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Magnetoelastic interactions in a soft ferromagnetic body with a nonlinear law of magnetization: Some applications

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ABSTRACT

Linearized equations and boundary conditions of a magnetoelastic ferromagnetic body are obtained with the nonlinear law of magnetization. Magnetoelastic interactions in a multi-domain ferromagnetic materials are considered for magneto soft materials, i.e. the case when the magnetic field intensity vector and magnetization vector are parallel. As a special case, the following two problems are considered: (1) the magnetoelastic stability of a ferromagnetic plate-strip in a homogeneous transverse magnetic field; (2) the stress–strain state of a ferromagnetic plane with a moving crack in a transverse magnetic field. It is shown that the modeling of magnetoelastic equations with a nonlinear law of magnetization provides qualitative and quantitative predictions on physical quantities including critical loads and stresses. In particular, it is shown that the critical magnetic field in plate stability problems found with the nonlinear law of magnetization is in better agreement with the experimental finding than the one found with a linear law. Furthermore, it is also shown that the stress concentration factor around a crack predicted with the nonlinear law of magnetization is more accurate than the one obtained with a linear counterpart. Numerical results are presented for above mentioned two problems and for various forms of nonlinear laws of magnetization.

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1. Introduction

Recent years have witnessed an increased interest in the investigation of problems of magnetoelasticity for ferromagnetic materials. The general theory of magnetoelasticity for ferromagnetic body has been developed by many authors (Akhiezer et al., 1968; Brown, 1966; Dorfmann and Ogden, 2004; Eringen and Maugin, 1990; Landau and Lifshitz, 1995; Maugin, 1988; Moon, 1984; Pao and Yeh, 1973; Steigmann, 2004; Tiersten, 1964). These theories can be applied to investigate the magnon–phonon interaction (magnetoacoustic resonance) effects. This coupling interaction is pronounced when the wave frequency is near or above the magnetic resonance frequency, which is usually higher than 10^9 Hz (for example, magnetoacoustic resonance in a material yttrium–iron–garnet is observed when the frequency of spin waves is around 10^{10} Hz). A theory presented by Pao and Yeh (1973) was applied to investigate the magnetoelastic stability of thin structural elements (Moon, 1984; Maugin, 1988) and investigation of stresses around the cracks (Bagdasarian and Hasanyan, 2000; Hasanyan and Philiposyan, 2001; Shindo, 1977; Shindo et al., 2000). A material is called soft ferromagnetic when the magnetic field intensity vector $\mathbf{H}$ and magnetization vector $\mathbf{M}$ are parallel in the rigid body state, i.e. $\mathbf{M} = \chi(|\mathbf{H}|)\mathbf{H}$ (χ is called the magnetic susceptibility of the material). A soft magnetic material is characterized by small hysteresis losses (narrow hysterisis loop for *H*–*M* curves) and low remnant magnetization. Nickel–iron alloys, which are widely used as core materials for motors, generators, inductors and transformers, are a typical example.

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In a large number of works related to the theory of magnetoelasticity of ferromagnetic materials, it has been assumed that the magnetization of a material depends linearly ($\chi = \text{const.}$) upon the applied magnetic field. This assumption is valid for a ferromagnetic in a domain of a very weak magnetic field, or for most nonferromagnetics in strong magnetic fields. With the application of ferromagnetic structures in a strong magnetic fields (higher than 1 T), such as a structure of the first wall in a fusion reactor, the magnetic field generated in the structures may be close to the region of saturation. For example, from the requirements for the international thermonuclear experimental reactor (ITER), the toroidal magnetic field intensity is $\sim$5 Tesla or more. In such cases, it is imperative to consider nonlinear dependence of magnetization on the magnetic field (i.e. $\chi \neq \text{const.}$). The influence of a nonlinear law of magnetization on stability and vibration, wave propagation, stress–strain state and other processes is critical. The results related to the magnetoelastic stability and vibration of thin-walled bodies, the stress–strain state (SSS) and also the wave propagation of ferromagnetic materials with a linear law of magnetization are of special interest by many authors (Bagdasaryan and Hasanyan, 1995; Hasanyan and Philiposyan, 2001; Lin and Yeh, 2002; Maugin, 1988; Moon and Pao, 1969; Nishida et al., 1984; Sabir and Maugin, 1996; Shindo, 1977; Shindo et al., 2000; Zhao and Lee, 2004). Moon and Pao (1969) were one of the first to investigate theoretically and experimentally the problem of buckling of a ferromagnetic plate in a transversal magnetic field. They found that the theoretical value of the critical magnetic field when the plate losses its stability is almost 1.5 times high as the one which is given by experiments. Many researchers (Hasanyan and Philiposyan, 2001; Van de Ven, 1983) try to explain the source of the differences between the theoretical and experimental results. Note that all these investigations have been carried out using a linear law of magnetization.

In plate stability problem, we show that the theoretical value of critical magnetic field, obtained using a nonlinear law of magnetization is in better agreement with the experimental predictions. Furthermore, we show also that the stress concentration around the crack in a ferromagnetic body strongly depends on the nonlinear law of magnetization.

2. Magnetoelastic equations and boundary conditions

It is assumed that the elastic dielectric medium with an ordered magnetic structure is in an external stationary magnetic field with the magnetic intensity $\mathbf{H}^0$ and the magnetic induction vector $\mathbf{B}^0 = \mu_0 \mathbf{H}^0$, where $\mu_0 = 4\pi \times 10^{-7}$ N/A^2 is the universal magnetic constant. The medium surrounding the body is assumed to be vacuum. Under the influence of the magnetic field $\mathbf{H}^0$ the total force $\mathbf{f}$ and body couple $\mathbf{c}$ (per unit volume) acting on the body are as follows (Brown, 1966; Tiersten, 1964):

$$\mathbf{f} = \mu_0 (\mathbf{M} \cdot \nabla)\mathbf{H}, \quad \mathbf{c} = \mu_0 \mathbf{M} \times \mathbf{H}, \tag{2.1}$$

where $\mathbf{H}$ and $\mathbf{M}$ are magnetic field intensity and magnetization (the magnetic moment of a unit volume) inside the body, ∇ the gradient operator. The magnetic field intensity $\mathbf{H}$ and the magnetization $\mathbf{M}$ inside the magnetized body are related to the external magnetic field $\mathbf{H}^0$ through a set of magnetoelastic field equations, constitutive equations, and boundary conditions. The vectors $\mathbf{H}$ and $\mathbf{M}$ are connected with the magnetic induction vector B by the relation $\mathbf{B} = \mu_0(\mathbf{H} + \mathbf{M})$ and satisfy (in quasi-stationary approximation) the Maxwell equations.

Let a particle of magnetizable and deformable solid originally at $(X_1 X_2 X_3)$ be moved, after deformation to (x_1, x_2, x_3) at time t. Both systems $(X_1 X_2 X_3)$ and (x_1, x_2, x_3) are Cartesian components referring to a common frame. The function $x_i = x_i(X_1, X_2, X_3, t)$ or their inverses describe the deformation for the body as a whole.

The Maxwell's equations in system of coordinates (x_1, x_2, x_3) are

$$\frac{\partial B_i}{\partial x_i} = 0, \quad e_{ijk} \frac{\partial H_j}{\partial x_k} = 0. \tag{2.2}$$

The general field equations of magnetoelasticity are derived by substituting the body force $\mathbf{f}$ and body couple $\mathbf{c}$, as defined in (2.1), and the rate of energy supply

$$\varepsilon = \mu_0 M_i \upsilon_j \frac{\partial H_j}{\partial x_i} + \mu_0 \rho H_i \frac{d}{dt}\left(\frac{M_i}{\rho}\right)$$

into the equations of balance of linear momentum, angular momentum, and energy, respectively. The results, expressed in terms of the current position vector x_i are (see Pao and Yeh, 1973; Tiersten, 1964)

$$\frac{d\rho}{dt} + \rho \frac{\partial \upsilon_i}{\partial x_i} = 0, \tag{2.3}$$

$$\frac{\partial t_{ij}}{\partial x_i} + \mu_0 M_i \frac{\partial H_j}{\partial x_i} = \rho \frac{dv_i}{dt}, \tag{2.4}$$

$$e_{mij} t_{ij} + c_m = 0, \tag{2.5}$$

$$\rho \frac{dU}{dt} = t_{ij} \frac{\partial \upsilon_j}{\partial x_i} + \mu_0 \rho H_i \frac{d}{dt}\left(\frac{M_i}{\rho}\right), \tag{2.6}$$

where $d/dt = \partial/\partial t + \upsilon_k(\partial/\partial x_k)$ and $\upsilon_k = dx_k/dt$; t_{ij} is the magnetoelastic stresses; e_{mij} is the permutation symbol with $e_{mij} = 1$ or -1 depending on whether the indices are in a cyclic or an anticyclic order, respectively and $e_{mij} = 0$ otherwise; ρ is the mass density of the medium in the deformed state; U is the internal energy per unit mass.

