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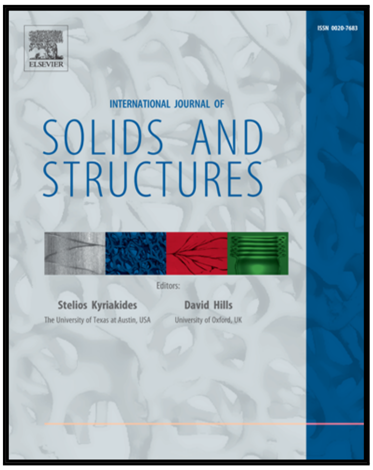

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Randomly Oriented Cracks in a Transversely Isotropic Material

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Abstract

Our work focuses on the calculation of the overall elastic properties of a transversely isotropic material containing multiple randomly oriented circular cracks. We first propose a new methodology to estimate (approximately) the contribution of a single arbitrarily oriented crack in an infinite transversely isotropic media into the overall elastic moduli. This effect is described by a forth rank compliance contribution tensor which serves as the basic building block for various homogenization schemes aimed at calculation of the overall elastic properties of the materials containing multiple inhomogeneities. In this paper we use the Mori-Tanaka-Benveniste scheme which coincides with non-interaction approximation for the case of crack-like inhomogeneities. The approach is illustrated by examples of Berea sandstone and wet bovine dentin.

Keywords: transverse isotropy, cracks, effective elastic properties, randomly oriented

1. Introduction

In this paper, we calculate the overall elastic properties of a transversely isotropic material containing multiple randomly oriented cracks. Our work is motivated by needs in geophysics and biomechanics, where the cracks are typically embedded in microcracked anisotropic background. Dentin, cortical bone, and Berea Sandstone shown in Figure 1 serve as examples. To the best of our knowledge, there are no explicit analytical expressions for the overall elastic properties of a transversely-isotropic material containing randomly oriented cracks. Our approach is based on the results of Guerrero et al (2007), who developed the method to evaluate crack opening displacement tensor (see Kachanov, 1993) for an arbitrarily oriented crack in a transversely-isotropic material of elliptic type (Sevostianov and Kachanov, 2008).

The problem about microcracks effect on the overall material properties has been first discussed by Bristow (1960), who introduced crack density parameter and expressed the effective elastic moduli and overall electric conductivity in its terms (in the framework of the non-interaction approximation, NIA). For circular (penny-shaped) cracks, of radii a_k and random orientation distribution, their concentration is represented by the *scalar* crack density:

$$\rho = \frac{1}{V}\sum_k a_k^3 \quad \text{(in 2-D case of rectilinear cracks of lengths } 2l_k\text{, } \rho = \frac{1}{A}\sum_k l_k^2\text{)} \tag{1.1}$$

This parameter was generalized by Budiansky and O'connell (1976) to planar cracks of the elliptical shapes, of areas S_k and perimeters P_k – under rather restrictive assumption that all ellipses have the same eccentricity – as

$$\rho = \frac{1}{V}\frac{2}{\pi}\sum_k \left(\frac{A^2}{P}\right)^{(k)} \tag{1.2}$$

For an arbitrary non-random orientation distribution of circular cracks, the *crack density tensor* was introduced by Kachanov (1980) (see also his review of 1992):

$$\boldsymbol{\alpha} = \frac{1}{V}\sum_k \left(a^3 \boldsymbol{nn}\right)^{(k)} \qquad \text{(in 2-D case, } \boldsymbol{\alpha} = \frac{1}{A}\sum_k \left(l^2 \boldsymbol{nn}\right)^{(k)}\text{)} \tag{1.3}$$

where $\boldsymbol{n}$ is a unit normal to a crack; the scalar crack density is its trace: $\rho = \alpha_{ii}$. He also identified the fourth-rank tensor

$$\boldsymbol{\beta} = \frac{1}{V}\sum_k \left(a^3 \boldsymbol{nnnn}\right)^{(k)} \tag{1.4}$$

