol{Q}}^T, \widehat{\boldsymbol{Q}}\boldsymbol{Q}_0\boldsymbol{A}\boldsymbol{Q}_0^T\widehat{\boldsymbol{Q}}^T, \widehat{\boldsymbol{Q}}\boldsymbol{Q}_0\boldsymbol{B}\boldsymbol{Q}_0^T\widehat{\boldsymbol{Q}}^T\Big).\end{aligned} \tag{2.10}$$

Since the above equation is true for all proper orthogonal $\widehat{\boldsymbol{Q}}$, and in view of

$$tr(\boldsymbol{E}_1\boldsymbol{Q}_0\boldsymbol{A}\boldsymbol{Q}_0^T) = \zeta_2, \quad tr(\boldsymbol{E}_2\boldsymbol{Q}_0\boldsymbol{A}\boldsymbol{Q}_0^T) = \zeta_1, \quad tr(\boldsymbol{E}_3\boldsymbol{Q}_0\boldsymbol{A}\boldsymbol{Q}_0^T) = \zeta_3,$$

$$tr(\boldsymbol{E}_1\boldsymbol{Q}_0\boldsymbol{B}\boldsymbol{Q}_0^T) = \xi_2, \quad tr(\boldsymbol{E}_2\boldsymbol{Q}_0\boldsymbol{B}\boldsymbol{Q}_0^T) = \xi_1 \quad \text{and} \quad tr(\boldsymbol{E}_3\boldsymbol{Q}_0\boldsymbol{B}\boldsymbol{Q}_0^T) = \xi_3,$$

we have,

$$\begin{aligned}W_e &= W_f(\lambda_1, \lambda_2, \lambda_3, \zeta_1, \zeta_2, \zeta_3, \xi_1, \xi_2, \xi_3) \\ &= W_f(\lambda_2, \lambda_1, \lambda_3, \zeta_2, \zeta_1, \zeta_3, \xi_2, \xi_1, \xi_3).\end{aligned} \tag{2.11}$$

The remainder of Eq. (2.9) follows in a similar fashion.

If the two families of $\boldsymbol{a}$ and $\boldsymbol{b}$ fibres are *mechanically equivalent*, then $\widetilde{W}$ must be symmetric with respect to interchanges of $\boldsymbol{a}$ and $\boldsymbol{b}$. Hence we have the symmetry

$$\widetilde{W}(\lambda_1, \lambda_2, \lambda_3, \xi_1, \xi_2, \zeta_1, \zeta_2) = \widetilde{W}(\lambda_1, \lambda_2, \lambda_3, \zeta_1, \zeta_2, \xi_1, \xi_2). \tag{2.12}$$

The commonly used invariants mentioned in Section 1 can be written explicitly in terms of the physical variables, i.e.,

$$I_1 = \lambda_1^2+\lambda_2^2+\lambda_3^2, \quad I_2 = \lambda_1^2\lambda_2^2+\lambda_1^2\lambda_3^2+\lambda_2^2\lambda_3^2, \quad I_3=(\lambda_1\lambda_2\lambda_3)^2,$$
$$I_4 = \lambda_1^2\zeta_1+\lambda_2^2\zeta_2+\lambda_3^2\zeta_3, \quad I_5 = \lambda_1^4\zeta_1+\lambda_2^4\zeta_2+\lambda_3^4\zeta_3,$$
$$I_6 = \lambda_1^2\xi_1+\lambda_2^2\xi_2+\lambda_3^2\xi_3, \quad I_7 = \lambda_1^4\xi_1+\lambda_2^4\xi_2+\lambda_3^4\xi_3. \tag{2.13}$$

Note that, for a particular value of $\boldsymbol{C}$, where two or more of the principal stretches have the same value, some of the tensors $\boldsymbol{E}_1$, $\boldsymbol{E}_2$ and $\boldsymbol{E}_3$ are not unique; however, it can be easily shown via Eq. (2.13) that the classical invariants have unique values for the corresponding non-unique values of ζ_1, ζ_2, ξ_1 and ξ_2. In view of this non-unique property, care must be taken in formulating the strain energy function (an example is given in Section 5.1). However, if a strain energy function is written in terms of combinations of variables, such as that given in (2.13), then automatically, it has a unique value for a particular value of $\boldsymbol{C}$. The inverse of Eq. (2.13) is given in the Appendix B.

For an incompressible material, $\lambda_1\lambda_2\lambda_3 = 1$, the number of variables is reduce to 6 and we can express

$$W_e = W(\lambda_1,\lambda_2,\zeta_1,\zeta_2,\xi_1,\xi_2) = \widetilde{W}\left(\lambda_1,\lambda_2,\frac{1}{\lambda_1\lambda_2},\zeta_1,\zeta_2,\xi_1,\xi_2\right). \tag{2.14}$$

2.1. Ground state conditions

In the reference state $\boldsymbol{U} = \boldsymbol{I}$, $\lambda_1 = \lambda_2 = \lambda_3 = 1$, any orthonormal set of vectors can represent the principal directions of $\boldsymbol{U}$. For simplicity, we let $\boldsymbol{a} = \boldsymbol{e}_3$ and $\boldsymbol{b} = \boldsymbol{e}_2$. Hence, $\zeta_3 = 1$, $\zeta_1 = \zeta_2 = 0$ and $\xi_2 = 1$, $\xi_1 = \xi_3 = 0$ in this state. To be consistent with the classical linear theory of compressible orthotropic elasticity, appropriate for infinitesimal deformations, we must have the non-zero second derivative relations

$$\begin{aligned}
&\frac{\partial^2\widetilde{W}}{\partial\lambda_1^2}(1,1,1,0,0,0,1) = \hat{\lambda}+2\mu,\\
&\frac{\partial^2\widetilde{W}}{\partial\lambda_2^2}(1,1,1,0,0,0,1) = \hat{\lambda}+2\mu+2\alpha_2+4\mu_2+\beta_2,\\
&\frac{\partial^2\widetilde{W}}{\partial\lambda_3^2}(1,1,1,0,0,0,1) = \hat{\lambda}+2\mu+2\alpha_1+4\mu_1+\beta_1,\\
&\frac{\partial^2\widetilde{W}}{\partial\lambda_1\partial\lambda_2}(1,1,1,0,0,0,1) = \hat{\lambda}+\alpha_2,\\
&\frac{\partial^2\widetilde{W}}{\partial\lambda_1\partial\lambda_3}(1,1,1,0,0,0,1) = \hat{\lambda}+\alpha_1,\\
&\frac{\partial^2\widetilde{W}}{\partial\lambda_2\partial\lambda_3}(1,1,1,0,0,0,1) = \hat{\lambda}+\alpha_1+\alpha_2+\beta_3,
\end{aligned} \tag{2.15}$$

where $\hat{\lambda}$, μ, μ_1, μ_2, α_1, α_2, β_1, β_2 and β_3 are ground state elastic constants.

In the case of an incompressible material we must have the relation

$$\begin{aligned}
&\frac{\partial^2 W}{\partial\lambda_1^2}(1,1,0,0,0,1) = 4\mu+4\mu_1+\beta_1,\\
&\frac{\partial^2 W}{\partial\lambda_2^2}(1,1,0,0,0,1) = 4\mu+2\mu_1+4\mu_2+\beta_1+\beta_2-2\beta_3,\\
&\frac{\partial^2 W}{\partial\lambda_1\partial\lambda_2}(1,1,0,0,0,1) = 2\mu+2\mu_1+\beta_1-\beta_3.
\end{aligned} \tag{2.16}$$

3. Stress

The Biot stress $\boldsymbol{T}^{(1)}$ for a compressible material is given by the relation

$$\boldsymbol{T}^{(1)} = \frac{\partial W_e}{\partial \boldsymbol{U}}. \tag{3.1}$$