In addition to Eqs. (2.2)–(2.6), the following boundary conditions on the deformed surface should be fulfilled:

$$n_i[B_i = 0], \quad e_{ijk}n_j[H_k] = 0, \quad n_i[t_{ij} + T_{ij}^M] = 0, \tag{2.7}$$

where n_i is a unit normal to the deformed surface of discontinuity; $[A] = A^+ - A^-$ is a jump of function A from the negative side $(-)$ to the positive side $(+)$ of the deformed surface of discontinuity; Maxwell's tensor $T_{ij}^M = B_iH_j - 0.5\delta_{ij}H_kH_k$, where δ_{mk} is the Kronecker symbol with $\delta_{mk} = 1$ when $m = k$ and $\delta_{mk} = 0$ otherwise;

It is obvious from Eq. (2.5) that the tensor t_{im} is nonsymmetrical. It becomes symmetrical only if the magnetic moments $c_k = 0$. Substituting the values of c_k from Eq. (2.1) into Eq. (2.4) yield

$$e_{imk}(t_{im} + \mu_0 M_i H_m) = 0,$$

hence the symmetry of the tensor $t_{im} + \mu_0 M_i H_m$ follows.

From (2.6), the following constitutive equations for a magnetoelastic media is obtained

$$t_{ij} = \rho \frac{\partial U}{\partial(\partial x_j/\partial X_k)} \frac{d}{dt}\left(\frac{\partial x_j}{\partial X_k}\right), \tag{2.8}$$

$$\mu_0 H_i = \frac{\partial U}{\partial \mu_i}, \tag{2.9}$$

where $\mu_i = M_i/\rho$.

Since U must be invariant in rigid body rotation, the theorem on invariant functions of several vectors (see Akhiezer et al., 1968; Tiersten, 1964) limits U to be a function of the lengths of the vectors, the scalar product of a pair of vectors and the determinants of their components taken three at a time. Thus, U must be reduce at most to a function of

$$E_{IJ} = \frac{1}{2}\left(\frac{\partial x_k}{\partial X_I}\frac{\partial x_k}{\partial X_J} - \delta_{IJ}\right) \quad \text{and} \quad \mu_i = M_i/\rho. \tag{2.10}$$

The final form of the energy density function should be

$$U = U(E_{IJ}, \mu_i). \tag{2.11}$$

For detailed derivations of Eq. (2.11), see Akhiezer et al., 1968 and Tiersten, 1964.

Using Eqs. (2.9)–(2.11) and Eq. (2.8) can be written in the following compact form:

$$t_{ij} = \rho \frac{\partial x_i}{\partial X_K}\frac{\partial U}{\partial E_{KL}}\frac{\partial x_j}{\partial X_L} + \mu_0 M_j H_i. \tag{2.12}$$

Next, the expression for the specific intrinsic energy of the deformable elastic nonconductive magnetosoft ferromagnetic body is chosen in the following form:

$$U(E_{IJ}, \mu_i) = U^{el}(E_{IJ}) + U^m(\mu_i), \tag{2.13}$$

where U^{el} and U^m are elastic and magnetic energy, respectively.

In the case of a soft ferromagnetic material the following statement can be proved:

The vectors $\mathbf{H}$ *and* $\boldsymbol{\mu}$ *are parallel in the basic state if and only if the magnetic energy* U^m *satisfies the condition*

$$U^m(\mu_1, \mu_2, \mu_3) = U^m(|\boldsymbol{\mu}|), \quad |\boldsymbol{\mu}| = \sqrt{\mu_1^2 + \mu_2^2 + \mu_3^2} \tag{2.14}$$

i. e. the function U^m *depends only on the module of the vector* $\boldsymbol{\mu}$.

The sufficient condition can be deduced from Eqs. (2.10) and (2.14) as follows:

$$H_i = \frac{\partial U^m}{\partial \mu_i} = \frac{\partial U^m}{\partial|\mu|}\frac{1}{|\mu|}\mu_i \Rightarrow \mathbf{H} \uparrow\uparrow \mu \quad (\mathbf{H} \text{ is parallel to } \boldsymbol{\mu}). \tag{2.15}$$

The *necessary condition* should be proved. For simplicity, it is assumed that the function does not depend on the component μ_3. Then from (2.14) and the condition $\mathbf{H} \uparrow\uparrow \boldsymbol{\mu}$, it follows that $\mathbf{H} = \psi^*(\mu_1, \mu_2) \cdot \boldsymbol{\mu}$, where $\psi^*(\mu_1, \mu_2)$ is a continuous differentiable scalar function. Taking into account (2.14) and (2.10)

$$H_1 = \frac{\partial U^m}{\partial \mu_1} = \psi^*(\mu_1, \mu_2) \cdot \mu_1, \tag{2.16}$$

$$H_2 = \frac{\partial U^m}{\partial \mu_2} = \psi^*(\mu_1, \mu_2) \cdot \mu_2, \tag{2.17}$$

It follows from Eqs. (2.16) and (2.17) that

$$\mu_1 \frac{\partial \psi^*}{\partial \mu_2} = \mu_2 \frac{\partial \psi^*}{\partial \mu_1} \Rightarrow \psi^* = \psi^*(|\mu|),$$

i.e., the function ψ^* depends only on the module of the vector μ. The function U^m also depends only on the module of the vector μ. It is proved next. In polar coordinates ($\mu_1 = |\mu|\sin\theta$, $\mu_2 = |\mu|\cos\theta$), Eqs. (2.16) and (2.17) can be expressed in the following way:

$$\frac{\partial U^m}{\partial|\mu|} - ctg\theta\frac{\partial U^m}{\partial\theta} = \psi^*(|\mu|)\cdot|\mu|,$$
$$\frac{\partial U^m}{\partial|\mu|} + tg\theta\frac{\partial U^m}{\partial\theta} = \psi^*(|\mu|)\cdot|\mu|,$$

From these two equations the relationship $(tg\theta + ctg\theta)(\partial U^m/\partial\theta) = 0$ can be obtained.

Since $(tg\theta + ctg\theta)\neq 0$, it follows that $\partial U^m/\partial\theta = 0$ (i.e. U^m does not depend on the direction of magnetization vector μ). In other words, U^m depends only on the modulus of magnetization vector μ:

$$U^m(\mu_1, \mu_2) = U^m(|\mu|). \tag{2.18}$$

In general case (when $\mu_3\neq 0$), the proof can be done in spherical system of coordinates.

It follows from the above statement that

$$\mathbf{H} = \varphi_*(|\boldsymbol{\mu}|)\cdot\mu \quad \text{or} \quad \mathbf{H} = \varphi_1(|\mathbf{M}|)\cdot M, \varphi_1(|M|) = \varphi\left(\frac{1}{\rho}|M|\right)\cdot\frac{1}{\rho}. \tag{2.19}$$

Eq. (2.19) can be expressed also in a following form:

$$\mathbf{M} = \chi(|\mathbf{H}|)\mathbf{H}. \tag{2.20}$$

The function $\chi(|\mathbf{H}|)$ is the magnetic susceptibility. From the proof it is clear that the susceptibility χ for magnetosoft materials will depend only on the modulus of a magnetic field. Fig. 1 shows the typical dependence of $\chi(|\mathbf{H}|)$ on the modulus of a magnetic field (curve 1 is for superpermalloy and curve 2 for soft iron).

