International Journal of Solids and Structures 46 (2009) 2929–2938

Contents lists available at ScienceDirect

International Journal of Solids and Structures

journal homepage: www.elsevier.com/locate/ijsolstr

Exact slip-buckling analysis of two-layer composite columns

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ARTICLE INFO

Article history:
Received 1 September 2008
Received in revised form 3 February 2009
Available online 2 April 2009

Keywords:
Stability
Buckling
Exact analysis
Composites
Slip
Non-linear
Critical load
Elastic
Layers

ABSTRACT

A mathematical model for slip-buckling has been proposed and its analytical solution has been found for the analysis of layered and geometrically perfect composite columns with inter-layer slip between the layers. The analytical study has been carried out to evaluate exact critical forces and to compare them to those in the literature. Particular emphasis has been placed on the influence of interface compliance on decreasing the bifurcation loads. For this purpose, a preliminary parametric study has been performed by which the influence of various material and geometric parameters on buckling forces have been investigated.

1. Introduction

Layered columns arise in a wide range of applications. Slender columns made of composite materials are widely used in aerospace engineering, civil engineering, shipbuilding, and in other branches of industry because of their high load-carrying capacity and convenient strength-to-weight ratio. The behaviour of these structures largely depends on the type of the connection between the layers.

Since absolutely stiff connection between the layers can hardly be realized in practice, an inter-layer slip develops. If the slip has a sufficient magnitude, it significantly affects the mechanical behaviour of the composite system. Consequently, the inter-layer slip has to be taken into consideration in what is called partial interaction analysis of composite structures. Accordingly, there are many published papers in which composite beams and beam-columns are analysed analytically and numerically, see e.g., Ayoub (2005), Čas et al. (2004a,b, 2007), Dall'Asta and Zona (2004), Gara et al. (2006), Schnabl et al. (2007a,b), Ranzi et al. (2003), and Ranzi and Zona (2007). An extensive literature review on linear and non-linear analysis of layered structures is given by Leon and Viest (1998) and Schnabl et al. (2007b).

The strength of straight layered columns depends to a great extent on their buckling resistance and cohesion between the layers. It is therefore of practical importance to employ analytical formulations of such a problem. There have been relatively few analytical investigations of this problem and to date only a few exact slip-buckling models of composite columns have been developed. Rassam and Goodman (1970) derived a simplified governing equations for buckling behaviour of layered wood columns with both equal and unequal layer thicknesses. Buckling parameter for a wide range of geometric and physical parameters of a three layered wood column is presented in design charts. Subsequently, an analytical solution of buckling problem is derived by Girhammar and Gopu (1993). Their solution is based on the so-called "modified second-order theory" and approximate buckling length coefficients. As it is well known, the above-mentioned theory neglects the influence of extensional strains on buckling loads of Euler columns. An extension and generalization of the latter theory is presented in Girhammar and Pan (2007), where exact buckling length coefficients are used. Recent papers by Xu and Wu (2007a,b,c) have presented an interesting approach to the solution of slip-buckling and vibration problem of composite beam-columns when shear deformation is taken into account. If shear deformation is neglected, the equations for buckling load obtained by Xu and Wu (2007a,b,c) are the same as presented in Girhammar and Pan (2007).

The goal of this paper is the exact formulation of slip-buckling problem of geometrically perfect two-layer composite columns. As a result, exact analytical solutions are derived. However, in

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doi:10.1016/j.ijsolstr.2009.03.020

contrast to other researchers (Girhammar and Gopu, 1993; Girhammar and Pan, 2007; Xu and Wu, 2007a,b,c), a linearized stability theory is employed (Keller, 1970). Therefore, a solution of slip-buckling problem is obtained without simplification of the governing equations. The critical buckling forces are determined from the solution of a linear eigenvalue problem, i.e., $\det \boldsymbol{K} = 0$ (see, Planinc and Saje, 1999).

In the numerical examples critical buckling loads are compared to those of Girhammar and Pan (2007). Afterwards, the exact solution is used to investigate the effect of the inter-layer slip on the buckling of a two-layer column for various boundary conditions. A preliminary parametric study is conducted, by which an influence of different geometric and material parameters on buckling forces of geometrically perfect two-layer composite column is investigated.

2. Analytical model — model description

2.1. Assumptions

A formulation of the planar Euler–Bernoulli two-layer composite column used in this paper is based on the following assumptions: (1) the column is geometrically perfect and straight; (2) the axial load is loaded eccentrically at a distance e from the reference axis; (3) material is linear elastic; (4) displacements, strains and rotations are finite (each of the layers satisfies the assumptions of geometrically exact Reissner beam theory); (5) the effect of shear deformations is negligible; (6) strains vary linearly over each layer, e.i. the "Bernoulli hypothesis" is assumed; (7) the layers are continuously connected and the slip modulus of the connection is constant; (8) shapes of the cross-sections are symmetric with respect to the plane of deformation and remain unchanged in the form and size during deformation; (9) friction between the layers is not considered. An additional assumption (10) is that an inter-layer tangential slip can occur at the interface between the layers, but no transverse separation (uplift) between them is possible.

2.2. Governing equations

We consider an initially straight, planar, two-layer composite column of undeformed length L. Layers as shown in Fig. 1 are marked by letters a and b. The column is placed in the (X, Z) plane of spatial Cartesian coordinate system with coordinates (X, Y, Z) and unit base vectors $\boldsymbol{E}_X, \boldsymbol{E}_Y$ and $\boldsymbol{E}_Z = \boldsymbol{E}_X \times \boldsymbol{E}_Y$. The undeformed reference axis of the layered column is common to both layers and is defined as an intersection of the (X, Z)-plane and their contact plane. It is parametrized by the undeformed arc-length x. Local coordinate system (x, y, z) is assumed to coincide initially with spatial coordinates, and then it follows the deformation of the column. Thus, $x^a \equiv x^b \equiv x \equiv X, y^a \equiv y^b \equiv y \equiv Y$, and $z^a \equiv z^b \equiv z \equiv Z$ in the undeformed configuration. The *geometrically perfect composite column* is subjected to a conservative compressive axial force P centrally located at both ends in such way that homogeneous strain and stress state at primary configuration of the column is achieved. For further details an interested reader is referred to e.g., Schnabl et al. (2007a,b).

2.2.1. Kinematic equations

The deformed configurations of the reference axes of layers a and b are defined by vector-valued functions (see Fig. 1)

$$
\begin{aligned}
\boldsymbol{R}_0^a &= X^a \boldsymbol{E}_X + Y^a \boldsymbol{E}_Y + Z^a \boldsymbol{E}_Z = (x^a + u^a)\boldsymbol{E}_X + y^a \boldsymbol{E}_Y + w^a \boldsymbol{E}_Z, \\
\boldsymbol{R}_0^b &= X^b \boldsymbol{E}_X + Y^b \boldsymbol{E}_Y + Z^b \boldsymbol{E}_Z = (x^b + u^b)\boldsymbol{E}_X + y^b \boldsymbol{E}_Y + w^b \boldsymbol{E}_Z,
\end{aligned} \tag{1}
$$

where superscripts a and b denote that quantities are related to layer a and b, respectively. Functions u^a and w^a denote the components of the displacement vector of layer a at the reference axis with respect to the base vectors $\boldsymbol{E}_X$ and $\boldsymbol{E}_Z$. Similarly, functions u^b and w^b are related to layer b. The geometrical components u^a, w^a, u^b, and w^b of the vector-valued functions $\boldsymbol{R}_0^a$ and $\boldsymbol{R}_0^b$ are related to the deformation variables with the equations derived by Reissner (1972):

