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Semi-analytical solution for mode I penny-shaped crack in a soft inhomogeneous layer

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ABSTRACT

An axisymmetric elastostatics problem for a penny-shaped crack placed in the middle of a inhomogeneous (FGM) elastic layer is considered. It is assumed that the elastic modulus of the layer varies through the thickness symmetrically with respect to the crack plane. Several specific distributions of the moduli variations have been analysed. We report a semi-analytical approximate solution for the determination of the stress intensity factor for the distributions considered. The obtained solution is accurate enough and can be applied in engineering applications for the analysis of crack propagation in FGM and hydrofracture growth in elastic reservoirs.

1. Introduction

The work is aimed at investigation of the fracture characteristics of elastic materials with inhomogeneous layers weakened by cracks. The classical problem for the disk-like crack in homogeneous isotropic elastic medium was studied (Sneddon, 1946) by the method of dual integral equations and (Sack, 1946) with the use of spherical harmonic functions. For two-dissimilar media the problem with an interface crack has been studied in a number of studies, among them in Arin and Erdogan (1971), Erdogan (1965), Erdogan and Arin (1972), Kassir and Bregman (1972), Lowengrub and Sneddon (1972), and Willis (1972).

The development of advanced materials, such as functionally graded materials, FGM, necessitates investigation of fracture propagation in media with non-uniform elastic properties. Particular formulations for FGM layers with continuous variations of elastic properties have been considered by Selvadurai (2000), for the case when the shear moduli of the bonded half-spaces vary in accordance with the exponential law $G(z) = G_1 + G_2 e^{\pm z\varsigma}$ (where the z-axis is perpendicular to the interface and the parameter ς characterises the rate of exponent decay/grow). The disk-like crack at a bonded plane (the interface between two half-spaces) with localised elastic inhomogeneity has been considered and the mode I stress intensity factors for different shear moduli distributions were calculated for the case of uniform remote tension applied perpendicular to the crack plane. The method used has been based on the Hankel transform followed by numerical solving a system of the Fredholm equation of the second kind.

It should be noted that the exponential form for elastic moduli is convenient for mathematical manipulations, however other forms present certain interest as for FGM as in other applications, for instance (Mendelsohn, 1984) for investigations of hydrofracture development (Savitski and Detournay, 2002) in inhomogeneous reservoirs surrounded by the rock layers with different (but constant) elastic properties. Thus, geological observations of Bazhenov shale formation structures (e.g. Strahov, 1970) and lab tests of velocity anisotropy of different shale formations (e.g., Vernik and Liu, 1997) demonstrate diversity in elastic moduli through the layer thickness. For the plane case such formulations are found, e.g., in Erdogan and Gupta (1971a,b) and more general in Delale and Erdogan (1988); for penny-shaped cracks in dissimilar layer one can mention Arin and Erdogan (1971). This study partly employs the above-mentioned formulations but assumes that the layer is spatially inhomogeneous through its thickness and deals with a mode I penny-shaped crack.

The lack of general analytical solutions for the problems involving cracks in functionally graded materials is emphasised by Eischen (1987). It should be noted that the methods of contact

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mechanics for FGM can also be applied for the crack problems. For instance, one can use the piecewise linear approximation of elastic moduli for constructing the kernel transform as suggested (Ke and Wang, 2007; Liu et al., 2008) for the contact problems for half-space and half-plane with arbitrary variations of elastic properties. We will further expand the techniques developed in a number of previous studies (Aizikovich and Alexandrov, 1984; Aizikovich, 1995; Aizikovich et al., 2002; Aizikovich et al., 2011; Vasiliev et al., 2012) for contact problems for the case of a penny-shaped crack located within a functionally graded layer. We construct a semi-analytical solution that depends on a single dimensionless parameter characterising the ratio of the crack radius to the layer thickness and examine the accuracy of such approximate solution. The analysis is conducted for a soft layer, although no restrictions is imposed for the case when the layer is stiffer that the surrounded media.

2. Formulation of the problem

Let us consider a mode I disk-like crack of certain radius R in isotropic inhomogeneous space. The crack lies in the plane $z = 0$ of the cylindrical coordinate system (r, ϕ, z) and its centre is located at the origin.

It is assumed that the elastic modulus of the space is an even function of z of the following form

$$E(z) = E_\infty \begin{cases} f(z), & 0 \leqslant |z| \leqslant H \\ 1, & |z| > H \end{cases} \tag{1.1}$$

where $f(z)$ is an arbitrary function (continuous or piecewise continuous), H is the thickness of the inhomogeneous layer and E_∞ is the modulus of the material outside the layer, see Fig. 1.

Additionally the following conditions are satisfied to provide positiveness of Young's modulus

$$\min_{z\in(0;\infty)} \Delta(z) \geqslant c_1 > 0, \quad \max_{z\in(0;\infty)} \Delta(z) \leqslant c_2 < \infty, \quad \lim_{z\to\infty} \Delta(z) = const$$

$$\begin{aligned}\Delta(z) &= 2\mathrm{M}(z)(\Lambda(z) + \mathrm{M}(z))(\Lambda(z) + 2\mathrm{M}(z))^{-1} = G(z)(1 - \nu(z))^{-1} \\ &= \tfrac{1}{2}E(z)\left(1 - \nu(z)^2\right)^{-1}\end{aligned} \tag{1.2}$$

Here $G(z)$ is the shear modulus, $\Lambda(z)$ and $\mathrm{M}(z)$ are the Lame coefficients, $\nu(z)$ is Poisson's ratio of the inhomogeneous spaces, c_1, c_2 are certain constants.

Let us further analyse the case when the crack surfaces are loaded by normal pressure $p(r) > 0$. Given symmetry of the elastic properties with respect to the z-axis and the loading conditions one can suggest that the stress/strain/displacement fields are independent of the angular coordinate φ, which leads to the following 2D axisymmetric boundary value problem for the half-space

$$\begin{aligned}&\tau_{rz}(r,0) = 0, \quad 0 < r < \infty \\ &\sigma_z(r,0) = -p(r), \quad 0 \leqslant r \leqslant R; \; w(r,0) = 0, \; r > R\end{aligned} \tag{1.3}$$

where $\sigma_z(r,z)$, $\tau_{rz}(r,z)$ are the normal and shear components of the stress tensor respectively and $w(r,z)$ is the normal component of displacements. It is also assumed that the displacements and the stresses are continuous across the planes $|z| = H$ and vanish at infinity.