Experimental investigations shown that the magnetic susceptibility of magnetosoft ferromagnetic materials can be approximated by the following formulae (Bozort, 1951):

(1) *Dreifous form:*

$$\chi(H) = (\beta/\mu_0 H)arctg(\alpha H), \tag{2.21}$$

where

$$\beta = 2B_s/\pi, \alpha = (\mu_{ri} - 1)\mu_0/\beta,$$

and B_s denotes the induction saturation, μ_{ri} is the initial relative magnetic permeability of the material. Eq. (2.21) is a good approximation to the curves 1–2 in Fig. 1 for large magnetic fields.

(2) Instead of Eq. (2.21) sometimes *the Rayleigh dependence* is used

$$\chi(H) = \mu_r + b_r H, \tag{2.22}$$

which is the linear approximation of the Dreifous form and is applicable if $H < H_c$. Here, H_c is the coercitive force and μ_r is the magnetic susceptibility of the material. Eq. (2.22) is a good approximation of the curves 1–2 in Fig. 1 for moderate magnetic fields.

(3) *Linear dependence*. If the coefficient of nonlinearity $b_r = 0$, then from (2.22) one can get

$$\chi(H) = \text{const.} = \mu_r - 1. \tag{2.23}$$

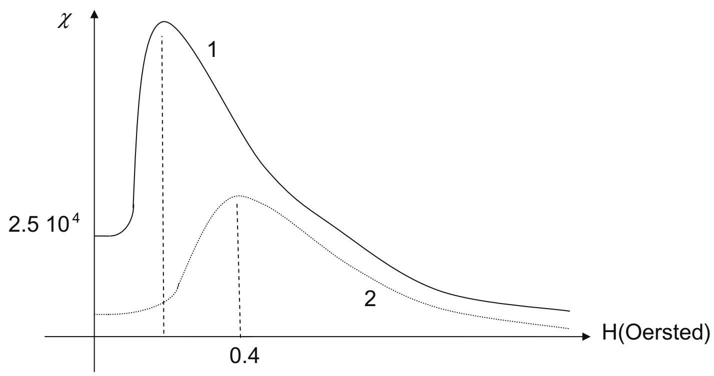

Fig. 1. Dependence of magnetic susceptibility on magnetic field.

Eq. (2.23) is a good approximation to the curves 1–2 in Fig. 1 for weak magnetic fields.

This type of relation is used in the constitutive equations for soft ferromagnetic materials in weak magnetic fields or for nonferomagnetic materials even in strong magnetic fields.

(4) The model of *magneto rigid* (magnetically saturated) materials is

$$\chi(H) = M_s/H, \tag{2.24}$$

where $M_s = B_s/\mu_0$ is the saturation magnetization. Eq. (2.24) is a good approximation to the curves 1–2 in Fig. 1 for large magnetic fields.

The numerical values of coefficients $\alpha,\ \beta,\ M_s,\ \kappa_0$ and b_r for different ferromagnetic materials can be found in Bozort (1951).

Thus the specific intrinsic energy for soft ferromagnetic elastic materials we represented in the following form:

$$U(\varepsilon_{ij}, M_i) = U^{el}(\varepsilon_{ij}) + U^m(|\mu|). \tag{2.25}$$

From (2.8) and (2.9), the constitutive equations of a magnetoelastic media (stress–strain-magnetic field relations) are obtained if specific intrinsic energy for soft ferromagnetic elastic materials is provided.

3. The linearized equations and boundary conditions

In general the equations and boundary conditions of magnetoelasticity are nonlinear. These equations and boundary conditions can be linearized by replacing transformation $x_i = x_i(X_1, X_2, X_3, t)$ by

$$x_i = X_J\delta_{iJ} + u_i(X_1, X_2, X_3, t),$$

where $u_i(X_1, X_2, X_3, t)$ is the displacement vector and δ_{iJ} is the Kronecker delta which shifts a vector at x_i parallels from x_i to X_J. Let us decompose magnetic field characteristics in the following form:

$$\mathbf{B} = \mathbf{B}_0 + \mathbf{b}, \quad \mathbf{H} = \mathbf{H}_0 + \mathbf{h}, \quad \mathbf{M} = \mathbf{M}_0 + \mathbf{m}. \tag{3.1}$$

Here, $\mathbf{B}_0,\ \mathbf{M}_0$ and $\mathbf{H}_0$ are the magnetic induction vector, the magnetization and the magnetic field intensity, respectively, in a rigid state, $\mathbf{b},\ \mathbf{m}$ and $\mathbf{h}$ are the perturbations to the mentioned quantities due to the deformation of the body. The values of $\mathbf{B}_0,\ \mathbf{M}_0$ and $\mathbf{H}_0$ are determined from the solution of the following magnetostatic problem:

(a) *Equations in the domain occupied by the body (internal domain)*:

$$rot\,\mathbf{H}_0 = 0, \quad div\,\mathbf{B}_0 = 0, \tag{3.2}$$

where $\mathbf{B}_0 = \mu_0(\mathbf{H}_0 + \mathbf{M}_0) = \mu_0[1 + \chi(\mathbf{H}_0)] \cdot \mathbf{H}_0$, $H_0 = |\mathbf{H}_0|$.

(b) *Equations in the external domain (domain outside of the body):*

$$rot\,\mathbf{H}_0^{(e)} = 0, \quad div\,\mathbf{B}_0^{(e)} = 0, \tag{3.3}$$

where $\mathbf{B}_0^{(e)} = \mu_0\mathbf{H}_0^{(e)},\ \mathbf{M}_0^{(e)} = 0$.

(c) *Conditions on the surface of the nondeformed body:*

$$\mathbf{n}_0 \cdot [\mathbf{B}_0] = 0, \quad \mathbf{n}_0 \times [\mathbf{H}_0] = 0, \tag{3.4}$$

(d) *Conditions at infinity:*

$$\mathbf{B}_0^{(e)} \to \mathbf{B}^0 \quad \text{when } r = (x_1^2 + x_2^2 + x_3^2)^{1/2} \to \infty. \tag{3.5}$$

Assuming that $|\partial u_i/\partial X_J| \ll 1$, $|b_i|/|\mathbf{B}_0| \ll 1$, $|m_i|/|\mathbf{M}_0| \ll 1$ $|h_i|/|\mathbf{H}_0| \ll 1$ the characteristics of the stress–strain state of a body (the displacement vector components u_k and stress tensor components t_{im}) and the vectors $\mathbf{b},\ \mathbf{m}$ and $\mathbf{h}$ are being determined from (2.2)–(2.6). Assuming that the deformations and the absolute values of $\mathbf{b},\ \mathbf{m}$ and $\mathbf{h}$ are small, one can linearize these equations and boundary conditions similar to Pao and Yeh (1973). As a result, the following linear equations and boundary conditions for the magnetoelastic body can be obtained from (2.2)–(2.4)

$$t_{ji,j} + \mu_0(M_{0j}H_{0i,j} + M_{0j}h_{i,j} + m_jH_{0j,i}) - \mu_0M_{0j}H_{0i,k}u_{k,j} - \bar{t}_{ij,k}u_{k,j} = \rho_0\frac{\partial^2 u_i}{\partial t^2}, \tag{3.6}$$

$$\rho = \rho_0(1 - u_{k,k}),\ \ e_{ijk}[h_{k,j} - H_{0j,m}u_{m,k}] = 0,\ \ b_{i,i} - B_{0j,k}u_{k,j} = 0 \quad (i,j,k = 1,2,3). \tag{3.7}$$

For isotropic materials the following applies:

$$\begin{aligned} t_{ij} &= \sigma_{ij} + \mu_0H_{0i}M_{0j} + \mu_0(H_{0i}m_{0j} + H_{0j}m_{0i}), \\ \sigma_{ij} &= \lambda\delta_{ij}u_{k,k} + \mu(u_{i,j} + u_{j,i}), \bar{t}_{ij} = \mu_0H_{0i}M_{0j} \end{aligned} \tag{3.8}$$

where λ and μ are the Lame coefficients ($\mu = E/2(1+\nu)$, E and ν are Young's modulus and Poisson's ratio, respectively).