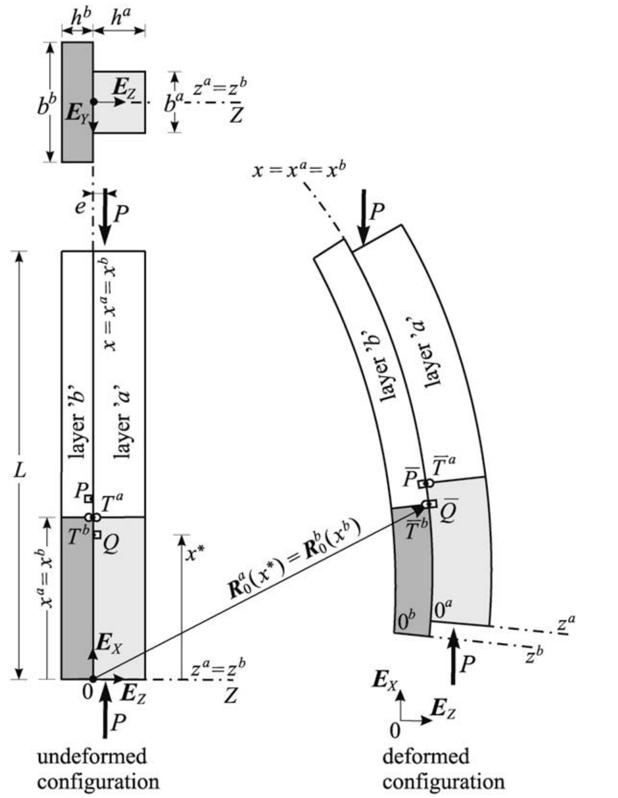

Fig. 1. Geometry and notation for a straight geometrically perfect two-layer composite column.

layer a:

$$
\begin{aligned}
&1 + u^{a\prime} - (1 + \varepsilon^a)\cos\varphi^a = 0, \\
&w^{a\prime} + (1 + \varepsilon^a)\sin\varphi^a = 0, \\
&\varphi^{a\prime} - \kappa^a = 0,
\end{aligned} \tag{2}
$$

layer b:

$$
\begin{aligned}
&1 + u^{b\prime} - (1 + \varepsilon^b)\cos\varphi^b = 0, \\
&w^{b\prime} + (1 + \varepsilon^b)\sin\varphi^b = 0, \\
&\varphi^{b\prime} - \kappa^b = 0.
\end{aligned} \tag{3}
$$

Here, the prime $(')$ denotes the derivative with respect to x. In (2) and (3) the deformation variables ε^a and ε^b are the extensional strains of the reference axes of layers a and b; κ^a and κ^b are the pseudocurvatures (Vratanar and Saje, 1999); whereas φ^a and φ^b are the rotations of layers' reference axes.

2.2.2. Equilibrium equations

The composite column is subjected to a force P at both ends. Furthermore, each layer of the two-layer composite column is subjected to interlayer contact tractions measured per unit of layer's undeformed length which are defined by

$$
\begin{aligned}
\boldsymbol{p}^a &= p_X^a \boldsymbol{E}_X + p_Z^a \boldsymbol{E}_Z, \\
\boldsymbol{p}^b &= p_X^b \boldsymbol{E}_X + p_Z^b \boldsymbol{E}_Z.
\end{aligned} \tag{4}
$$

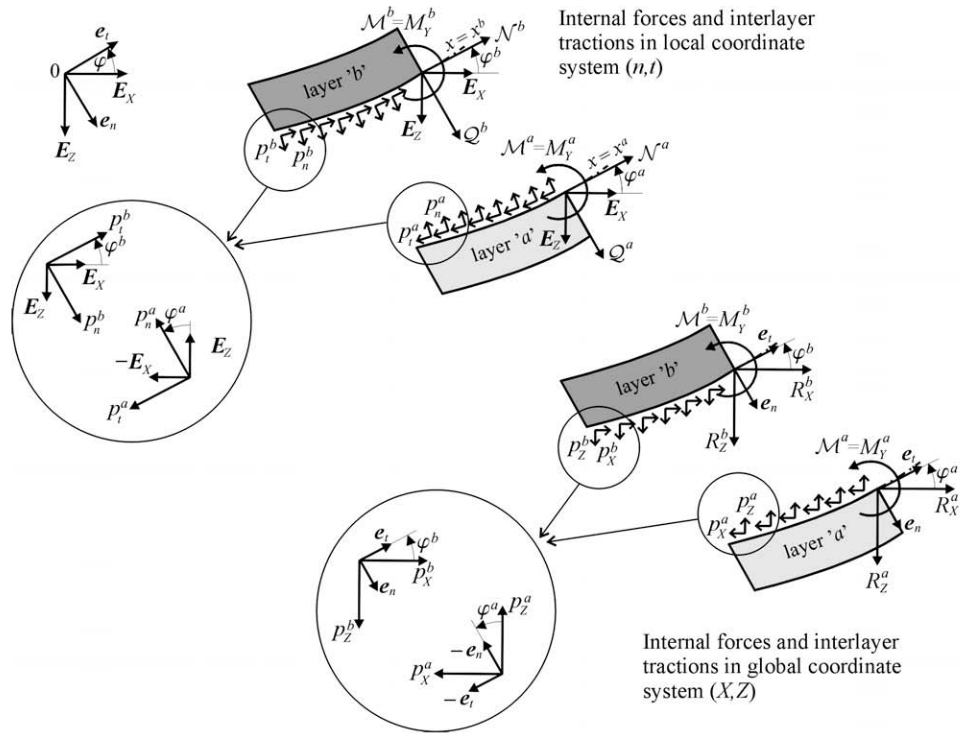

Fig. 2. Interlayer contact tractions and generalized equilibrium internal forces and moments with respect to the fixed global and rotated local coordinate system.

In order to write constitutive equations in usually used coordinate systwm, it is suitable to express the (X, Z) components of the interlayer contact tractions with the tangential and normal components of the interlayer tractions p_t^a, p_t^b, p_n^a, and p_n^b (see Fig. 2):

$$\begin{aligned}
p_X^a &= p_t^a \cos\varphi^a + p_n^a \sin\varphi^a,\\
p_Z^a &= -p_t^a \sin\varphi^a + p_n^a \cos\varphi^a,\\
p_X^b &= p_t^b \cos\varphi^b + p_n^b \sin\varphi^b,\\
p_Z^b &= -p_t^b \sin\varphi^b + p_n^b \cos\varphi^b.
\end{aligned} \tag{5}$$

Using (4) and (5), the equilibrium equations of each layer are, see e.g. Reissner (1972) and Čas et al. (2007):
layer a:

$$\begin{aligned}
&R_X^{a\prime} + p_X^a = R_X^{a\prime} + p_t^a \cos\varphi^a + p_n^a \sin\varphi^a = 0,\\
&R_Z^{a\prime} + p_Z^a = R_Z^{a\prime} - p_t^a \sin\varphi^a + p_n^a \cos\varphi^a = 0,\\
&M_Y^{a\prime} - (1+\varepsilon^a)\mathcal{Q}^a + m_Y^a = 0,
\end{aligned} \tag{6}$$

layer b:

$$\begin{aligned}
&R_X^{b\prime} + p_X^b = R_X^{b\prime} + p_t^b \cos\varphi^b + p_n^b \sin\varphi^b = 0,\\
&R_Z^{b\prime} + p_Z^b = R_Z^{b\prime} - p_t^b \sin\varphi^b + p_n^b \cos\varphi^b = 0,\\
&M_Y^{b\prime} - (1+\varepsilon^b)\mathcal{Q}^b + m_Y^b = 0,
\end{aligned} \tag{7}$$

where

$$\begin{aligned}
\mathcal{N}^a &= R_X^a \cos\varphi^a - R_Z^a \sin\varphi^a,\\
\mathcal{Q}^a &= R_X^a \sin\varphi^a + R_Z^a \cos\varphi^a,\\
\mathcal{M}^a &= M_Y^a,\\
\mathcal{N}^b &= R_X^b \cos\varphi^b - R_Z^b \sin\varphi^b,\\
\mathcal{Q}^b &= R_X^b \sin\varphi^b + R_Z^b \cos\varphi^b,\\
\mathcal{M}^b &= M_Y^b.
\end{aligned} \tag{8}$$

$R_X^a, R_Z^a, R_X^b, R_Z^b, M_Y^a$, and M_Y^b in (6)–(8) represent the generalized equilibrium internal forces and moments of a cross-section of layers a and b with respect to the fixed coordinate basis. On the other hand, $\mathcal{N}^a, \mathcal{Q}^a, \mathcal{N}^b$, and $\mathcal{Q}^b$ represent the equilibrium axial and shear internal forces of the layers' cross-sections with respect to the rotated local coordinate system. Functions $\mathcal{M}^a$ and $\mathcal{M}^b$ are the equilibrium bending moments.