It has been shown (Aizikovich and Alexandrov, 1984) that under conditions (1.2) the following relationship between the normal stresses and the normal displacements on the surface of the half-space ($z = 0$) is satisfied

$$\begin{aligned}&w(r,0) = \Delta^{-1}(0)\int_0^R q(\rho)\rho d\rho \int_0^\infty L(\gamma)J_0(\gamma\rho)J_0(\gamma r)d\gamma \\ &q(r) = \sigma_z(r,0) = \int_0^\infty Q(\alpha)J_0(\alpha r)\alpha d\alpha, Q(\alpha) = \int_0^R q(\rho)J_0(\alpha\rho)\rho d\rho\end{aligned} \tag{1.4}$$

Here $J_0(r)$ is the Bessel function and the function Δ is defined in (1.2), γ is the dimensional parameter of the Hankel transform.

The function $L(\gamma)$ is found numerically by the method of modulating functions, detail in Babeshko et al. (1987). It has the following asymptotics as shown by Aizikovich and Alexandrov (1984) (provided that the conditions specified by Eqs. (1.1) and (1.2) are valid)

$$L(\gamma) = A + B|\gamma| + \bar{\bar{o}}(\gamma^2), \quad \gamma \to 0 \tag{1.5}$$

$$L(\gamma) = 1 + D|\gamma|^{-1} + \bar{\bar{o}}(\gamma^{-2}), \quad \gamma \to \infty \tag{1.6}$$

where $A = \Delta(0)\Delta^{-1}(|H|)$ and B, D are constants. It should be noted that for a multilayer media this function possesses the following properties (Aizikovich and Alexandrov, 1982)

$$L(\alpha) = A + o(\alpha), A = D_1^{-1}D_2^{-1}\cdots D_{n-1}^{-1}, \quad \alpha \to 0 \tag{1.7}$$

$$L(\alpha) = 1 + B\left(\alpha^2h_1^2 + \alpha h_1\right)Me^{-2\alpha h_1} + o(e^{-2\alpha h_1}), \quad \alpha \to \infty \tag{1.8}$$

Here

$$M = \frac{4(\tilde{\Delta}_1 + k_2)^2 - 1}{(D_1+1)^2 - (k_1D_1 - k_2)^2}, \quad n \geqslant 2; \quad \tilde{\Delta}_1 = \frac{D_1}{2(1-\nu_1)};$$

$$k_j = \frac{1-\nu_j}{2(1-\nu_j)}, \quad D_k = \frac{E_{k+1}(1-\nu_k^2)}{E_k(1-\nu_{k+1}^2)}$$

and h_1 is the thickness of the upper layer, E_i and the Young moduli and the Poisson's coefficients of the j^{th} layer respectively.

It is evident that the second terms in (1.6) and (1.8) are different at $\alpha \to \infty$, which emphasise the difference in solutions of the integral equations for FGM and layered media. The properties (1.5) and (1.7) mean that the value $L(0)$ does not depend on the variation of the Lame coefficients but rather determined by their values at $z = 0$ and $|z| = H$.

Using the approach (Ishlinsky, 1986) and taking into account (1.3) one can present (1.4) in the form

$$\int_0^R \delta(\rho)\rho d\rho \int_0^\infty \frac{\gamma^2}{L(\gamma)}J_0(\gamma\rho)J_0(\gamma r)d\gamma = -\Delta^{-1}(0)p(r), \quad 0 \leqslant r \leqslant R \tag{1.9}$$

where $\delta(r) = -w(r,0)$ is the function that describe the shape of the crack (crack opening displacements, COD). This function should satisfy the following condition

$$\delta(R) = 0 \tag{1.10}$$

By taking into account the above relationships one can reduce the problem to the following dual integral equations for an auxiliary function $\Delta_1(\beta)$

$$\begin{cases} \int_0^\infty \frac{\Delta_1(\beta)}{L(\beta\lambda)}J_1(\beta r)d\beta = \Delta^{-1}(0)p^*, & 0 \leqslant r \leqslant 1 \\ \int_0^\infty \Delta_1(\beta)\beta J_1(\beta r)d\beta = 0, & r > 1 \end{cases} \tag{1.11}$$

Here the new unknown function $\Delta_1(\beta)$ is linked with the unknown crack opening displacements by the following relationship $\delta(r) = \int_0^\infty \Delta_1(\alpha)J_0(\alpha r)d\alpha$, and $J_1(r)$ is the Bessel function of the first order. In the right hand side of (1.11) we introduce a dimensionless load p^* as detailed in Appendix A. Further the asterisk at the notation for the dimensionless loads will be removed for compactness.

It is convenient to denote the reciprocal of $L(u)$ as $F(u)$ in the kernel of (1.8) and bear in mind the asymptotic behaviour of $F(u)$ yielding from (1.5) and (1.6)

$$\begin{aligned}&F(\gamma) = A^{-1} - BA^{-2}|\gamma| + \bar{\bar{o}}(\gamma^2), \quad \gamma \to 0 \\ &F(\gamma) = 1 - D|\gamma|^{-1} + \bar{\bar{o}}(\gamma^{-2}), \quad \gamma \to \infty\end{aligned} \tag{1.12}$$

3. Closed form solution of the dual integral equation

We further apply an approximate analytical method for the solution of dual integral equation (1.11) whose kernels satisfy (1.12). The method has been fully detailed in the previous papers by Aizikovich with co-authors (Aizikovich and Alexandrov, 1984; Aizikovich et al., 2002, 2009; Aizikovich, 1982; Aizikovich and Vasiliev, 2013), see also Appendix A. Here it is necessary to emphasise that the asymptotics (1.12) makes it possible to approximate $F(\gamma)$ by a rational expression of a special kind.

Let us introduce the following definitions.

Definition 1. The function $L(\gamma)$ belongs to the class Π_N if it can be presented in the form

$$L_N^{\Pi}(\gamma) = \prod_{i=1}^{N}(\gamma^2 + A_i^2)(\gamma^2 + B_i^2)^{-1}, \ (B_i - B_k)(A_i - A_k) \neq 0, \quad i \neq k \tag{2.1}$$

where A_i, B_i $(i = 1, 2, \ldots, N)$ are complex constants.