as a second parameter that plays a relatively minor role for traction-free cracks. The extra compliances ΔS_{ijkl} due to circular cracks of arbitrary orientations $\boldsymbol{n}^k$ and radii a_k derived, in the NIA, by Kachanov (1980) has the following form

$$\Delta S_{ijkl} = \frac{32\left(1-\nu_0^2\right)}{3\left(2-\nu_0\right)\, E_0}\left[\frac{1}{4}\left(\delta_{ik}\alpha_{jl} + \delta_{il}\alpha_{jk} + \delta_{jk}\alpha_{il} + \delta_{jl}\alpha_{ik}\right) - \frac{\nu_0}{2}\beta_{ijkl}\right] \tag{1.5}$$

where tensors $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are defined by (1.3) and (1.4).

The second-rank crack density tensor $\boldsymbol{\alpha}$ and fourth rank tensor $\boldsymbol{\beta}$ are the proper crack density parameters for circular cracks: the expression (1.5) covers all orientation distributions of cracks in a unified way. For example, in the case of random orientations (overall isotropy),

$$\boldsymbol{\alpha} = (\rho/3)\boldsymbol{I}\,,\;\; \beta_{ijkl} = (\rho/15)\left(\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\right); \tag{1.6}$$

the effective Young's and shear moduli are as follows (first given by Bristow 1960):

$$\frac{E_0}{E} = 1 + \frac{16\left(1-\nu_0^2\right)\left(10-3\nu_0\right)}{45\left(2-\nu_0\right)}\rho\,, \quad \frac{G_0}{G} = 1 + \frac{32\left(5-\nu_0\right)}{45\left(2-\nu_0\right)}\rho \tag{1.7}$$

For anisotropic materials, the first analytical results have been obtained by Hoenig (1977, 1978), who derived formulas (in the integral form) for the stress intensity factors and crack opening displacements for an elliptical crack in generally anisotropic media and showed that the integrals can be evaluated in closed form only for a transversely-isotropic material if the crack is parallel to the plane of isotropy. He also calculated the change in moduli for transversely-isotropic elastic media with cracks parallel to the plane of isotropy. Later, results of Hoenig (1977, 1978) on elliptical crack in a 3-D anisotropic material have been repeated or specified for some particular cases (very often without proper citation). Fabrikant (1989) used the method of potential functions to calculate crack opening displacement tensor for a single circular crack embedded in a transversely isotropic material parallel to the plane of isotropy. The closed form analytical expressions for a spheroidal inhomogeneity embedded in a transversely-isotropic matrix (with a circular crack considered as a limiting case) with the rotation axis of the spheroid and the symmetry axis of transverse isotropy being aligned have been obtained by various methods in papers of Withers, (1989), Yu et. al., (1994), and Sevostianov et al., (2005). These results were generalized to the case of piezoelectric materials by Dunn and Wienecke (1997), Levin et al. (2000) and Mikata (2000). Kanaun (2007) used the Fourier transform method and reduced integral equation for crack opening displacements to evaluation of two convergent integrals. The only case when these results can be written in closed explicit form is when the crack is parallel to the plane of isotropy of a transversely isotropic material. Kanaun and Levin (2009) presented an integral equation for anisotropic medium with elliptical cracks and its solution with constant and linear polynomial external fields. They also considered the problem about overall properties of an anisotropic media with multiple cracks.

In 2-D, the main result has been obtained by Tsukrov and Kachanov (2000) who derived a closed-form analytical solution for an arbitrarily oriented rectilinear crack in an anisotropic matrix. They showed that the second rank crack opening displacement tensor that relates the average crack opening displacement (displacement discontinuity) vector to vector of uniform traction applied at the crack faces, for such a crack is independent of the crack orientation if the coordinate system coincides with the principal directions of anisotropy. This results inspired Guerrero et al. (2007) to consider crack opening displacement tensor for an arbitrarily oriented circular crack in a three-dimensional transversely isotropic medium of elliptic type (Sevostianov and Kachanov,2008; see Appendix). The authors showed that the independence of the crack opening displacement tensor that is exact for 2-D cracks can be used as a good approximation for 3-D cracks even at very high extent of anisotropy if the anisotropy is of elliptic type.