The boundary conditions (2.7) can be linearized and expressed as follows (for the details see Maugin, 1988; Pao and Yeh, 1973):

$$n_{0i}[t_{ij}+t^M_{ij}]=0,\quad n_{0i}[b_i]-u_{m,i}n_{0m}[B_{0i}]=0,\quad e_{ijk}\{n_{0j}[h_k]-n_{0m}u_{m,j}[H_{0k}]\}=0, \tag{3.9}$$

where

$$t^M_{ij}=B_{0i}H_{0j}+B_{0i}h_j+B_{0j}h_i-0.5\mu_0\delta_{ij}(H^2_{0k}+2H_{0k}h_k).$$

After linearization of (2.20)

$$\mathbf{m}=\hat{a}\cdot\mathbf{h},\quad \mathbf{b}=\mu_0(I+\hat{a})\cdot\mathbf{h}, \tag{3.10}$$

where $\hat{I}$ the identity matrix and the elements of the matrix $\hat{a}$ are determined as follows:

$$a_{ij}=\chi\delta_{ij}+\frac{H_{0i}H_{0j}}{H_0}\frac{d\chi}{dH_0},\quad H_0=\sqrt{H^2_{01}+H^2_{02}+H^2_{03}}. \tag{3.11}$$

The final equations of motion (3.6) for an isotropic magnetoelastic media with a nonlinear law of magnetization (2.20) can be express as

$$\begin{aligned}&\Delta u_i+\frac{1}{1-2\nu}u_{j,ij}+\frac{\mu_0}{\mu}f_i=\frac{\rho_0}{\mu}\frac{\partial^2u_i}{\partial t^2},\\ &div[(\hat{I}+\hat{a})h]-B_{0j,k}u_{k,j}=0,\\ &e_{ijk}[h_{k,j}-H_{0j,m}u_{m,k}]=0.\end{aligned} \tag{3.12a- -c}$$

The first component of a body force $(\mu_0/\mu)f_i$ $(i=1,2,3)$ is expresses as

$$\frac{\mu_0}{\mu}f_1=b_{11}h_{1,1}+b_{12}h_{2,2}+b_{13}h_{3,3}+b_{14}h_{1,2}+b_{15}h_{1,3}+b_{16}h_{2,3}+\frac{\mu_0}{\mu}[2M_{0j}H_{01,j}+H_{0j}M_{01,j}]-\frac{\mu_0}{\mu}M_{0j}H_{01,k}u_{k,j}-\frac{1}{\mu}\bar{t}_{1j,k}u_{k,j}, \tag{3.13}$$

where

$$\begin{aligned}&b_{11}=\chi H_{01}+2a_{11}H_{01},\quad b_{12}=a_{22}H_{01}+a_{12}H_{02},\quad b_{13}=\chi H_{01}+a_{13}H_{03},\quad b_{14}=\chi H_{02}+3a_{12}H_{01}+a_{11}H_{02},\\ &b_{15}=\chi H_{03}+3a_{13}H_{01}+a_{11}H_{03}, b_{16}=2a_{23}H_{01}+a_{13}H_{02}+a_{12}H_{03}\end{aligned}$$

Expressions for $(\mu_0/\mu)f_2$ and $(\mu_0/\mu)f_3$ can be derived from $(\mu_0/\mu)f_1$ by cyclic permutation $1\rightarrow2\rightarrow3\rightarrow1$.

For the domain outside the body (considered to be vacuum), the magnetic field equations should satisfy the following equations:

$$\frac{\partial b^{(e)}_i}{\partial x_i}=0,\quad e_{ijk}\frac{\partial h^{(e)}_j}{\partial x_k}=0. \tag{3.14}$$

When χ = const., Eqs. (3.12a–c) and boundary conditions (3.9) are analogous to the equations and boundary conditions obtained in Pao and Yeh (1973).

Another model for a soft ferromagnetic material with a nonlinear law of magnetization is developed by Zheng and Wang (2001).

To illustrate the preceding theory, in the next sections the following two problems will be considered:

- *the magnetoelastic stability of a ferromagnetic plate-strip in homogeneous transverse magnetic field;*
- *stress–strain state of ferromagnetic plane with a moving crack in a transverse magnetic field.*

4. Stability of a ferromagnetic plate-strip in a homogeneous transverse magnetic field

Assume that an isotropic homogeneous plate-strip of constant thickness $2h$ in the direction $0x_2$ and infinite length in the direction $0x_1$ is located in an external uniform magnetic field $B_0=(0,B_0,0)$, $B_0=\text{const.}$ (see Fig. 2). Equilibrium equations are solved to determine whether a nontrivial equilibrium configuration exists for a deformed plate. There is a critical value for the externally applied magnetic induction B_0 corresponding to this equilibrium state when the plate buckles. A plane-strain problem will be considered. In this case, the nonzero components of the displacement and magnetic field components are

$$u_1=u_1(x_1,x_2),\quad u_2=u_2(x_1,x_2),\quad u_3=0,$$

$$\mathbf{h}^{(i,e)}=(h^{(i,e)}_1(x_1,x_2),h^{(i,e)}_2(x_1,x_2),0),\mathbf{h}^{(i,e)}=grad\,\Phi^{(i,e)}. \tag{4.1}$$

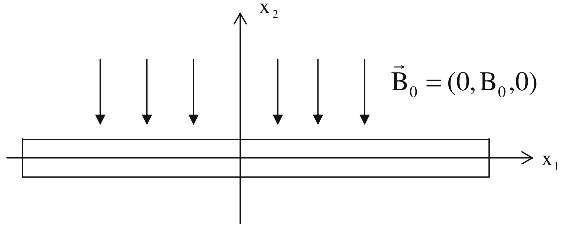

Fig. 2. Soft ferromagnetic plate-layers in a transversal magnetic field.

The solution of the system of Eqs. (3.12)–(3.14) can be represented in the following form:

$$u_1(x_1,x_2) = u(x_2)\cos(kx_1), \quad u_2(x_1,x_2) = \upsilon(x_2)\sin(kx_1)$$

$$\Phi^{(i)}(x_1,x_2) = A\cosh(k\gamma x_2)\sin(kx_1) \quad \text{where } \gamma^2 = (1+a_{11})/(1+a_{22}), \tag{4.2}$$

when $|x_2| < h$ and

$$\Phi^{(e)}(x_1,x_2) = a_1\, sign(x_2)\exp[-kx_2\, sign(x_2)]\sin(kx_1)$$

when $|x_2| > h$. In Eqs. (4.1) and (4.2) $u(x_2)$ and $\upsilon(x_2)$ are

$$u(x_2) = b_1\sinh(kx_2) + b_2[(3-4\nu)\sinh(kx_2) + kx_2\cosh(kx_2)] + Q_1^0 A\sinh(k\gamma x_2),$$
$$\upsilon(x_2) = b_1\cosh(kx_2) + b_2 kx_2\sinh(kx_2) + Q_2^0 A\cosh(k\gamma x_2),$$

where

$$Q_1^0 = \gamma\{\lambda_1[(1-2\nu)/(2-2\nu) - \gamma^2] + \lambda_2\gamma^2/(2-2\nu)\}/(1-\gamma^2)^2,$$
$$Q_2^0 = \gamma^2\{\lambda_2[1-(1-2\nu)\gamma^2/(2-2\nu)] - \lambda_1/(2-2\nu)\}/(1-\gamma^2)^2,$$
$$\lambda_1 = 2\mu_0\chi(H_0)H_0/\mu,$$
$$\lambda_2 = \mu_0\chi(H_0)H_0[2(\chi + H_0\chi')(1+\chi) - H_0\chi\chi']/[\mu\chi(1+\chi)],$$
$$a_{11} = \chi, \quad a_{22} = \chi + H_0\chi', \quad \chi\prime = d\chi/dH_0.$$

The parameter $k = \pi/l$ is the wave number and l is the wavelength in the direction $0x_1$.