Definition 2. The function $L(\gamma)$ belongs to the class Σ_M if it can be presented as

$$L_M^{\Sigma}(\gamma) = \sum_{k=1}^{M} C_k \gamma (\gamma^2 + D_k^2)^{-1} \tag{2.2}$$

where C_k are real constants, and D_k $(k = 1, 2, \ldots, M)$ are complex constants.

Definition 3. The function $F(\alpha)$ belongs to the class $S_{N,M}(\Pi_N, \Sigma_M)$ if can be presented in the form

$$F(\alpha\lambda) = \begin{cases} \prod_{i=1}^{N}(\alpha^2 + A_i^2\lambda^{-2})(\alpha^2 + B_i^2\lambda^{-2})^{-1} \equiv L_N^{\Pi}(\alpha\lambda) \in \Pi_N \\ \sum_{k=1}^{M} C_k \alpha\lambda^{-1}(\alpha^2 + D_k^2\lambda^{-2})^{-1} \equiv L_M^{\Sigma}(\alpha\lambda) \in \Sigma_M \\ L_N^{\Pi}(\alpha\lambda) + L_M^{\Sigma}(\alpha\lambda) \in S_{N,M} \end{cases} \tag{2.3}$$

$A_i,\ B_i,\ C_k,\ D_k$ $(i = 1, 2, \ldots, N), (k = 1, 2, \ldots, M)$ are certain constants and $(A_i - A_k)(B_i - B_k) \neq 0$, if $i \neq k$.

Dual integral equation (1.11) can be rewritten in the operator form as follows

$$\Pi_N\delta + \Sigma_\infty\delta = p \tag{2.4}$$

where the operators Π_N and Σ_∞ correspond to the functions $F(\alpha) \in \Pi_N$ and $F(\alpha) \in \Sigma_\infty$ respectively. Following (Aizikovich, 1982), one can show that regardless of the right hand side of Eq. (1.11) satisfying (1.12), the operator Π_N is invertible, which yields

$$p = \Pi_N^{-1}\delta \tag{2.5}$$

The later formula implies that the norm of the operator Σ_∞ is small and one can consider (2.5) as an approximate solution of (2.4) based on the following theorems presented in (Aizikovich, 1982).

Theorem 1. *Let $p^*(x)$ be an odd function that can be expanded into the Bessel series, then Π_N has its inverse and the following estimate takes place*

$$\|\delta^N\|_{C_{1/2}^{(0)-}(-1,1)} \leqslant m(\Pi_N) M_p(-1,1), \quad m(\Pi_N) = const$$

Theorem 2. *Under the conditions specified in (1.2), Eq. (1.11) possesses a unique solution in the space $C_{1/2}^{(0)-}(-1,1)$ for any $p^*(x)$ satisfying Theorem 1 for $0 < \lambda < \lambda^*$ and $\lambda > \lambda^0$, where λ^* and λ^0 are some fixed values of λ, and the following estimate holds*

$$|\delta^*(x)| C_{1/2}^{(0)-}(-1,1) \leqslant m(\Pi_N, \Sigma_\infty) M_{p^*}(-1,1).$$

It also follows from the analysis presented in (Aizikovich, 1982) that for $\lambda \to 0$ and $\lambda \to \infty$ the operator $\Pi_N^{-1}\Sigma_\infty$ in Eq. (2.4) is a contraction operator, i.e. the analytical solution of the form (2.5) is a bilateral asymptotic solution of Eq. (2.4) for $\lambda \to 0$ and $\lambda \to \infty$. It should be noted that the error of the approximate solution does not exceed the error of approximation of the functions $F(\alpha)$ from the class Π_N.

By applying the methods of operational calculus as by Alexandrov (1973), one obtains the general solution of the dual integral equation.

Let $F(\alpha) \in \Pi_N$, then by assuming the uniform pressure $p(r) \equiv p$ on the crack surfaces, one can present the crack opening displacements in the form

$$\delta(r) = -\frac{2}{\pi}\frac{p}{\Delta(0)}\left[L_N(0)\sqrt{1-r^2} + \sum_{i=1}^{N} C_i \tilde{b}_i \int_r^1 \frac{sh\tilde{b}_i t dt}{\sqrt{t^2 - r^2}}\right] \tag{2.6}$$

The constants C_i are determined from the following system of linear algebraic equations

$$\begin{aligned} &\sum_{i=1}^{N} C_i P(\tilde{a}_k; \tilde{b}_i) + \frac{1 + \tilde{a}_k}{\tilde{a}_k^2} L_N(0) = 0, \ k = 1, 2, \ldots, N \\ &P(A; B) = \frac{BchB + AshB}{A^2 - B^2} \end{aligned} \tag{2.7}$$

where $\tilde{a}_k = A_k\lambda^{-1}$, $\tilde{b}_i = B_i\lambda^{-1}$.

The normal stresses on the crack plane outside the crack are determined as follows

$$p(r) = -\frac{p}{\Delta(0)}\int_0^\infty \alpha\Delta_1(\alpha)F_N(\lambda\alpha)J_0(\alpha r)d\alpha, \quad r > 1 \tag{2.8}$$

After transformations of (2.8), one finally finds the following expression for the stress distribution along the radial direction

$$\begin{aligned} p(r) = 2\pi^{-1}p\Bigg\{ &\arcsin\left(\frac{1}{r}\right) - \left[\frac{L(0)}{\Delta(0)} + \sum_{n=1}^{N} C_n sh(\tilde{b}_n)\right] \times \frac{1}{\sqrt{r^2-1}} \\ &- \sum_{k=1}^{N} L_N^k(\tilde{a}_k)\left[\frac{L(0)}{\Delta(0)} + \sum_{n=1}^{N} C_n \frac{\tilde{a}_k^2 sh(\tilde{b}_n)}{\tilde{a}_k^2 - \tilde{b}_k^2}\right] \times \int_1^r \frac{\exp(-\tilde{a}_k(t-1))}{\sqrt{r^2 - t^2}} dt \Bigg\}, \quad r > 1 \end{aligned} \tag{2.9}$$

Here $L_N^k(\tilde{a}_k) = \prod_{i=1,\, k}^{N} \frac{\tilde{a}_k^2 + \tilde{a}_i^2}{\tilde{a}_k^2 + \tilde{b}_i^2}$ is the product of the rational factors except the k^{th} one.