2.2.3. Constitutive equations

To relate the equilibrium internal forces $\mathcal{N}^a, \mathcal{Q}^b, \mathcal{N}^a$, and $\mathcal{Q}^b$ and equilibrium internal moments $\mathcal{M}^a$ and $\mathcal{M}^b$ to a material model, the following set of equations which assure the balance of equilibrium and constitutive cross-sectional forces and bending moments of the composite column have been introduced. Due to the assumption that the transverse shear deformations are neglected, the constitutive equations of a two-layer composite column are

$$\begin{aligned}
&\mathcal{N}^a - \mathcal{N}_C^a(x, \varepsilon^a, \kappa^a) = \mathcal{N}^a - \int_{\mathcal{A}^a} \sigma_C^a(D^a)\,\mathrm{d}A^a = 0,\\
&\mathcal{M}^a - \mathcal{M}_C^a(x, \varepsilon^a, \kappa^a) = \mathcal{M}^a - \int_{\mathcal{A}^a} z^a \sigma_C^a(D^a)\,\mathrm{d}A^a = 0,\\
&\mathcal{N}^b - \mathcal{N}_C^b(x, \varepsilon^b, \kappa^b) = \mathcal{N}^b - \int_{\mathcal{A}^b} \sigma_C^b(D^b)\,\mathrm{d}A^b = 0,\\
&\mathcal{M}^b - \mathcal{M}_C^b(x, \varepsilon^b, \kappa^b) = \mathcal{M}^b - \int_{\mathcal{A}^b} z^b \sigma_C^b(D^b)\,\mathrm{d}A^b = 0.
\end{aligned} \tag{9}$$

The constitutive functions $\mathcal{N}_C^a, \mathcal{M}_C^b, \mathcal{N}_C^b$, and $\mathcal{M}_C^b$ introduced in (9) are dependent only on deformation variables $\varepsilon^a, \kappa^a, \varepsilon^b$, and κ^b and are subordinated to the adopted linear elastic constitutive model defined by stress–strain relations

$$\begin{aligned}
\sigma_C^a(D^a) &= E^a D^a = E^a(\varepsilon^a + z^a\kappa^a),\\
\sigma_C^b(D^b) &= E^b D^b = E^b(\varepsilon^b + z^b\kappa^b),
\end{aligned} \tag{10}$$

where σ_C^a and σ_C^b are the longitudinal normal stresses of layers a and b; D^a, D^b are the mechanical extensional strains in longitudinal direction in layers a and b; and E^a, E^b are elastic moduli of layers a and b.

By introducing (10) into (9) the well-known constitutive equations of linear elastic columns can be rewritten as

$$\begin{aligned}
&\mathcal{N}^a - E^a \int_{\mathcal{A}^a} (\varepsilon^a + z^a\kappa^a)\mathrm{d}A^a = \mathcal{N}^a - C_{11}^a\varepsilon^a - C_{12}^a\kappa^a = 0,\\
&\mathcal{M}^a - E^a \int_{\mathcal{A}^a} z^a(\varepsilon^a + z^a\kappa^a)\mathrm{d}A^a = \mathcal{M}^a - C_{21}^a\varepsilon^a - C_{22}^a\kappa^a = 0,\\
&\mathcal{N}^b - E^b \int_{\mathcal{A}^b} (\varepsilon^b + z^b\kappa^b)\mathrm{d}A^b = \mathcal{N}^b - C_{11}^b\varepsilon^b - C_{12}^b\kappa^b = 0,\\
&\mathcal{M}^b - E^b \int_{\mathcal{A}^b} z^b(\varepsilon^b + z^b\kappa^b)\mathrm{d}A^b = \mathcal{M}^b - C_{21}^b\varepsilon^b - C_{22}^b\kappa^b = 0,
\end{aligned} \tag{11}$$

in which material and geometric constants are marked by $C_{11}^a, C_{12}^a, \ldots, C_{22}^b$; e.g., $C_{11}^a = E^aA^a$, where A^a denotes the cross-sectional area of layer a, see e.g. Kryžanowski et al. (2008) and Rodman et al. (2008).

Moreover, the contact constitutive law must also be introduced. In the presented analysis the linear constitutive law of bond slip between the layers is assumed:

$$p_t^a(x) = \mathcal{H}(\Delta(x)) = K\Delta(x). \tag{12}$$

In the above equation constant K determines the inter-layer-slip modulus.

Remark 1. Considering strain and stress state at primary configuration as homogeneous (i.e., $\varepsilon^a = \varepsilon^b = \varepsilon$ and $\kappa^a = \kappa^b = 0$) and denoting by e an eccentricity of axial load P from the reference axis (see, Fig. 1), the following two constitutive equations are obtained from (11), for two-layer composite column, respectively

$$\begin{aligned}
&-P = \mathcal{N}^a + \mathcal{N}^b = C_{11}^a\varepsilon + C_{11}^b\varepsilon = (C_{11}^a + C_{11}^b)\varepsilon,\\
&-Pe = \mathcal{M}^a + \mathcal{M}^b = C_{21}^a\varepsilon + C_{21}^b\varepsilon = (C_{21}^a + C_{21}^b)\varepsilon.
\end{aligned}$$

These equations represent a system of two equtions for two unknown functions e and ε. The solution for e is

$$e = \frac{C_{21}^a + C_{21}^b}{C_{11}^a + C_{11}^b}.$$

Since $C_{11}^a, C_{12}^b C_{21}^a$, and C_{21}^b, depend on geometric and material parameters, the point of application of P of two-layer composite column coincides with the centre of gravity of equivalent fully composite column only when layers have the same material properties, i.e., $E^a = E^b$.

2.2.4. Constraining equations

Once the layers are connected, the upper layer b is constrained to follow the deformation of the lower layer a and vice versa. As already stated, the layers can slip along each other but their transverse separation (uplift) or penetration is not allowed. This fact is expressed by a kinematic-constraint requirement

$$\boldsymbol{R}_0^a(x^*) = \boldsymbol{R}_0^b(x), \tag{13}$$

where x and x^* are coordinates of two distinct particles of layers a and b in the undeformed configuration which are in the deformed configuration in contact and thus their vector-valued functions $\boldsymbol{R}_0^b(x)$ and $\boldsymbol{R}_0^a(x^*)$ coincide (see, Fig. 1). Eq. (13) can be written equivalently in componential form as

$$\begin{aligned}
&x^* + u^a(x^*) = x + u^b(x),\\
&w^a(x^*) = w^b(x).
\end{aligned} \tag{14}$$

The relative displacement (slip) that occurs between the two particles of layers a and b which are in contact in the undeformed configuration is denoted by Δ and is defined as the difference of their deformed arc-lengths s^a and s^b, see e.g. Čas et al. (2004a,b). Then,

$$\Delta(x) + s^b(x) = \Delta(0) + s^a(x) \rightarrow \Delta(x) = \Delta(0) + \int_0^x (\varepsilon^a(\xi) - \varepsilon^b(\xi))\mathrm{d}\xi. \tag{15}$$

By differentiating (14), adding the results with (2) and (3), the following relations by which the rotations and pseudocurvatures of layers are constrained to each other are obtained as

$$\varphi^a(x^*) = \varphi^b(x), \tag{16}$$

$$\kappa^a(x^*)\frac{1+\varepsilon^b(x)}{1+\varepsilon^a(x^*)} = \kappa^b(x). \tag{17}$$

Remark 2. The relation (16) is exact, and may be obtained by the following derivation. Differentiation of (14) with respect to x gives

$$\left(1 + \frac{\mathrm{d}u^a(x^*)}{\mathrm{d}x^*}\right)\frac{\mathrm{d}x^*}{\mathrm{d}x} = 1 + \frac{\mathrm{d}u^b(x)}{\mathrm{d}x},$$

$$\frac{\mathrm{d}w^a(x^*)}{\mathrm{d}x^*}\frac{\mathrm{d}x^*}{\mathrm{d}x} = \frac{\mathrm{d}w^b(x)}{\mathrm{d}x}.$$

