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Reflection and transmission of elastic waves at an elastic/porous solid saturated by two immiscible fluids

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Abstract

Wave propagation in a porous elastic medium saturated by two immiscible fluids is investigated. It is shown that there exist three dilatational waves and one transverse wave propagating with different velocities. It is found that the velocities of all the three longitudinal waves are influenced by the capillary pressure, while the velocity of transverse wave does not at all. The problem of reflection and refraction phenomena due to longitudinal and transverse wave incident obliquely at a plane interface between uniform elastic solid half-space and porous elastic half-space saturated by two immiscible fluids has been analyzed. The amplitude ratios of various reflected and refracted waves are found to be continuous functions of the angle of incidence. Expression of energy ratios of various reflected and refracted waves are derived in closed form. The amplitude ratios and energy ratios have been computed numerically for a particular model and the results obtained are depicted graphically. It is verified that during transmission there is no dissipation of energy at the interface. Some particular cases have also been reduced from the present formulation.

Keywords: Poroelasticity; Capillary effects; Unsaturated porous medium; Boundary conditions

1. Introduction

The problems of reflection and refraction of elastic waves from the discontinuity between two elastic half-spaces are of great interest in various fields e.g. geophysics, seismology and petroleum engineering. These problems not only provide better information about the internal composition of the Earth but are

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Nomenclature

α	velocity of longitudinal wave in elastic half-space
β	velocity of transverse wave in elastic half-space
λ, μ	Lame's parameters
τ_e	stress in elastic half-space
τ_s	stress in porous elastic half-space
$\langle\tau_i\rangle$	pressures in fluid phase i
$\mathbf{u}_e$	displacement vector in the porous elastic solid
ϕ_e, ψ_e	potentials in elastic half-space
ρ_e	density of the elastic half-space
K_i	bulk modulus of fluid phase i
v_i	velocity of fluid phase i
v_s	velocity of solid phase
α_s	volume fraction of the solid phase
α_i	volume fraction of the fluid phase i
G_{fr}	shear modulus of the porous solid
K	intrinsic permeability
K_{fr}	frame or drained bulk modulus
K_{ri}	relative permeability of fluid phase i
μ_i	viscosity of fluid phase i
P_i^*	intrinsic averaged pressure of fluid phase i
$\langle\rho_i\rangle$	volume averaged density of fluid phase i
$\langle\rho_s\rangle$	volume averaged density of porous solid
k	wavenumber
$\mathbf{u}_i$	displacement vector in the fluid phase i
$\mathbf{u}_s$	displacement vector in the porous elastic solid
V	phase velocity
ω	angular frequency
X_1, X_2, X_3	velocities of compressional waves in porous solid
X_4	velocity of transverse wave in porous solid
ϕ, ψ, η	scalar potentials
$\mathbf{H}, \mathbf{G}, \mathbf{J}$	vector potentials
I	unit tensor matrix
$\mathbf{r}, \mathbf{n}$	Position vector, unit normal vector
P_{cap}	capillary pressure
ι	$\sqrt{-1}$.

also very helpful in exploration of valuable materials beneath the Earth surface e.g. water, oils, minerals, hydro-carbon etc. Many problems of reflection and refraction of elastic waves from boundaries have been discussed by the researchers in the past and have appeared in the open literature (see Ewing et al., 1957; Achenbach, 1973; Aki and Richards, 1980; Sheriff and Geldart, 1995; Udias, 1999, among several others).

Biot (1956a,b) formulated the dynamical equations and constitutive relations for a fluid saturated porous elastic solid. Since then a number of dynamical problems in the porous media have been investigated by the researchers. Notable among them are Deresiewicz (1960, 1962, 1964, 1967), Geertsma and Smith (1961), Biot (1962), Yew and Jogi (1976), Plona (1980), Berryman (1981), Hajra and Mukhopadhyay (1982), de la Cruz and Spanos (1985), Wu et al. (1990), de la Cruz et al. (1992), Sharma and Gogna (1992), Cieszko and Kubik (1998), Gurevich and Schoenberg (1999) among several others. Basically, there are three widely acceptable theories of porous media: one is Biot's theory, second is mixture theory and third is the averaging theory or hybrid mixture theory. Literature is extensive on mixture theories. Some of the references are Morland (1972), Bedford and Drumheller (1978), Bowen (1980, 1982) and Hassanizadeh and Gray (1990). Bowen (1982) has shown that Biot's theory and mixture theory are equivalent, if the coupling parameter between the state variable of the solid skeleton and the pore fluid, introduced by Biot, is neglected. Mixture theory for porous media saturated by fluids includes the concept of volume fraction to characterize the microstructure of the medium. Hybrid mixture theory (HMT) is an upscaling approach based on thermodynamical principals and is used to develop deterministic models of porous media. Recently, Lu and Hanyga (2005) presented a comprehensive comparison of these theories.

Biot (1956a,b) developed constitutive relations and the equations of motion, including inertial terms for a liquid saturated porous medium. He has also discussed the propagation of plane harmonic elastic waves and shown that in an isotropic, homogeneous fluid saturated porous medium, there are two *P*-waves namely fast *P*-wave and slow *P*-wave, and one *S*-wave propagating with different velocities. The problems of wave propagation in porous elastic medium saturated by two or more fluids together are also of great interest in seismology. Tuncay and Corapcioglu (1997) developed a theory of wave propagation in isotropic poroelastic media saturated by two immiscible Newtonian fluids. They applied the volume average technique to explore the wave propagation characteristics in a linearly elastic medium saturated by two immiscible Newtonian fluids. The equations for low frequency wave propagation in a poroelastic medium saturated by two immiscible fluids are developed. They have shown that in such a medium, there exist three compressional waves and one transverse wave. The first two compressional waves and one transverse wave are similar to those predicted by Biot (1956a,b), while the third compressional wave arises due to the pressure difference between the fluid phase and is dependent on the slope of the capillary pressure–saturation relation. The third compressional wave was also predicted by Garg and Nayfeh (1986) and Santos et al. (1990b). Santos et al. (1990a) proposed a method to determine the elastic constants for isotropic porous media saturated by two fluids. Gray and Schrefler (2001) derived the equilibrium effective stress acting on the solid phase of a porous medium containing two immiscible fluid phases and obtained the relation between the fluid pressures at the fluid–fluid interface. Wei and Muraleetharan (2002a,b) developed a continuum theory of multiphase porous media that is capable of rigorously characterizing the interactions among coexisting components. They have also developed a macroscale model where the state of a porous medium is described by macrostate variables measurable through experiments. Recently, Lu and Hanyga (2005) developed a linear isothermal dynamic model for a porous medium saturated by two immiscible fluids. They obtained equation of motion in the frequency domain and calculated wave velocities and attenuations for three *P*-waves and one *S*-wave. Thigpen and Berryman (1985) presented a continuum theory of mixtures for a porous elastic solid saturated by immiscible viscous fluids. Bedford and Drumheller (1983) gave an extensive survey of continuum theories of mixtures of immiscible constituents. Schanz and Diebels (2003) presented the governing equations for the mixture theory based on the Theory of Porous Media (TPM-mixture theory extended by the concept of volume fractions) under the assumption of linear theory in terms of small displacements and small deformation gradients. They have also derived linear constitutive equations (Hooke's law) and showed that the structure of governing differential equations in Biot's Theory and in Theory of Porous Media is the same and hence the wave forms predicted by these theories are equal. Hanyga (2004) developed a general dynamical model for porous media saturated by two immiscible fluids. The central idea of his theory is the use of one energy and one entropy for the porous medium as well as the use of the concept of volume fraction from the mixture theory.

In the present paper we first discuss the low frequency wave propagation in a porous elastic medium saturated by two immiscible fluids using the theory proposed by Tuncay and Corapcioglu (1997). We then present the calculations of reflection and refraction coefficients for longitudinal and transverse waves propagating through the uniform elastic half-space and striking obliquely at the plane interface of elastic/porous half-space. These coefficients are then used to find the expressions of energy ratios of various reflected and refracted waves at the interface. Numerical computations have been performed for a particular model to study the nature of dependence of these coefficients and of energy ratios on angle of incidence of the incident wave. The energy conservation at the interface is rigorously verified. The variations of velocities of existing waves with some fluid parameters have been presented. Some problems of reflection and refraction phenomenon have also been deduced as particular case of this formulation.

2. Field equations and constitutive relations

Following Tuncay and Corapcioglu (1997), the equations of motion in the absence of body forces for low frequency wave propagation in a porous elastic medium saturated by two immiscible liquids, are given by

$$\begin{aligned}\langle\rho_s\rangle\frac{\partial^2\mathbf{u}_s}{\partial t^2} &= \nabla\left(\left(a_{11}+\frac{1}{3}G_{fr}\right)\nabla\cdot\mathbf{u}_s + a_{12}\nabla\cdot\mathbf{u}_1 + a_{13}\nabla\cdot\mathbf{u}_2\right)\\ &\quad + \nabla\cdot(G_{fr}\nabla\mathbf{u}_s) + c_1(v_1-v_s)+c_2(v_2-v_s),\end{aligned} \tag{1}$$

$$\langle\rho_1\rangle\frac{\partial^2\mathbf{u}_1}{\partial t^2} = \nabla(a_{21}\nabla\cdot\mathbf{u}_s + a_{22}\nabla\cdot\mathbf{u}_1 + a_{23}\nabla\cdot\mathbf{u}_2) - c_1(v_1-v_s), \tag{2}$$

$$\langle\rho_2\rangle\frac{\partial^2\mathbf{u}_2}{\partial t^2} = \nabla(a_{31}\nabla\cdot\mathbf{u}_s + a_{32}\nabla\cdot\mathbf{u}_1 + a_{33}\nabla\cdot\mathbf{u}_2) - c_2(v_2-v_s), \tag{3}$$

where

$$\begin{aligned}&a_{11}=K_{fr},\quad a_{12}=a_{21}=K_1\alpha_s S_1(A_2+K_2)/D,\quad a_{13}=a_{31}=K_2\alpha_s(1-S_1)(A_2+K_1),\\ &a_{22}=K_1S_1^2(1-\alpha_s)\left(K_2+\frac{A_2}{S_1}\right)\Big/D,\quad a_{23}=a_{32}=K_1K_2S_1(1-S_1)(1-\alpha_s)/D,\\ &a_{33}=K_2(1-S_1)^2(1-\alpha_s)\left(K_s+\frac{A_2}{(1-S_1)}\right)\Big/D,\quad D=K_1(1-S_1)+A_2+K_2S_1,\\ &c_1=(1-\alpha_s)^2S_1^2\mu_1/KK_{r1},\quad c_2=(1-\alpha_s)^2(1-S_1)^2\mu_2/KK_{r2},\end{aligned}$$

where K_i and v_i are the bulk modulus and velocity of fluid phase i, v_s is the velocity of solid phase, α_s is the volume fraction of the solid phase, K is the intrinsic permeability of the medium and K_{ri} is the relative permeability of fluid phase i. G_{fr} is the shear modulus of the porous solid whereas K_{fr} is frame or drained bulk modulus. $\mathbf{u}_s$ is the the displacement vector in the porous elastic solid and $\mathbf{u}_i$ is the displacement vector in the fluid phase i, $\langle\rho_s\rangle$ is the volume averaged density of porous solid, $\langle\rho_i\rangle$ and μ_i are the volume averaged density and viscosity of fluid phase i respectively, $S_i=\alpha_i/(1-\alpha_s)$, $(i=1,2)$ with $S_1+S_2=1$ and $A_2=\frac{dP_{cap}}{dS_1}S_1(1-S_1)$, where $P_{cap}=P_1^*-P_2^*$, P_i^* is the intrinsic averaged pressures of fluid phase i and α_i is the volume fraction for the fluid phase i.