In the present work, we use results of Guerrero (2007) to calculate the effective properties of a transversely – isotropic material with multiple randomly oriented microcracks. For this goal, we first approximate the transversely isotropic material by best-fit elliptic transverse-isotropy using formulas (A.4), then we calculate the crack opening displacement for a single arbitrarily oriented crack, using results of Fabrikant (1989) for the crack in the plane of isotropy, then we evaluate fourth rank compliance contribution tensor and use it in Non-Interaction Approximation (that coincides for cracks with Mori-Tanaka-Benveniste scheme) to find the effective elastic moduli of the transversely isotropic material containing multiple randomly oriented circular cracks. We illustrate the approach on examples of two materials: bovine dentin and Berea sandstone.

2. Compliance contribution tensor of a flat cracks

Compliance contribution tensors have been first introduced in the context of pores and cracks by Horii and Nemat-Nasser (1983) (see also detailed discussion in the book of Horii and Nemat-Nasser, 1993). Components of this tensor were calculated for 2-D pores of various shape and 3-D ellipsoidal pores in isotropic material by Kachanov (1994). Sevostianov et al. (2005) calculated components of this tensor for a spheroidal pore in a transversely-isotropic matrix when axis of rotation of the spheroid coincides with the axis of transverse isotropy. Following this works, we consider a homogeneous elastic material (matrix), with the compliance tensor $\boldsymbol{S}^0$ containing an inhomogeneity, of volume V_*, of a different material with the compliance tensor

$\boldsymbol{S}^1$. The compliance contribution tensor of the inhomogeneity is a fourth-rank tensor $\boldsymbol{H}$ that gives the extra strain (per reference volume V) due to its presence:

$$\Delta\boldsymbol{\varepsilon} = \frac{V_*}{V}\boldsymbol{H}:\boldsymbol{\sigma}^\infty, \qquad \text{or, in components, } \Delta\varepsilon_{ij} = \frac{V_*}{V}H_{ijkl}\sigma_{kl}^\infty \tag{2.1}$$

where σ_{kl}^∞ are remotely applied stresses that are assumed to be uniform within V in the absence of the inhomogeneity ("homogeneous boundary conditions", Hashin (1983). For a pore or a crack, the additional strain due to its presence is calculated as an integral over its boundary $\partial\Omega$

$$\Delta\varepsilon_{ij} = \frac{-1}{2V}\int_{\partial\Omega}\left(u_i n_j + u_j n_i\right)dS \tag{2.2}$$

where $\boldsymbol{u}$ and $\boldsymbol{n}$ denote displacements on the pore boundary and a unit normal to it (directed inwards the pore). The representation (2.3) directly follows from application of the divergence theorem to a solid containing a pore (see, for example, Kachanov et. al. 1994).

In the limit of a *crack*, pore boundary degenerates into a two-sided surface Γ, with opposite directions of normals and displacements discontinuous across Γ. Therefore, the last formula takes the form of the surface integral

$$\Delta\varepsilon_{ij} = \frac{1}{2V}\int_{\Gamma}\left(\left[u_i\right]n_j + \left[u_j\right]n_i\right)dS \tag{2.3}$$

where $[\boldsymbol{u}] = \boldsymbol{u}^+ - \boldsymbol{u}^-$ is the displacement discontinuity vector on Γ where the "+" sign denotes the side of Γ corresponding to the chosen positive direction of $\boldsymbol{n}$; note that products $[u_i]n_j$ do not depend on the choice of this direction since its reversal changes signs of both multipliers. If the crack is *flat* ($\boldsymbol{n} = const$), of area S, the expression (2.3) reduces to

$$\Delta\varepsilon_{ij} = \frac{1}{2V}\left(b_i n_j + b_j n_i\right)S \tag{2.4}$$

where $\boldsymbol{b} = \left\langle \boldsymbol{u}^+ - \boldsymbol{u}^- \right\rangle$ is the average displacement discontinuity vector.