Applaying Eqs. (2) and (3), and rearranging, the following relations result

$$\begin{aligned}
&(1 + \varepsilon^a(x^*))\cos\varphi^a(x^*)\frac{\mathrm{d}x^*}{\mathrm{d}x} = (1 + \varepsilon^b(x))\cos\varphi^b(x),\\
&(1 + \varepsilon^a(x^*))\sin\varphi^a(x^*)\frac{\mathrm{d}x^*}{\mathrm{d}x} = (1 + \varepsilon^b(x))\sin\varphi^b(x).
\end{aligned}$$

By mutual division of these equations, the relation (16) is derived explicitly as

$$\tan\varphi^a(x^*) = \tan\varphi^b(x) \rightarrow \varphi^a(x^*) = \varphi^b(x).$$

In addition to the above presented constraining equations, the equilibrium of the interlayer contact tractions of the particles in contact is expressed in vector-valued function form as

$$\boldsymbol{p}^a(x) + \boldsymbol{p}^b(x) = \boldsymbol{0}, \tag{18}$$

and by substituting (4) and (5) to (18) in componential form as

$$\begin{aligned}
&p_X^a + p_X^b = p_t^a\cos\varphi^a + p_n^a\sin\varphi^a + p_t^b\cos\varphi^b + p_n^b\sin\varphi^b = 0,\\
&p_Z^a + p_Z^b = -p_t^a\sin\varphi^a + p_n^a\cos\varphi^a - p_t^b\sin\varphi^b + p_n^b\cos\varphi^b = 0.
\end{aligned} \tag{19}$$

A complete set of non-linear governing equations of two-layer composite beam Eqs. (2)–(19) consists of 32 equations for 32 unknown functions $u^a, u^b, w^a, w^b, \varphi^a, \varphi^b, \varepsilon^a, \varepsilon^b, \kappa^a, \kappa^b, R_X^a, R_X^b, R_Z^a, R_Z^b, M_Y^a, M_Y^b, \mathcal{N}^a, \mathcal{N}^b, \mathcal{Q}^a, \mathcal{Q}^b, \mathcal{M}^a, \mathcal{M}^b, p_X^a, p_X^b, p_Z^a, p_Z^b, p_t^a, p_t^b, p_n^a, p_n^b, \Delta$, and, x^*.

2.3. Linearized equations

In order to investigate the stability of boundary value problem, non-linear equations which govern the behaviour of that problem have to be introduced. The non-linear stability problems are considerably more difficult to solve than linear problems. Therefore, approximation methods should be used. One of the most applicable methods for stability analysis of non-linear systems is what we call *linearized theory of stability* or *linear theory of stability*. It is founded on the fact that the bifurcation (critical) points of the non-linear system coincide with the critical points of its equivalent linearized system (Keller, 1970). The application of the linearized stability theory, regarding the existence and uniqueness of the solution of Reissner's elastica, is presented by Flajs et al. (2003).

The linearized theory of stability is based upon the variation of a functional $\mathcal{F}$, which will here be made in the sense of the continuous linear *Gateaux* operator or directional derivative, defined as follows (Hartmann, 1985)

$$\delta\mathcal{F}(\boldsymbol{x}, \delta\boldsymbol{x}) = \lim_{\alpha\to 0}\frac{\mathcal{F}(\boldsymbol{x} + \alpha\delta\boldsymbol{x}) - \mathcal{F}(\boldsymbol{x})}{\alpha} = \frac{\mathrm{d}}{\mathrm{d}\alpha}\bigg|_{\alpha=0}\mathcal{F}(\boldsymbol{x} + \alpha\delta\boldsymbol{x}), \tag{20}$$

where the $\boldsymbol{x}$ and δx represent the generalized displacement field and its increment, respectively, and α is an arbitrary small scalar parameter. $\delta\mathcal{F}(\boldsymbol{x},\delta\boldsymbol{x})$ is also called linearization or linear approximation of $\delta\mathcal{F}$ at $\boldsymbol{x}$. Accordingly, it is convenient for Eqs. (2)–(19) to be re-written in compact form as $\boldsymbol{\mathcal{F}} = \{\mathcal{F}_1, \mathcal{F}_1, \ldots, \mathcal{F}_{32}\}^T$, and their arguments as $\boldsymbol{x} = \{u^a, u^b, w^a, w^b, \ldots, p_n^a, p_n^b, \Delta, x^*\}^T$.

After the linearization of the governing Eqs. (2)–(19) has been completed, linearized equations are evaluated at an arbitrary configuration of the two-layer composite column. In order to apply linearized equations to the two-layer composite column buckling problem, these equations have to be evaluated at the primary configuration of the column. The fundamental or primary configuration of the column is an arbitrary deformed configuration in which the composite column remains straight. In this case, the primary configuration is defined as

$$\begin{aligned}
&\varepsilon^a = \varepsilon^b = -\frac{1}{C_{11}^a + C_{11}^b} P,\\
&\kappa^a = \kappa^b = 0,\\
&u^a = u^b = u^a(0) - \frac{x}{C_{11}^a + C_{11}^b} P\\
&w^a = w^b = 0,\\
&\varphi^a = \varphi^b = 0,\\
&x^* = x,\\
&\Delta = 0,\\
&R_X^a = \mathcal{N}^a = -\frac{C_{11}^a}{C_{11}^a + C_{11}^b} P,\\
&R_X^b = \mathcal{N}^b = -\frac{C_{11}^b}{C_{11}^a + C_{11}^b} P\\
&R_Z^a = \mathcal{Q}^a = 0,\\
&R_Z^b = \mathcal{Q}^b = 0,\\
&M_Y^a = \mathcal{M}^a = -\frac{C_{21}^a}{C_{11}^a + C_{11}^b} P,\\
&M_Y^b = \mathcal{M}^b = -\frac{C_{21}^b}{C_{11}^a + C_{11}^b} P,
\end{aligned} \tag{21}$$

$$\begin{aligned}
&p_X^a = p_t^a = 0,\\
&p_X^b = p_t^b = 0,\\
&p_Z^a = p_n^a = 0,\\
&p_Z^b = p_n^b = 0.
\end{aligned} \tag{22}$$

Finally, the linearized system of equilibrium Eqs. (2)–(19) when written at the primary configuration (21) of the composite column is easily derived in the following form

$$\begin{aligned}
&\delta\mathcal{F}_1 = \delta u^{a\prime} - \delta\varepsilon^a = 0,\\
&\delta\mathcal{F}_2 = \delta u^{b\prime} - \delta\varepsilon^b = 0,\\
&\delta\mathcal{F}_3 = \delta w' + (1+\varepsilon)\delta\varphi = 0,\\
&\delta\mathcal{F}_4 = \delta\varphi' - \delta\kappa = 0,\\
&\delta\mathcal{F}_5 = \delta R_X^{a\prime} - \delta p_t = 0,\\
&\delta\mathcal{F}_6 = \delta R_X^{b\prime} + \delta p_t = 0,\\
&\delta\mathcal{F}_7 = \delta R_Z' = 0,\\
&\delta\mathcal{F}_8 = \delta M_Y' + R_X \delta w' - (1+\varepsilon)\delta R_Z = 0,\\
&\delta\mathcal{F}_9 = \delta R_X^a - C_{11}^a \delta\varepsilon^a - C_{12}^a \delta\kappa = 0,\\
&\delta\mathcal{F}_{10} = \delta R_X^b - C_{11}^b \delta\varepsilon^b - C_{12}^b \delta\kappa = 0,\\
&\delta\mathcal{F}_{11} = \delta M_Y - C_{21}^a \delta\varepsilon^a - C_{21}^b \delta\varepsilon^b - (C_{22}^a + C_{22}^b)\delta\kappa = 0,\\
&\delta\mathcal{F}_{12} = \delta\Delta - \delta u^a + \delta u^b = 0,\\
&\delta\mathcal{F}_{13} = \delta p_t - K\delta\Delta = 0,\\
&\delta\mathcal{F}_{14} = \delta x^* + \delta u^a - \delta x - \delta u^b = 0,
\end{aligned} \tag{23}$$

where

$$\begin{aligned}
&\varepsilon = -\frac{1}{C_{11}^a + C_{11}^b} P,\\
&\delta w = \delta w^a = \delta w^b,\\
&\delta\varphi = \delta\varphi^a = \delta\varphi^b,\\
&\delta\kappa = \delta\kappa^a = \delta\kappa^b,\\
&R_X = -P,\\
&\delta R_Z = \delta R_Z^a + \delta R_Z^b,\\
&\delta M_Y = \delta M_Y^a + \delta M_Y^b,\\
&\delta p_t = \delta p_t^a = \delta p_t^b.
\end{aligned} \tag{24}$$