Introducing the scalar potentials ϕ, ψ and η; vector potentials $\mathbf{H}$, $\mathbf{G}$ and $\mathbf{J}$ through Helmholtz representation of vector as follows

$$\mathbf{u}_s=\nabla\phi+\nabla\times\mathbf{H},\quad \nabla\cdot\mathbf{H}=0, \tag{4}$$

$$\mathbf{u}_1=\nabla\psi+\nabla\times\mathbf{G},\quad \nabla\cdot\mathbf{G}=0, \tag{5}$$

$$\mathbf{u}_2=\nabla\eta+\nabla\times\mathbf{J},\quad \nabla\cdot\mathbf{J}=0, \tag{6}$$

Inserting these values of vectors $\mathbf{u}_s$, $\mathbf{u}_1$ and $\mathbf{u}_2$ into Eqs. (1)–(3), we obtain the following equations

$$\langle\rho_s\rangle\frac{\partial^2\phi}{\partial t^2} = a_{11}^*\nabla^2\phi + a_{12}\nabla^2\psi + a_{13}\nabla^2\eta + c_1\left(\frac{\partial\psi}{\partial t} - \frac{\partial\phi}{\partial t}\right) + c_2\left(\frac{\partial\eta}{\partial t} - \frac{\partial\phi}{\partial t}\right), \tag{7}$$

$$\langle\rho_1\rangle\frac{\partial^2\psi}{\partial t^2} = a_{21}\nabla^2\phi + a_{22}\nabla^2\psi + a_{23}\nabla^2\eta + c_1\left(\frac{\partial\phi}{\partial t} - \frac{\partial\psi}{\partial t}\right), \tag{8}$$

$$\langle\rho_2\rangle\frac{\partial^2\eta}{\partial t^2} = a_{31}\nabla^2\phi + a_{32}\nabla^2\psi + a_{33}\nabla^2\eta + c_2\left(\frac{\partial\phi}{\partial t} - \frac{\partial\eta}{\partial t}\right), \tag{9}$$

$$\langle\rho_s\rangle\frac{\partial^2\mathbf{H}}{\partial t^2} = G_{\text{fr}}\nabla^2\mathbf{H} + c_1\left(\frac{\partial\mathbf{G}}{\partial t} - \frac{\partial\mathbf{H}}{\partial t}\right) + c_2\left(\frac{\partial\mathbf{J}}{\partial t} - \frac{\partial\mathbf{H}}{\partial t}\right), \tag{10}$$

$$\langle\rho_1\rangle\frac{\partial^2\mathbf{G}}{\partial t^2} = c_1\left(\frac{\partial\mathbf{H}}{\partial t} - \frac{\partial\mathbf{G}}{\partial t}\right), \tag{11}$$

$$\langle\rho_2\rangle\frac{\partial^2\mathbf{J}}{\partial t^2} = c_2\left(\frac{\partial\mathbf{H}}{\partial t} - \frac{\partial\mathbf{J}}{\partial t}\right), \tag{12}$$

where $a_{11}^* = a_{11} + \frac{4}{3}G_{\text{fr}}$.

3. Wave propagation

Let us consider a plane wave propagating in the positive direction of a unit vector $\mathbf{n}$ in the form given by

$$\{\phi, \psi, \eta\} = \{A, B, C\}\exp\{\imath k(\mathbf{n}\cdot\mathbf{r} - Vt)\}, \tag{13}$$

where k is the wavenumber, V is the phase velocity of the wave and A, B and C denote the amplitudes and $\mathbf{r}$ is the position vector. Substituting Eq. (13) into Eqs. (7)–(9), we get the following equations

$$(a_{11}^*k^2 - \imath(c_1 + c_2)\omega - \langle\rho_s\rangle\omega^2)A + (a_{12}k^2 + \imath c_1\omega)B + (a_{13}k^2 + \imath c_2\omega)C = 0, \tag{14}$$

$$(a_{21}k^2 + \imath c_1\omega)A + (k^2a_{22} - \imath c_1\omega + \langle\rho_1\rangle\omega^2)B + a_{23}k^2C = 0, \tag{15}$$

$$(a_{31}k^2 + \imath c_2\omega)A + a_{32}k^2B + (a_{33}k^2 - \imath c_2\omega - \langle\rho_2\rangle\omega^2)C = 0, \tag{16}$$

where $\omega = kV$ is the angular frequency. A non-trivial solution of Eqs. (14)–(16) exists if the determinant of the coefficient matrix vanishes:

$$Z_1X^3 + Z_2X^2 + Z_3X + Z_4 = 0, \tag{17}$$

where $X = \frac{\omega^2}{k^2}$ and the coefficients of Eq. (17) are given by,

$$\begin{aligned} Z_1 = {} & \frac{c_1c_2(\langle\rho_1\rangle + \langle\rho_2\rangle + \langle\rho_s\rangle) - \langle\rho_s\rangle\langle\rho_1\rangle\langle\rho_2\rangle\omega^2}{\omega^2} \\ & - \frac{\imath c_2\langle\rho_1\rangle(\langle\rho_2\rangle + \langle\rho_s\rangle) + c_1\langle\rho_2\rangle(\langle\rho_1\rangle + \langle\rho_s\rangle)}{\omega}, \end{aligned} \tag{18}$$

$$\begin{aligned} Z_2 = {} & -\frac{a_{11}^*(c_1c_2 - \langle\rho_1\rangle\langle\rho_2\rangle\omega^2) + 2c_1c_2(a_{12} + a_{13} + a_{23})}{\omega^2} \\ & - \frac{a_{22}(c_1c_2 - \langle\rho_s\rangle\langle\rho_2\rangle\omega^2) + a_{33}(c_1c_2 - \langle\rho_s\rangle\langle\rho_1\rangle\omega^2)}{\omega^2} \\ & - \imath\frac{a_{11}(c_2\langle\rho_1\rangle + c_1\langle\rho_2\rangle) + 2a_{12}c_1\langle\rho_2\rangle + 2a_{13}c_2\langle\rho_1\rangle}{\omega} \\ & - \imath\frac{a_{22}(c_2(\langle\rho_2\rangle + \langle\rho_s\rangle) + c_1\langle\rho_2\rangle) + a_{33}(c_2\langle\rho_1\rangle + c_1(\langle\rho_s\rangle + \langle\rho_1\rangle))}{\omega}, \end{aligned} \tag{19}$$

$$Z_3 = -a^*_{11}(a_{22}\langle\rho_2\rangle + a_{33}\langle\rho_1\rangle) - a^2_{12}\langle\rho_2\rangle - a^*_{13}\langle\rho_1\rangle + \langle\rho_s\rangle(a_{22}a_{33} - a^2_{23})$$
$$- \imath\frac{a_{11}(a_{22}c_2 + a_{33}c_1) - a^2_{12}c_2 - 2a_{12}(a_{21}c_2 - a_{33}c_1) - a^2_{13}c_1 + 2a_{13}(a_{22}c_2 - a_{23}c_1)}{\omega}$$
$$- \imath\frac{(c_1 + c_2)(a_{22}a_{33} - a^2_{23})}{\omega}, \tag{20}$$
$$Z_4 = a^*_{11}(a_{22}a_{33} - a^2_{23}) - a^2_{12}a_{33} + a_{13}(2a_{12}a_{23} - a_{13}a_{22}). \tag{21}$$

Similarly, we can take

$$\{\mathbf{H}, \mathbf{G}, \mathbf{J}\} = \{\mathbf{A}, \mathbf{B}, \mathbf{C}\}\exp\{\imath k(\mathbf{n}\cdot\mathbf{r} - Vt\}). \tag{22}$$

Substituting Eq. (22) into Eqs. (10)–(12) and putting the determinant of the coefficient matrix equal to zero, we obtain

$$X^2(Z_5X + Z_6) = 0, \tag{23}$$

where

$$Z_5 = \frac{c_1c_2(\langle\rho_1\rangle + \langle\rho_2\rangle + \langle\rho_s\rangle) - \langle\rho_s\rangle\langle\rho_1\rangle\langle\rho_2\rangle\omega^2}{\omega^2}$$
$$- \imath\frac{c_2\langle\rho_1\rangle(\langle\rho_2\rangle + \langle\rho_s\rangle) + c_1\langle\rho_2\rangle(\langle\rho_1\rangle + \langle\rho_s\rangle)}{\omega}, \tag{24}$$
$$Z_6 = \frac{-G_{\mathrm{fr}}(c_1c_2 - \langle\rho_1\rangle\langle\rho_2\rangle\omega^2)}{\omega^2} + \imath\frac{G_{\mathrm{fr}}(c_2\langle\rho_1\rangle + c_1\langle\rho_2\rangle)}{\omega}, \tag{25}$$

The coefficients given by (18)–(21), (24), (25) are the same coefficients as obtained by Tuncay and Corapcioglu (1997) for the relevant analysis.

Using Eq. (13) into Eqs. (4)–(6), we obtain

$$\{\mathbf{u}_s, \mathbf{u}_1, \mathbf{u}_2\} = \{A, B, C\}\imath k\mathbf{n}\exp\{\imath k(\mathbf{n}\cdot\mathbf{r} - Vt)\}.$$

This shows that the displacement vectors $\mathbf{u}_s$, $\mathbf{u}_1$ and $\mathbf{u}_2$ have the same direction as that of $\mathbf{n}$. Therefore, three waves with velocities given by Eq. (17) are compressional (longitudinal). The three roots of Eq. (17) denoted by X_1, X_2 and X_3 would then represent the velocities of first, second and third longitudinal waves respectively. Similarly, substituting Eq. (22) into Eqs. (4)–(6) and taking scalar product with $\mathbf{n}$, we find that $\mathbf{u}_s \cdot \mathbf{n} = 0$. Thus, the wave propagating with velocity X_4 (which is given by the root of Eq. (23)) is a transverse wave. It can be seen that, if we put $A_2 = 0$, the coefficient Z_4 vanishes. This shows that one of the longitudinal wave is associated with the pressure difference between two fluid phases, i.e. capillary pressure.