Substituting solution (4.2) into the boundary conditions (3.9) yield a system of linear homogeneous algebraic equations for the unknown coefficients $b_1,\ b_2,\ a_1$ and A. The critical magnetic field when the plate-strip losses its stability is determined from the condition of the determinant of the algebraic equations being zero. Under the assumption $\gamma kh \ll 1$, the following equation is derived for the critical magnetic field:

$$\begin{aligned}&4dQ_2^0(1-\nu)kh[2(1-\nu)\gamma^2/(1-2\nu) - 1] - 4dQ_1^0(1-\nu)kh/(1-2\nu) - 4d_1L(1-\nu)kh4d_1e_1(1-\nu)\gamma^2 kh\\ &\quad +8(kh)^3[\gamma^2 1 + a_{11})kh + 1] = 0,\end{aligned} \tag{4.3}$$

where

$$L = \mu_0 H_0 a_{11}/\mu, \quad e_1 = \mu_0 H_0 a_{22}\chi/\mu, \quad d_1 = -\chi(H_0)/(\chi(H_0)+1).$$

From (4.3), the critical magnetic field H_{0*} for instability of a plate-strip can be determined. The critical value of the external magnetic field $B_{0*}^{(e)}$ can be determined from

$$B_{0*}^{(e)}/\mu_0 = [1+\chi(H_{0*})]H_{0*}. \tag{4.4}$$

Let us consider some particular cases:

(1) *Linear dependence*: $\chi(H_0) = \text{const.} = \mu_r - 1$ .From (4.3) and (4.4), one can obtain

$$B_{0*}^{(e)2}(\mu_r - 1)^2/(\mu_0\mu\mu_r^2) = (kh)^2(\mu_r kh + 1)/[3(1-\nu)(\mu_r+1)]. \tag{4.5}$$

Assuming $kh\mu_r \gg 1$, from (4.5)

$$B_{0*}^{(e,Linear)} = \mu_0\mu_r H_{0*}^L = \sqrt{\frac{E\mu_0(kh)^3}{3(1-\nu^2)}} \quad \text{or} \quad H_{0*}^L = \sqrt{\frac{E(kh)^3}{3(1-\nu^2)\mu_r^2}}\frac{1}{\sqrt{\mu_0}}. \tag{4.6}$$

Expression (4.6) coincides with analogous results obtained by authors: Hasanyan and Philiposyan (2001), Maugin (1988), Moon and Pao (1969) and Van de Ven (1983).

(2) *Rayleigh form*: $\chi(H_0) = \mu_r + b_r H_0$.From Eq. (4.3), it is easy to get

$$-4(\mu_0 H_{0*}^2/E)\gamma^2 kh(1-\nu^2)(\mu_r + b_r H_{0*})^2(\mu_r + 2b_r H_{0*}) + (8/3)(kh)^3[kh\gamma^2(1+\mu_r+2b_r H_{0*})+1] = 0. \tag{4.7}$$

Under the assumption $kh(\mu_r + b_r H_{0*}) \gg 1$ one obtains

$$(\mu_0 H_{0*}^2/E)(\mu_r + b_r H_{0*})^2 = (kh)^3/(3-3\nu^2) \tag{4.8}$$

or the following expression for critical magnetic field H_{0*}:

$$H_{0*} = \frac{\mu_r}{2b_r}\left[-1+\sqrt{1+4b_r H_{0*}^L/\mu_r}\right].$$

The critical value of the external magnetic field $B_{0*}^{(e)}$ can be expressed as

$$B_{0*}^{(e,Nonlinear)} = \mu_0[\mu_r + b_r H_{0*}]H_{0*} = \frac{\mu_r}{2b_r}\left[-1+\sqrt{1+4b_r H_{0*}^L/\mu_r}\right]\mu_0[\mu_r + b_r H_{0*}]. \tag{4.9}$$

From (4.9), it follows that $H_{0*} \to H_{0*}^L$ when $b_r \to 0$, i.e. the critical magnetic field to be found from a nonlinear law of magnetization coincides with one found on the base of linear law of magnetization. A critical magnetic field determined theoretically for a cantilever beam (with a linear law of magnetization) is almost 1.5 times larger than a critical magnetic field found from experiments (Moon and Pao, 1969). By comparing (4.6) and (4.9), it appears that $H_{0*} < H_{0*}^L$ when $b_r > 0$. It means that the critical magnetic field found from nonlinear law of magnetization is always smaller than that found from linear law of magnetization. In other words the critical value of a magnetic field found by formulae (4.9) is much closer to the experimental results than the critical value of magnetic field found by formulae (4.6). For example, when the material of a ferromagnetic plate is a pure iron ($\mu_r = 150;\ b_r = 2\times10^5$ m/A) and $kh = 10^{-2}$ the critical value of a magnetic field is $B_{0*}^{(e,Linear)} = 0.58$ T with the linear law of magnetization, and $B_{0*}^{(e,Nonlinear)} = 0.44$ T with the nonlinear law of magnetization. The experimental value of the critical magnetic field is $B_{0*}^{(e,Experimental)} = 0.36$ T (Moon and Pao, 1969). From this primary observation, it is clear that the critical magnetic field found with nonlinear law of magnetization is in better agreement with experimental results than one with the linear law of magnetization.

(3) *Dreifous form*: $\chi(H_0) = (\beta/\mu_0 H_0)arctg(\alpha H_0)$.In this case, it is very difficult to obtain an analytic formula for H_{0*}. Numerical analysis shows that the critical magnetic field decreases with B_s increasing and monotonically increases with μ_i increasing. Ignoring the volume forces ($\lambda_1 = \lambda_2 = 0$) does not lead to a large change of the critical magnetic field, in fact it remains almost the same. Also in this case, the critical magnetic field with nonlinear law of magnetization is in better agreement with an experiment results. Notice that for a very weak magnetic field from (4.3) the results reported in Hasanyan and Philiposyan (2001), Maugin (1988), Moon and Pao (1969) and Van de Ven (1983) can be reproduced.

5. Stress–strain state of a ferromagnetic plane with a moving crack

The problem of a stress–strain state of a ferromagnetic plane with a moving crack is discussed in this paragraph. A soft magnetic ferroelastic body which is modeled with a nonlinear law of magnetization, immersed in a magnetic field perpendicular to a crack line is considered. Assuming that the processes in moving coordinates are stationary Fourier transform method is used to reduce the mixed boundary value problem to the pairs of dual integral equations which are solved analytically. The magnetoelastic stress intensity factor and its dependency on the crack velocity, material constants and nonlinear law of magnetization are obtained.

5.1. Formulation of the problem

Let a magnetoelastic plane with a finite crack of length $2a$ be located in a magnetic field $\mathbf{H}^0 = (0, H_0, 0), H_0 = \text{const}$. The crack is moving with the constant velocity $V < c_R$(c_R is a speed of propagation of Rayleigh waves for a considered media) and is located in a plane x_1ox_2 along a line $x_2 = 0$, and $-a + Vt < x_1 < a + Vt$ (see Fig. 3).