The proposed alternative formulation requires the symmetric components $\left(\frac{\partial W_e}{\partial \boldsymbol{U}}\right)_{ij}$ of $\frac{\partial W_e}{\partial \boldsymbol{U}}$ relative to the basis $\{\boldsymbol{e}_i\}$. They are:

$$\left(\frac{\partial W_e}{\partial \boldsymbol{U}}\right)_{ii} = \frac{\partial\widetilde{W}}{\partial\lambda_i} \quad (i \quad \text{not summed}) \tag{3.2}$$

and the shear components (Shariff, 2008)

$$\left(\frac{\partial W_e}{\partial \boldsymbol{U}}\right)_{ij} = \frac{1}{\lambda_i-\lambda_j}\left(\left(\frac{\partial\widetilde{W}}{\partial\zeta_i}-\frac{\partial\widetilde{W}}{\partial\zeta_j}\right)\boldsymbol{e}_i\bullet\boldsymbol{A}\boldsymbol{e}_j+\left(\frac{\partial\widetilde{W}}{\partial\xi_i}-\frac{\partial\widetilde{W}}{\partial\xi_j}\right)\boldsymbol{e}_i\bullet\boldsymbol{B}\boldsymbol{e}_j\right)$$
$$i\neq j,\ i,j=1,2, \tag{3.3}$$

$$\left(\frac{\partial W_e}{\partial \boldsymbol{U}}\right)_{\alpha 3} = \frac{1}{\lambda_\alpha-\lambda_3}\left(\frac{\partial\widetilde{W}}{\partial\zeta_\alpha}\boldsymbol{e}_\alpha\bullet\boldsymbol{A}\boldsymbol{e}_3+\frac{\partial\widetilde{W}}{\partial\xi_\alpha}\boldsymbol{e}_\alpha\bullet\boldsymbol{B}\boldsymbol{e}_3\right), \quad \alpha=1,2. \tag{3.4}$$

It is assumed that $\widetilde{W}$ has sufficient regularity to ensure that, as λ_i and λ_α approach λ_j and λ_3, respectively, Eqs. (3.3) and (3.4) have limits. Relations (3.2) and (3.3) can be used for transversely isotropic materials by letting $\boldsymbol{B} = \boldsymbol{I}$ and for isotropic materials by letting $\boldsymbol{A} = \boldsymbol{B} = \boldsymbol{I}$. The Cauchy stress is given by the relation

$$J\boldsymbol{\sigma} = \boldsymbol{F}\boldsymbol{T}^{(2)}\boldsymbol{F}^T, \tag{3.5}$$

where $\boldsymbol{T}^{(2)} = 2\frac{\partial\widetilde{W}}{\partial\boldsymbol{C}}$ is the second Piola–Kirchhoff stress tensor. Since

$$\boldsymbol{T}^{(1)} = \frac{1}{2}(\boldsymbol{T}^{(2)}\boldsymbol{U}+\boldsymbol{U}\boldsymbol{T}^{(2)}), \tag{3.6}$$

we cannot explicitly express $\boldsymbol{\sigma}$ in terms of $\boldsymbol{T}^{(1)}$. Hence we require the symmetric components $\left(\frac{\partial W_e}{\partial \boldsymbol{C}}\right)_{ij}$ of $\frac{\partial W_e}{\partial \boldsymbol{C}}$ relative to the basis $\{\boldsymbol{e}_i\}$. These components are obtained in a similar fashion to the components of $\frac{\partial W_e}{\partial \boldsymbol{U}}$, i.e.,

$$\left(\frac{\partial W_e}{\partial \boldsymbol{C}}\right)_{ii} = \frac{1}{2\lambda_i}\frac{\partial\widetilde{W}}{\partial\lambda_i} \quad (i \quad \text{not summed}) \tag{3.7}$$

and the shear components

$$\left(\frac{\partial W_e}{\partial \boldsymbol{C}}\right)_{ij} = \frac{1}{\lambda_i^2-\lambda_j^2}\left(\left(\frac{\partial\widetilde{W}}{\partial\zeta_i}-\frac{\partial\widetilde{W}}{\partial\zeta_j}\right)\boldsymbol{e}_i\bullet\boldsymbol{A}\boldsymbol{e}_j+\left(\frac{\partial\widetilde{W}}{\partial\xi_i}-\frac{\partial\widetilde{W}}{\partial\xi_j}\right)\boldsymbol{e}_i\bullet\boldsymbol{B}\boldsymbol{e}_j\right)$$
$$i\neq j,\ i,j=1,2, \tag{3.8}$$

$$\left(\frac{\partial W_e}{\partial \boldsymbol{C}}\right)_{\alpha 3} = \frac{1}{\lambda_\alpha^2-\lambda_3^2}\left(\frac{\partial\widetilde{W}}{\partial\zeta_\alpha}\boldsymbol{e}_\alpha\bullet\boldsymbol{A}\boldsymbol{e}_3+\frac{\partial\widetilde{W}}{\partial\xi_\alpha}\boldsymbol{e}_\alpha\bullet\boldsymbol{B}\boldsymbol{e}_3\right), \quad \alpha=1,2. \tag{3.9}$$

It is explicit in Eqs. (3.3) and (3.4) that the Biot (or the second Piola–Kirchhoff) stress is coaxial with $\boldsymbol{U}$ when the preferred directions $\boldsymbol{a}$ and $\boldsymbol{b}$ are parallel to any two of the principal directions. This explicitness may not be as transparent if the strain energy function is expressed in terms of the classical invariants (2.13) (or possibly most types of invariants found in the literature).

In the case of an incompressible material the Biot, second Piola–Kirchhoff and Cauchy stresses are given by

$$\boldsymbol{T}^{(1)} = \frac{\partial W_e}{\partial \boldsymbol{U}} - p\boldsymbol{U}^{-1}, \quad \boldsymbol{T}^{(2)} = 2\frac{\partial W_e}{\partial \boldsymbol{C}} - p\boldsymbol{C}^{-1},$$
$$\boldsymbol{\sigma} = 2\boldsymbol{F}\frac{\partial W_e}{\partial \boldsymbol{C}}\boldsymbol{F}^T - p\boldsymbol{I}, \tag{3.10}$$

where p is the Lagrange multiplier associated with the incompressible constraint $\lambda_1\lambda_2\lambda_3 = 1$.

4. Applications

In this section we use the principal axes techniques to obtain results for three types of deformation and reveal the mathematical simplicity of the proposed formulation.

4.1. Homogeneous triaxial deformation

We consider a homogeneous deformation so that the deformation tensor $\boldsymbol{F}$ is constant. Specifically, we consider the pure homogeneous deformation defined by

$$x_1 = \lambda_1 X_1, \quad x_2 = \lambda_2 X_2, \quad x_3 = \lambda_3 X_3, \tag{4.1}$$

where x_i and X_i are the Cartesian components of $\boldsymbol{x}$ and $\boldsymbol{X}$, respectively. For this deformation $\boldsymbol{F} = \boldsymbol{U}$ and the principal axes of the deformation coincide with the Cartesian coordinate directions and are fixed as the values of the stretches change. Thus, $\boldsymbol{F} \equiv diag(\lambda_1, \lambda_2, \lambda_3)$.

The Cartesian components of the Cauchy stress have the expression

$$J\sigma_{11} = \lambda_1 \frac{\partial \widetilde{W}}{\partial \lambda_1}, \quad J\sigma_{22} = \lambda_2 \frac{\partial \widetilde{W}}{\partial \lambda_2}, \quad J\sigma_{33} = \lambda_3 \frac{\partial \widetilde{W}}{\partial \lambda_3}, \tag{4.2}$$

$$J\sigma_{12} = \frac{2\lambda_1\lambda_2}{\lambda_1^2 - \lambda_2^2}\left(\left(\frac{\partial \widetilde{W}}{\partial \zeta_1} - \frac{\partial \widetilde{W}}{\partial \zeta_2}\right)\boldsymbol{e}_1 \bullet \boldsymbol{A}\boldsymbol{e}_2 + \left(\frac{\partial \widetilde{W}}{\partial \xi_1} - \frac{\partial \widetilde{W}}{\partial \xi_2}\right)\boldsymbol{e}_1 \bullet \boldsymbol{B}\boldsymbol{e}_2\right),$$
$$J\sigma_{\alpha 3} = \frac{2\lambda_\alpha\lambda_3}{\lambda_2^2 - \lambda_3^2}\left(\frac{\partial \widetilde{W}}{\partial \zeta_\alpha}\boldsymbol{e}_\alpha \bullet \boldsymbol{A}\boldsymbol{e}_3 + \frac{\partial \widetilde{W}}{\partial \xi_\alpha}\boldsymbol{e}_\alpha \bullet \boldsymbol{B}\boldsymbol{e}_3\right), \quad \alpha = 1, 2. \tag{4.3}$$