Following Tuncay and Corapcioglu (1997), the stress in porous solid is given by

$$\langle\tau_s\rangle = (a_{11}\nabla\cdot\mathbf{u}_s + a_{12}\nabla\cdot\mathbf{u}_1 + a_{13}\nabla\cdot\mathbf{u}_2)I + G_{\mathrm{fr}}\left(\nabla\mathbf{u}_s + (\nabla\mathbf{u}_s)^{\mathrm{T}} - \frac{2}{3}\nabla\cdot\mathbf{u}_sI\right), \tag{26}$$

and the pressures in fluids are given by

$$\langle\tau_1\rangle = (a_{21} + \nabla\cdot\mathbf{u}_s + a_{22}\nabla\cdot\mathbf{u}_1 + a_{23}\nabla\cdot\mathbf{u}_2)I, \tag{27}$$
$$\langle\tau_2\rangle = (a_{31} + \nabla\cdot\mathbf{u}_s + a_{32}\nabla\cdot\mathbf{u}_1 + a_{33}\nabla\cdot\mathbf{u}_2)I, \tag{28}$$

where I is unit tensor matrix.

The general displacement of the solid and fluids in x and z directions are given by

$$u_{sx} = \frac{\partial\phi}{\partial x} - \frac{\partial H}{\partial z}, \quad u_{sz} = \frac{\partial\phi}{\partial z} + \frac{\partial H}{\partial x}, \tag{29}$$

$$u_{1x} = \frac{\partial\psi}{\partial x} - \frac{\partial G}{\partial z}, \quad u_{1z} = \frac{\partial\psi}{\partial z} + \frac{\partial G}{\partial x}, \tag{30}$$

$$u_{2x} = \frac{\partial\eta}{\partial x} - \frac{\partial J}{\partial z}, \quad u_{2z} = \frac{\partial\eta}{\partial z} + \frac{\partial J}{\partial x}, \tag{31}$$

where u_{sx}, u_{sz}, u_{1x}, u_{1z}, u_{2x} and u_{2z} denote the displacement components in porous solid, first fluid and second fluid respectively. The suffixes x and z denote their directions. H, G and J are the y-components of the vectors $\mathbf{H}$, $\mathbf{G}$ and $\mathbf{J}$ respectively. Taking $\{\mathbf{H},\mathbf{G},\mathbf{J}\}(\mathbf{x},\mathbf{z},\mathbf{t}) = \{\mathbf{H},\mathbf{G},\mathbf{J}\}(\mathbf{x},\mathbf{z})\exp(-\imath\omega\mathbf{t})$, we get the following solutions of Eqs. (11) and (12)

$$\mathbf{G} = \frac{\imath c_1 \mathbf{H}}{\imath c_1 + \langle\rho_1\rangle\omega}, \quad \mathbf{J} = \frac{\imath c_2 \mathbf{H}}{\imath c_2 + \langle\rho_2\rangle\omega}, \tag{32}$$

Considering Helmholtz representation of displacement vector in the uniform elastic medium, we have the components

$$u_{ex} = \frac{\partial\phi_e}{\partial x} - \frac{\partial\psi_e}{\partial z}, \quad u_{ez} = \frac{\partial\phi_e}{\partial z} + \frac{\partial\psi_e}{\partial x} \tag{33}$$

where u_{ex} and u_{ez} are the components of displacement vector $\mathbf{u}_e$ in the elastic solid along x and z directions respectively. It can be shown that ϕ_e and ψ_e satisfy the following wave equations

$$\nabla^2\phi_e = \frac{1}{\alpha^2}\frac{\partial^2\phi_e}{\partial t^2}, \quad \nabla^2\psi_e = \frac{1}{\beta^2}\frac{\partial^2\psi_e}{\partial t^2}, \tag{34}$$

where $\alpha^2 = \frac{\lambda+2\mu}{\rho_e}$ and $\beta^2 = \frac{\mu}{\rho_e}$ are the velocities of longitudinal and transverse waves respectively; λ and μ are the Lame's parameters and ρ_e is the density of the elastic solid.

The stress–strain relation in uniform elastic medium is given by

$$\tau_e = \lambda\nabla\cdot\mathbf{u}_e I + 2\mu(\nabla\mathbf{u}_e + (\nabla\mathbf{u}_e)^{\mathrm{T}}), \tag{35}$$

where the subscript 'e' denotes the quantities in the uniform elastic medium.

4. Problem formulation and boundary conditions

We consider a plane interface along x-axis between a uniform elastic solid half-space and a porous elastic solid half-space saturated by two immiscible fluids. The z-axis is chosen vertical to the interface and pointing downward into the porous half-space so that, the uniform elastic half-space occupies the region $-\infty < Z < 0$ and porous half-space occupies the region $0 < Z < \infty$. We aim to attempt a reflection and refraction problem in the two-dimensional x–z plane and the incident wave is assumed to incident obliquely at the interface, after traveling through the uniform elastic half-space.

We assume that the two half-space separated by a plane interface along $z = 0$ are in perfect contact. Therefore, the boundary conditions at the plane interface are the continuity of stress components, displacement components and a condition preventing the flow of two fluids of porous solid into the uniform elastic solid. The mathematical form of these boundary conditions at the interface $z = 0$ are:

$$\begin{aligned}&(\tau_e)_{zz} = \tau_{zz} + \langle\tau_1\rangle + \langle\tau_2\rangle, \quad (\tau_e)_{zx} = \tau_{zx},\\ &\dot{u}_{ex} = \dot{u}_{sx}, \quad \dot{u}_{ez} = \dot{u}_{sz}, \quad \dot{u}_{sz} = \dot{u}_{1z}, \quad \dot{u}_{sz} = \dot{u}_{2z},\end{aligned} \tag{36}$$

where superposed dot represents the temporal derivative.

5. Reflection and refraction of waves at a plane interface

5.1. Incident longitudinal wave

Let a train of longitudinal wave with amplitude A_{e0} traveling through the uniform elastic medium be incident at the interface $z = 0$ and making an angle θ_0 with the normal to the interface. This incident wave will give rise to the following reflected and refracted waves.

In the elastic half-space: (i) a reflected longitudinal wave with amplitude A_{e1} making an angle θ_1 with the normal to the interface; (ii) a reflected transverse wave with amplitude B_e making an angle θ_2 with the normal to the interface.

In the porous half-space: (i) three refracted longitudinal waves with amplitudes A_{s1}, A_{s2} and A_{s3} and making angles γ_1, γ_2 and γ_3 with the normal to the interface; (ii) a refracted transverse wave with amplitude B_{s1} making an angle γ_4 with the normal to the interface. Following Hajra and Mukhopadhyay (1982), the form of potentials in the half-spaces are taken as:

In the elastic half-space:

$$\begin{aligned}\phi_e = {} & A_{e0}\exp\{\imath k_0(x\sin\theta_0 + z\cos\theta_0 - \alpha t)\}\\ & + A_{e1}\exp\{\imath k_0(x\sin\theta_1 - z\cos\theta_1 - \alpha t)\},\end{aligned} \tag{37}$$

$$\psi_e = B_e\exp\{\imath k_1(x\sin\theta_2 - z\cos\theta_2 - \beta t)\}, \tag{38}$$

where $k_0 = \omega/\alpha$ and $k_1 = \omega/\beta$ are the wavenumbers.

In the porous half-space:

$$\phi = A_{s1}\exp\{\imath[k_{s1}(x\sin\gamma_1 + z\cos\gamma_1) - \omega t]\}, \tag{39}$$

$$\psi = A_{s2}\exp\{\imath[k_{s2}(x\sin\gamma_2 + z\cos\gamma_2) - \omega t]\}, \tag{40}$$

$$\eta = A_{s3}\exp\{\imath[k_{s3}(x\sin\gamma_3 + z\cos\gamma_3) - \omega t]\}, \tag{41}$$

$$H = B_{s4}\exp\{\imath[k_{s4}(x\sin\gamma_4 + z\cos\gamma_4) - \omega t]\}. \tag{42}$$

Using Eqs. (26)–(31), (33) into the above boundary conditions (36) and inserting the expressions of potentials given by (37)–(42) and the Snell's law is given by

$$\frac{\sin\theta_0}{\alpha} = \frac{\sin\theta_1}{\alpha} = \frac{\sin\theta_2}{\beta} = \frac{\sin\gamma_1}{X_1} = \frac{\sin\gamma_2}{X_2} = \frac{\sin\gamma_3}{X_3} = \frac{\sin\gamma_4}{X_4},$$

we obtain a set of six equations into six unknown. This set of equations can be written in matrix form as

$$PR = Q, \tag{43}$$

where $P = \{b_{ij}\}$ is a matrix of order 6×6, with zero entries except for

$$b_{11} = -\left(\frac{\lambda}{\mu} + 2\cos^2\theta_0\right), \quad b_{12} = 2\sin\theta_0\sqrt{\frac{\alpha^2}{\beta^2} - \sin^2\theta_0},$$

$$b_{13} = \left(\frac{a_{11}+a_{21}+a_{31}}{\mu} + \frac{G_{\mathrm{fr}}}{3\mu}\right)\frac{\alpha^2}{X_1^2} - \frac{G_{\mathrm{fr}}}{\mu}\sin^2\theta_0, \quad b_{14} = \left(\frac{a_{12}+a_{22}+a_{32}}{\mu}\right)\frac{\alpha^2}{X_2^2},$$

$$b_{15} = \left(\frac{a_{13}+a_{23}+a_{33}}{\mu}\right)\frac{\alpha^2}{X_3^2}, \quad b_{16} = \frac{G_{\mathrm{fr}}}{\mu}\sin\theta_0\sqrt{\frac{\alpha^2}{X_4^2} - \sin^2\theta_0},$$

$$b_{21} = \sin 2\theta_0, \quad b_{22} = \frac{\alpha^2}{\beta^2} - 2\sin^2\theta_0, \quad b_{23} = \frac{G_{\mathrm{fr}}}{\mu}\sin\theta_0\sqrt{\frac{\alpha^2}{X_1^2} - \sin^2\theta_0},$$

$$b_{26} = -\frac{G_{\mathrm{fr}}}{2\mu}\left(\frac{\alpha^2}{X_4^2} - 2\sin^2\theta_0\right), \quad b_{31} = \sin\theta_0, \quad b_{32} = \sqrt{\frac{\alpha^2}{\beta^2} - \sin^2\theta_0}, \quad b_{33} = -\sin\theta_0,$$

$$b_{36} = \sqrt{\frac{\alpha^2}{X_4^2} - \sin^2\theta_0}, \quad b_{41} = \cos\theta_0, \quad b_{42} = -\sin\theta_0, \quad b_{43} = \sqrt{\frac{\alpha^2}{X_1^2} - \sin^2\theta_0},$$

$$b_{46} = \sin\theta_0, \quad b_{53} = \sqrt{\frac{\alpha^2}{X_1^2} - \sin^2\theta_0}, \quad b_{54} = -\sqrt{\frac{\alpha^2}{X_2^2} - \sin^2\theta_0}, \quad b_{56} = \left(\frac{\langle\rho_1\rangle\omega}{\imath c_1 + \langle\rho_1\rangle\omega}\right)\sin\theta_0,$$

$$b_{63} = \sqrt{\frac{\alpha^2}{X_1^2} - \sin^2\theta_0}, \quad b_{65} = -\sqrt{\frac{\alpha^2}{X_3^2} - \sin^2\theta_0}, \quad b_{66} = \left(\frac{\langle\rho_2\rangle\omega}{\imath c_2 + \langle\rho_1\rangle\omega}\right)\sin\theta_0,$$