Eq. (23) constitute a linear system of 14 algebraic-differential equations of the first order with constant coefficients for 14 unknown functions of $x: \delta u^a, \delta u^b, \delta w, \delta\varphi, \delta\varepsilon^a, \delta\varepsilon^b, \delta\kappa, \delta R_X^a, \delta R_X^b, \delta R_Z, \delta M_Y, \delta p_t, \delta\Delta$, and δx^* along with the corresponding natural and essential boundary conditions which may be written in the following general form, see e.g. Čas et al. (2004b):

$x = 0$:

$$\begin{aligned}
&s_1^0 \delta R_X^a(0) + s_2^0 \delta u^a(0) = 0,\\
&s_3^0 \delta R_X^b(0) + s_4^0 \delta u^b(0) = 0,\\
&s_5^0 \delta R_Z(0) + s_6^0 \delta w(0) = 0,\\
&s_7^0 \delta M_Y(0) + s_8^0 \delta\varphi(0) = 0,
\end{aligned} \tag{25}$$

$x = L$:

$$\begin{aligned}
&s_1^L \delta R_X^a(L) + s_2^L \delta u^a(L) = 0,\\
&s_3^L \delta R_X^b(L) + s_4^L \delta u^b(L) = 0,\\
&s_5^L \delta R_Z(L) + s_6^L \delta w(L) = 0,\\
&s_7^L \delta M_Y(L) + s_8^L \delta\varphi(L) = 0,
\end{aligned} \tag{26}$$

where $s_i \in \{0, 1\}$ are parameters that determine different combinations of boundary conditions of the two-layer composite column. The superscript "0" and "L" of s indentifies its value at $x = 0$ and $x = L$, respectively.

2.4. Analytical solution for critical buckling load

Due to the simple form of Eq. (23) and boundary conditions (25) and (26) a critical buckling load can be determined analytically. After the systematic elimination of the primary unknowns, the set of linearized equations (23) is reduced to a system of three higher-order linear homogeneous ordinary differential equations with constant coefficients for unknown functions $\delta w, \delta u^a$, and $\delta\Delta$, which uniquely describe an arbitrary deformed configuration of the linearized column

$$\begin{aligned}
&A\,\delta w^{IV} + B\,\delta w'' + C\,\delta\Delta' = 0,\\
&D\,\delta u^{a\prime\prime} + E\,\delta w''' - K\,\delta\Delta = 0,\\
&F\,(\delta u^{a\prime\prime} - \delta\Delta'') + G\,\delta w''' + K\,\delta\Delta = 0,
\end{aligned} \tag{27}$$

where

$$\begin{aligned}
&A = -\frac{1}{1+\varepsilon}\left(C_{22} - \frac{C_{12}^a C_{21}^a}{C_{11}^a} - \frac{C_{12}^b C_{21}^b}{C_{11}^b}\right),\\
&B = R_X,\\
&C = K\left(\frac{C_{21}^a}{C_{11}^a} - \frac{C_{21}^b}{C_{11}^b}\right),\\
&D = C_{11}^a,
\end{aligned}$$

$$E = -\frac{C_{12}^a}{1+\varepsilon},\quad F = C_{11}^b,\quad G = -\frac{C_{12}^b}{1+\varepsilon}. \tag{28}$$

The aforementioned system of differential equations (27) may further be simplified and as a result only a fifth-order non-homogeneous linear differential equation with constant coefficients for unknown δw is derived

$$H\,\delta w^V + I\,\delta w''' + J\,\delta w' = S\,\mathcal{C}_1, \tag{29}$$

where $\mathcal{C}_1$ is an unknown integration constant and H, I, J, S are constants defined from

$$H = \frac{FA}{C},\quad I = \frac{ABF - CEF + ACG - A(A+F)K}{AC},\quad J = -\frac{KB}{C}(F+1),\quad S = -\left(\frac{KF}{D} + K\right). \tag{30}$$

The corresponding general solution of (29) is the superposition of the complementary solution $\delta w_H(x)$ which is the general solution of the associated homogeneous equations and the particular solution $\delta w_P(x)$ satisfying Eq. (29)

$$\delta w(x) = \delta w_H(x) + \delta w_P(x). \tag{31}$$

The homogeneous solution of (29) is obtained by solving the corresponding characteristic polynomial of Eq. (29), which is derived if δw in (29) is replaced by e^{rx}. Division of the derived equation by e^{rx} gives, see e.g. Coddington and Levinson, 1955

$$Hr^5 + Ir^3 + Jr = 0. \tag{32}$$

The solution of (32) is investigated parametrically for different geometric and material parameters and as a result three real ($\lambda_1 = 0, \lambda_2$ and λ_3) and two complex roots ($\lambda_5 = \beta i, \lambda_6 = -\beta i$) are obtained. According to the superposition principle, the solution of the corresponding homogeneous equation to (29) is therefore

$$\delta w_H(x) = \mathcal{C}_1 \sin\beta x + \mathcal{C}_2 \cos\beta x + \mathcal{C}_3\, e^{\lambda_2 x} + \mathcal{C}_4\, e^{\lambda_3 x} + \mathcal{C}_5. \tag{33}$$

On the other hand, a particular solution is obtained by the method of undetermined coefficients and is in this simple case given as

$$\delta w_P(x) = \mathcal{C}_6 \frac{S}{J} x. \tag{34}$$

Consequently, the general solution of (29) is

$$\delta w(x) = \mathcal{C}_1 \sin\beta x + \mathcal{C}_2 \cos\beta x + \mathcal{C}_3\, e^{\lambda_2 x} + \mathcal{C}_4\, e^{\lambda_3 x} + \mathcal{C}_5 + \mathcal{C}_6 \frac{S}{J} x. \tag{35}$$

Using the $\delta\mathcal{F}_{12}$ of (23) and substituting (35) into the last two equations of (27) we obtain the solution for δu^a and $\delta\Delta$ as

$$\begin{aligned}\delta u^a(x) = {} & \mathcal{C}_1 \frac{(M\beta^2 - N)\cos\beta x}{\beta} + \mathcal{C}_2 \frac{(N - M\beta^2)\sin\beta x}{\beta} \\ & + \mathcal{C}_3 \frac{(M\lambda_2^2 + N)e^{\lambda_2 x}}{\lambda_2} + \mathcal{C}_4 \frac{(M\lambda_3^2 + N)e^{\lambda_3 x}}{\lambda_3} \\ & + \mathcal{C}_5 N x + \mathcal{C}_6 \left(\frac{SNx}{J} + \frac{O}{2}x^2\right) + \mathcal{C}_7 x + \mathcal{C}_8,\end{aligned} \tag{36}$$

$$\begin{aligned}\delta\Delta(x) = {} & \mathcal{C}_1\, \beta(R - P\beta^2)\cos\beta x - \mathcal{C}_2\, \beta(R - P\beta^2)\sin\beta x + \mathcal{C}_3\, \lambda_2(P\lambda_2^2 \\ & + R)e^{\lambda_2 x} + \mathcal{C}_4\, \lambda_3(P\lambda_3^2 + R)e^{\lambda_3 x} + \mathcal{C}_6,\end{aligned} \tag{37}$$

where

$$M = -\frac{CE + AK}{CD},\quad N = -\frac{KB}{CD},\quad O = -\frac{K}{D},\quad P = -\frac{A}{C},\quad R = -\frac{B}{C}. \tag{38}$$