On specializing a triaxial deformation to a biaxial deformation applied on a thin sheet that lies on the (X_1,X_2)-plane with the Cauchy stress component $\sigma_{33} = 0$, we have,

$$J\sigma_{33} = \lambda_3 \frac{\partial \widetilde{W}}{\partial \lambda_3} = 0, \tag{4.4}$$

where λ_3 is implicitly related to λ_2 and λ_1. In the case of an incompressible material, we have, for biaxial deformation, the stress–strain relations

$$\sigma_{11} = \lambda_1 \frac{\partial W}{\partial \lambda_1}, \quad \sigma_{22} = \lambda_2 \frac{\partial W}{\partial \lambda_2}, \tag{4.5}$$

$$\sigma_{12} = \frac{2\lambda_1\lambda_2}{\lambda_1^2 - \lambda_2^2}\left(\left(\frac{\partial W}{\partial \zeta_1} - \frac{\partial W}{\partial \zeta_2}\right)\boldsymbol{e}_1 \bullet \boldsymbol{A}\boldsymbol{e}_2 + \left(\frac{\partial W}{\partial \xi_1} - \frac{\partial W}{\partial \xi_2}\right)\boldsymbol{e}_1 \bullet \boldsymbol{B}\boldsymbol{e}_2\right), \tag{4.6}$$

$$\sigma_{\alpha 3} = \frac{2\lambda_\alpha\lambda_3}{\lambda_\alpha^2 - \lambda_3^2}\left(\frac{\partial W}{\partial \zeta_\alpha}\boldsymbol{e}_\alpha \bullet \boldsymbol{A}\boldsymbol{e}_3 + \frac{\partial W}{\partial \xi_\alpha}\boldsymbol{e}_\alpha \bullet \boldsymbol{B}\boldsymbol{e}_3\right), \quad \alpha = 1, 2, \tag{4.7}$$

where we have used the relations $\lambda_\alpha \frac{\partial W}{\partial \lambda_\alpha} = \lambda_\alpha \frac{\partial \widetilde{W}}{\partial \lambda_\alpha} - \lambda_3 \frac{\partial \widetilde{W}}{\partial \lambda_3}$ ($\alpha = 1,2$) and $p = \lambda_3 \frac{\partial \widetilde{W}}{\partial \lambda_3}$ to obtain the relations (4.5)–(4.7).

When the preferred directions $\boldsymbol{a}$ and $\boldsymbol{b}$ are taken to be perpendicular to $\boldsymbol{e}_3$, we have,

$$\sigma_{\alpha,3} = 0, \quad \alpha = 1, 2. \tag{4.8}$$

In this case, it is explicit in Eq. (4.6) that σ_{12} vanishes if $\boldsymbol{a}$ or $\boldsymbol{b}$ is along one of the coordinate axes or for an mechanically equivalent material when $\zeta_\alpha = \xi_\alpha$ ($\alpha = 1,2$) and $\boldsymbol{e}_1 \bullet \boldsymbol{A}\boldsymbol{e}_2 = -\boldsymbol{e}_1 \bullet \boldsymbol{B}\boldsymbol{e}_2$. In this case the Biot stress is coaxial with $\boldsymbol{U}$ and $\boldsymbol{\sigma}$ is coaxial with the left stretch tensor $\boldsymbol{V}$.

When the material is inextensible in the preferred directions we have the constraints $\boldsymbol{a} \bullet \boldsymbol{C}\boldsymbol{a} = 1$ and $\boldsymbol{b} \bullet \boldsymbol{C}\boldsymbol{b} = 1$. In this case,

$$\boldsymbol{\sigma} = 2\boldsymbol{F}\frac{\partial W_e}{\partial \boldsymbol{C}}\boldsymbol{F}^T - pI + q\boldsymbol{F}\boldsymbol{a} \otimes \boldsymbol{F}\boldsymbol{a} + r\boldsymbol{F}\boldsymbol{b} \otimes \boldsymbol{F}\boldsymbol{b}, \tag{4.9}$$

where q and r are the Lagrange multipliers associated with the constraints $\boldsymbol{a} \bullet \boldsymbol{C}\boldsymbol{a} = 1$ and $\boldsymbol{b} \bullet \boldsymbol{C}\boldsymbol{b} = 1$, respectively. The Cartesian components of the Cauchy stress are

$$\begin{aligned}\sigma_{11} &= \lambda_1 \frac{\partial W}{\partial \lambda_1} + q\lambda_1^2\zeta_1 + r\lambda_1^2\xi_1,\\ \sigma_{22} &= \lambda_2 \frac{\partial W}{\partial \lambda_2} + q\lambda_2^2\zeta_2 + r\lambda_2^2\xi_2,\\ \sigma_{12} &= \frac{2\lambda_1\lambda_2}{\lambda_1^2 - \lambda_2^2}\left(\frac{\partial W}{\partial \zeta_1}\boldsymbol{e}_1 \bullet \boldsymbol{A}\boldsymbol{e}_2 + \frac{\partial W}{\partial \xi_1}\boldsymbol{e}_1 \bullet \boldsymbol{B}\boldsymbol{e}_2\right)\\ &\quad + q\lambda_1\lambda_2\boldsymbol{e}_1 \bullet \boldsymbol{A}\boldsymbol{e}_2 + r\lambda_1\lambda_2\boldsymbol{e}_1 \bullet \boldsymbol{B}\boldsymbol{e}_2.\end{aligned} \tag{4.10}$$

The rest of the stress components have zero values since $\boldsymbol{a}$ and $\boldsymbol{b}$ are perpendicular to $\boldsymbol{e}_3$.

4.2. Extension and inflation of a thick-walled tube

Here, we examine a non-homogeneous deformation which has several applications. We consider an incompressible thick-walled circular cylindrical tube with initial geometry defined by

$$A \leqslant R \leqslant B, \quad 0 \leqslant \Theta \leqslant 2\pi, \quad 0 \leqslant Z \leqslant L, \tag{4.11}$$

where A, B, L are positive constants and R, Θ, Z are cylindrical polar coordinates. The resulting deformation is described by the equations

$$r^2 - a^2 = \frac{1}{\lambda_z}(R^2 - A^2), \quad \theta = \Theta, \quad z = \lambda_z Z, \tag{4.12}$$

where a is the internal radius of the deformed tube, r, θ and z are cylindrical polar coordinates in the deformed configuration, and λ_z (constant) is the axial stretch.

The principal stretches are given by

$$\lambda_1 = \frac{1}{\lambda\lambda_z}, \quad \lambda_2 = \lambda = \frac{r}{R}, \quad \lambda_3 = \lambda_z, \tag{4.13}$$

where we have introduced the notation λ. It can be easily shown that the principal directions are in the directions

$$\boldsymbol{e}_1 = \boldsymbol{E}_R, \quad \boldsymbol{e}_2 = \boldsymbol{E}_\Theta, \quad \boldsymbol{e}_3 = \boldsymbol{E}_Z, \tag{4.14}$$

where $\boldsymbol{E}_R$, $\boldsymbol{E}_\Theta$, $\boldsymbol{E}_Z$ are the base vectors for the R, Θ, Z cylindrical coordinate system.

Consider the case when the preferred directions are in the directions

$$\boldsymbol{a} = cos(\alpha)\boldsymbol{E}_\Theta + sin(\alpha)\boldsymbol{E}_Z \quad \text{and}$$
$$\boldsymbol{b} = -sin(\alpha)\boldsymbol{E}_\Theta + cos(\alpha)\boldsymbol{E}_Z, \tag{4.15}$$

where $0 \leqslant \alpha \leqslant \frac{\pi}{2}$ and $\zeta_1 = 0$. With λ and λ_z as the independent variables the strain energy function $\widetilde{W}$ can be expressed as

$$W_t(\lambda, \lambda_z, \zeta_2, \xi_2) = \widetilde{W}\left(\frac{1}{\lambda\lambda_z}, \lambda, \lambda_z, 0, \zeta_2, 0, \xi_2\right). \tag{4.16}$$

The components of the Cauchy stress in the cylindrical coordinate system are:

$$\sigma_{\theta\theta} - \sigma_{rr} = \lambda\frac{\partial W_t}{\partial \lambda}, \quad \sigma_{zz} - \sigma_{rr} = \lambda_z\frac{\partial W_t}{\partial \lambda_z}, \tag{4.17}$$

$$\sigma_{\theta z} = \frac{2cs\lambda_z\lambda}{\lambda^2 - \lambda_z^2}\left(\frac{\partial W_t}{\partial \zeta_2} - \frac{\partial W_t}{\partial \xi_2}\right), \quad \sigma_{r\theta} = \sigma_{rz} = 0, \tag{4.18}$$

where $c = cos(\alpha)$ and $s = sin(\alpha)$. It is clear that when $\alpha = 0$ or $\alpha = \frac{\pi}{2}$ the shear stress $\sigma_{\theta z}$ is zero.