The mode I stress intensity factor is found from (2.9) as follows

$$K_I = \lim_{r \to 1+0} \sqrt{r-1}\,p(r) = -\frac{\sqrt{2}}{\pi}\frac{p}{\Delta(0)}\left(L(0) + \Delta(0)\sum_{i=1}^{N} C_i sh\tilde{b}_i\right) \tag{2.10}$$

It is evident that the solution obtained possesses the inverse square root singularity at the crack front, which confirms the previous results for cracks in FGM, e.g. Parameswaran and Shukla (1999) and Selvadurai (2000). This fact also allows one to use the concept of linear fracture mechanics for estimations of fracture propagation, for instance the Irwin's criteria based on the fracture toughness.

4. Results

For numerical approximation of the function $L(\gamma)$, $\gamma = \lambda R$ by the functions from the class Π_N the following procedure is applied.

Firstly, let us transform the function $L(\gamma)$ by using the substitution $u = \gamma^2/(\gamma^2 + C^2)$ from the interval $(0, \infty)$ to the interval $(0, 1)$

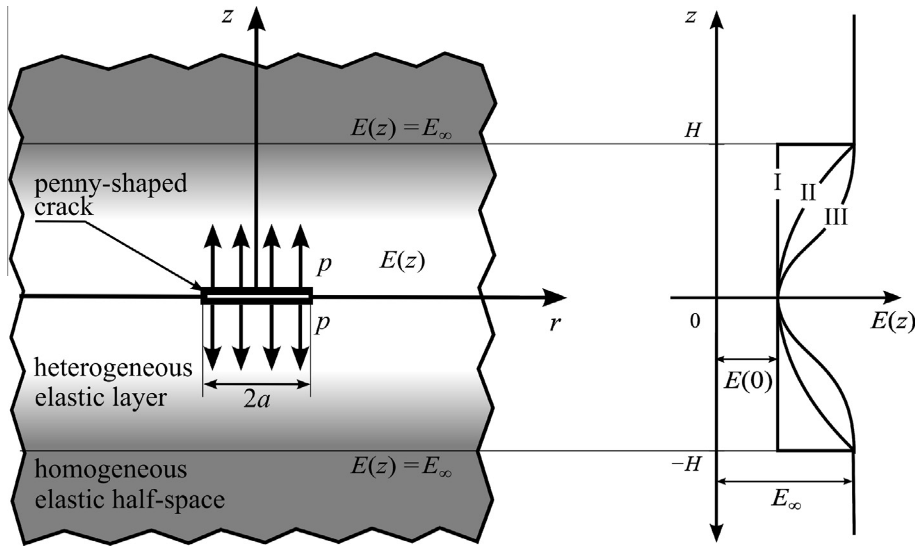

Fig. 1. Geometry of the problem: curves marked I, II, III present examples of possible Young's modulus $E(z)$ variation in the inhomogeneous elastic layer.

$\left(\gamma = C\sqrt{u(u-1)^{-1}}\right)$. Here C is a positive constant that should be chosen to optimise the approximation of the function $L(\gamma)$ as shown below.

Then, let us approximate the function $\sqrt{L(\gamma)}$ and $\sqrt{L^{-1}(\gamma)}$ on the interval $(0,1)$ by the Bernstein polynomial of the N-order as follows

$$\sqrt{L_N(\gamma)} = \sum_{i=0}^{N} a_i u^i, \quad \sqrt{L_N^{-1}(\gamma)} = \sum_{i=0}^{N} b_i u^i \quad L_N(\gamma) = \frac{\sqrt{L_N(\gamma)}}{\sqrt{L_N^{-1}(\gamma)}}$$
$$= \left(\sum_{i=0}^{N} a_i^* \gamma^{2i}\right)\left(\sum_{i=0}^{N} b_i^* \gamma^{2i}\right)^{-1}$$

where the coefficients a_i^*, b_i^* are determined by the substitution $u = \gamma^2/(\gamma^2 + C^2)$.

By finding the roots of both the numerator and the denominator one obtains the following approximation

$$L(\lambda\gamma) \approx L_N(\lambda\gamma) = \prod_{i=1}^{N} \frac{\lambda^2\gamma^2 + A_i^2}{\lambda^2\gamma^2 + B_i^2} \tag{3.1}$$

In order to estimate the error of approximation one can introduce the maximum ratio of the difference between the exact and approximate values of the function $L(\gamma)$

$$L_{err} = \max_{0 \leqslant \gamma < \infty} \frac{|L(\lambda\gamma) - L_N(\lambda\gamma)|}{L(\lambda\gamma)} \tag{3.2}$$

The error L_{err} is further minimised by varying the free parameter C. For some simple cases of monotonic variations of the elastic modulus it is possible to construct quite accurate approximation even for $N = 1$.

Let us consider four specific monotonic distributions of the elastic moduli along the normal to the crack surfaces as shown in Fig 2.

For simplicity Poisson's ratio is assumed to be constant ($\nu = 0.25$), and the elastic modulus varies as in (1.1) where the function $f_n(z)$ is specified by the following four different shapes shown in Fig. 2

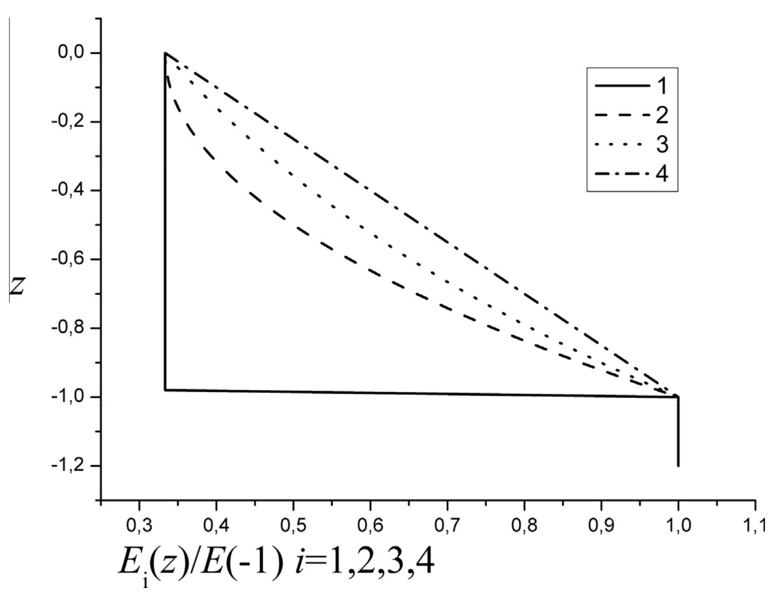

Fig. 2. Dependencies of the Young's modulus in the direction perpendicular to the crack: four specific distributions specified by (3.3).