$Q = \{q_i\}$ and $R = \{R_i\}$ are column matrices of order 6×1. Their entries are given by

$$q_1 = \frac{\lambda}{\mu} + 2\cos^2\theta_0, \quad q_2 = \sin 2\theta_0, \quad q_3 = -\sin\theta_0, \quad q_4 = \cos\theta_0, \quad q_5 = q_6 = 0,$$

$$R_1 = \frac{A_{\mathrm{e}1}}{A_{\mathrm{e}0}}, \quad R_2 = \frac{B_{\mathrm{e}}}{A_{\mathrm{e}0}}, \quad R_3 = \frac{A_{\mathrm{s}1}}{A_{\mathrm{e}0}}, \quad R_4 = \frac{A_{\mathrm{s}2}}{A_{\mathrm{e}0}}, \quad R_5 = \frac{A_{\mathrm{s}3}}{A_{\mathrm{e}0}}, \quad R_6 = \frac{B_{\mathrm{s}1}}{A_{\mathrm{e}0}}.$$

Following Achenbach (1973), the rate at which the energy is transferred per unit area of the surface is given by the scalar product of surface traction and the particle velocity. Thus, for the uniform elastic solid, the rate of transmission of energy per unit area denoted by P_{e}^* at the interface $z = 0$ is given by

$$P_{\mathrm{e}}^* = (\tau_{\mathrm{e}})_{xz}\dot{u}_{\mathrm{e}x} + (\tau_{\mathrm{e}})_{zz}\dot{u}_{\mathrm{e}z}, \tag{44}$$

For the fluid saturated porous solid, the rate of of transmission of energy per unit area denoted by P_{p}^* at the interface $z = 0$ is given by

$$P_{\mathrm{p}}^* = (\tau_{\mathrm{s}})_{xz}\dot{u}_{\mathrm{s}x} + (\tau_{\mathrm{s}})_{zz}\dot{u}_{\mathrm{s}z} + \tau_1\dot{u}_{1z} + \tau_2\dot{u}_{2z}. \tag{45}$$

Using the appropriate potentials given by (37)–(42) and using Eqs. (26)–(35) into the expressions given by (44) and (45), one can obtain the average energy transmission per unit surface area. The energy ratios denoted by E_i, $(i = 1, 2, \ldots 6)$ give the time rate of average energy transmission for the respective wave to that of the incident wave. The expressions of these energy ratios E_i for the reflected P, reflected SV, first refracted P, second refracted P, third refracted P and refracted SV-waves respectively are given by

$$E_1 = R_1^2, \quad E_2 = R_2^2 \frac{1}{\cos\theta_0}\sqrt{\frac{\alpha^2}{\beta^2} - \sin^2\theta_0},$$

$$E_3 = R_3^2 \frac{\beta^2}{X_1^2\cos\theta_0}\left(\frac{G_{\rm fr}}{3\mu} + \frac{a_{11}}{\mu}\right)\sqrt{\frac{\alpha^2}{X_1^2} - \sin^2\theta_0}, \quad E_4 = R_4^2 \frac{a_{22}\beta^2}{\mu X_2^2\cos\theta_0}\sqrt{\frac{\alpha^2}{X_2^2} - \sin^2\theta_0},$$

$$E_5 = R_5^2 \frac{a_{33}\beta^2}{\mu X_3^2\cos\theta_0}\sqrt{\frac{\alpha^2}{X_3^2} - \sin^2\theta_0}, \quad E_6 = R_6^2 \frac{G_{\rm fr}\beta^2}{2\mu X_4^2\cos\theta_0}\sqrt{\frac{\alpha^2}{X_4^2} - \sin^2\theta_0}.$$

5.2. Incident transverse wave

Next, to discuss the reflection and refraction of transverse wave at the interface $z = 0$, we shall assume the same geometry of the problem as considered earlier in case of incident longitudinal wave. Suppose a train of transverse wave traveling through the uniform elastic solid becomes incident at the interface $z = 0$ at an angle θ_0 with the normal to the interface. This wave would give rise to exactly the same reflected and refracted waves as described in case of incident longitudinal wave. We take the following potentials in the elastic medium.

$$\psi_{\rm e} = \overline{B}_{\rm e0}\exp\{\imath k_0(x\sin\theta_0 + z\cos\theta_0 - \beta t)\},$$
$$+ \overline{B}_{\rm e1}\exp\{\imath k_0(x\sin\theta_2 - z\cos\theta_2 - \beta t)\}, \tag{46}$$

$$\phi_{\rm e} = \overline{A}_{\rm e}\exp\{\imath k_1(x\sin\theta_1 - z\cos\theta_1 - \alpha t)\}, \tag{47}$$

where $k_0 = \omega/\beta$ and $k_1 = \omega/\alpha$ are the wavenumbers, $\overline{B}_{\rm e0}$ is the amplitude of incident transverse wave with θ_0 as the angle of incidence, $\overline{B}_{\rm e1}$ is the amplitude of reflected transverse wave, $\overline{A}_{\rm e}$ is the amplitude of reflected longitudinal wave.

In the porous solid the potentials are taken as follows

$$\phi = \overline{A}_{\rm s1}\exp\{\imath[k_{\rm s1}(x\sin\gamma_1 + z\cos\alpha_1) - \omega t]\}, \tag{48}$$

$$\psi = \overline{A}_{\rm s2}\exp\{\imath[k_{\rm s2}(x\sin\gamma_2 + z\cos\alpha_2) - \omega t]\}, \tag{49}$$

$$\eta = \overline{A}_{\rm s3}\exp\{\imath[k_{\rm s3}(x\sin\gamma_3 + z\cos\alpha_3) - \omega t]\}, \tag{50}$$

$$H = \overline{B}_{\rm s1}\exp\{\imath[k_{\rm s4}(x\sin\gamma_4 + z\cos\gamma_4) - \omega t]\}. \tag{51}$$

where $\overline{A}_{\rm s1}$, $\overline{A}_{\rm s2}$ and $\overline{A}_{\rm s3}$ are the the amplitudes of the three refracted longitudinal waves and $\overline{B}_{\rm s1}$ is the amplitude of the refracted transverse wave, $k_{\rm sn} = \omega/X_n$, $(n = 1, 2, 3, 4)$ are the wavenumbers of respective waves.

Substituting the potentials (46)–(51) with the help of (26)–(35), into the boundary conditions given by (36) and making use of Snell's law given by

$$\frac{\sin\theta_0}{\beta} = \frac{\sin\theta_1}{\alpha} = \frac{\sin\theta_2}{\beta} = \frac{\sin\gamma_1}{X_1} = \frac{\sin\gamma_2}{X_2} = \frac{\sin\gamma_3}{X_3} = \frac{\sin\gamma_4}{X_4},$$

we find that amplitude ratios satisfy the relation

$$\overline{PR} = \overline{Q} \tag{52}$$

where $\overline{P} = \{\overline{b}_{ij}\}$ is a matrix of order 6×6, whose non-zero elements are given by

$$\overline{b}_{11} = -\left[\left(\frac{\lambda}{\mu}+2\right)\frac{\beta^2}{\alpha^2} - 2\sin^2\theta_0\right], \quad \overline{b}_{12} = \sin 2\theta_0,$$

$$\overline{b}_{13} = \left(\frac{a_{11}+a_{21}+a_{31}}{\mu} + \frac{G_{\mathrm{fr}}}{3\mu}\right)\frac{\beta^2}{X_1^2} - \frac{G_{\mathrm{fr}}}{\mu}\sin^2\theta_0, \quad \overline{b}_{14} = \left(\frac{a_{12}+a_{22}+a_{32}}{\mu}\right)\frac{\beta^2}{X_2^2},$$

$$\overline{b}_{15} = \left(\frac{a_{13}+a_{23}+a_{33}}{\mu}\right)\frac{\beta^2}{X_3^2}, \quad \overline{b}_{16} = \frac{G_{\mathrm{fr}}\sin\theta_0}{2\mu}\sqrt{\frac{\beta^2}{X_4^2} - \sin^2\theta_0},$$

$$\overline{b}_{21} = -2\sin\theta_0\sqrt{\frac{\beta^2}{\alpha^2} - \sin^2\theta_0}, \quad \overline{b}_{22} = -\cos 2\theta_0,$$

$$\overline{b}_{23} = -\frac{G_{\mathrm{fr}}}{\mu}\sin\theta_0\sqrt{\frac{\beta^2}{X_1^2} - \sin^2\theta_0}, \quad \overline{b}_{26} = \frac{G_{\mathrm{fr}}}{2\mu}\left(\frac{\beta^2}{X_4^2} - 2\sin^2\theta_0\right),$$

$$\overline{b}_{31} = \sin\theta_0, \quad \overline{b}_{32} = \cos\theta_0, \quad \overline{b}_{33} = -\sin\theta_0, \quad \overline{b}_{36} = \sqrt{\frac{\beta^2}{X_4^2} - \sin^2\theta_0},$$

$$\overline{b}_{41} = \sqrt{\frac{\beta^2}{\alpha^2} - \sin^2\theta_0}, \quad \overline{b}_{42} = -\sin\theta_0, \quad \overline{b}_{43} = \sqrt{\frac{\beta^2}{X_1^2} - \sin^2\theta_0}, \quad \overline{b}_{46} = \sin\theta_0,$$

$$\overline{b}_{53} = \sqrt{\frac{\beta^2}{X_1^2} - \sin^2\theta_0}, \quad \overline{b}_{54} = -\sqrt{\frac{\beta^2}{X_2^2} - \sin^2\theta_0}, \quad \overline{b}_{56} = \left(\frac{\langle\rho_1\rangle\omega}{\imath c_1 + \langle\rho_1\rangle\omega}\right)\sin\theta_0,$$

$$\overline{b}_{63} = \sqrt{\frac{\beta^2}{X_1^2} - \sin^2\theta_0}, \quad \overline{b}_{65} = -\sqrt{\frac{\beta^2}{X_3^2} - \sin^2\theta_0}, \quad \overline{b}_{66} = \left(\frac{\langle\rho_2\rangle\omega}{\imath c_2 + \langle\rho_2\rangle\omega}\right)\sin\theta_0,$$

$\overline{Q} = \{\overline{q}_i\}$ and $\overline{R} = \{\overline{R}_i\}$ are column matrices of order 6×1. Their entries are given by

$$\overline{q}_1 = \sin 2\theta_0, \quad \overline{q}_2 = \cos 2\theta_0, \quad \overline{q}_3 = \cos\theta_0, \quad \overline{q}_4 = \sin\theta_0, \quad \overline{q}_5 = \overline{q}_6 = 0,$$

$$\overline{R}_1 = \frac{\overline{A}_{\mathrm{e}}}{\overline{B}_{\mathrm{e0}}}, \quad \overline{R}_2 = \frac{\overline{B}_{\mathrm{e1}}}{\overline{B}_{\mathrm{e0}}}, \quad \overline{R}_3 = \frac{\overline{A}_{\mathrm{s1}}}{\overline{B}_{\mathrm{e0}}}, \quad \overline{R}_4 = \frac{\overline{A}_{\mathrm{s2}}}{\overline{B}_{\mathrm{e0}}}, \quad \overline{R}_5 = \frac{\overline{A}_{\mathrm{s3}}}{\overline{B}_{\mathrm{e0}}}, \quad \overline{R}_6 = \frac{\overline{B}_{\mathrm{s1}}}{\overline{B}_{\mathrm{e0}}}.$$