By considering the symmetry of the problem, the equation of equilibrium with negligible body forces reduces to

$$\frac{d\sigma_{rr}}{dr} + \frac{1}{r}(\sigma_{rr} - \sigma_{\theta\theta}) = 0. \tag{4.19}$$

The above equation is to be solved in conjunction with the boundary conditions

$$\sigma_{rr} = \begin{cases} -P & \text{on } r = a \\ 0 & \text{on } r = b \end{cases} \tag{4.20}$$

corresponding to the pressure $P \geqslant 0$ on the inside of the tube and zero traction on the outside. b is the external radius of the deformed tube.

Integrating Eq. (4.19), using the first part of Eq. (4.17) and the relation

$$\frac{dr}{r} = \frac{d\lambda}{\lambda(1-\lambda^2\lambda_z)} \tag{4.21}$$

we get the equation for the pressure in terms of λ, i.e.,

$$P = \int_a^b \lambda \frac{\partial W_t}{\partial \lambda}\frac{dr}{r} = \int_{\lambda_b}^{\lambda_a} (\lambda^2\lambda_z - 1)^{-1}\frac{\partial W_t}{\partial \lambda} d\lambda, \tag{4.22}$$

where $\lambda_a = \frac{a}{A}$ and $\lambda_b = \frac{b}{B}$. From the first part of Eq. (4.12) we derive the relations

$$\lambda_a^2\lambda_z - 1 = \frac{R^2}{A^2}(\lambda^2\lambda_z - 1) = \frac{B^2}{A^2}(\lambda_b^2\lambda_z - 1) \tag{4.23}$$

which relates λ_b with λ_a. Eq. (4.22) provides an expression for the pressure P as a function of λ_a when λ_z is fixed. The axial load N needed to hold λ_z fixed can be obtained by the relation

$$N = 2\pi \int_a^b \sigma_{zz} r dr + \pi P a^2, \tag{4.24}$$

where the pressure contributes to the axial load of the deformed tube with closed ends. Using the relation

$$\int_a^b \sigma_{rr} r dr = \frac{Pa^2}{2} - \int_a^b \frac{1}{2}\lambda \frac{\partial W_t}{\partial \lambda} r\, dr \tag{4.25}$$

and Eqs. (4.13) and (4.21), we have,

$$\frac{N}{2\pi} = \int_a^b \left(\lambda_z \frac{\partial W_t}{\partial \lambda_z} - \frac{\lambda}{2}\frac{\partial W_t}{\partial \lambda}\right) r dr + Pa^2. \tag{4.26}$$

If material is inextensible in the preferred direction the components of the Cauchy stress have the relations

$$\begin{aligned}
\sigma_{zz} - \sigma_{rr} &= \lambda_z \frac{\partial W_t}{\partial \lambda_z} + q\lambda_z^2 s^2 + t\lambda_z^2 c^2, \\
\sigma_{\theta\theta} - \sigma_{rr} &= \lambda \frac{\partial W_t}{\partial \lambda} + q\lambda^2 c^2 + t\lambda^2 s^2, \\
\sigma_{z\theta} &= \lambda\lambda_z cs\left(\frac{2}{\lambda^2 - \lambda_z^2}\left(\frac{\partial W_t}{\partial \zeta_2} - \frac{\partial W_t}{\partial \xi_2}\right) + q - t\right).
\end{aligned} \tag{4.27}$$

The pressure P on the inside of the tube required to maintain the deformation is given by

$$P = \int_a^b \left(\lambda \frac{\partial W_t}{\partial \lambda} + q\lambda^2 c^2 + t\lambda^2 s^2\right)\frac{dr}{r}. \tag{4.28}$$

The expression for axial load N is given by

$$\frac{N}{2\pi} = \int_a^b \left(\lambda_z \frac{\partial W_t}{\partial \lambda_z} - \frac{\lambda}{2}\frac{\partial W_t}{\partial \lambda} + q\lambda_z^2 s^2 + t\lambda_z^2 c^2 - \frac{q}{2}\lambda^2 c^2 - \frac{t}{2}\lambda^2 s^2\right) r dr + Pa^2. \tag{4.29}$$

4.3. Simple shear

In Sections 4.1 and 4.2, results for homogeneous and non-homogeneous deformations, where the principal directions are fixed during deformation, are given. In this section we give results for a simple shear deformation where the principal directions of $\boldsymbol{U}$ change continuously during deformation. For simplicity, we only consider incompressible materials.

Let the axes of $\boldsymbol{x}$ and $\boldsymbol{X}$ to coincide and the deformation can be described by the equations

$$x_1 = X_1 + \gamma X_2, \quad x_2 = X_2, \quad x_3 = X_3, \tag{4.30}$$

where *the amount of shear* $\gamma \geqslant 0$. Let θ denote the orientation (in the anticlockwise sense relative to the X_1 axis) of the in plane Lagrangean principal axes. The angle θ is restricted according by the following (Shariff, 2008)

$$\frac{\pi}{4} \leqslant \theta < \frac{\pi}{2}. \tag{4.31}$$

The principal directions have components

$$e_1 = \begin{bmatrix} c \\ s \\ 0 \end{bmatrix}, \quad e_2 = \begin{bmatrix} -s \\ c \\ 0 \end{bmatrix}, \quad e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \tag{4.32}$$

where $c = cos(\theta)$ and $s = sin(\theta)$. It can be easily shown (Shariff, 2008) that the principal stretches take the values

$$\lambda_1 = \frac{\gamma + \sqrt{\gamma^2+4}}{2} \geqslant 1, \quad \lambda_2 = \frac{1}{\lambda_1} = \frac{\sqrt{\gamma^2+4} - \gamma}{2} \leqslant 1, \\ \lambda_3 = 1 \tag{4.33}$$

and

$$c = \frac{1}{\sqrt{1+\lambda_1^2}}, \quad s = \frac{\lambda_1}{\sqrt{1+\lambda_1^2}}, \quad c^2 - s^2 = -\gamma cs. \tag{4.34}$$

Without loss of generality, we consider $\sigma_{33} = 0$, since incompressibility allows the superposition of an arbitrary hydrostatic stress without effecting the deformation.

The Cartesian components of stress take the form

$$\begin{aligned}
\sigma_{11} &= 2[l_1(s^2(1+\gamma^2) + \gamma cs) + l_2(c^2(1+\gamma^2) - \gamma cs) - 2l_4 cs - l_3] \\
\sigma_{12} &= 2[l_1(\gamma s^2 + cs) + l_2(\gamma c^2 - cs) + l_4\gamma cs] \\
\sigma_{22} &= 2(l_1 s^2 + l_2 c^2 + 2l_4 cs - l_3), \\
\sigma_{13} &= 2(l_5(c + \gamma s) + l_6(-s + \gamma c)), \\
\sigma_{23} &= 2(l_5 s + l_6 c),
\end{aligned} \tag{4.35}$$

where

$$\begin{aligned}
l_1 &= \frac{1}{2\lambda_1}\frac{\partial \widetilde{W}}{\partial \lambda_1}, \quad l_2 = \frac{1}{2\lambda_2}\frac{\partial \widetilde{W}}{\partial \lambda_2}, \quad l_3 = \frac{1}{2\lambda_3}\frac{\partial \widetilde{W}}{\partial \lambda_3}, \\
l_4 &= \frac{1}{\lambda_1^2 - \lambda_2^2}\left(\left(\frac{\partial \widetilde{W}}{\partial \zeta_1} - \frac{\partial \widetilde{W}}{\partial \zeta_2}\right)\boldsymbol{e}_1 \bullet \boldsymbol{A}\boldsymbol{e}_2 + \left(\frac{\partial \widetilde{W}}{\partial \xi_1} - \frac{\partial \widetilde{W}}{\partial \xi_2}\right)\boldsymbol{e}_1 \bullet \boldsymbol{B}\boldsymbol{e}_2\right), \\
l_5 &= \frac{1}{\lambda_1^2 - \lambda_3^2}\left(\frac{\partial \widetilde{W}}{\partial \zeta_1}\boldsymbol{e}_1 \bullet \boldsymbol{A}\boldsymbol{e}_3 + \frac{\partial \widetilde{W}}{\partial \xi_1}\boldsymbol{e}_1 \bullet \boldsymbol{B}\boldsymbol{e}_3\right), \\
l_6 &= \frac{1}{\lambda_2^2 - \lambda_3^2}\left(\frac{\partial \widetilde{W}}{\partial \zeta_2}\boldsymbol{e}_2 \bullet \boldsymbol{A}\boldsymbol{e}_3 + \frac{\partial \widetilde{W}}{\partial \xi_2}\boldsymbol{e}_2 \bullet \boldsymbol{B}\boldsymbol{e}_3\right).
\end{aligned} \tag{4.36}$$