For a flat crack of any shape, second-rank *displacement discontinuity tensor* $\boldsymbol{B}$ can be introduced that relates $\boldsymbol{b}$ to the vector of uniform traction $\boldsymbol{n}\cdot\boldsymbol{\sigma}$ induced at the crack site, in absence of the crack, by $\boldsymbol{\sigma}^\infty$:

$$\boldsymbol{b} = \boldsymbol{n} \cdot \boldsymbol{\sigma}^{\infty} \cdot \boldsymbol{B} \tag{2.5}$$

As follows from dimensional considerations, tensor $\boldsymbol{B}$ is proportional to a linear dimension of the crack.

In the coordinate system of the crack ($\boldsymbol{n}, \boldsymbol{\tau}_1, \boldsymbol{\tau}_2$) where $\boldsymbol{\tau}_1$ and $\boldsymbol{\tau}_2$ are two orthogonal unit vectors in the crack plane, component $B_{nn} \equiv B_N$ characterizes the normal compliance of the crack and $B_{\tau\tau}$ its shear compliance in the direction $\boldsymbol{\tau}$: they give the normal and shear components of $\boldsymbol{b}$ produced by uniform tractions of unit intensity applied in the same directions (note that $B_N S$ is the volume of the crack subjected to uniform pressure of unit intensity). The off-diagonal component $B_{n\tau} = B_{\tau n}$ characterizes coupling of the normal and shear modes (if the matrix is isotropic, $B_{n\tau} = 0$) and $B_{\tau_1\tau_2}$ - coupling between the two in-plane directions.

Since tensor $\boldsymbol{B}$ is symmetric (as follows from application of the reciprocity theorem to the normal and shear loadings on a crack), three orthogonal *principal directions of the crack compliance* exist: application of a uniform traction in one of them does not generate components of $\boldsymbol{b}$ in the other two directions. If the matrix is isotropic, $\boldsymbol{n}$ is one of them, and the other two, $\boldsymbol{t}$ and $\boldsymbol{s}$, lie in the crack plane:

$$\boldsymbol{B} = B_N \boldsymbol{nn} + B_{tt} \boldsymbol{tt} + B_{ss} \boldsymbol{ss} \tag{2.6}$$

In the case of anisotropic matrix, the representation (2.6) still holds but the mutually orthogonal vectors $\boldsymbol{n}$, $\boldsymbol{s}$ and $\boldsymbol{t}$ are not necessarily normal/parallel to the crack.

In the limit of a crack, the product $V_* \boldsymbol{H}$ in (2.1) is an indeterminacy $0 \cdot \infty$; formulas (2.4) shows that, in this limit,

$$V_* \boldsymbol{H} = S\, \boldsymbol{nBn} \tag{2.7}$$

with symmetrization with respect to $ij \leftrightarrow kl$, $i \leftrightarrow j$, $k \leftrightarrow l$ imposed on $ijkl$ components of $\boldsymbol{H}$. In particular, $V_* H_{nnnn} = S\, B_N$. In particular, for a circular crack,

$$V_{*}\boldsymbol{H} = \frac{16\left(1-\nu_0^2\right)}{3E_0} a^3 \left(\boldsymbol{nnnn} + \frac{1}{1-\nu_0/2} \boldsymbol{n}\left(\boldsymbol{I} - \boldsymbol{nn}\right)\boldsymbol{n} \right) \tag{2.8}$$

with the above-mentioned symmetrization imposed.