In-plane nontrivial displacements are

$$u_1 = u_1(x_1, x_2, t_1), \quad u_2 = u_2(x_1, x_2, t_1), \quad u_3 = 0.$$

A moving coordinate system ($oxyz$) attached to the center of the moving crack is chosen such that (Nishida et al., 1984)

$$x = x_1 - Vt_1, \quad t = t_1, \quad y = x_2, \quad z = x_3. \tag{5.1}$$

It is assumed that the crack propagation occurs during an interval of time when in the moving system of coordinates the magnetoelastic state is time-invariant. Magnetic field $\mathbf{H}_0$, magnetization $\mathbf{M}_0$ and magnetic induction $\mathbf{B}_0$ characterizing undeformed state of a body according (3.2)–(3.5) are given as

$$\mathbf{H}_0 = \mathbf{H}^0, \quad \mathbf{M}_0 = \chi(H_0)\cdot\mathbf{H}_0, \quad \mathbf{B}_0 = \mu_0(1+\chi(H_0))\cdot\mathbf{H}_0.$$

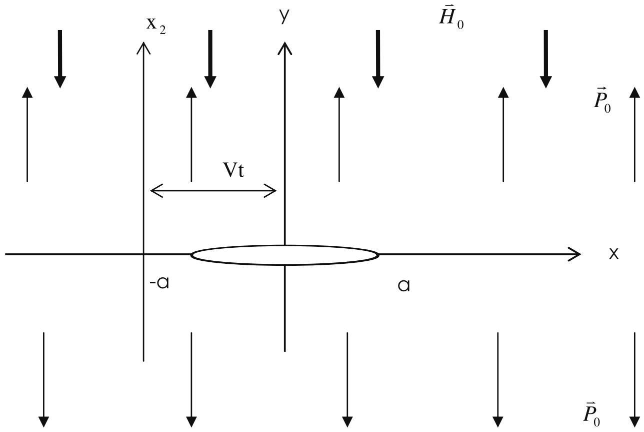

Fig. 3. Ferromagnetic plane with a crack in a magnetic field.

The displacement u_i and magnetic potential Φ characterizing the stress–strain state of a ferromagnetic body are determined from Eqs. (3.12–3.14). Introducing u_1 and u_2 such that

$$u_1 = \varphi_{,x_1} + \psi_{,x_2}, \quad u_2 = \varphi_{,x_2} - \psi_{,x_1}, \tag{5.2}$$

where φ and ψ are potential functions from Eq. (3.12) the following two decoupled equations with respect to φ and ψ are obtained:

$$c_1^2\nabla^2\varphi - \varphi_{,t_1t_1} + \delta_1^2\Phi_{,x_2} = 0, \quad c_2^2\nabla^2\psi - \psi_{,t_1t_1} + \delta_2^2\Phi_{,x_1} = 0, \tag{5.3}$$

where

$$\nabla^2 = \partial^2/\partial x_1^2 + \partial^2/\partial x_2^2, \quad \delta_1^2 = \frac{\gamma^2\lambda_2 - \lambda_1}{\gamma^2 - 1}c_2^2, \quad \delta_2^2 = -\frac{\gamma^2(\lambda_2 - \lambda_1)}{\gamma^2 - 1}c_2^2, \quad c_1^2 = 2(1-v)/(1-2v)c_2^2,$$
$$\lambda_1 = 2\chi(H_0)(1+\chi(H_0))h_c^2, \quad \lambda_2 = \lambda_1[2(\chi(H_0) + H_0\chi'(H_0))/\chi(H_0) - H_0\chi'(H_0)/(1+\chi(H_0))]/2,$$
$$h_c^2 = \mu_0 H_0^2/\mu, \quad c_2^2 = \mu/\rho.$$

Using coordinate transformations (5.1), Eqs. (5.3) and (3.12) become

$$s_1^2\varphi_{,xx} + \varphi_{,yy} + r_1\Phi_{,y} = 0, \quad s_2^2\psi_{,xx} + \psi_{,yy} + r_2\Phi_{,y} = 0, \quad \gamma^2\Phi_{,xx} + \Phi_{,yy} = 0, \tag{5.4}$$

where $s_i^2 = 1 - M_i^2,\ M_i = V/c_i,\ r_i = \delta_i/c_i^2\ (i = 1,2)$. The Mach numbers $M_i < 1$ since the crack is propagating at subsonic speed.

The boundary conditions of the problem are

$$u_2(x,0) = 0 \quad \text{when } |x| > a, \tag{5.5}$$
$$\Phi(x,0) = 0 \quad \text{when } |x| > a, \tag{5.6}$$
$$\Phi_{,x}(x,0) + d_1 \cdot u_{y,x}(x,0) = 0 \quad \text{when } |x| < a, \tag{5.7}$$
$$u_{2,x}(x,0) + u_{1,y}(x,0) + L\cdot\Phi_{,x} = 0 \quad \text{when } |x| < \infty, \tag{5.8}$$
$$\frac{2v}{1-2v}u_{1,x}(x,0) + \frac{2(1-v)}{1-2v}u_{2,y}(x,0) - e_1\cdot\Phi_{,y}(x,0) = -P_{0mec} + P_{0mag} = -P_0 \quad \text{when } |x| < a, \tag{5.9}$$

where $P_{0mag} = b_c^2 \cdot \chi(H_0)\cdot(\chi(H_0) - 2)/(\chi(H_0)+1)^2;\ \mu\cdot P_{0mec}$ is a mechanical force acting on the surface of the crack, $e_1 = h_c^2(1+\chi(H_0))[\chi^2(H_0) - 2\chi(H_0)],\ d_1 = -\chi(H_0)/(\chi(H_0)+1),\ L = \lambda_1/2,\ b_c^2 = (\chi(H_0)+1)^2h_c^2$.

5.2. The solution methodology

By applying the Fourier transform method to Eq. (5.4) the potential functions are readily obtained

$$\varphi(x,y) = \frac{1}{\pi}\int_{-\infty}^{\infty}\left\{A(\alpha)\exp[-\beta_1 y] + \frac{\delta_1}{c_1^2}\frac{|\alpha|}{\alpha^2 - \beta_1^2}C(\alpha)\exp[-|\alpha|y]\right\}\exp[-i\alpha x]d\alpha, \tag{5.10}$$

$$\psi(x,y)=\frac{1}{\pi}\int_{-\infty}^{\infty}B(\alpha)\exp[-\beta_2 y]\exp[-i\alpha x]d\alpha, \tag{5.11}$$

$$\Phi(x,y)=\frac{1}{\pi}\int_{-\infty}^{\infty}C(\alpha)\exp[-\gamma|\alpha|y]\exp[-i\alpha x]d\alpha, \tag{5.12}$$

where A, B and C are unknown functions to be defined from boundary conditions (5.5)–(5.9) and $\beta_i^2=\alpha^2(1-M_i^2)$.

The unknowns A, B and C can be rewritten as function of a new quantity $D(\alpha)$:

$$B(\alpha)=\frac{i}{\alpha^2-\beta_2^2}\{-2\alpha-d_1[\alpha Q_2^*(\alpha)+i\gamma|\alpha|Q_1^*(\alpha)-\alpha L]\}D(\alpha)\}, \tag{5.13}$$

$$A(\alpha)=\frac{1}{\beta_1}\left\{1+d_1Q_2^*(\alpha)+\frac{\alpha}{\alpha^2-\beta_2^2}[-2\alpha-d_1(\alpha Q_2^*(\alpha)+i\gamma|\alpha|Q_1^*(\alpha)-\alpha L)]D(\alpha)\right\}, \tag{5.14}$$

where

$$Q_1^*(\alpha)=-i\frac{|\alpha|}{\alpha}Q_1^0,\quad Q_2^*(\alpha)=-Q_1^0,\quad Q_1^0=\frac{\delta_1^*}{c_1^2}\frac{\gamma}{\gamma^2-M_1^2}+\frac{\delta_2^*}{c_2^2}\frac{\gamma}{\gamma^2-M_2^2}$$

From the boundary conditions (5.5)–(5.9), the determination of unknown $D(\alpha)$ leads to the following dual integral equations:

$$\frac{1}{\pi}\int_{-\infty}^{\infty}D(\alpha)\exp[-ix\alpha]d\alpha=0,\quad |x|>a, \tag{5.15}$$

$$\frac{1}{\pi}\int_{-\infty}^{\infty}D(\alpha)[f_e(\alpha)+d_1f_m(\alpha)]\exp[-ix\alpha]d\alpha=P_0,\quad |x|<a, \tag{5.16}$$

where

$$f_e(\alpha)=|\alpha|R_*(M_1^2,M_2^2),\quad R_*(M_1^2,M_2^2)=\frac{4\sqrt{1-M_1^2}\sqrt{1-M_2^2}-(2-M_2^2)^2}{M_2^2\sqrt{1-M_1^2}},\quad f_m(\alpha)=|\alpha|R_m(M_1^2,M_2^2),$$

$$R_m(M_1^2,M_2^2)=\frac{(2-M_2^2)}{\sqrt{1-M_1^2}}\left[Q_2^0-\frac{1}{M_2^2}(Q_2^0+Q_1^0\gamma-L)\right]+\frac{2\sqrt{1-M_2^2}}{M_2^2}(Q_2^0+Q_1^0\gamma-L)-\frac{2\nu}{1-2\nu}Q_1^0-\frac{2(1-\nu)}{1-2\nu}Q_2^0\gamma+e_1\gamma.$$