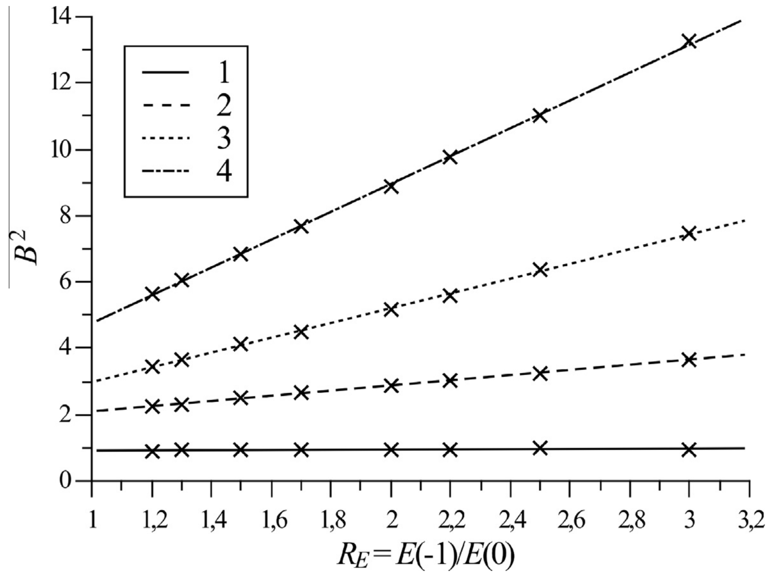

Fig. 3. Dependence B^2 (3.4) for distribution laws specified in (3.3).

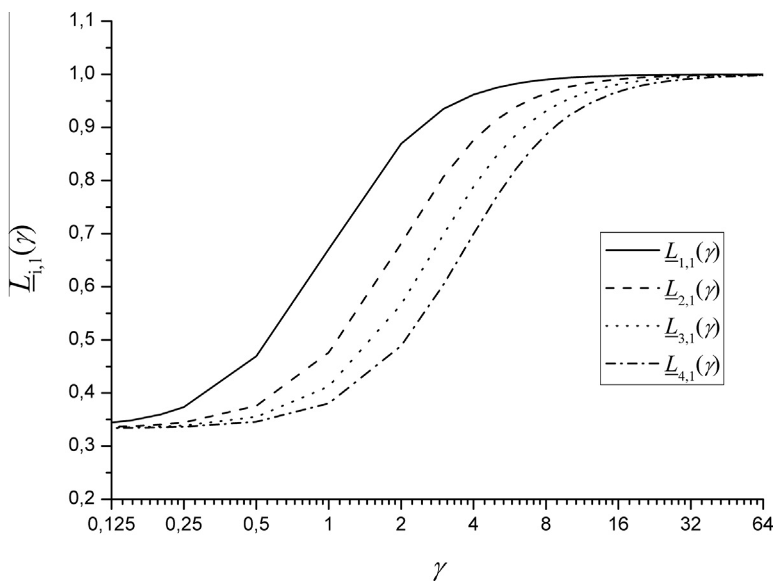

Fig. 4. Dependencies of the kernels on the parameter γ.

$$
\begin{aligned}
& f_1(\bar{z}) = \frac{1}{R_E}, \quad f_2(\bar{z}) = \frac{1}{R_E} + \frac{R_E - 1}{R_E} z^2 \\
& f_3(\bar{z}) = \frac{1}{R_E} + \frac{R_E - 1}{R_E} \frac{\exp(-z) - 1}{\exp(1) - 1}, \quad f_4(\bar{z}) = \frac{1}{R_E} - \frac{R_E - 1}{R_E} \bar{z} \\
& R_E = \frac{E_\infty}{E(0)} = \frac{E(-1)}{E(0)}, \quad 1.2 \leqslant R_E \leqslant 3
\end{aligned}
\tag{3.3}
$$

The dimensionless parameter R_E characterises the degree of inhomogeneity of the soft layer.

In numerical calculations the value of $L_1(0) = 1/R_E = E(0)/E(-1)$ has been varied and the errors have been minimised. As a result, a linear dependence of the magnitude of B^2 from R_E has been determined. Therefore, the ratio of the elastic modulus on the crack plane to its remote value varies in the range $1.2 \leqslant R_E \leqslant 3$ and the parameter is found as follows

$$
B_i^2(R_E) = a_i + b_i R_E \tag{3.4}
$$

The values of the parameters a_i, b_i, c_i, d_i are shown in the table below for the elastic modulus distributions specified in (3.3)

i	a_i	b_i
1	0.88334	0.03085
2	1.31739	0.78188
3	0.75099	2.22931
4	0.55121	4.20334

Approximations of all four elastic distributions investigated are shown in Fig. 3. The number indicated in the legend corresponds to the law of change of Young's modulus, crosses represent the values of B^2 found after minimisation of the errors. The curves have been constructed by (3.3) after which the coefficients a_i, b_i are determined.

Approximate analytical expressions for the kernel transformation for (1.11) can be obtained from the data in the table and (3.4) as follows

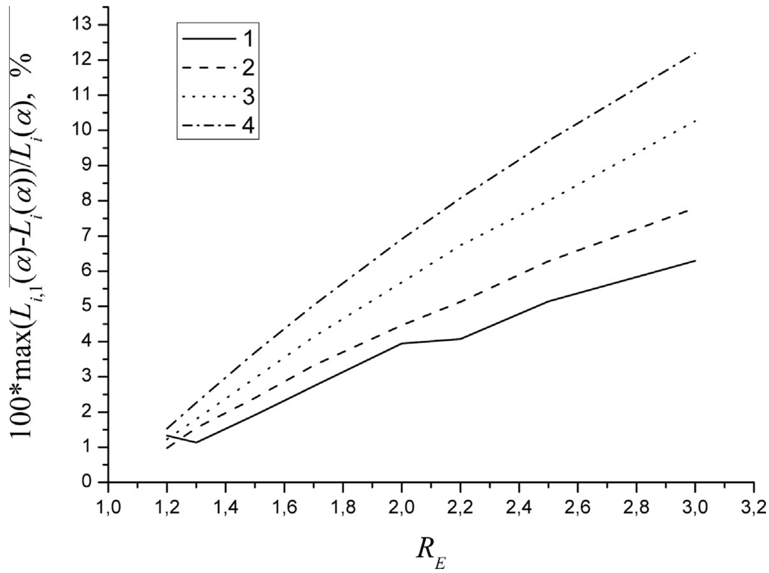

Fig. 5. The dependences of the maximum relative errors $(L - L_N)/L$ from R_E for all four distributions for the case when the approximations contain one term.