To consider the partitioning of energy at the interface between different reflected and refracted waves, we proceed exactly similar to that in case of incident longitudinal wave. The expressions of energy ratios $\overline{E}_i, (i = 1, 2, \ldots, 6)$ for the reflected-P, reflected-SV, first refracted-P, second refracted-P, third refracted-P and refracted-SV waves are given as follows:

$$\overline{E}_1 = \overline{R}_1^2\frac{1}{\cos\theta_0}\sqrt{\frac{\beta^2}{\alpha^2} - \sin^2\theta_0}, \quad \overline{E}_2 = \overline{R}_2^2,$$

$$\overline{E}_3 = \overline{R}_3^2\left(\frac{a_{11}}{\mu} + \frac{G_{\mathrm{fr}}}{3\mu}\right)\frac{\beta^2}{X_1^2\cos\theta_0}\sqrt{\frac{\beta^2}{X_1^2} - \sin^2\theta_0}, \quad \overline{E}_4 = \overline{R}_4^2\frac{\beta^2 a_{22}}{\mu X_2^2\cos\theta_0}\sqrt{\frac{\beta^2}{X_2^2} - \sin^2\theta_0},$$

$$\overline{E}_5 = \overline{R}_5^2\frac{\beta^2 a_{33}}{\mu X_3^2\cos\theta_0}\sqrt{\frac{\beta^2}{X_3^2} - \sin^2\theta_0}, \quad \overline{E}_6 = \overline{R}_6^2\frac{G_{\mathrm{fr}}}{2\mu}\frac{\beta^2}{X_4^2\cos\theta_0}\sqrt{\frac{\beta^2}{X_4^2} - \sin^2\theta_0}.$$

6. Particular cases

(a) When one fluid out of two immiscible fluids is neglected then the problem reduces to the problem of reflection and transmission of elastic waves at a plane interface between a uniform elastic half-space and that of porous elastic half-space saturated by a fluid. In this case, we have $S_1 = A_2 = 0$, so that $a_{12} = a_{22} = a_{32} = 0$ and $c_1 = 0$. With these values, one can verify that the boundary condition $\dot{u}_{sz} = \dot{u}_{1z}$ at the interface $z = 0$ is automatically satisfied and the remaining five boundary conditions which can be deduced from the matrix equation (43), can now be written as

$$\sum_{j=1}^{5} b_{ij}R_j = q_i, \quad (i = 1, 2, 3, 4, 5), \tag{53}$$

where

$$b_{11} = -\left(\frac{\lambda}{\mu} + 2\cos^2\theta_0\right), \quad b_{12} = 2\sin\theta_0\sqrt{\frac{\alpha^2}{\beta^2} - \sin^2\theta_0},$$

$$b_{13} = \left(\frac{a_{11} + a_{31}}{\mu} + \frac{G_{\rm fr}}{3\mu}\right)\frac{\alpha^2}{X_1^2} - \frac{G_{\rm fr}}{\mu}\sin^2\theta_0, \quad b_{14} = \left(\frac{a_{13} + a_{33}}{\mu}\right)\frac{\alpha^2}{X_3^2},$$

$$b_{15} = \frac{G_{\rm fr}}{\mu}\sin\theta_0\sqrt{\frac{\alpha^2}{X_4^2} - \sin^2\theta_0}, \quad b_{21} = \sin 2\theta_0, \quad b_{22} = \frac{\alpha^2}{\beta^2} - 2\sin^2\theta_0,$$

$$b_{23} = \frac{G_{\rm fr}}{\mu}\sin\theta_0\sqrt{\frac{\alpha^2}{X_1^2} - \sin^2\theta_0}, \quad b_{25} = -\frac{G_{\rm fr}}{2\mu}\left(\frac{\alpha^2}{X_4^2} - 2\sin^2\theta_0\right),$$

$$b_{31} = \sin\theta_0, \quad b_{32} = \sqrt{\frac{\alpha^2}{\beta^2} - \sin^2\theta_0}, \quad b_{33} = -\sin\theta_0, \quad b_{35} = \sqrt{\frac{\alpha^2}{X_4^2} - \sin^2\theta_0},$$

$$b_{41} = \cos\theta_0, \quad b_{42} = -\sin\theta_0, \quad b_{43} = \sqrt{\frac{\alpha^2}{X_1^2} - \sin^2\theta_0}, \quad b_{45} = \sin\theta_0,$$

$$b_{53} = \sqrt{\frac{\alpha^2}{X_1^2} - \sin^2\theta_0}, \quad b_{54} = -\sqrt{\frac{\alpha^2}{X_3^2} - \sin^2\theta_0}, \quad b_{55} = \left(\frac{\langle\rho_2\rangle\omega}{\imath c_2 + \langle\rho_2\rangle\omega}\right)\sin\theta_0,$$

and $q_1 = \frac{\lambda}{\mu} + 2\cos^2\theta_0$, $q_2 = \sin 2\theta_0$, $q_3 = -\sin\theta_0$, $q_4 = \cos\theta_0$, $q_5 = 0$. Eq. (43) gives the expressions of reflection and transmission coefficients for the relevant problem. Similarly, in case of incident transverse wave, one can deduce from Eq. (52), the followings

$$\sum_{j=1} \overline{b}_{ij}\overline{R}_j = \overline{q}_i, \quad (i = 1, 2, 3, 4, 5) \tag{54}$$

where

$$\overline{b}_{11} = -\left[\left(\frac{\lambda}{\mu} + 2\right)\frac{\beta^2}{\alpha^2} - 2\sin^2\theta_0\right], \quad \overline{b}_{12} = \sin 2\theta_0,$$

$$\overline{b}_{13} = \left(\frac{a_{11} + a_{31}}{\mu} + \frac{G_{\rm fr}}{3\mu}\right)\frac{\beta^2}{X_1^2} - \frac{G_{\rm fr}}{\mu}\sin^2\theta_0, \quad \overline{b}_{14} = \left(\frac{a_{13} + a_{33}}{\mu}\right)\frac{\beta^2}{X_3^2},$$

$$\overline{b}_{15} = \frac{G_{\rm fr}}{2\mu}\sin\theta_0\sqrt{\frac{\beta^2}{X_4^2} - \sin^2\theta_0}, \quad \overline{b}_{21} = -2\sin\theta_0\sqrt{\frac{\beta^2}{\alpha^2} - \sin^2\theta_0},$$

$$\bar{b}_{22} = -\cos 2\theta_0, \quad \bar{b}_{23} = -\frac{G_{\rm fr}}{\mu}\sin\theta_0\sqrt{\frac{\beta^2}{X_1^2} - \sin^2\theta_0}, \quad \bar{b}_{25} = \frac{G_{\rm fr}}{2\mu}\left(\frac{\beta^2}{X_4^2} - 2\sin^2\theta_0\right),$$

$$\bar{b}_{31} = \sin\theta_0, \quad \bar{b}_{32} = \cos\theta_0, \quad \bar{b}_{33} = -\sin\theta_0, \quad \bar{b}_{35} = \sqrt{\frac{\beta^2}{X_4^2} - \sin^2\theta_0},$$

$$\bar{b}_{41} = \sqrt{\frac{\beta^2}{\alpha^2} - \sin^2\theta_0}, \quad \bar{b}_{42} = -\sin\theta_0, \quad \bar{b}_{43} = \sqrt{\frac{\beta^2}{X_1^2} - \sin^2\theta_0}, \quad \bar{b}_{45} = \sin\theta_0,$$

$$\bar{b}_{53} = \sqrt{\frac{\beta^2}{X_1^2} - \sin^2\theta_0}, \quad \bar{b}_{54} = -\sqrt{\frac{\beta^2}{X_3^2} - \sin^2\theta_0}, \quad \bar{b}_{55} = \left(\frac{\langle\rho_2\rangle\omega}{\iota c_2 + \langle\rho_2\rangle\omega}\right)\sin\theta_0,$$

and

$$\bar{q}_1 = \sin 2\theta_0 \quad \bar{q}_2 = \cos 2\theta_0, \quad \bar{q}_3 = \cos\theta_0, \quad \bar{q}_4 = \sin\theta_0, \quad \bar{q}_5 = 0.$$

(b) If both the fluids in the porous half-space are neglected then the problem reduces to the classical problem of reflection and transmission of longitudinal wave and transverse wave at a plane interface between two different uniform elastic half-spaces in perfect contact. In this case, we have $\alpha_1 = \alpha_2 = 0$, so that $S_1 = S_2 = A_2 = 0$. Using these values, one can deduce from Eqs. (43) and (52), the well known expressions of reflection and transmission coefficients for the relevant problem in each case.

7. Numerical results and discussion

In order to study the problem in greater detail, we have computed the reflection and transmission coefficients and the energy ratios numerically for a particular model. For this purpose, we have taken the values of relevant elastic parameters as follows:

In the uniform elastic half-space:

$$\lambda = 2.238 \times 10^{11}\ \text{dyne/cm}^2, \quad \mu = 2.992 \times 10^{11}\ \text{dyne/cm}^2, \quad \rho_{\rm e} = 2.65\ \text{gm/cm}^3,$$
$$\alpha = 5.57\ \ \text{km/s}, \quad \beta = 3.36\ \text{km/s}.$$

In the porous elastic half-space:

$$K_{\rm fr} = 0.4 \times 10^{11}\ \text{dyne/cm}^2, \quad K_{\rm s} = 0.95 \times 10^{11}\ \text{dyne/cm}^2, \quad K_1 = 0.1375 \times 10^{11}\ \text{dyne/cm}^2,$$
$$K_2 = 0.1156 \times 10^{11}\ \text{dyne/cm}^2, \quad G_{\rm fr} = 0.55 \times 10^{11}\ \text{dyne/cm}^2, \quad \langle\rho_{\rm s}\rangle = 2.6\ \text{gm/cm}^3,$$
$$\langle\rho_1\rangle = 0.82\ \text{gm/cm}^3, \quad \langle\rho_2\rangle = 0.92\ \text{gm/cm}^3.$$

Note that we have considered the porous medium saturated by two immiscible non-viscous fluids throughout the numerical computations.