The magnetoelastic stress $t_{22}^c(x,0)/\mu=t_{22}(x,0)/\mu+t_{22}^M(x,0)/\mu$ is expressed through $D(\alpha)$ as follows:

$$t_{22}^c(x,0)=P_{0mag}+\frac{1}{\pi}\int_{-\infty}^{\infty}D(\alpha)[f_e(\alpha)+d_1f_m^*(\alpha)]\exp[-ix\alpha]d\alpha, \tag{5.17}$$

where

$$f_m^*(\alpha)=f_e(\alpha)-e_1\cdot\gamma|\alpha|-R_0\cdot\gamma|\alpha|,\quad R_0=b_c^2\frac{2a_{22}-2\chi(H_0)}{\chi(H_0)+1},\quad a_{22}=\chi(H_0)+H_0\cdot\chi'(H_0).$$

Let us

$$E(s)=\frac{P_0}{F_{em}(M_1^2,M_2^2)}\cdot\frac{s}{\sqrt{a^2-s^2}}, \tag{5.18}$$

where

$$F_{em}(M_1^2,M_2^2)=R_*(M_1^2,M_2^2)+d_1R_m(M_1^2,M_2^2).$$

Then the solution of dual integral equations (5.15) and (5.16) can be represented as

$$D(\alpha)=\frac{1}{\pi}\int_{-a}^{a}E(s)\exp[-i\alpha s]ds. \tag{5.19}$$

When $y=0$, the magnetoelastic stress $t_{yy}^T(x,y)/\mu$ have the following form:

$$t_{22}^c(x,0)/\mu=P_{0mag}+\frac{k_1}{\pi}\int_{-a}^{a}\frac{s}{\sqrt{a^2-s^2}}\cdot\frac{1}{s-x}ds=P_{0mag}+k_1\begin{cases}1-\frac{x}{\sqrt{x^2-a^2}}, & x>a,\\ 1+\frac{x}{\sqrt{x^2-a^2}}, & x<-a,.\\ 1, & |x|\leqslant a.\end{cases} \tag{5.20}$$

The normalized stress intensity factor can be expressed as

$$k_1=a^{1/2}\cdot\lim_{x\to a+0}\sqrt{2(x-a)}t_{22}^c(x,0)=\frac{F_{em}^*(M_1^2,M_2^2)\cdot a^{1/2}}{F_{em}(M_1^2,M_2^2)}P_0, \tag{5.21}$$

where

$$F^{*}_{em}(M_1^2, M_2^2) = R_*(M_1^2, M_2^2) + d_1 R_m(M_1^2, M_2^2) - e_1 - \gamma R_0.$$

From (5.21), when $V = 0$, i.e. the crack is stationary the following is obtained:

$$\left.\frac{k_1}{P_0 a^{1/2}}\right|_{V=0} = \frac{1 + d_1(1-\nu)\{(\gamma+1)\eta Q_1^0 - R_0\gamma + [\lambda_1/2 - (\gamma-1)Q_1^0]/\eta\}}{1 + d_1(1-\nu)\{e_1\gamma + (\gamma+1)\eta Q_1^0 + [\lambda_1/2 - (\gamma-1)Q_1^0]/\eta\}}, \tag{5.22}$$

where $\eta = 2(1-\nu)/(1-2\nu)$. In a case of magnetosoft material with a linear law of magnetization ($\chi(H_0) = \chi = \text{const.}$), when the velocity of a crack is equal to zero ($V = 0$) it follows from (5.21) or (5.22) that

$$k_1 \approx a^{1/2} P_0 [1 - \chi(1-\nu) b_c^2]^{-1}, \tag{5.23}$$

where $b_c^2 = (\chi+1)^2 h_c^2$.

Result (5.23) first time was derived by Shindo (1977).

Notice that for a magnetosoft material with a linear law of magnetization the intensity factor $k_1 \to \infty$ when $b_c^2 \to 1/\chi(1-\nu^2)$ (see formulae (5.23) and Shindo (1976)). At the same time, from Eq. (5.21), the denominator of Eq. (5.21) cannot be zero when the magnetization law is given by $\chi(H) = M_s/H$.

It follows from abovementioned that the nonlinear law of magnetization can have qualitative and quantitative influence on magnetoelastic quantities.

It follows from Eq. (5.21) that for a magnetosoft material with a linear law of magnetization ($\chi(H_0) = \chi = \text{const.}$) the stress intensity factor has the following form:

$$\begin{aligned} k_1 = & -a^{1/2} P_0 \left\{ R_*(M_1^2, M_2^2) + \frac{L_1}{M_2^2}\left[\frac{2 - M_2^2}{\sqrt{1 - M_1^2}} - 2\sqrt{1 - M_2^2}\right] + \frac{\delta_1}{c_1^2 M_1^2}\left[2 - \frac{2 - M_2^2}{\sqrt{1 - M_1^2}}\right]\right\} \\ & \times \left\{ R_*(M_1^2, M_2^2) + \frac{L_1}{M_2^2}\left[\frac{2 - M_2^2}{\sqrt{1 - M_1^2}} - 2\sqrt{1 - M_2^2}\right] + \frac{\delta_1}{c_1^2 M_1^2}\left[2 - \frac{2 - M_2^2}{\sqrt{1 - M_1^2}}\right] - \frac{\chi^3}{\mu_r^2} b_c^2 \right\}^{-1}, \end{aligned} \tag{5.24}$$

where

$$L_1 = (\chi b_c^2 \mu_r) \quad \text{and} \quad \frac{\delta_1}{c_1^2} = \frac{1 - 2\nu}{1 - \nu} \cdot L_1.$$

5.3. Numerical results

Numerical calculations have been carried out for a normalized stress intensity factor $\overline{k_1} = k_1/a^{1/2}P_0$. In particular, the dependence of the stress intensity factor on the magnetic field $h_c^2 = \mu_0 H_0^2/\mu = H^2 \times 10^{-17}$ and normalized velocity of the crack $\bar{\upsilon} = V^2/c_1^2$ is found.

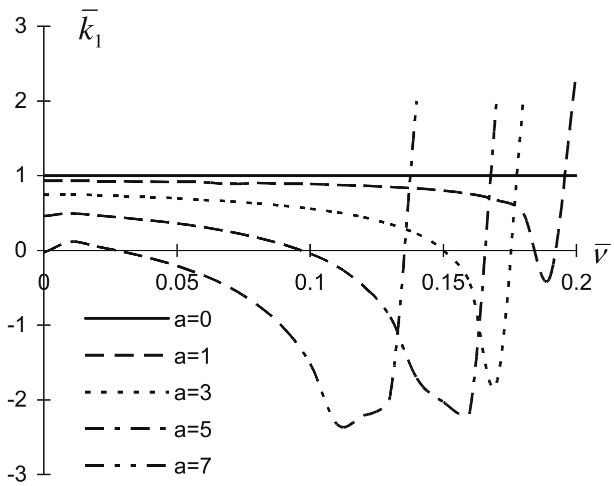

Fig. 4. Dependence of $\overline{k_1}$ on nondimensional speed of movement $\bar{\nu}$, for selected magnetic fields $h_c^2 = a \times 10^{-16}$.

From Eq. (5.21), it appears that if $h_c^2 \neq 0$ the intensity factor $\overline{k_1} \to \infty$ when $V \to V_* < c_2$ (where V_* is the velocity leading the denominator of Eq. (5.21) to zero). If $h_c^2 = 0$ the stress intensity factor $\overline{k_1}$ does not depend on the speed of a moving crack. These conclusions show that the intensity factor essentially depends on an external magnetic field, the speed of a moving crack and physical parameters of the problem.