In general, the Poynting relation $\sigma_{11} - \sigma_{22} = \gamma\sigma_{12}$ (generally associated with isotropic theory) does not hold. Poynting relation is a relation between stress components and the deformation which is independent of the choice of (isotropic) constitutive equation. It is interesting to see if this *universal* relation holds for orthotropic materials under certain conditions. From (4.35)

$$\sigma_{11} - \sigma_{22} = \gamma\sigma_{12} - 2l_4 cs(4+\gamma^2), \tag{4.37}$$

Hence, from (4.37), we see that Poynting relation holds if and only if $l_4 = 0$; no conditions are required for l_5 or l_6. An example of a case when $l_4 = 0$ for an *arbitrary* strain energy function is, when the one of the preferred directions $\boldsymbol{a}$ or $\boldsymbol{b}$ is parallel to $\boldsymbol{e}_1$ or $\boldsymbol{e}_2$. For example, if the components of $\boldsymbol{a}$ and $\boldsymbol{b}$ are

$$\begin{bmatrix} \frac{1}{2} \\ \frac{\sqrt{3}}{2} \\ 0 \end{bmatrix} \tag{4.38}$$

and

$$\begin{bmatrix} -\frac{\sqrt{3}}{2} \\ \frac{1}{2} \\ b_3 \end{bmatrix}, \tag{4.39}$$

respectively, where b_3 is any third component of $\boldsymbol{b}$, then Poynting relation holds at the particular strain when $\gamma = \frac{2}{\sqrt{3}}$.

We note that l_5 and l_6 appear in σ_{13} and σ_{23} only. In view of this, we have the relations

$$s\sigma_{13} + c\sigma_{23} = 2(l_5(2cs + \gamma s^2)) \tag{4.40}$$

and

$$c\sigma_{13} - s\sigma_{23} = 2(l_6(\gamma c^2 - 2cs)). \tag{4.41}$$

Since $2cs + \gamma s^2$ and $\gamma c^2 - 2cs = \frac{3\lambda_1^2 - 1}{\lambda_1^2 + 1}$ are both positive and arbitrary,

$$\lambda_1 \sigma_{13} + \sigma_{23} = 0 \tag{4.42}$$

and

$$\sigma_{13} - \lambda_1 \sigma_{23} = 0 \tag{4.43}$$

if and only if $l_5 = 0$ and $l_6 = 0$, respectively. Since l_5 and l_6 can be zero for an arbitrary strain energy, the relations (4.42) and (4.43) are "deformation-dependent" *universal* relations, i.e., they are *independent* of the choice of orthotropic constitutive equation. We use the term "deformation-dependent" since the relations hold at particular strains and at particular directions of $\boldsymbol{a}$ and $\boldsymbol{b}$. An example of $l_5 = 0$ and $l_6 \neq 0$ at a particular strain is when both $\boldsymbol{a}$ and $\boldsymbol{b}$ are perpendicular to $\boldsymbol{e}_1$ but not perpendicular to $\boldsymbol{e}_2$ and $\boldsymbol{e}_3$. An example of $l_6 = 0$ and $l_5 \neq 0$ is when both $\boldsymbol{a}$ and $\boldsymbol{b}$ are perpendicular to $\boldsymbol{e}_2$ but not perpendicular to $\boldsymbol{e}_1$ and $\boldsymbol{e}_3$. Since in the above two examples, either $l_6 \neq 0$ or $l_5 \neq 0$, and since $\widetilde{W}$ is arbitrary, the shear stresses σ_{13} and σ_{23} are generally non-zero. The author believe that the universal relations (4.42) and (4.43) do not exist in the literature and may not be straightforward to derive using the classical invariants.

In the case when the preferred directions are perpendicular to the direction $\boldsymbol{e}_3$, the shear components $\sigma_{13} = \sigma_{23} = 0$ and we have the relations

$$\begin{aligned} &\frac{\partial \lambda_1}{\partial \gamma} = s^2, \quad \frac{\partial \lambda_2}{\partial \gamma} = -c^2, \\ &\frac{\partial \zeta_1}{\partial \gamma} = 2\lambda_1 s c^3 \boldsymbol{e}_1 \bullet \boldsymbol{A}\boldsymbol{e}_2, \quad \frac{\partial \zeta_2}{\partial \gamma} = -\frac{\partial \zeta_1}{\partial \gamma}, \\ &\frac{\partial \xi_1}{\partial \gamma} = 2\lambda_1 s c^3 \boldsymbol{e}_1 \bullet \boldsymbol{B}\boldsymbol{e}_2, \quad \frac{\partial \xi_2}{\partial \gamma} = -\frac{\partial \xi_1}{\partial \gamma}. \end{aligned} \tag{4.44}$$

In general, the Poynting relation does not hold. Since a simple shear deformation depends on γ, the strain energy function can be considered as a function of γ, i.e., $W_e = \widehat{W}(\gamma)$. Using Eq. (4.44), we can easily deduce (after some algebra) that, for $\boldsymbol{a}$ and $\boldsymbol{b}$ perpendicular to $\boldsymbol{e}_3$,

$$\sigma_{12} = \widehat{W}'(\gamma). \tag{4.45}$$

4.3.1. Inextensible fibres

For a material that is inextensible in the preferred directions $\boldsymbol{a}$ and $\boldsymbol{b}$, the last two terms of Eq. (4.9) take the simple forms

$$\begin{aligned} \boldsymbol{Fa} \otimes \boldsymbol{Fa} &= \sum_{i,j} \lambda_i \lambda_j (\boldsymbol{e}_i \bullet \boldsymbol{A}\boldsymbol{e}_j) \hat{\boldsymbol{e}}_i \otimes \hat{\boldsymbol{e}}_j, \quad \boldsymbol{Fb} \otimes \boldsymbol{Fb} \\ &= \sum_{i,j} \lambda_i \lambda_j (\boldsymbol{e}_i \bullet \boldsymbol{B}\boldsymbol{e}_j) \hat{\boldsymbol{e}}_i \otimes \hat{\boldsymbol{e}}_j, \end{aligned} \tag{4.46}$$

where $\hat{\boldsymbol{e}}_i$ are the Eulerian principal directions with components

$$\hat{e}_1 = \begin{bmatrix} s \\ c \\ 0 \end{bmatrix}, \quad \hat{e}_2 = \begin{bmatrix} -c \\ s \\ 0 \end{bmatrix}, \quad \hat{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}. \tag{4.47}$$

5. Functional form of incompressible W_e

In order to refine the functional form of W_e for an incompressible material, we consider the polynomial expansion

$$W_e = \sum_{\alpha,\beta,\gamma} C_{\alpha,\beta,\gamma} (\lambda_1^\alpha \lambda_2^\beta \lambda_3^\gamma - 1), \tag{5.1}$$

where the terms $C_{\alpha,\beta,\gamma}$ are functions of ζ_i and ξ_i, and α, β, and γ are non-negative integers. Note that we do not intend to use the above polynomial form as a constitutive model or as an "Nth" order approximation; we only use it to obtain a reduced functional form of W_e.