First, we solve Eqs. (17) and (23) numerically to obtain the values of velocities of three dilatational waves and one transverse wave propagating in the porous medium. Since, we are considering the porous medium saturated by non-viscous fluids, so we put $\mu_i = 0$, which implies $c_i = 0$. We shall make use of these values while solving Eq. (17) numerically. We shall apply Cardon's method to solve Eq. (17). Applying the transformation $Y = X + \frac{a}{3}$, we obtain

$$Y^3 + 3hY + g = 0, \tag{55}$$

where

$$h=\frac{1}{9}(3b-a^2),\quad g=\frac{1}{27}(2a^2-9ab+27c),$$

$$a=-\left[\frac{a_{11}^*}{\rho_s}+\frac{a_{22}}{\rho_1}+\frac{a_{33}}{\rho_2}\right],$$

$$b=\left[\frac{a_{11}^*}{\rho_s}\left(\frac{a_{22}}{\rho_1}+\frac{a_{33}}{\rho_2}\right)+\left(\frac{a_{12}}{\rho_1}\right)^2\frac{\rho_1}{\rho_s}+\left(\frac{a_{13}}{\rho_2}\right)^2\frac{\rho_2}{\rho_s}-\frac{a_{22}}{\rho_1}\frac{a_{33}}{\rho_2}+\left(\frac{a_{23}}{\rho_2}\right)^2\frac{\rho_2}{\rho_1}\right],$$

$$c=-\left[\frac{a_{11}^*}{\rho_s}\left(\frac{a_{22}}{\rho_1}\frac{a_{33}}{\rho_2}-\frac{a_{23}^2}{\rho_1\rho_2}\right)-\frac{a_{12}^2}{\rho_s\rho_1}\frac{a_{33}}{\rho_2}+\frac{a_{13}}{\rho_s}\left(2\frac{a_{12}}{\rho_1}\frac{a_{23}}{\rho_2}-\frac{a_{13}}{\rho_2}\frac{a_{22}}{\rho_1}\right)\right].$$

For all the three roots to be real $\Delta(=g^2+4h^3)<0$. Assuming $\Delta<0$, we obtain three roots of Eq. (55) as follows:

$$Y_n=2\sqrt{-h}\cos\left(\frac{\phi_0+2\pi(n-1)}{3}\right),\quad n=1,2,3,$$

where $\phi_0=\tan^{-1}\left(\frac{\sqrt{|\Delta|}}{-g}\right)$. Hence we get

$$X_n=\sqrt{Y_n-\frac{a}{3}},\quad (n=1,2,3). \tag{56}$$

Eq. (56) gives the expressions of velocities of three dilatational waves.

From Eq. (23), the velocity of transverse wave is given by

$$X_4=\sqrt{\frac{G_{\text{fr}}}{\rho_s}}. \tag{57}$$

We have computed these velocities for different values of α_s, α_1 and $Pr(=\frac{dP_{\text{cap}}}{dS_1})$. Fig. 1 depicts the variations of velocities with Pr for $\alpha_s=0.08$ and $\alpha_1=0.04$. We note that longitudinal wave with velocity X_1 decreases monotonically with Pr, the longitudinal wave with velocity X_2 first remains constant in the range $0<Pr\leqslant 2.8$ and then decreases monotonically with Pr and the longitudinal wave with velocity X_3 first increases very slowly in the range $0<Pr\leqslant 2.8$ and then increases fast with Pr. Fig. 2 shows the variations of velocities with α_s, when $\alpha_1=0.04$ and $Pr=0.30$ Pa s and Fig. 3 shows the variations of velocities with α_1, when $\alpha_s=0.08$ and $Pr=0.30$ Pa s. We note from Figs. 2 and 3 that the velocities of all the three longitudinal waves are strongly influenced by the parameters α_s and α_1. However the velocity of transverse wave X_4 remains unchanged and is not influenced by any of these parameters at all as was expected beforehand. As mentioned by Tuncay and Corapcioglu (1997), the longitudinal wave associated with the pressure difference between the fluids has lowest phase velocity and high attenuation coefficient, therefore, the longitudinal wave with velocity X_3 is that very wave and exists due to the presence of second fluid in the porous medium. The other two longitudinal waves with velocities X_1 and X_2 are similar to the velocities of 'P fast' and 'P slow' waves exist in porous medium saturated by one liquid and discussed extensively by Biot (1956a,b).

Next, we have solved the matrix equations given by (43) and (52) by applying the method of *Matrix Inversion* using a FORTRAN-77 computer program. The values of reflection and transmission coefficients are computed against the angles of incidence for both longitudinal and transverse incident waves. The variations of these amplitude and energy ratios with the angle of incidence are shown graphically through Figs. 4–11.

The angular dependence of reflection/transmission coefficients for an incident longitudinal wave are shown in Figs. 4 and 5. We notice from Fig. 4 that the amplitude ratio R_1 has value 0.78 near normal

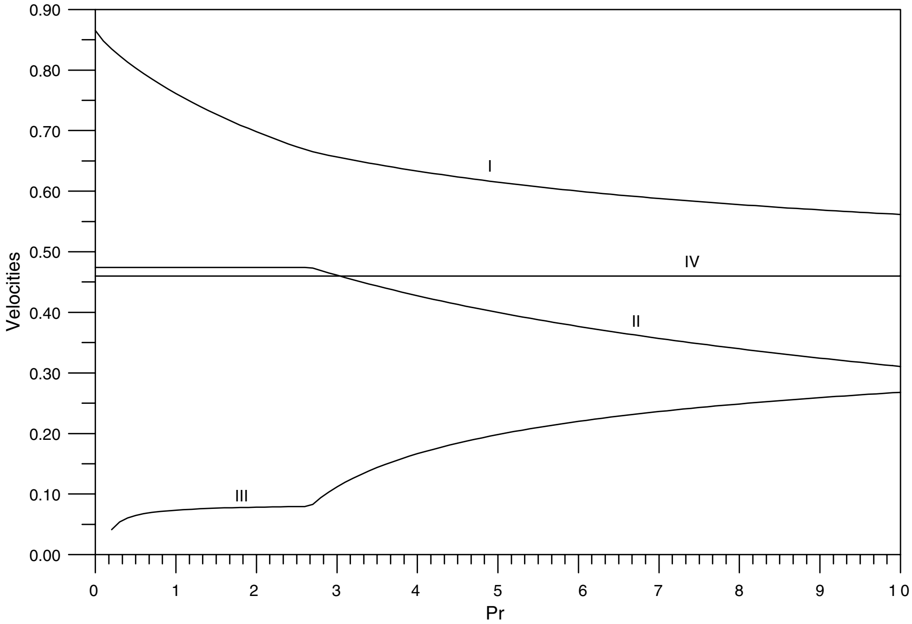

Fig. 1. Variations of velocities with $Pr\left(=\frac{dP_{cap}}{dS_1}\right)$. (Curve I: X_1, Curve II: X_2, Curve III: X_3, Curve IV: X_4).

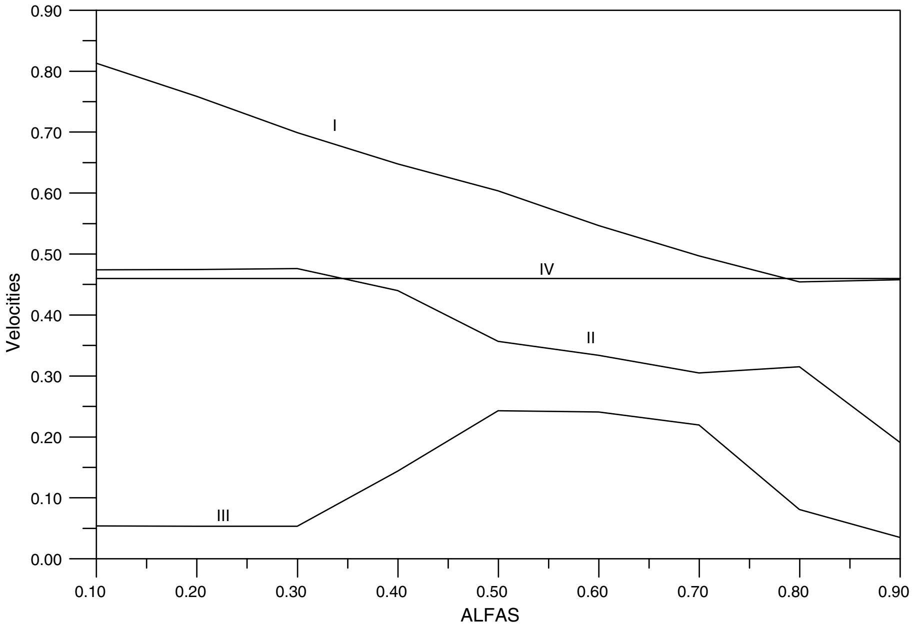

Fig. 2. Variations of velocities with ALFAS ($=\alpha_s$). (Curve I: X_1, Curve II: X_2, Curve III: X_3, Curve IV: X_4).

incidence and thereafter decreases with increase of θ_0 achieving its value nearly zero at $\theta_0 = 75°$. As θ_0 increases beyond 75°, the value of R_1 increases sharply and approaches to its maximum value equal to 1.0 at

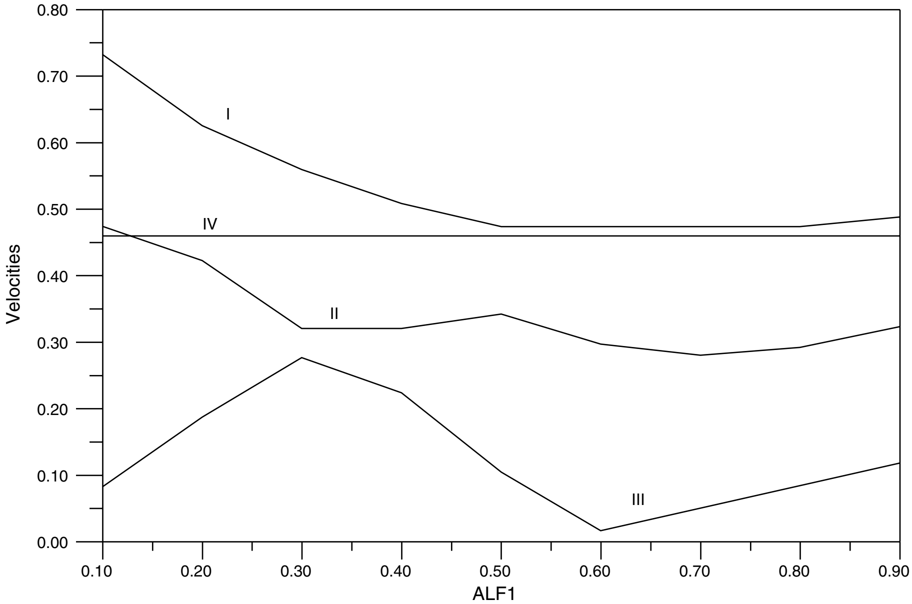

Fig. 3. Variations of velocities with ALF1 ($=\alpha_1$). (Curve I: X_1, Curve II: X_2, Curve III: X_3, Curve IV: X_4).