Synergistic implications of the interaction of the elastic and magnetic fields on the intensity factor for different type of law of magnetizations are displayed in Figs. 4–7.

In Fig. 4, effects of the normalized speed of moving crack $\bar{\upsilon} = V^2/c_1^2$ on the normalized intensity factor $\overline{k_1}$ for various values of normalized magnetic fields h_c^2 are shown. The law of magnetization is set by Eq. (2.23), i.e. a magnetosoft material with the linear law of magnetization. The numerical results are carried out for $\mu_r = 10^5,\ \nu = 0.35$. As shown in Fig. 4, the external magnetic field ($h_c^2 \neq 0$) essentially changes the value $\overline{k_1}$ in comparison with a pure elastic case ($h_c^2 = 0$).

In Figs. 5 and 6, the dependence of $\overline{k_1}$ on a normalized magnetic field is given for various values of normalized speed of a moving crack. In the numerical simulations it is assumed $\nu = 0.3$. The law of magnetization is set by Eq. (2.22). Fig. 7 is set for

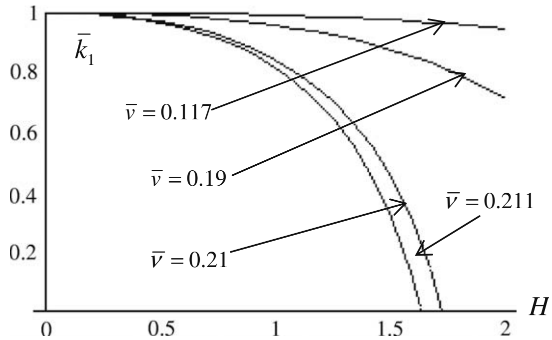

Fig. 5. Dependence of $\overline{k_1}$ on magnetic field $h_c^2 = H^2 \times 10^{-17}$ for selected values of speed of movement when $\mu_r = 10^5, b_r = 10^3$.

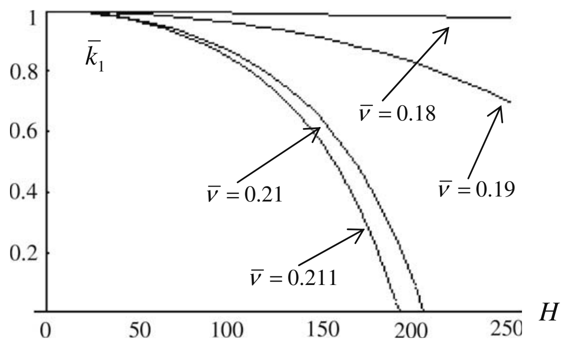

Fig. 6. Dependence of $\overline{k_1}$ on magnetic field $h_c^2 = H^2 \times 10^{-17}$ for selected values of nondimensional speed of movement when $\mu_r = 10^5, b_r = 10^3$.

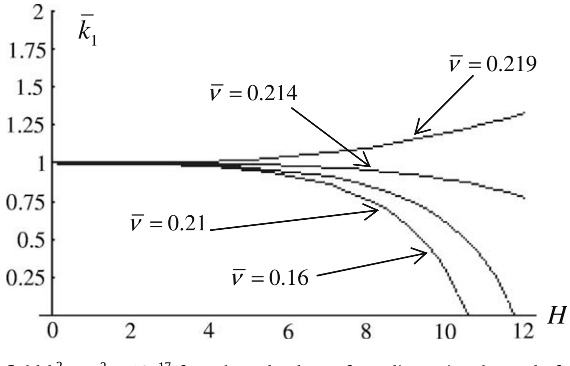

Fig. 7. Dependence of $\overline{k_1}$ on magnetic field $h_c^2 = H^2 \times 10^{-17}$ for selected values of nondimensional speed of movement when $B_s = 5\ \mathrm{T}, \mu_{ri} = 5 \times 10^3$.

Table 1
Dependence of $\overline{k_1}$ on dimensionless magnetic field H for parameters $b = 10^8, \kappa = 10^5$ when speed of cracks movement $\bar{\upsilon} = 0$

	$\chi(H) = (b/H)arctg(\kappa H/b)$	$\chi(H) = \text{const.} = \kappa$
$H = 0$	1	1
$H = 10$	1.07	1.079
$H = 20$	1.405	1.398
$H = 30$	2.846	2.517
$H = 40$	−6.553	92.25
$H = 50$	−1.25	−2.46

the nonlinear law of magnetization (2.21). In all cases, the normalized intensity factor $\overline{k_1}$ essentially depends on enclosed magnetic field and the speed of a moving crack.

From the numerical simulations, it is possible to conclude that:

(a) With an increase of the magnetic field, the intensity factor $\overline{k_1}$ first decreases, passing through zero to become negative, but then sharply increases (for $V^2/c_1^2 = \bar{\upsilon} < 0.21$). Negative intensity factor is physically impossible under tensile stress field. The crack surfaces will approach each other, and, if the compressive stress is large enough, the crack will close. However, only circumstances in which the crack remains open are considered here. This observation implies that the magnetic field can retard the propagation of a crack in ferromagnetic materials. Analog type of phenomena for energy release rate was observed and discussed by Li and Kardomateas (2007) in a problem related to the investigation of stress fields in a piezoelectromagneto-elastic anisotropic bimaterials with an interface crack.
(b) The Fig. 7 shows that for normalized velocity $V^2/c_1^2 = \bar{\upsilon} = 0.219$ the stress intensity factor is increases as magnetic field increases, a tendency is opposite for other velocities. This phenomenon is due to fact that the velocity of a crack $V^2/c_1^2 = \bar{\upsilon} = 0.219$ is close to the velocity of a propagation of Rayleigh waves (in discussed case the Rayleigh waves speed is $\bar{\upsilon}_R = 0.28$). The intensity factor $\overline{k_1} \to \infty$ with the increase of the magnetic field for $\bar{\upsilon} \approx \bar{\upsilon}_R$.
(c) An applied magnetic field makes the stress intensity factor and magnetoelastic stresses velocity dependent. A magnetic field can increase or it can decrease the stress intensity factor (depends on crack speed). At critical magnetic field strengths, the stresses in the vicinity of the crack change sign. Magnetoelastic behavior of ferromagnetic body is highly sensitive to material properties.
(d) Figs. 5–7 shows also that the intensity factor $\overline{k_1}$ essentially different for different laws of magnetization.

Note that the linear law expressed by Eq. (2.23) takes place at a rather weak magnetic field. The nonlinear laws (2.21) and (2.22) can be used for a strong magnetic field.

Table 1 shows the dependence of the intensity factor $\overline{k_1}$ on the normalized magnetic field $H = 10^9 h_c^2$ for two different laws of magnetizations: ***Dreifous form and linear law.*** It clearly appears that the nonlinear law of magnetization has a strong influence on the intensity factor $\overline{k_1}$ starting from $H = 30$. The simulations based on the linear law of magnetization and on the nonlinear law of magnetization give the same results for the intensity factor $\overline{k_1}$ for a small magnetic field.

6. Conclusions

The equations of magnetoelasticity for magnetosoft materials with a nonlinear law of magnetization are presented. The following two problems are considered on the basis of these equations: (1) the magnetoelastic stability of a ferromagnetic plate-layer in a homogeneous transverse magnetic field, (2) the stress–strain state of a ferromagnetic plane with a moving crack in a transverse magnetic field. The nonlinear law of magnetization should be taken into account when dealing with stability problem of plates in strong external magnetic field. The nonlinear law of magnetization has a strong influence on the magnetoelastic characteristics in the crack propagation problems.

The present model can be used in magnetoelasticity studies where soft ferromagnetic materials are used in the presence of a strong magnetic field. Possible applications include the use of an applied magnetic field for increasing the stress intensity factor during machining (which should assist chip formation and increase machinability), and applying a magnetic field to lower stress intensity factors in structural components (which should effectively increase toughness).

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