When $\delta w, \delta u^a$, and $\delta\Delta$ are known functions of x, the remaining quantities of the column $\delta u^b, \delta\varphi, \delta R_X^a, \delta R_X^b, \delta R_Z, \delta M_Y$, and δx^* and thus the general solution of the system of Eq. (23) can easily be obtained. In order to properly consider the boundary conditions (25), (26), it is suitable to express $\delta\varphi, \delta R_X^a, \delta R_X^b, \delta R_Z, \delta M_Y$ with (35)–(37) and their derivatives. Finally, the unknown integration constants $\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3, \mathcal{C}_4, \mathcal{C}_5, \mathcal{C}_6, \mathcal{C}_7$, and $\mathcal{C}_8$ are determined from the boundary conditions (25), (26). Applying (35)–(37) to (23) and (25), (26) and rearranging the following system of eight homogeneous linear algebraic equations for eight unknown constants is obtained. These equations can be expressed in a matrix form as

$$\boldsymbol{Kc} = \boldsymbol{0}, \tag{39}$$

where $\boldsymbol{K}$ and $\boldsymbol{c}$ denote a tangent matrix of the current equilibrium state on the fundamental path and a vector of unknown constants, respectively. A non-trivial solution of (39) is obtained only if determinant of the system matrix $\boldsymbol{K}$ is zero, see e.g. Planinc and Saje (1999)

$$\det \boldsymbol{K} = 0. \tag{40}$$

The condition (40) represents a linear eigenvalue problem and its solution, i.e. the lowest eigenvalue corresponds to the smallest critical buckling load, P_{cr}, of the column. The explicit form of matrix $\boldsymbol{K}$ and the analytical solution for the lowest buckling load, P_{cr}, can easiliy be determined but are unfortunately too cumbersome to be presented as closed-form expressions. This general stability criterion applies to all kinds of boundary conditions which are embedded in the general boundary conditions given in (25) and (26). The critical buckling loads for geometrically perfect two-layer composite columns with various forms of boundary conditions will be presented in the next section. For further details on determination of critical points and their classification an interested reader is referred to Planinc and Saje (1999).

3. Numerical examples

Numerical examples will demonstrate the applicability of the presented exact analytical model to predict critical buckling loads for various composite columns with partial interaction between the layers. Thus, the analytical model presented in the paper will be numerically evaluated through the analysis of two examples: (i) a comparison of the analytical results with existing results in the literature; (ii) a preliminary parametric analysis of the affect of various parameters on critical buckling loads of geometrically perfect two-layer composite column.

3.1. Exact critical buckling loads and comparison with existing results in the literature

This example presents a comparison of the analytical results for critical buckling loads of geometrically perfect two-layer composite columns with interlayer slip with existing buckling loads in the literature, proposed by Girhammar and Gopu (1993), Girhammar and Pan (2007), Xu and Wu (2007a,c), and Čas et al. (2007).

In order to compare critical buckling loads of the presented analytical model to the above-mentioned buckling models, the critical buckling loads of two-layer timber columns with different types of end conditions have been evaluated. Four kinds of two-layer Euler column end conditions: clamped-free column (C-F), clamped-

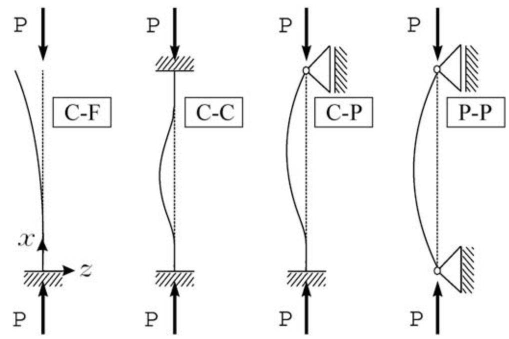

Fig. 3. Original and deflected (buckled) configurations of classical Euler columns for different end conditions.

clamped column (C-C), clamped-pinned column (C-P) and pinned–pinned column (P–P) have been considered, see Fig. 3.

In accordance to the boundary conditions (25) and (26) the classical boundary conditions of two-layer Euler columns and the corresponding non-zero values of parameters s_i and effective length coefficient, μ, are summarized in Table 1.

The results for critical buckling loads of the presented analytical model are compared to those obtained with what is called "modified second-order theory" which has been proposed by Girhammar and Gopu (1993), Girhammar and Pan (2007), Xu and Wu (2007a,c), and to those obtained numerically by Čas et al. (2007).

Hence, a simple but indicative example of the two-layer column with different kinds of column end conditions is considered. The mechanical and geometric properties of the two-layer composite column are characterized by the following parameters: elastic moduli of layers a and $b, E^a = E^b = 800\ \mathrm{kN/cm^2}$; interlayer-slip modulus $K \in [10^{-10}\ \mathrm{kN/cm^2} \leqslant K \leqslant 10^{10}\ \mathrm{kN/cm^2}]$; length of the column $L = 500\ \mathrm{cm}$; layer heights $h^a = h^b = 10\ \mathrm{cm}$; and widths $b^a = b^b = 20\ \mathrm{cm}$.

Table 1
Classical two-layer column boundary conditions and effective length factors β_E of Euler columns.

Classical cases	Non-zero values of s_i	Effective length coefficient
C-F	$s_2^0 = s_4^0 = s_6^0 = s_8^0 = 1$ $s_1^L = s_3^L = s_5^L = s_7^L = 1$	$\beta_E = 2$
C-C	$s_2^0 = s_4^0 = s_6^0 = s_8^0 = 1$ $s_1^L = s_3^L = s_6^L = s_8^L = 1$	$\beta_E = 0.5$
C-P	$s_2^0 = s_4^0 = s_6^0 = s_8^0 = 1$ $s_1^L = s_3^L = s_6^L = s_7^L = 1$	$\beta_E = 0.699$
P–P	$s_2^0 = s_4^0 = s_6^0 = s_7^0 = 1$ $s_1^L = s_3^L = s_6^L = s_7^L = 1$	$\beta_E = 1$

C = clamped (fixed); F = free; P = pinned

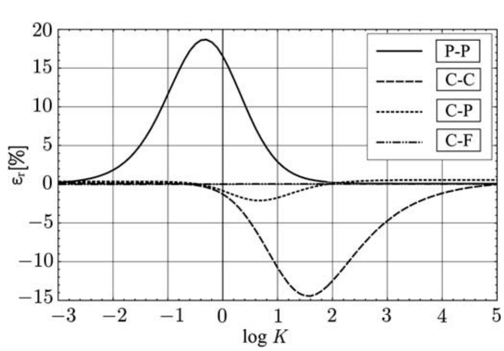

Fig. 4. Comparison of critical buckling loads of geometrically perfect two-layer composite column for different analytical models, end conditions, and different Ks.

Critical buckling loads as a function of K and different end conditions have been computed by the presented analytical model and compared to the results of Girhammar and Pan (2007). In Fig. 4, a relative error which is defined as

$$\varepsilon_r[\%] = \frac{P_{cr} - P^*_{cr}}{P_{cr}} \times 100, \tag{41}$$

is shown as a function of K for different end conditions where P^*_{cr} represents a critical buckling load obtained with the formula proposed by Girhammar and Gopu (1993), Girhammar and Pan (2007), Xu and Wu (2007a,c).

Positive errors indicate that formula derived by Girhammar and Pan (2007) underestimates the critical buckling loads of geometrically perfect two-layer composite columns. It is also interesting to note that the discrepancy between the exact buckling loads and buckling loads obtained by the "modified second-order theory" is interlayer-slip modulus and boundary conditions dependent. Of the values shown in Fig. 4, the maximum discrepancy is for the pinned–pinned column (P–P) and is about 18.5%, while for the clamped-free column (C-F) it is negligible. It is also apparent from Fig. 4 that critical force, P_{cr}, in C-C column case obtained by Girhammar and Pan (2007) is as much as approximately 14.5% higher than the exact ones. Thus, in the C-C column case the buckling load calculated by Girhammar and Pan (2007), is rather conservative. On the other hand, the exact critical buckling loads are practically identical with the numerically obtained critical loads, see Čas et al. (2007).