3. Isolated arbitrarily oriented penny-shaped crack in a transversely isotropic material.

Exact explicit analytical results for an arbitrarily oriented crack in an anisotropic material are available in the context of 2-D problem only. Mauge and Kachanov (1994) considered 2D orthotropic material and derived crack opening displacement tensor $\boldsymbol{B}$ for an arbitrarily oriented crack of length $2l$. Tsukrov and Kachanov (2000) showed that, in the coordinate system coinciding with the axes of orthotropy of the matrix, tensor $\boldsymbol{B}$ is independent of the crack orientation and has the following form:

$$\boldsymbol{B} = l^2 C[(1+\mathrm{D})\boldsymbol{e_1 e_1} + (1+D)\boldsymbol{e}_2\boldsymbol{e}_2] \tag{3.1}$$

where

$$C = \frac{\pi l^2}{4} \frac{\sqrt{E_1^0} + \sqrt{E_2^0}}{\sqrt{E_1^0 E_2^0}} \sqrt{\frac{1}{G_{12}^0} - \frac{2\nu_{12}^0}{E_1^0} + \frac{2}{\sqrt{E_1^0 E_2^0}}} \tag{3.2}$$

$$D = \frac{\pi l^2}{2} \frac{\sqrt{E_1^0} - \sqrt{E_2^0}}{\sqrt{E_1^0} + \sqrt{E_2^0}} \tag{3.3}$$

Independence of tensor $\boldsymbol{B}$ of the crack orientation means that, according to (2.7), compliance contribution tensor $\boldsymbol{H}$ of a crack reflects the crack orientation only through the unit normal vector $\boldsymbol{n}$.

Guerrero et. al. (2007) showed that, with good accuracy, tensor $\boldsymbol{B}$ can be considered as independent of the crack orientation, if (a) material is elliptically transversely isotropic (see appendix) and (b) coordinate axes are chosen along the axes of elastic symmetry of the matrix. In this case, we can use solution of Fabrikant (1989) for components of tensor $\boldsymbol{B}$ for a circular crack in the plane of isotropy of a transversely isotropic material:

$$B_{nn} = B_{33} = \frac{8aG}{3\pi(C_{11}C_{33} - C_{13}^2)\sqrt{C_{44}/C_{33}}} \tag{3.4}$$

$$B_{ss} = B_{\tau\tau} = B_{22} = B_{11} = \frac{16aG}{3\pi[\sqrt{C_{44}C_{66}}G + (C_{11}C_{33} - C_{13}^2)\sqrt{C_{44}/C_{33}}]} \tag{3.5}$$

Where a is the radius of the crack and

$$G=\sqrt{C_{44}^2+C_{11}C_{33}-(C_{13}+C_{44})^2+2C_{44}\sqrt{C_{11}C_{33}}} \tag{3.6}$$

(here, we corrected some minor typos). Then, compliance contribution tensor for an arbitrarily oriented circular flat crack can be calculated using (2.7). Taking $V_*=\gamma a^3\dfrac{4}{3}\pi$, $S=\pi a^2$,

$$H_{ijkl}=n_i B_{jk} n_l\cdot\frac{3}{4\gamma a}\quad(\gamma\to 0) \tag{3.7}$$

Cartesian components of tensor H_{ijkl} for a circular crack arbitrarily oriented in a transversely-isotropic material of elliptic type can now be written in the coordinate system coinciding with the axes of anisotropy of the matrix as follows

$$H_{1111}=\frac{4G}{\gamma\pi[\sqrt{C_{44}C_{66}}G+(C_{11}C_{33}-C_{13}^2)\sqrt{C_{44}/C_{33}}]}c\,\mathrm{os}^2\theta\sin^2\varphi \tag{3.16}$$

$$H_{2222}=\frac{4G}{\gamma\pi[\sqrt{C_{44}C_{66}}G+(C_{11}C_{33}-C_{13}^2)\sqrt{C_{44}/C_{33}}]}\sin^2\theta\sin^2\varphi \tag{3.17}$$

$$H_{3333}=\frac{2G}{\gamma\pi(C_{11}C_{33}-C_{13}^2)\sqrt{C_{44}/C_{33}}}\cos^2\varphi\ ;\ H_{1122}=H_{1133}=H_{2211}=H_{3322}=H_{3311}=0 \tag{3.18}$$