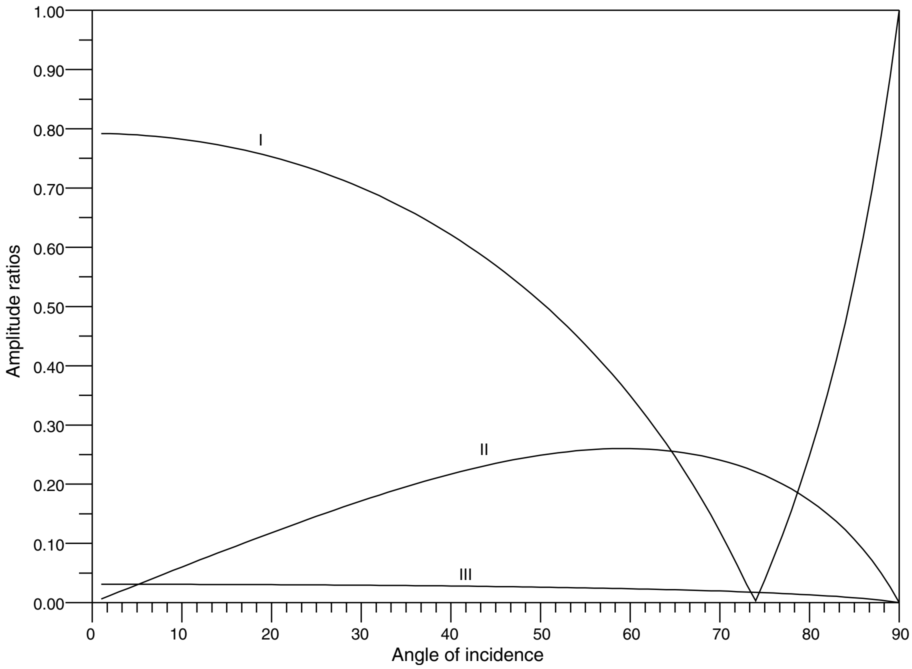

Fig. 4. Variations of amplitude ratios with angle of incidence of longitudinal wave. (Curve I: R_1, Curve II: R_2, Curve III: R_3).

$\theta_0 = 90°$. The amplitude ratio R_2 has value nearly zero at normal incidence and goes on increasing with increase of θ_0 achieving its maximum value near 60° and then decreases to the value zero as θ_0 approaches

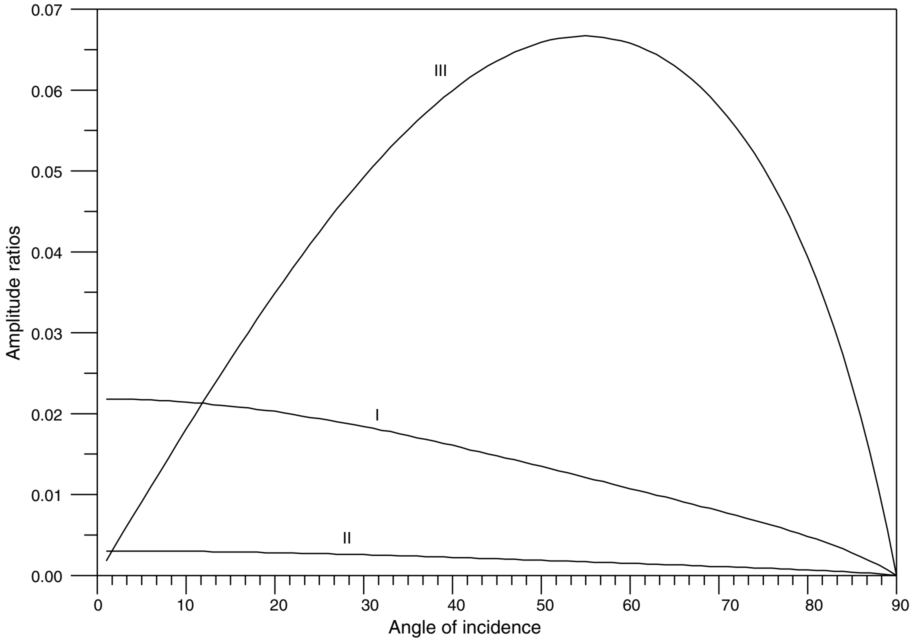

Fig. 5. Variations of amplitude ratios with angle of incidence of longitudinal wave. (Curve I: $R_4 \times 10$, Curve II: R_5, Curve III: R_6).

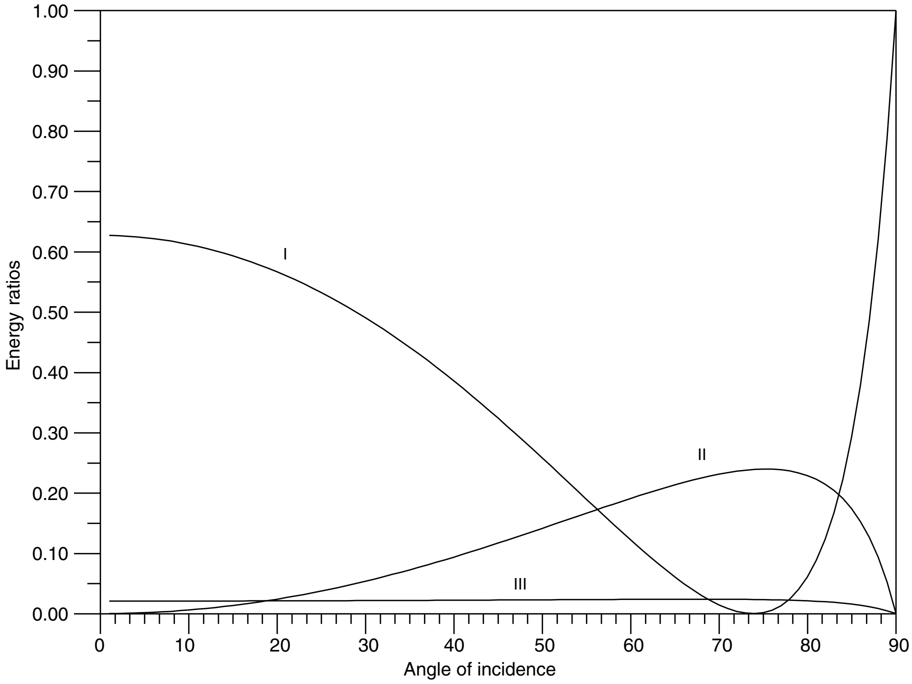

Fig. 6. Variations of energy ratios with angle of incidence of longitudinal wave. (Curve I: E_1, Curve II: E_2, Curve III: E_3).

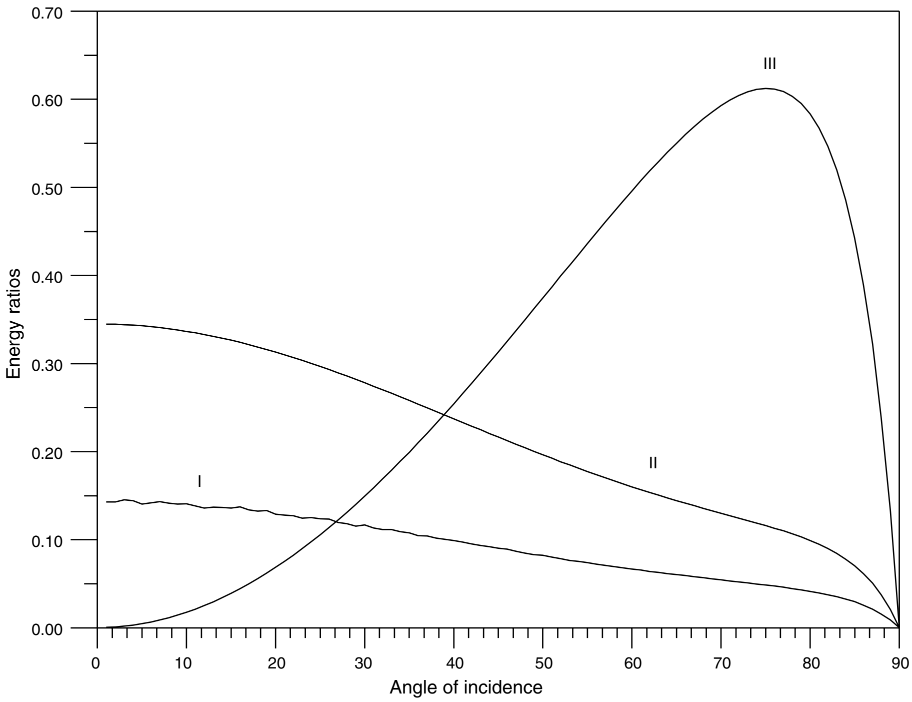

Fig. 7. Variations of energy ratios with angle of incidence of longitudinal wave. (Curve I: $E_4 \times 10^3$, Curve II: E_5, Curve III: E_6).

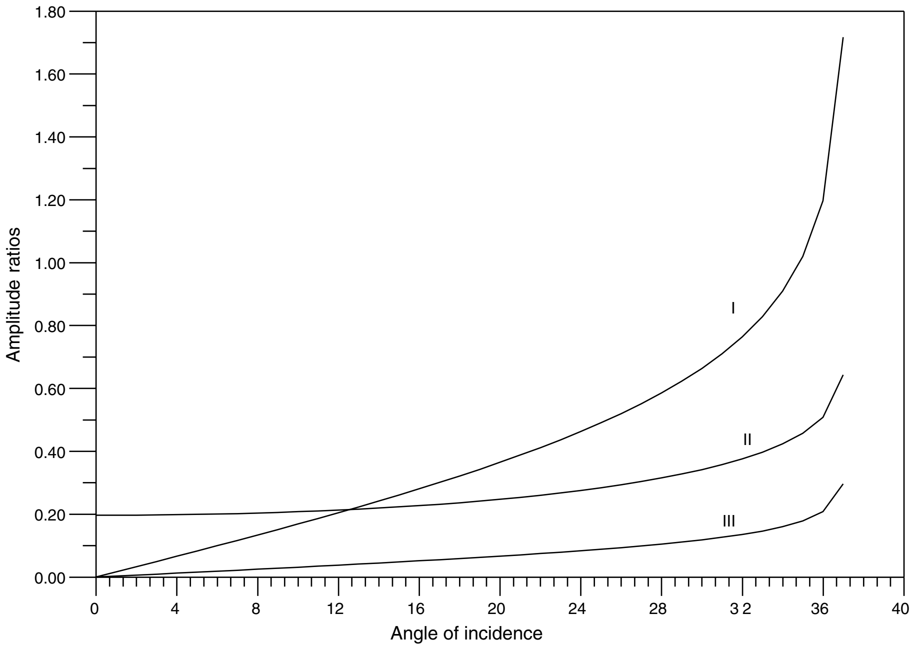

Fig. 8. Variations of amplitude ratios with angle of incidence of transverse wave. (Curve I: $\overline{R}_1$, Curve II: $\overline{R}_2$, Curve III: $\overline{R}_3 \times 10$).