The critical buckling loads of two-layer pinned–pinned composite column are presented in detail in Table 2.

As anticipated, there is a general trend showing that critical buckling load, P_{cr}, of two-layer pinned–pinned column decreases by decreasing the inter-layer stiffness K. The discrepancy is the largest for values of inter-layer slip modulus K which usually exists in actual practice. Hence, a large effect of inter-layer slip is evident especially when actual buckling loads of two-layer composite column are compared with the one for an equivalent solid column, obtained by e.g. Flajs et al. (2003). Note also that in the limiting case when there is absolutely stiff connection ($\Delta = 0; K \to \infty$) or there exists no connection between the layers ($\Delta = \Delta_{max} \neq 0; K \to 0$), the exact buckling loads of geometrically perfect two-layer composite columns converge perfectly to the analytical buckling loads of the corresponding solid column. In these special cases only minor disagreement is observed between the critical buckling loads obtained by the present method and the analytical buckling loads obtained by Girhammar and Gopu (1993), Girhammar and Pan (2007), Xu and Wu (2007a,c).

From this example we can confirm (see Fig. 4) that partial interaction between the layers has a considerable influence on critical buckling loads of geometrically perfect two-layer composite columns.

3.2. Preliminary parametric analysis of the affect of various parameters on critical buckling loads of geometrically perfect two-layer composite column

This section presents a preliminary parametric study performed on a geometrically perfect two-layer composite column subjected to a concentrated compressive axial force P, see Fig. 3, with the aim to investigate the influence of boundary conditions, material and geometric parameters, such as inter-layer slip modulus K, depth-to-depth ratios h^a/h^b, column slenderness λ, etc., on critical buckling loads of the geometrically perfect two-layer composite column.

The critical buckling loads have been calculated for different kinds of boundary conditions, different values of parameters K, λ, and h^a/h^b. The results are presented in Figs. 5 and 6.

Table 2
Comparison of buckling loads of pinned–pinned two-layer composite column with different models and different *K*s.

P_{cr} [kN]				
K [kN/cm^2]	Girhammar and Gopu (1993)*	Present	ε_r [%]	Čas et al. (2007)**
10^{-10}	105.2757803	105.3104375	0.0329	105.3104374***
10^{-5}	105.2767803	105.3134393	0.0331	–
10^{-3}	105.3757486	105.6099848	0.2218	105.615
10^{-2}	106.2726240	108.2487730	1.8256	–
10^{-1}	114.9688693	130.0907979	11.624	130.117
1	181.2273375	217.1489159	16.542	217.190
10^{1}	345.2976517	355.6165146	2.9017	355.617
10^{2}	411.4338134	412.4908988	0.2563	412.530
10^{3}	420.1087924	420.6795391	0.1357	–
10^{5}	421.0931467	421.6487510	0.1317	421.617
10^{10}	421.1031210	421.6587338	0.1317	421.6587339***

* Girhammar and Pan (2007), Xu and Wu (2007a,c).
** Numerical solution.
*** Flajs et al. (2003)(Analytical solution for $K = 0, \infty$).

In Fig. 5 the critical buckling load, P_{cr}, of the geometrically perfect two-layer composite columns with partial inter-layer connection between the layers is calculated for various inter-layer slip moduli K and for different column slenderness λ which is defined as

$$\lambda = \frac{\beta_E L \sqrt{b^a h^a + b^b h^b}}{\sqrt{b^a \int_0^{h^a} z^2 \mathrm{d}z + b^b \int_{-h^b}^{0} z^2 \mathrm{d}z}}, \tag{42}$$

where β_E represents the effective length coefficient of Euler columns with stiff connection between the layers. Effective length coefficients, β_E, are given in Table 1 for different types of end conditions along with schematic illustrations of the buckling modes in Fig. 3. Variation in column slenderness has been achieved by considering a range of column lengths.

In Fig. 5 it can be observed that by decreasing the column slenderness, λ, and increasing the inter-layer slip modulus, K, the critical buckling load, P_{cr}, increases in all cases of boundary conditions. The influence of K on critical buckling loads is considerable and is the biggest in the P–P column case where the difference between critical buckling loads P_{cr} for $\lambda = 60$ ranges between 215.0 and 861.7 kN. It is interesting to note that the multiplication factor by which P_{cr} changes with K does not depend on column slenderness. Furthermore, the results shown in Fig. 5 indicate that critical buckling loads depend on boundary conditions. Consequently, for $K = 10$ kN/cm^2 and $L = 800$ cm a critical force P_{cr} is in the C-F case 151.02 kN, in the P–P case 519.29 kN, in the C-P case 821.09 kN and finally in the C-C case 1220.7 kN.

A preliminary parametric study has also been conducted to asses the effect of depth-to-depth ratios h^a/h^b and K on critical buckling loads of geometrically perfect layered composite columns. For this purpose, critical forces have been calculated for various h^a/h^b, Ks and different end conditions. When treating various structural stability problems it is often useful to express the buckling load, P_{cr}, in the form of the Euler formula with a suitable modification of the column length. Thus, the critical load of a layered geometrically perfect composite column with interlayer slip may be expressed in terms of the classical Euler formula for solid column as

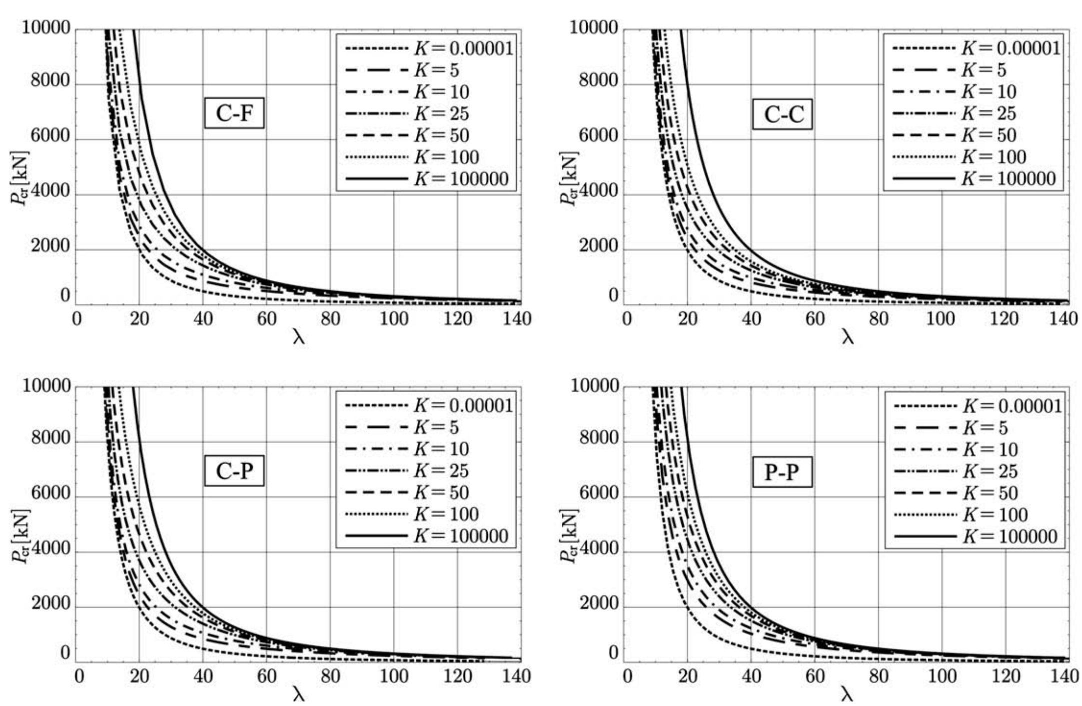

Fig. 5. Illustrations of critical buckling load, P_{cr}, for different Ks, λ, and different types of end conditions. These diagrams are applicable for this specific example only.