$$\underline{L}_{i,1}(\lambda\gamma) = \frac{\lambda^2\gamma^2 + A_{i,1}^2}{\lambda^2\gamma^2 + B_{i,1}^2} = \frac{\lambda^2\gamma^2 + a_i/R_E + b_i}{\lambda^2\gamma^2 + a_i + b_i R_E} \quad i = 1,\ 2,\ 3,\ 4$$

For $R_E = 3$ and $\lambda = H/a = 1$ the approximations for $\underline{L}_{i,1}(\lambda\gamma)$ are shown below and plotted in Fig 4.

$$\underline{L}_{1,1}(\gamma) = \frac{\gamma^2+0.8833/3+0.0309}{\gamma^2+0.8833+0.0309*3} = \frac{\gamma^2+0.3253}{\gamma^2+0.9759}$$
$$\underline{L}_{2,1}(\gamma) = \frac{\gamma^2+1.3174/3+0.7819}{\gamma^2+1.3174+0.7819*3} = \frac{\gamma^2+1.2210}{\gamma^2+3.6630}$$
$$\underline{L}_{3,1}(\gamma) = \frac{\gamma^2+0.7510/3+2.2293}{\gamma^2+0.7510+2.2293*3} = \frac{\gamma^2+2.4796}{\gamma^2+7.4389}$$
$$\underline{L}_{4,1}(\gamma) = \frac{\gamma^2+0.5512/3+4.2033}{\gamma^2+0.5512+4.2033*3} = \frac{\gamma^2+4.3871}{\gamma^2+13.1612}$$

Fig. 5 shows the dependence of the maximum relative errors $(L_i - L_{i,n})/L_i$ from R_E for all four distributions for the case when the approximations contain one term. Fig. 6 shows similar errors when 29 terms are used in the approximation for L. Comparison of the results presented in Fig 5 and 6 shows that for the approximation of the integral equation kernels for distributions 1–4 is accurate to within 12% error when the ratio of elastic modulus at the surface to its remote value varies in the range from 1.2 to 3. If the number of approximation terms increases up to 29 this reduce the error up to 3 times.

The approximation of the kernel transform (3.1) by a single term allows one to express all the components of the solution (for all distributions considered) in terms of the parameter $L_1(0) = 1/R_E = E(0)/E(-1)$. In this case $L_1(0) = A^2B^{-2}$. The stress intensity factor is found as the following limit

$$K_I = \lim_{r\to 1+0} \sqrt{r-1}\,p(r) = -\frac{\sqrt{2}}{\pi} p\Delta(0)^{-1}\left(L_1(0) + C\,shB\lambda^{-1}\right) \tag{3.5}$$

The constant C is determined from the following linear algebraic equation

$$C\frac{B\lambda^{-1}chB\lambda^{-1} + A\lambda^{-1}shB\lambda^{-1}}{A^2\lambda^{-2} - B^2\lambda^{-2}} + \frac{1 + A\lambda^{-1}}{A^2\lambda^{-2}} L_1(0) = 0, \tag{3.6}$$

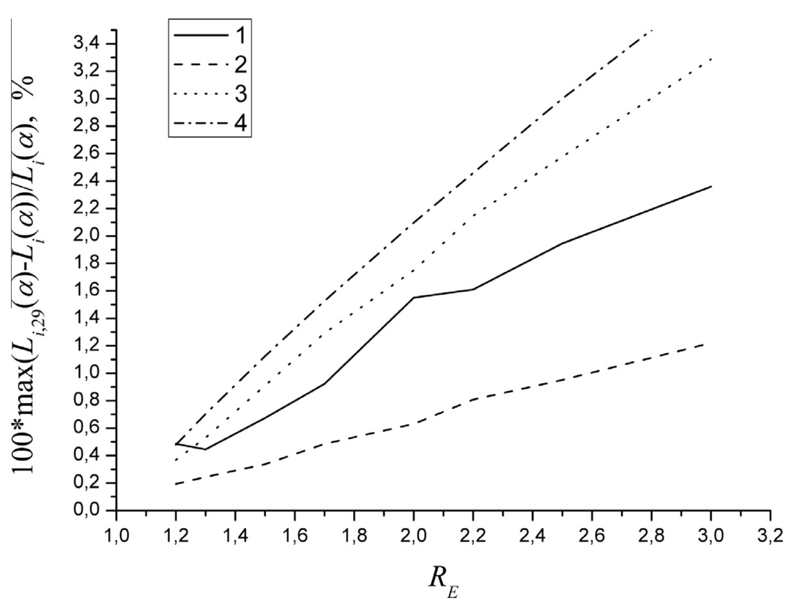

Fig. 6. Maximum errors for the case of 29 terms used in the approximation for L.

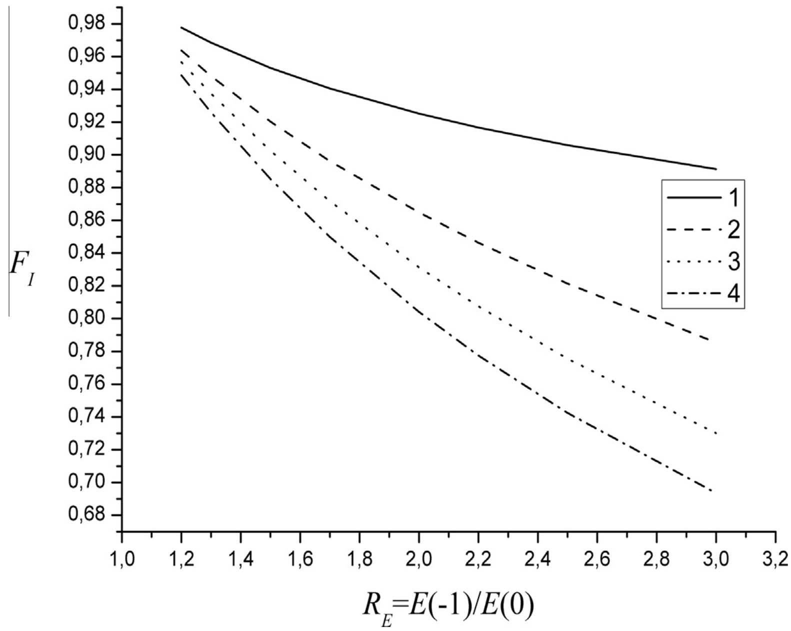

Fig. 7. The dependence of the stress intensity factor from R_E.