$$H_{1331}=\frac{G}{\gamma\pi[\sqrt{C_{44}C_{66}}G+(C_{11}C_{33}-C_{13}^2)\sqrt{C_{44}/C_{33}}]}\cos^2\varphi+\frac{G}{2\gamma\pi(C_{11}C_{33}-C_{13}^2)\sqrt{C_{44}/C_{33}}}c\,\mathrm{os}^2\theta\sin^2\varphi$$

$$;\ H_{1331}=H_{3113}=H_{1313}=H_{3131} \tag{3.19}$$

$$H_{1212}=\frac{G}{\gamma\pi[\sqrt{C_{44}C_{66}}G+(C_{11}C_{33}-C_{13}^2)\sqrt{C_{44}/C_{33}}]}\sin^2\varphi \tag{3.20}$$

$$H_{2332}=\frac{G}{\gamma\pi[\sqrt{C_{44}C_{66}}G+(C_{11}C_{33}-C_{13}^2)\sqrt{C_{44}/C_{33}}]}\cos^2\varphi+\frac{G}{2\gamma\pi(C_{11}C_{33}-C_{13}^2)\sqrt{C_{44}/C_{33}}}\sin^2\theta\sin^2\varphi \tag{3.21}$$

$$H_{1332}=\frac{G}{2\gamma\pi(C_{11}C_{33}-C_{13}^2)\sqrt{C_{44}/C_{33}}}c\,\mathrm{os}\theta\sin\theta\cos^2\varphi\quad ;$$

$$H_{1332}=H_{3132}=H_{1323}=H_{3123}=H_{3213}=H_{3231}=H_{2313}=H_{2331} \tag{3.22}$$

$$H_{3112}=\frac{G}{\gamma\pi[\sqrt{C_{44}C_{66}}G+(C_{11}C_{33}-C_{13}^2)\sqrt{C_{44}/C_{33}}]}c\,\mathrm{os}\varphi\sin\varphi\sin\theta \tag{3.23}$$

$$H_{1223} = \frac{G}{\gamma\pi[\sqrt{C_{44}C_{66}}G + (C_{11}C_{33} - C_{13}^2)\sqrt{C_{44}/C_{33}}]}\cos\varphi\sin\varphi\cos\theta \tag{3.24}$$

$$H_{1112} = \frac{2G}{\gamma\pi[\sqrt{C_{44}C_{66}}G + (C_{11}C_{33} - C_{13}^2)\sqrt{C_{44}/C_{33}}]}\sin^2\varphi\sin\theta\cos\theta\ ;\ H_{1112} = H_{1211} = H_{1121} = H_{2111} \tag{3.25}$$

$$H_{1113} = \frac{2G}{\gamma\pi[\sqrt{C_{44}C_{66}}G + (C_{11}C_{33} - C_{13}^2)\sqrt{C_{44}/C_{33}}]}\sin\varphi\cos\varphi\cos\theta\ ;\ H_{1113} = H_{1311} \tag{3.26}$$

$$H_{2223} = \frac{2G}{\gamma\pi[\sqrt{C_{44}C_{66}}G + (C_{11}C_{33} - C_{13}^2)\sqrt{C_{44}/C_{33}}]}\sin\varphi\cos\varphi\sin\theta\ ;\ H_{2223} = H_{2322} \tag{3.27}$$

$$H_{2221} = \frac{2G}{\gamma\pi[\sqrt{C_{44}C_{66}}G + (C_{11}C_{33} - C_{13}^2)\sqrt{C_{44}/C_{33}}]}\sin^2\varphi\sin\theta\cos\theta \tag{3.28}$$

$$H_{3331} = \frac{G}{\gamma\pi(C_{11}C_{33} - C_{13}^2)\sqrt{C_{44}/C_{33}}}c\,\mathrm{os}\,\theta\sin\varphi\cos\varphi \tag{3.29}$$

$$H_{3332} = \frac{G}{\gamma\pi(C_{11}C_{33} - C_{13}^2)\sqrt{C_{44}/C_{33}}}\sin\theta\sin\varphi\cos\varphi \tag{3.30}$$