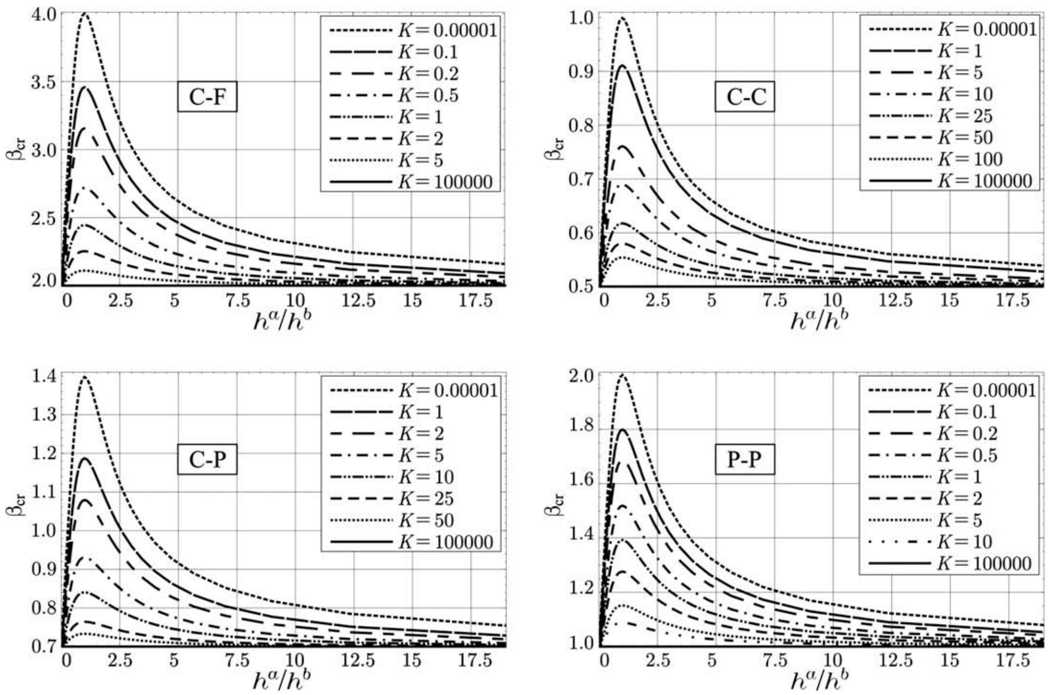

Fig. 6. Illustrations of critical effective length parameter, β_{cr}, for different Ks, h^a/h^b, and different types of end conditions. These diagrams are applicable for this specific example only.

$$P_{cr} = \frac{\pi^2 (EJ)_s}{(\beta_{cr} L)^2}, \tag{43}$$

in which $(EJ)_s$ is the flexural rigidity of the corresponding solid column and β_{cr} denotes the critical effective length parameter of the geometrically perfect two-layer composite column which depends entirely on the particular buckling mode, inter-layer contact stiffness K, and depth-to-depth ratio h^a/h^b and should not be confused with the effective length coefficient β_E that gives the distance between the points of inflection in a solid column. The effective length coefficient β_{cr} is obtained by a comparison of the critical force P_{cr} calculated with the presented exact model and the Euler critical force, P_E, for a solid column

$$\beta_{cr} = \sqrt{\frac{P_E}{P_{cr}}} \beta_E. \tag{44}$$

The critical effective length coefficient, β_{cr}, against the depth-to-depth ratio, h^a/h^b, is shown in Fig. 6 for different Ks and different end conditions. In all four kinds of end conditions, the parametric study reveals that minimum critical forces and maximum critical effective length coeffecients are obtained when layers have approximately equal depths, i.e., $h^a/h^b \approx 1$. In Fig. 6 it is also shown that β_{cr} is higher for smaller values of K and can be in case of fully flexible connection ($K = 10^{-5}\ \mathrm{kN/cm^2}$) as much as about two times higher than in the case of absolutely stiff connection between the layers. Conseuentlly, the corresponding critical forces can be four times smaller in comparison with the critical forces of the geometrical and material equivalent solid column. The effect of the h^a/h^b ratio on the β_{cr} becomes much less pronounced for higher values of K. This effect becomes negligible in the case of the absolutely stiff connection where β_{cr} equals β_E. Similarly, this effect may also be neglected for composite columns where the depth of one layer is very small compared to the depth of the other one. For example, for $h^a/h^b = 3$ and $K = 1\ \mathrm{kN/cm^2}$, the effective length parameter, β_{cr}, is in the C-F column case 2.248, in the C-C column case 0.716, in the C-P column case 0.960, and in the P–P column case 1.206, while for $h^a/h^b = 19$ and $K = 1\ \mathrm{kN/cm^2}$, the $\beta_{cr}[\mathrm{C-F}] = 2.018; \beta_{cr}[\mathrm{C-C}] = 0.528; \beta_{cr}[\mathrm{C-P}] = 0.728$; and $\beta_{cr}[\mathrm{P-P}] = 1.018$.

Partial interaction between the layers has a considerable influence on critical buckling load of geometrically perfect two-layer composite column and hence should be taken into consideration when composite columns with inter-layer slip are analysed.

4. Conclusions

A mathematical model for slip-buckling has been proposed and its analytical solution has been found for the analysis of layered and geometrically perfect composite columns with inter-layer slip between the layers. The analytical study has been carried out for evaluating exact critical forces and comparing them to those in the literature. Particular emphasis has been given to the influence of interface compliance on decreasing the bifurcation loads. For this purpose, a preliminary parametric study has been performed by which the influence of various material and geometric parameters on buckling forces have been investigated. A detailed parametric analysis by which the influence of various non-dimensional parameters on buckling forces of geometrically perfect composite columns will be the topic of the future analysis. Based on the analytical results and the preliminary parametric study undertaken the following conclusions are drawn:

(1) The present formulation of slip-buckling problem is applicable for the geometrically perfect composite columns only, it is relatively easy to comprehend and agrees well with the classical results for Euler columns.
(2) Since the solution of slip-buckling problem has been obtained without simplification of the governing equations, the results for buckling forces can be considered more accurate than those proposed by other researchers, e.g. Girhammar and Pan (2007), Xu and Wu (2007a), and others, who used what is called modified second-order theory in which the influnece of extensional strains on buckling loads in not explicitly incorporated.

(3) The discrepancy between the exact buckling loads and buckling loads obtained by the "modified second-order theory" depends on interlayer-slip modulus and boundary conditions. The maximum discrepancy is for the pinned–pinned column (P–P), and is about 18.5%, whereas for the clamped-free column (C-F), it is negligible. A critical force, P_{cr}, obtained by Girhammar and Pan (2007) can be in C-C column case about 14.5% higher than the exact one. In the C-C column case the buckling load calculated by Girhammar and Pan (2007) is rather conservative.

(4) The preliminary parametric study has confirmed that reduced stiffness between the layers can promote buckling which can lead to a drastic reduction of bifurcation load. Thus, partial interaction between the layers should be taken into consideration when composite columns with interlayer slip are considered.

(5) By decreasing the column slenderness, λ, and increasing the inter-layer slip modulus, K, the critical buckling load, P_{cr}, increases in all cases of boundary conditions. The influence of K on critical buckling loads is considerable, and is the biggest in P–P column case. The ratios between P_{cr} for $K = 10\ \mathrm{kN/cm^2}$ and $K \to \infty$ are 3.67, 3.16, 2.44, and 1.86 for C-F, P–P, C-P, and C-C cases, respectively.

(6) In all four kinds of boundary conditions, the minimum critical forces or maximum effective length parameters are obtained when the layers have approximately equal depths, i.e. $h^a/h^b \approx 1$. β_{cr} is higher for smaller values of K and can be, in the case of fully flexible connection ($K = 10^{-5}\ \mathrm{kN/cm^2}$), as much as about two times higher than in the case of absolutely stiff connection. The corresponding critical forces can be four times smaller in comparison with the critical forces of the equivalent solid column. The effect of the h^a/h^b ratio on the β_{cr} becomes much less pronounced for higher values of K. This effect becomes negligible in the case of the absolutely stiff connection where β_{cr} equals β_E. The effect may be neglected for composite columns where the depth on one layer is very small compared to the depth of the other one.

Acknowledgement

The authors would like to thank Mrs. Romana Hudin for a detailed language review of this paper.

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