For an incompressible material, $\lambda_1 \lambda_2 \lambda_3 = 1$ and we can write (5.1) in the form

$$W_e = \sum_{r,s} C^{(1)}_{r,s} (\lambda_1^r \lambda_2^s - 1) + \sum_{r,s} C^{(2)}_{r,s} (\lambda_1^r \lambda_3^s - 1) + \sum_{r,s} C^{(3)}_{r,s} (\lambda_2^r \lambda_3^s - 1), \tag{5.2}$$

where r and s are non-negative integers, and $C^{(i)}_{r,s}$ are functions of ζ_i and ξ_i. To obtain the symmetry given in Eq. (2.9), certain conditions have to be imposed on the coefficients $C^{(i)}_{r,s}$. Before we do this, we write the expansion given in Eq. (5.2) in the form

$$\begin{aligned} W_e = &\sum_{r=0} C^{(1)}_{r,0} (\lambda_1^r - 1) + \sum_{s=1} C^{(1)}_{0,s} (\lambda_2^s - 1) + \sum_{r=0} C^{(2)}_{r,0} (\lambda_1^r - 1) \\ &+ \sum_{s=1} C^{(2)}_{0,s} (\lambda_3^s - 1) + \sum_{r=0} C^{(3)}_{r,0} (\lambda_2^r - 1) + \sum_{s=1} C^{(3)}_{0,s} (\lambda_3^s - 1) \\ &+ \sum_{r,s\neq 0} C^{(1)}_{r,s} (\lambda_1^r \lambda_2^s - 1) + \sum_{r,s\neq 0} C^{(2)}_{r,s} (\lambda_1^r \lambda_3^s - 1) \\ &+ \sum_{r,s\neq 0} C^{(3)}_{r,s} (\lambda_2^r \lambda_3^s - 1). \end{aligned} \tag{5.3}$$

To satisfy the symmetry given in Eq. (2.9), $C^{(i)}_{r,s}$ must take certain forms (as shown below), and since $\lambda_i^0 = 1$, we can re-write the above equation in the form

$$\begin{aligned} W_e = &\sum_{r=0} D_r(\zeta_1, \xi_1)(\lambda_1^r - 1) + \sum_{r=0} E_r(\zeta_1, \xi_1)(\lambda_1^r - 1) \\ &+ \sum_{r=0} D_r(\zeta_2, \xi_2)(\lambda_2^r - 1) + \sum_{r=0} E_r(\zeta_2, \xi_2)(\lambda_2^r - 1) \\ &+ \sum_{r=0} D_r(\zeta_3, \xi_3)(\lambda_3^r - 1) + \sum_{r=0} E_r(\zeta_3, \xi_3)(\lambda_3^r - 1) \\ &+ \sum_{r,s\neq 0} c_{r,s}(\zeta_1, \zeta_2, \xi_1, \xi_2)(\lambda_1^r \lambda_2^s - 1) \\ &+ \sum_{r,s\neq 0} c_{r,s}(\zeta_1, \zeta_3, \xi_1, \xi_3)(\lambda_1^r \lambda_3^s - 1) \\ &+ \sum_{r,s\neq 0} c_{r,s}(\zeta_2, \zeta_3, \xi_2, \xi_3)(\lambda_2^r \lambda_3^s - 1), \end{aligned} \tag{5.4}$$

where $c_{r,s}(x,y,z,t) = c_{r,s}(y,x,t,z)$ and $c_{r,s} = c_{s,r}$. From the above equation and in view of Weierstrass approximation theorem, we can write the strain energy function in the form

$$W_e = \sum_{i=1}^{3} \hat{f}(\lambda_i, \zeta_i, \xi_i) + \hat{g}(\lambda_1, \lambda_2, \zeta_1, \zeta_2, \xi_1, \xi_2) + \hat{g}(\lambda_1, \lambda_3, \zeta_1, \zeta_3, \xi_1, \xi_3) + \hat{g}(\lambda_2, \lambda_3, \zeta_2, \zeta_3, \xi_2, \xi_3), \quad (5.5)$$

where

$$\hat{f}(\lambda_i, \zeta_i, \xi_i) = \sum_{r=0} (D_r(\zeta_i, \xi_i) + E_r(\zeta_i, \xi_i))(\lambda_i^r - 1)$$

and

$$\hat{g}(\lambda_i, \lambda_j, \zeta_i, \zeta_j, \xi_i, \xi_j) = \sum_{r,s\neq 0} c_{r,s}(\zeta_i, \zeta_j, \xi_i, \xi_j)(\lambda_i^r \lambda_j^s - 1), \quad i \neq j. \quad (5.6)$$

The function $\hat{g}$ has the symmetry $\hat{g}(\lambda_i, \lambda_j, \zeta_i, \zeta_j, \xi_i, \xi_j) = \hat{g}(\lambda_j, \lambda_i, \zeta_j, \zeta_i, \xi_j, \xi_i),\ i \neq j$.

5.1. Semi-linear form and its extension

A general incompressible nonlinear (finite deformation) orthotropic strain energy function is more difficult to analyse than a (infinitesimal) linear one. For an incompressible material, the linear strain energy function has six ground state constants (see Eq. (2.16)), where their role are generally fully understood. However, more often, previously proposed nonlinear strain energy functions that contain all six invariants have constants (sometimes more than six) that are indirectly related to the ground state constants and generally, their role are not straightforward to analyse. A nonlinear strain energy function where its classical ground state constants are explicitly expressed is attractive in the sense that their role are easier to analyse. Using our proposed invariants its straightforward to extent the linear strain energy to a semi-linear form (for moderate strains) with only ground state constants, i.e., the terms in W_e given by (5.5) have the forms

$$f(\lambda_i, \zeta_i, \xi_i) = (\lambda_i - 1)^2 \left(\mu + 2\mu_1 \zeta_i + 2\mu_2 \xi_i + \frac{\beta_1}{2}\zeta_i^2 + \frac{\beta_2}{2}\xi_i^2 + \beta_3 \zeta_i \xi_i \right)$$

$$\hat{g}(\lambda_i, \lambda_j, \zeta_i, \zeta_j, \xi_i, \xi_j) = (\lambda_i - 1)(\lambda_j - 1)\big(\beta_1 \zeta_i \zeta_j + \beta_2 \xi_i \xi_j + \beta_3(\zeta_i \xi_j + \xi_i \zeta_j)\big), \quad i \neq j. \quad (5.7)$$

For larger strains, we propose an extension of the semi-linear form, where

$$f(\lambda_i, \zeta_i, \xi_i) = r(\lambda_i) \left(\mu + 2\mu_1 \zeta_i + 2\mu_2 \xi_i + \frac{\beta_1}{2}\zeta_i^2 + \frac{\beta_2}{2}\xi_i^2 + \beta_3 \zeta_i \xi_i \right)$$

$$\hat{g}(\lambda_i, \lambda_j, \zeta_i, \zeta_j, \xi_i, \xi_j) = s(\lambda_i)s(\lambda_j)\big(\beta_1 \zeta_i \zeta_j + \beta_2 \xi_i \xi_j + \beta_3(\zeta_i \xi_j + \xi_i \zeta_j)\big) \quad i \neq j, \quad (5.8)$$

where $r = s^2$. It is clear from (5.8) and (5.5) that the strain energy function has a unique value if two or more of the principal stretches have the same value. However, s may have constants that are not related to the ground state constants. We impose the conditions, for $x > 1$, $r'(x) > 0$, $r(x) > r(\frac{1}{x})$, $r'(x) + r'(\frac{1}{x}) > 0$ and for $x < 1$, $r'(x) < 0$ (see also Shariff (2000)). For stress free configurations, we impose $r'(1) = s(1) = r(1) = 0$. These conditions are satisfied if r take the semi-linear form (5.7). Note that, although the semi-linear form is valid for mildly moderate strains, useful information can be extracted from it, and in view of the ground-state-constant similarity between the semi-linear and the extended forms (see Eqs. (5.7) and (5.8)), this information can be used for the extended strain energy function. For example, consider three cases of simple shear deformations, where the directions $\boldsymbol{a}$ and $\boldsymbol{b}$ have the Cartesian components:

Case (i):

$$a = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}. \quad (5.9)$$

Case (ii):

$$a = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad b = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}. \quad (5.10)$$

Case (iii):

$$a = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad b = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}. \quad (5.11)$$