Note, that independence of tensor $\boldsymbol{B}$ can be considered only as an approximation. It does not hold exactly even in the case of isotropic matrix (with the exception of the case of zero Poisson's ratio). Moreover, the accuracy of this approximation is acceptable only if the material shows elastic properties of elliptic type. For general transverse isotropy with high deviation from isotropy, the accuracy is too low (Guerrero et al, 2008).

4. Overall elastic properties of a TI material containing multiple randomly oriented cracks.

Compliance contribution tensor serves as the basic building block for various micromechanical homogenization techniques. In the case of multiple inhomogeneities, the extra strain du to k-th inhomogeneity is $\Delta\boldsymbol{\varepsilon}^{(k)} = \left(V^{(k)*}/V\right)\boldsymbol{H}^{(k)} : \boldsymbol{\sigma}^0$ so that the extra compliance due to all the inhomogeneities is given by

$$\Delta\boldsymbol{S} = \frac{1}{V}\sum V^{(k)*}\boldsymbol{H}^{(k)} \tag{4.1}$$

It is advantageous to formulate the problem in terms of the elastic potential $f(\boldsymbol{\sigma}) = (1/2)S_{ijkl}\sigma_{ij}\sigma_{kl}$ such that $\varepsilon_{ij} = \partial f / \partial \sigma_{ij}$. The representation $S_{ijkl} = S^0_{ijkl} + \Delta S_{ijkl}$ where ΔS_{ijkl} are due to cracks implies similar representation for the potential: $f(\boldsymbol{\sigma}) = f_0(\boldsymbol{\sigma}) + \Delta f$ where, for general inhomogeneities, $\Delta f = \boldsymbol{\sigma} : (1/V)\sum_k V_k \boldsymbol{H}^{(k)} : \boldsymbol{\sigma}$. For flat cracks we have, with the account of (2.7):

$$\Delta f = \boldsymbol{\sigma} : (1/V)\sum_k (S\boldsymbol{nBn})^{(k)} : \boldsymbol{\sigma} \tag{4.2}$$

where S^k are crack areas or, substituting (2.6) and using the identity $\boldsymbol{\sigma} : \boldsymbol{nIn} : \boldsymbol{\sigma} = (\boldsymbol{\sigma} \cdot \boldsymbol{\sigma}) : \boldsymbol{nn}$,

$$\begin{aligned} 2\Delta f = (\boldsymbol{\sigma} \cdot \boldsymbol{\sigma}) : \underbrace{\frac{1}{V}\sum (SB_T\boldsymbol{nn})^{(k)}}_{\boldsymbol{\alpha}\text{-}type} + \boldsymbol{\sigma} : \underbrace{\frac{1}{V}\sum [S(B_N - B_T)\boldsymbol{nnnn}]^{(k)}}_{\boldsymbol{\beta}\text{-}type} : \boldsymbol{\sigma} \\ + \frac{1}{V}\sum \left[S\frac{B_{ss} - B_{tt}}{2}\left(\sigma_{ns}^2 - \sigma_{nt}^2\right)\right]^{(k)} \end{aligned} \tag{4.3}$$

This expression consists of three distinctly different terms:

- The first one is expressed in terms of a second-rank tensor of the $\boldsymbol{\alpha}$-type (for circular cracks, it reduces to $\boldsymbol{\alpha}$).
- The second term contains fourth-rank tensor of the $\boldsymbol{\beta}$-type. This term is small (compared to the $\boldsymbol{\alpha}$-term) if the differences between B_N and B_T is small, or if the differences $B_N - B_T$ fluctuate randomly from one crack to another (without correlation with either crack orientations $\boldsymbol{n}$ or crack areas S), i.e. shape "irregularities" (deviations from circles) are random;
- The third term (that vanishes for elastically axisymmetric cracks, $B_{tt} = B_{ss}$) contains differences $\sigma_{ns}^2 - \sigma_{nt}^2$ where σ_{ns} and σ_{nt} are shear stresses induced by the remote loading in directions $\boldsymbol{s}$ and $\boldsymbol{t}$. If shape "irregularities" are random the sum vanishes.