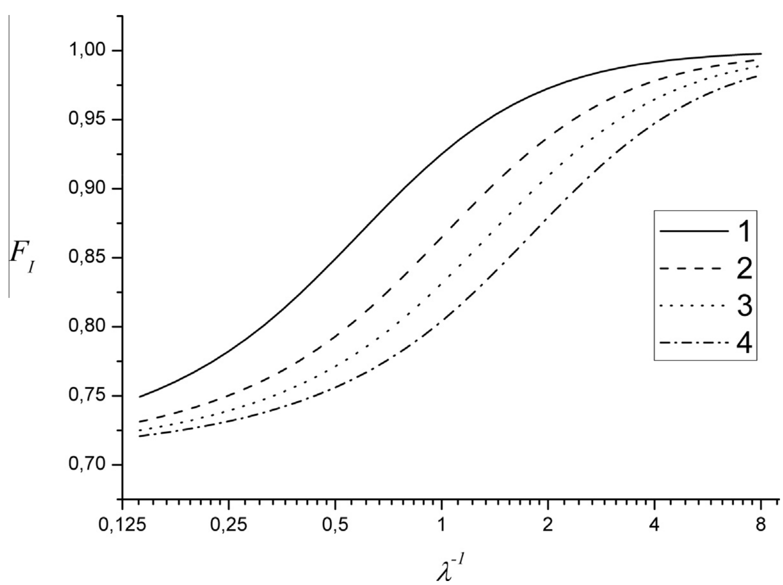

Fig. 8. The dependence of the dimensionless stress intensity factor from λ^{-1}.

Therefore, the stress intensity factor can be approximated as a function of $L_1(0)$ and the parameter B^2, which yields

$$
\begin{aligned}
F_I &= \frac{K_I}{K_{I0}} = L_1(0)\left(1 - shB\lambda^{-1}\frac{1+A\lambda^{-1}}{A^2\lambda^{-2}}\frac{A^2\lambda^{-2}-B^2\lambda^{-2}}{B\lambda^{-1}chB\lambda^{-1}+A\lambda^{-1}shB\lambda^{-1}}\right)\\
&= \frac{1}{R_E}\left(1+\frac{(\lambda+A)(R_E-1)}{B\,cth\,B\lambda^{-1}+A}\right) = \frac{1}{R_E}\left(1+\frac{(\lambda+\sqrt{a_i/R_E+b_i})(R_E-1)}{\sqrt{a_i+R_Eb_i}cth(\sqrt{(a_i+R_Eb_i)\lambda^{-1}})+\sqrt{a_i/R_E+b_i}}\right)\\
A &= \sqrt{B^2/R_E},\quad K_{I0} = -\frac{\sqrt{2}}{\pi}p\Delta(0)^{-1}
\end{aligned}
\tag{3.7}
$$

Fig. 7 shows the variation of the stress intensity factor, calculated by Eq. (3.7) with (3.3) for the case when the crack radius is equal to the thickness of the layer. The results are shown for all 4 distributions of the elastic modulus variations provided that the ratio R_E varies in the range from 1.2 to 3. The line numbers in the legend correspond to the distribution numbers as in Fig. 2.

By varying the parameter $\lambda = H/a$ at fixed value of $R_E = 2$ it is possible to obtain the dependencies of the dimensional stress intensity factor $F_I = K_I/K_{I0}$ (normalised by K_{I0} for a crack in the homogeneous space) from λ^{-1} by formula (3.7). The results are shown in Fig 8 for all 4 distributions of the elastic modulus variations (the line numbers in the legend correspond to the distribution numbers as in Fig. 2). The results for the step-like distribution (marked 1 in the figure) agree with those presented in the handbook by Murakami (1987).

5. Closure

The paper presents the approximate semi-analytical solution for a disk-like mode I crack in a functionally graded soft interlayer for four specific distributions of the elastic moduli. It is shown that the fracture characteristics (the mode I stress intensity factor) can be expressed in terms of the quantity R_E i.e. the ratio of the remote elastic modulus to its value on the plane of symmetry (crack plane). The accuracy of the solution has been found satisfactory for the four distributions given by Eqs. (3.3) when the ratio of the moduli, R_E, varies in the range from 1.2 to 3.

Simple analytical expressions obtained for the mode I stress intensity factor can serve for the analysis of fracture propagation in FGM or for the analysis of hydrofracture growth in elastic inhomogeneous reservoirs.

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Appendix A. Auxiliary transformations

Let us introduce the following notation for the integral operator

$$A[\delta] \equiv \int_0^R \delta(\rho)\rho d\rho \int_0^\infty \frac{\alpha^2}{L(\alpha)} J_0(\alpha\rho)J_0(\alpha r)d\alpha \tag{A.1}$$

Integration by parts yields:

$$A[\delta] = \delta(\rho)\rho \int_0^\infty \frac{\alpha^2}{L(\alpha)}\rho J_1(\alpha\rho)J_0(\alpha r)d\alpha\Big|_0^R - \int_0^R \delta'(\rho)d\rho \int_0^\infty \frac{\alpha\rho}{L(\alpha)} J_1(\alpha\rho)J_0(\alpha r)d\alpha \tag{A.2}$$

The first term in (A.2) vanishes due to (1.10). Thus, one obtains:

$$A[\delta] = -\int_0^R \delta'(\rho)d\rho \int_0^\infty \frac{\alpha\rho}{L(\alpha)} J_1(\alpha\rho)J_0(\alpha r)d\alpha = -\frac{p(r)}{\Delta(0)}, \quad 0 \leqslant r \leqslant R \tag{A.3}$$

It follows from (A.3) that the normal stresses on the crack plane are given by the following operator:

$$p(r) = \Delta(0)\int_0^R \delta'(\rho)d\rho \int_0^\infty \frac{\alpha\rho}{L(\alpha)} J_1(\alpha\rho)J_0(\alpha r)d\alpha, \quad r > R \tag{A.4}$$