We now partially analyse our extended constitutive equation. Using terminologies of Holzapfel and Ogden (2009), where they model passive myocardium via orthotropic elasticity, $\boldsymbol{a}$ and $\boldsymbol{b}$ are the *fibre* and *sheet* directions, respectively. Their *sheet-normal* direction is perpendicular to both $\boldsymbol{a}$ and $\boldsymbol{b}$. We note that in infinitesimal (or semi-linear) elasticity, when μ_1(or μ_2) = 0, β_1 (or β_2) = 0 and $\beta_3 = 0$, the material becomes transversely isotropic; when all of the ground state constants, except μ, are zero, the material becomes isotropic. The constants μ_1, β_1 and μ_2, β_2 are associated with stiffness of the fibre and sheet, respectively. β_3 is a constant associated with both the sheet and fibre. In view of Section 4.3, with some algebra, we found that the shear stresses σ_{12} of Cases (i) and (ii) are different in general. This difference is verified in the experimental data of Dokos et al. However, if the ground state constants $\mu_2 = \beta_2 = \beta_3 = 0$ (the material is transversely isotropic) then the shear stresses are the same. Note that in the past some authors modeled orthotropic materials by letting $W_e = W_o(I_1, I_2, I_4, I_6)$ (or $W_o(I_1, I_4, I_6)$ Holzapfel and Ogden, 2009). For this type of strain energy, the shear stresses for Cases (i) and (ii) are the same, i.e.,

$$\sigma_{12} = 2\gamma \left(\frac{\partial W_o}{\partial I_1} + \frac{\partial W_o}{\partial I_2} + \frac{\partial W_o}{\partial I_4} \right). \quad (5.12)$$

Hence we cannot capture the difference between the shear stresses of an orthotropic material with that of a transversely isotropic material with strain energy

$$W_m(I_1, I_2, I_4) = W_o(I_1, I_2, I_4, 1). \quad (5.13)$$

Note that $I_6 = 1$ for both Cases (i) and (ii). Even in infinitesimal elasticity the shear stress

$$\sigma_{12} = \gamma(\mu + \mu_1 + \mu_2) \quad (5.14)$$

in Case (i) and

$$\sigma_{12} = \gamma(\mu + \mu_1) \quad (5.15)$$

in Case (ii) are different; they are only the same if and only if $\mu_2 = 0$.

Let σ_{12f} and σ_{12s} be the shear stresses for Cases (i) and (iii), respectively. We have

$$\sigma_{12f} - \sigma_{12s} = (\mu_1 - \mu_2) \left\{ \frac{2cs(\lambda_1^2 r'(\lambda_1) + r'(\lambda_2))}{1 + \lambda_1^2} + \frac{8\gamma(cs)^2}{\lambda_1^2 - \lambda_2^2}(r(\lambda_1) - r(\lambda_2)) \right\} + (\beta_1 - \beta_2) \left\{ \frac{cs(\lambda_1^2 r'(\lambda_1) + r'(\lambda_2))}{2(1 + \lambda_1^2)} + \frac{2\gamma(cs)^2}{\lambda_1^2 - \lambda_2^2}(r(\lambda_1) - r(\lambda_2)) \right\}. \quad (5.16)$$

In view of the properties of r, it is clear from above that, if $\mu_1 > \mu_2$ and $\beta_1 > \beta_2$, (the fibre constants are larger than the sheet constants) the shear response when the fibre direction is extended is stiffer than when the sheet direction is extended; Dokos et al. (2002) data confirmed this behavior. However, in infinitesimal elasticity the shear stresses are the same, with their form given in Eq. (5.14).

In view of the discussion in the preceding paragraph, we indicate that the extended strain energy function (5.8) facilitate analyses of mechanical responses. Since s is single variable function, it is easier to analyse than multivariable functions. However, due to the scope of this paper, we shall not, in this communication, develop a specific form of (5.8) or discuss constitutive inequalities; this will be done in the near future.

6. Experimental advantage

In a triaxial test of an incompressible solid, the principal stretches λ_1 and λ_2 can be varied independently. Three of the invariants ζ_1, ζ_2, β_1 and β_2 can be varied independently by taking different samples, of the same material, with different preferred directions (relative to a principal direction (say)). Hence, it allows us to determine the functional form of W by doing tests that holds four out five invariants constant so that the dependence of W on the remaining invariant can be identified. We note in passing that the invariants I_1, I_2, I_4, I_5, I_6 and I_7 cannot be varied independently in a triaxial test.

In a triaxial deformation, where the deformation can be described by Eq. (4.1), we have,

$$\sigma_{11} - \sigma_{33} = \lambda_1 \frac{\partial W}{\partial \lambda_1}, \quad \sigma_{22} - \sigma_{33} = \lambda_2 \frac{\partial W}{\partial \lambda_2}. \tag{6.1}$$

Note in the case when a preferred direction is not parallel to one of the principal directions, some of the shear stresses have none zero values. It is assumed that the shear stresses can be controlled in the triaxial experiment; the author is not sure if this can be done practically. Care must be taken so that the data in all regions of the $(\lambda_1,\lambda_2,\zeta_1,\zeta_2,\xi_1,\xi_2)$ space are taken. Note that we are only concerned with the subset of the $(\lambda_1,\lambda_2,\zeta_1,\zeta_2,\xi_1,\xi_2)$ space where $0 < \lambda_1$, $0 < \lambda_2$, $0 \leqslant \zeta_1 \leqslant 1$, $0 \leqslant \zeta_2 \leqslant 1$, $\zeta_2 \leqslant 1 - \zeta_1$, $0 \leqslant \xi_1 \leqslant 1$, $0 \leqslant \xi_2 \leqslant 1$ and $\xi_2 \leqslant 1 - \xi_1$.

We shall take advantage of the symmetry given in Eq. (2.9) to obtain the functional form of W. Let $f_a(\lambda_1,\lambda_2,\zeta_1,\zeta_2,\xi_1,\xi_2)$ be the functional form constructed from the $\frac{\sigma_{11}-\sigma_{33}}{\lambda_1}$ data. In view of Eq. (6.1) we have

$$W(\lambda_1,\lambda_2,\zeta_1,\zeta_2,\xi_1,\xi_2) = \int f_a(\lambda_1,\lambda_2,\zeta_1,\zeta_2,\xi_1,\xi_2)d\lambda_1 + f_b(\lambda_2,\zeta_1,\zeta_2,\xi_2,\xi_1). \tag{6.2}$$

We now require the functional form of f_b. Due to the symmetry express in Eq. (2.9), it follows from Eq. (6.2) that

$$\int \frac{\partial f_a}{\partial \lambda_2}(\lambda_1,\lambda_2,\zeta_1,\zeta_2,\xi_1,\xi_2)d\lambda_1 + \frac{\partial f_b}{\partial \lambda_2}(\lambda_2,\zeta_1,\zeta_2,\xi_1,\xi_2) = f_a(\lambda_2,\lambda_1,\zeta_2,\zeta_1,\xi_2,\xi_1). \tag{6.3}$$

Hence we have

$$\begin{aligned} f_b(\lambda_2,\zeta_1,\zeta_2,\xi_1,\xi_2) = &-\int\left(\int \frac{\partial f_a}{\partial \lambda_2}(\lambda_1,\lambda_2,\zeta_1,\zeta_2,\xi_1,\xi_2)d\lambda_1\right)d\lambda_2 \\ &+ \int f_a(\lambda_2,\lambda_1,\zeta_2,\zeta_1,\xi_2,\xi_1)d\lambda_2 \\ &+ f_c(\zeta_1,\zeta_2,\xi_1,\xi_2). \end{aligned} \tag{6.4}$$

Let

$$g_a(\lambda_1,\lambda_2,\zeta_1,\zeta_2,\xi_1,\xi_2) = \int f_a(\lambda_1,\lambda_2,\zeta_1,\zeta_2,\xi_1,\xi_2)d\lambda_1 \tag{6.5}$$

and

$$\begin{aligned} g_b(\lambda_1,\lambda_2,\zeta_1,\zeta_2,\xi_1,\xi_2) = &-\int\left(\int \frac{\partial f_a}{\partial \lambda_2}(\lambda_1,\lambda_2,\zeta_1,\zeta_2,\xi_1,\xi_2)d\lambda_1\right)d\lambda_2 \\ &+ \int f_a(\lambda_2,\lambda_1,\zeta_2,\zeta_1,\xi_2,\xi_1)d\lambda_2. \end{aligned} \tag{6.6}$$

If we assume that $W(1,1,\zeta_1,\zeta_2,\xi_1,\xi_2) = 0$ in the undeformed configuration we have

$$f_c(\zeta_1,\zeta_2,\xi_1,\xi_2) = -g_a(1,1,\zeta_1,\zeta_2,\xi_1,\xi_2) - g_b(1,1,\zeta_1,\zeta_2,\xi_1,\xi_2). \tag{6.7}$$

Hence, the functional form of f_b is obtained and the functional form of W can be obtained from Eq. (6.2).