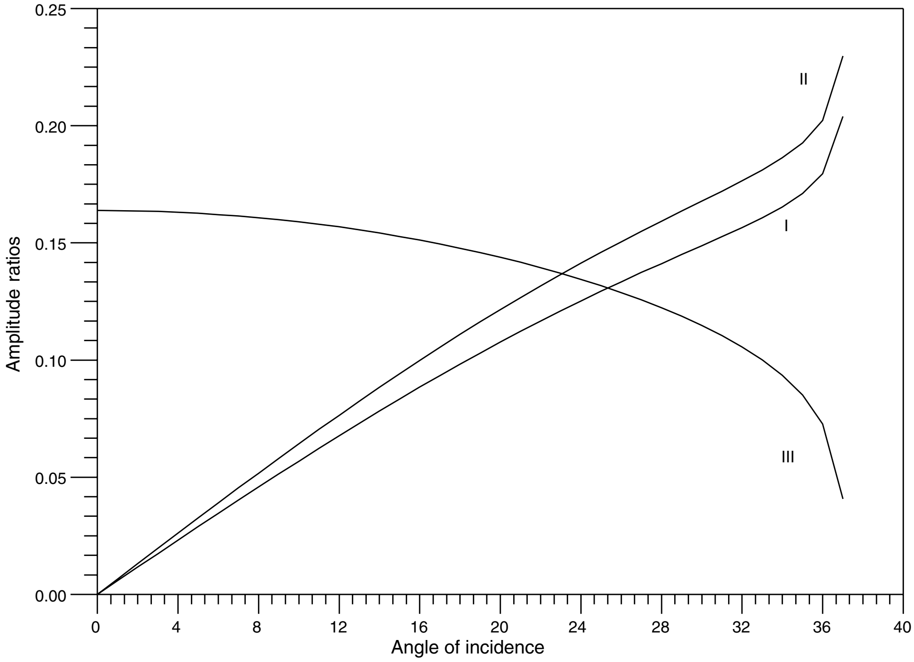

Fig. 9. Variations of amplitude ratios with angle of incidence of transverse wave. (Curve I: $\overline{R}_4 \times 10$, Curve II: $\overline{R}_5 \times 10^2$, Curve III: $\overline{R}_6$).

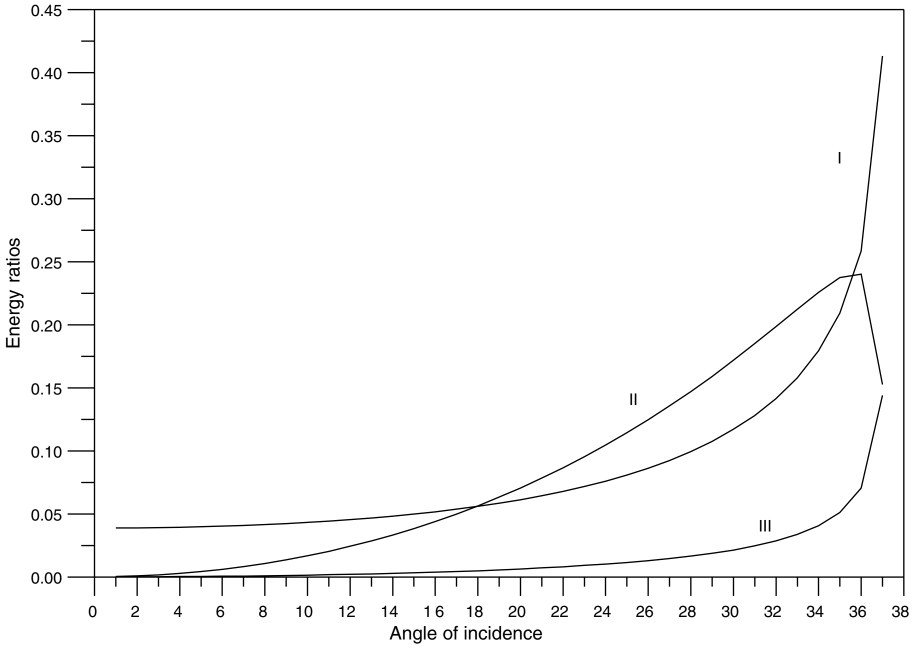

Fig. 10. Variations of energy ratios with angle of incidence of transverse wave. (Curve I: $\overline{E}_1$, Curve II: $\overline{E}_2$, Curve III: $\overline{E}_3 \times 10$).

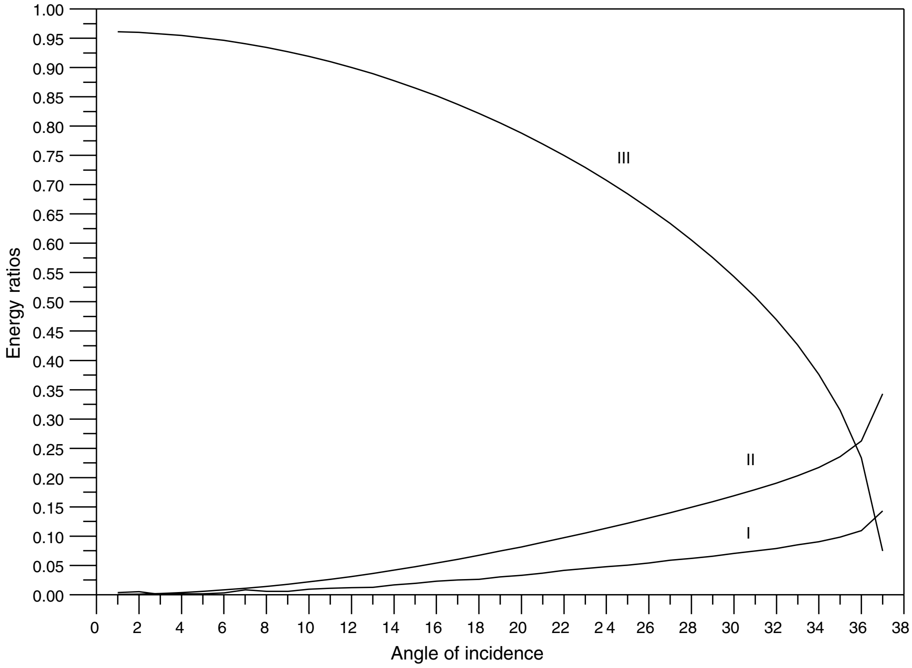

Fig. 11. Variations of energy ratios with angle of incidence of longitudinal wave. (Curve I: $\overline{E}_4 \times 10^3$, Curve II: $\overline{E}_5$, Curve III: $\overline{E}_6$).

to 90°. The value of amplitude ratio R_3 is found to be very small throughout the entire range of θ_0. Its maximum value is 0.0308 at $\theta_0 = 1°$ and then decreases with increase in θ_0, approaching to the value zero at $\theta_0 = 90°$. We note from Fig. 5 that the value of amplitude ratio R_4 has maximum value 0.002 at $\theta_0 = 1°$ and then decreases with θ_0 getting its value equal to zero at $\theta_0 = 90°$.

The variation of energy ratios of reflected and refracted waves with angle of incidence are depicted in Figs. 6 and 7. Fig. 6 reveals that energy ratio E_1 has maximum value at $\theta_0 = 1°$. Its value then decreases with increase of θ_0 and becomes zero at $\theta_0 = 75°$ after which its value increases sharply and reaching to its maximum value equal to 1.0 at grazing incidence. The energy ratio E_2 has value zero at $\theta_0 = 1°$ and after that its value increases continuously with increase of θ_0 attaining its maximum value at $\theta_0 = 76°$. Thereafter, its value starts decreasing and finally becomes zero at $\theta_0 = 90°$. The value of energy ratio E_3 is found to be very small but non-zero in the entire range of θ_0 except at grazing incidence where its value is zero. It retains almost constant value up to $\theta_0 = 80°$, after which its value starts decreasing and ultimately vanishes at $\theta_0 = 90°$. Fig. 7 shows that the value of energy ratio E_4 is very small and it decreases monotonically with increase of θ_0 and finally becomes zero at grazing incidence. The energy ratio E_5 begins from a value 0.35 near normal incidence and it then decreases with increase of θ_0 and finally becomes zero at $\theta_0 = 90°$ angle of incidence. The value of energy ratio E_6 has value zero at $\theta_0 = 1°$ and then increases with increase of θ_0 attaining its maximum value in the range $70° < \theta_0 < 80°$, thereafter its value decreases to vanish at $\theta_0 = 90°$. It is clear from Figs. 6 and 7 that at grazing incidence, no other reflected or transmitted waves appear, except the reflected longitudinal wave corresponding to the amplitude ratio R_1.

In the case of incident transverse wave at the interface, a critical angle θ_c is found between $37° < \theta_0 < 38°$. Figs. 8 and 9 show the variations of amplitude ratios with angle of incidence. Fig. 8 shows that the amplitude ratio $\overline{R}_1$ begins with value zero at $\theta_0 = 0°$ and its value increases with increase of θ_0. The behavior of $\overline{R}_2$

and $\overline{R}_3$ is almost similar. Both the amplitude ratios increase very slowly with increase of θ_0. From Fig. 9, we observe that the amplitude ratios $\overline{R}_4$ and $\overline{R}_5$ have value zero at normal incidence, thereafter their values increase almost linearly with θ_0 but at different rates. As the angle of incidence θ_0 comes close to the critical angle θ_c, both the amplitude ratios have a sudden jump in their values. The amplitude ratio $\overline{R}_6$ has maximum value at normal incidence and thereafter its value decreases as θ_0 increases. It is to be noted that the value of amplitude ratio R_4 is less than the values of amplitude ratio R_5 in the range $0° < \theta_0 \leqslant 37°$. The difference between their values increases with increase of θ_0.

Figs. 10 and 11 show the variation of reflected and refracted energy ratios with the angle of incidence of transverse wave. We note from these figures that all the energy ratios increase with increase of θ_0 with a slow rate except the energy ratios corresponding to reflected and refracted transverse waves. The energy ratios corresponding to these waves exhibit a reverse behavior in the range $0° < \theta_0 \leqslant 35°$. Beyond this range, their behavior is alike and decreasing. In the calculations of energy ratios, it has been verified that sum of energy ratios is equal to unity. This shows that there is no loss of energy during transmission of waves.

8. Conclusions

A mathematical analysis of reflection and refraction phenomenon of longitudinal and transverse waves traveling through a uniform elastic half-space and striking with varying angles at a plane interface between uniform elastic half-space and porous half-space saturated by two immiscible fluids. It is concluded that

(I) Three longitudinal waves and one transverse wave propagating with different velocities exist in a porous medium saturated by two immiscible fluids.
(II) Of three longitudinal waves, the two with velocities X_1 and X_2 corresponds to the P-fast and P-slow waves in Biot's theory, while the third longitudinal wave with velocity X_3 is related to the capillary pressure effect between the two fluids and it is found that $X_1 > X_2 > X_3$.
(III) When longitudinal or transverse wave is incident at the interface, the reflection and transmission coefficients are found to be the function of angle of incidence. However, the nature of dependence of these coefficients on angle of incidence is found to be different for different angle of incidence.
(IV) When transverse wave is incident normally, there appears only one reflected wave corresponding to amplitude ratio $\overline{R}_2$ and only one refracted wave corresponding to amplitude ratio $\overline{R}_6$. On the other hand, when longitudinal wave is incident normally, there exist two reflected waves corresponding to amplitude ratios R_1 and R_3 and two refracted waves corresponding to amplitude ratios R_4 and R_5. At grazing incidence of longitudinal wave, no other reflected or transmitted wave appears except a reflected wave corresponding to the amplitude ratio R_1.
(V) In both the problems, it is found that the sum of energy ratios is equal to unity. This shows that there is no dissipation of energy during transmission.

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