The application of the following operator

$$B[\psi] \equiv \int_0^r \rho\psi(\rho)d\rho \tag{A.5}$$

to both sides of (A.4) results in

$$\int_0^R \delta'(\rho)d\rho \int_0^\infty \frac{r\rho J_1(\alpha\rho)J_1(\alpha r)}{L(\alpha)}d\alpha = \frac{1}{\Delta(0)}\int_0^r p(\rho)\rho d\rho, \quad 0 \leqslant r \leqslant R \tag{A.6}$$

Let us introduce the following substitutions and notations

$$\alpha H = u; \quad \lambda = \frac{H}{R}; \quad r' = \frac{r}{R}; \quad \rho' = \frac{\rho}{R}; \quad \delta'(\rho' R) = \varphi'(\rho')$$
$$\frac{u}{\lambda} = \beta; \quad q^*(r) = \frac{1}{Rr}\int_0^r p(\rho)\rho d\rho = \frac{1}{Rr'}\int_0^{r'} p(R\rho')\rho' R d\rho', \ 0 \leqslant r' \leqslant 1 \tag{A.7}$$

Making use of (A.7) one can reduce the system to the following integral equations (hereafter the dash symbols are omitted)

$$\int_0^1 \varphi(\rho)d\rho \int_0^\infty \frac{\rho J_1(\beta\rho)J_1(\beta r)}{L(\beta\lambda)}d\beta = \frac{1}{\Delta(0)}q^*(r), \quad 0 \leqslant r \leqslant 1 \tag{A.8}$$

Here the following notation has been introduced for dimensionless loads:

$$p^*(r) = \frac{1}{r}\int_0^1 f(\rho)\rho d\rho \tag{A.9}$$

Taking into account that $\varphi(r) = 0$ for $r > 1$, and hence $\varphi'(r) = 0$, for $r > 1$, one finds the following relationship (based on the properties of the Hankel transform):

$$\int_0^\infty \Delta_1(\beta)\beta J_1(\beta\rho)d\rho = \begin{cases} \varphi'(\rho), & 0 < \rho \leqslant 1 \\ 0, & \rho > 1 \end{cases}, \quad \Delta_1(\beta) = \int_0^1 \varphi'(\rho)J_1(\beta\rho)\rho d\rho \tag{A.10}$$

Therefore the following dual integral equation can be derived from (A.8)–(A.10):

$$\begin{cases} \int_0^\infty \frac{\Delta_1(\beta)}{L(\beta\lambda)} J_1(\beta r)d\beta = \Delta^{-1}(0)p^*(r), & 0 \leqslant r \leqslant 1 \\ \int_0^\infty \Delta_1(\beta)\beta J_1(\beta r)d\beta = 0, & r > 1 \end{cases} \tag{A.11}$$

As soon as the function $\Delta_1(\beta)$ is found it is possible to find out $\delta(r)$ by the following integrals:

$$\delta'(r) = \int_0^\infty \Delta_1(\alpha)\alpha J_1(\alpha r)d\alpha \tag{A.12}$$

$$\delta(r) = \int_R^r \delta'(\rho)d\rho = \int_0^\infty \Delta_1(\alpha)\alpha \int_R^r J_1(\alpha\rho)d\alpha d\beta$$
$$= \int_0^\infty \Delta_1(\alpha)J_0(\alpha r)d\alpha, \text{ as } \delta(R) = 0$$

Let $p^*(r)$ be expanded into the Bessel series as follows

$$p^*(r) = \frac{1}{2}p\left[r + 2\sum_{k=1}^\infty c_k J_1(\mu_k r)\right] \tag{A.13}$$

Then the uniform load is expressed via the following limit

$$\lim_{\varepsilon\to 0}\frac{J_1(\varepsilon r)}{\varepsilon} = 2r \tag{A.14}$$

For the special case of uniform loads the crack opening displacements take the form:

$$\delta(r) = \frac{2}{\pi}\left(-\frac{p}{\Delta(0)}\right)\left[L_N(0)\sqrt{1-r^2} + \sum_{i=1}^N C_i\tilde{b}_i \int_r^1 \frac{sh\tilde{b}_i t dt}{\sqrt{t^2-r^2}}\right] \tag{A.15}$$

The normal stresses on the crack plane are given by the following expression:

$$p(r) = -\frac{p}{\Delta(0)}\int_0^\infty \alpha\Delta_1(\alpha)F_N(\lambda\alpha)J_0(\alpha r)d\alpha, \quad r > 1 \tag{A.16}$$

Evaluation of the integrals yields the following distribution of the normal stresses on the crack plane:

$$p(r) = 2\pi^{-1}p\left\{\arcsin\left(\frac{1}{r}\right) - \left[\frac{L(0)}{\Delta(0)} + \sum_{n=1}^{N} C_n sh(\tilde{b}_n)\right] \times \frac{1}{\sqrt{r^2-1}} - \sum_{k=1}^{N} L_N^k(\tilde{a}_k)\left[\frac{L(0)}{\Delta(0)} + \sum_{n=1}^{N} C_n \frac{\tilde{a}_k^2 sh(\tilde{b}_n)}{\tilde{a}_k^2 - \tilde{b}_n^2}\right] \times \int_1^r \frac{\exp(-\tilde{a}_k(t-1))}{\sqrt{r^2-t^2}} dt\right\},\ r > 1 \quad (A.17)$$

The stress intensity factor, SIF, is found from (A.17) by limiting transition

$$K_I = \lim_{r \to 1+0} \sqrt{r-1}\, p(r) = -\frac{\sqrt{2}}{\pi} \frac{p}{\Delta(0)} \left(L(0) + \Delta(0) \sum_{i=1}^{N} C_i sh\tilde{b}_i\right) \quad (A.18)$$

The energy relies rate, ERR, for the case of uniform load assume the form:

$$A = p_0 \int_0^1 2\pi r \delta(r) dr \quad (A.19)$$

which after evaluation of the integral becomes

$$A = 4p^2 \left[\frac{1}{3} L_N(0) \Delta^{-1}(0) + \sum_{i=1}^{N} C_i \tilde{b}_i^{-2} (\tilde{b}_i ch\tilde{b}_i - sh\tilde{b}_i)\right] \quad (A.20)$$

All the expressions in (A.15)–(A.20) are asymptotically accurate for COD, normal stresses on the crack plane, SIF and ERR respectively for the case of a penny-shaped crack in FGM.

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