Appendix A

If, for arbitrary $\boldsymbol{a}$ and $\boldsymbol{b}$, we choose the directions of the Cartesian X_1 and X_2 axes to be parallel to $\boldsymbol{a}$ and $\boldsymbol{b}$, respectively, we then have

$$\begin{aligned} &I_1 = tr\boldsymbol{C}, \quad I_2 = \frac{(tr\boldsymbol{C})^2 - tr\boldsymbol{C}^2}{2}, \\ &I_3 = det(\boldsymbol{C}), \quad I_4 = C_{11}, \quad I_5 = C_{1r}C_{r1}, \\ &I_6 = C_{22}, \quad I_7 = C_{2r}C_{r2}, \end{aligned} \tag{A1}$$

where C_{ij} are the components of $\boldsymbol{C}$ relative to the Cartesian basis. Since $\boldsymbol{C}$ has six independent components, there is a relation among the invariants I_{1-7}. However, this relation is a syzygy since no one invariant can be expressed as a polynomial in the remainder. Hence, the polynomial invariant set $\{I_{1-7}\}$ is a minimal integrity basis (Spencer, 1984) with a syzygy.

In the case of the proposed invariants, consider the (right-handed) orthonormal set of vectors $OT = \{\boldsymbol{a}_1,\boldsymbol{a}_2,\boldsymbol{a}_3\}$, where $\boldsymbol{a}_1 = \boldsymbol{a}$, $\boldsymbol{a}_2 = \boldsymbol{b}$ and $\boldsymbol{a}_3$ a unit vector perpendicular to $\boldsymbol{a}$ and $\boldsymbol{b}$. Then the components of the rotation matrix $\boldsymbol{A}$ from the basis OT to the basis $\{\boldsymbol{e}_1,\boldsymbol{e}_2,\boldsymbol{e}_3\}$ are given by

$$A_{ij} = \boldsymbol{a}_i \bullet \boldsymbol{e}_j, \tag{A2}$$

where $\boldsymbol{e}_j$ are the principal directions of $\boldsymbol{C}$. A_{ij} depends on the three *independent* Euler angles. Since the four invariants $\zeta_1 = (\boldsymbol{a} \bullet \boldsymbol{e}_1)^2$, $\zeta_2 = (\boldsymbol{a} \bullet \boldsymbol{e}_2)^2$, $\xi_1 = (\boldsymbol{b} \bullet \boldsymbol{e}_1)^2$, $\xi_2 = (\boldsymbol{b} \bullet \boldsymbol{e}_2)^2$ depend on three independent Euler angles, there exists a relation between the four invariants. In particular,

$$\boldsymbol{a} \bullet \boldsymbol{b} = \sum_{i=1}^{3}(\boldsymbol{a} \bullet \boldsymbol{e}_i)(\boldsymbol{b} \bullet \boldsymbol{e}_i) = 0, \tag{A3}$$

where

$$\boldsymbol{a} \bullet \boldsymbol{e}_3 = \pm\sqrt{1 - \zeta_1 - \zeta_2}, \quad \boldsymbol{b} \bullet \boldsymbol{e}_3 = \pm\sqrt{1 - \xi_1 - \xi_2}. \tag{A4}$$

It is clear from Eqs. (A3) and (A4) that we cannot express any one of the four invariants as a polynomial in the remaining three invariants. Hence the polynomial invariant set $\{\lambda_1,\lambda_2,\lambda_3,\zeta_1,\zeta_2,\xi_1,\xi_2\}$ is a minimal integrity basis with a syzygy.

Appendix B

Following the work of Shariff (2008), the inverse of Eq. (2.13) is,

$$\lambda_i = \frac{1}{\sqrt{3}}\left(I_1 + 2A\cos\left(\frac{\psi+2\pi i}{3}\right)\right)^{\frac{1}{2}}, \quad i = 1,2,3,$$

$$\zeta_1 = \frac{\lambda_2^2+\lambda_3^2}{\delta_1}(I_4-\lambda_3^2) - \frac{1}{\delta_1}(I_5-\lambda_3^4),$$

$$\zeta_2 = -\frac{\lambda_1^2+\lambda_3^2}{\delta_2}(I_4-\lambda_3^2) + \frac{1}{\delta_2}(I_5-\lambda_3^4), \tag{B1}$$

$$\xi_1 = \frac{\lambda_2^2+\lambda_3^2}{\delta_1}(I_6-\lambda_3^2) - \frac{1}{\delta_1}(I_7-\lambda_3^4),$$

$$\xi_2 = -\frac{\lambda_1^2+\lambda_3^2}{\delta_2}(I_6-\lambda_3^2) + \frac{1}{\delta_2}(I_7-\lambda_3^4),$$

where

$$A = \left(I_1^2 - 3I_2\right)^{\frac{1}{2}},$$

$$\psi = \cos^{-1}\frac{1}{2A^3}(2I_1^3 - 9I_1I_2 + 27I_3), \tag{B2}$$

$$\delta_1 = \left(\lambda_1^2-\lambda_3^2\right)\left(\lambda_2^2+\lambda_3^2\right) - \left(\lambda_1^4-\lambda_3^4\right),$$

$$\delta_2 = -\left(\lambda_2^2-\lambda_3^2\right)\left(\lambda_1^2+\lambda_3^2\right) + \left(\lambda_2^4-\lambda_3^4\right).$$

Eq. (B1) is only valid for $\lambda_1 \neq \lambda_2 \neq \lambda_3$. When two or more principal stretches have the same value the corresponding principal directions have non-unique values. In the case of $\lambda_1 = \lambda_2 \neq \lambda_3$ we have,

$$\zeta_3 = \frac{I_4-\lambda_1^2}{\lambda_3^2-\lambda_1^2} = \frac{I_5-\lambda_1^4}{\lambda_3^4-\lambda_1^4} \text{ and we choose } \zeta_1 = \zeta_2 = \frac{1-\zeta_3}{2}. \tag{B3}$$

$$\xi_3 = \frac{I_6-\lambda_1^2}{\lambda_3^2-\lambda_1^2} = \frac{I_7-\lambda_1^4}{\lambda_3^4-\lambda_1^4} \text{ and we choose } \xi_1 = \xi_2 = \frac{1-\xi_3}{2}. \tag{B4}$$

In the case of $\lambda_1 = \lambda_3 \neq \lambda_2$, we have,

$$\zeta_2 = \frac{I_4-\lambda_3^2}{\lambda_2^2-\lambda_3^2} = \frac{I_5-\lambda_3^4}{\lambda_2^4-\lambda_3^4} \text{ and we choose } \zeta_1 = \frac{1-\zeta_2}{2}. \tag{B5}$$

$$\xi_2 = \frac{I_6-\lambda_3^2}{\lambda_2^2-\lambda_3^2} = \frac{I_7-\lambda_3^4}{\lambda_2^4-\lambda_3^4} \text{ and we choose } \xi_1 = \frac{1-\xi_2}{2}. \tag{B6}$$

In the case of $\lambda_2 = \lambda_3 \neq \lambda_1$, we have

$$\zeta_1 = \frac{I_4-\lambda_3^2}{\lambda_1^2-\lambda_3^2} = \frac{I_5-\lambda_3^4}{\lambda_1^4-\lambda_3^4} \text{ and we choose } \zeta_2 = \frac{1-\zeta_1}{2}. \tag{B7}$$

$$\xi_1 = \frac{I_6-\lambda_3^2}{\lambda_1^2-\lambda_3^2} = \frac{I_7-\lambda_3^4}{\lambda_1^4-\lambda_3^4} \text{ and we choose } \xi_2 = \frac{1-\xi_1}{2}. \tag{B8}$$

In the case when $\lambda_1 = \lambda_2 = \lambda_3$ we choose $\zeta_1 = \zeta_2 = \xi_1 = 0$ and $\xi_2 = 1$.

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