Thus, taking into account approximate independence of tensor $\boldsymbol{B}$ of the crack orientation, one can write for randomly oriented circular cracks (accounting for (1.6)):

$$2\Delta f = \frac{8\rho G}{9(C_{11}C_{33}-C_{13}^2)\sqrt{C_{44}/C_{33}}}\boldsymbol{\sigma}_{ik}\boldsymbol{\sigma}_{kj}\,\delta_{ij}$$
$$+\frac{16\rho G}{9[\sqrt{C_{44}C_{66}}G+(C_{11}C_{33}-C_{13}^2)\sqrt{C_{44}/C_{33}}]}\left[\boldsymbol{\sigma}_{ik}\boldsymbol{\sigma}_{kj}\,\delta_{ij}-\frac{1}{5}\boldsymbol{\sigma}_{ij}\boldsymbol{\sigma}_{kl}\left(\delta_{ij}\delta_{kl}+\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk}\right)\right] \quad (4.4)$$

For microcracked materials, non-interaction approximation coincides with the method of Mori-Tanaka-Benveniste. Indeed, in this scheme, proposed by Mori and Tanaka (1973) and clarified by Benveniste (1986), each inhomogeneity is placed into a uniform field that is equal to its average over the matrix part of the composite and that generally differs from the remotely applied one. This method belongs to the class of "effective field" methods. The effective properties are calculated from the non-interaction approximation, by replacing the remotely applied field by the mentioned average one and microcracks do not vary the remotely applied field, so that it coincides with the one averaged over the matrix.

We now illustrate the procedure of calculation of the overall elastic properties on two examples – Berea sandstone and bovine dentine. The compliances of these materials are given in Table 1. We first approximate them by best fit elliptic transversely-isotropic constants using formulas (A.4). The results of the approximation are given in Table 2. As follows from the comparison of two tables, the errors of the approximation calculated by Euclidian norm

$$\delta = \sqrt{\frac{\left(C_{ijkl}-\tilde{C}_{ijkl}\right)\left(C_{ijkl}-\tilde{C}_{ijkl}\right)}{C_{pqrs}C_{pqrs}}} \quad (4.5)$$

where C_{ijkl} are elastic stiffnesses of the real material and $\tilde{C}_{ijkl}$ are their elliptic approximations are 3% and 4.6% for wet bovine dentin and Barea Sandstone, respectively. We verified the independence of tensor $\boldsymbol{B}$ on the crack orientation for these two materials numerically, following the procedure described by Guerrero et al (2007). Expression for Green's tensor in the form given by Pan and Chou (1976) was used for this aim. The crack was modeled by a strongly oblate spheroidal pore with aspect ratio 1:100 (Fig. 2) and two coordinate systems were introduced – global, associated with matrix material's symmetry and local, associated with the pore. To evaluate the integrals involved in the numerical procedure, the Gaussian quadrature method was used with 80 iterations. Figure 3 illustrates the obtained results for components B_{11} and B_{33}. It is seen that the plots can be approximated by horizontal lines for the entire range of variation of the rotation angle (with the exception of $\pi/2$, where indefiniteness $0/0$ appears) and we can use formulas (3.16)-(3.30) for components of the crack compliance contribution tensor. Now applying (4.4), (4.1), and (4.2) for the set of cracks sketched in Figure 4we can calculate

$$S_{eff} = S_0 + \rho \sum H \tag{4.6}$$

with nonzero components of $\left(\sum H\right)_{ijkl}$ given in Table 3. Figures 5 and 6 illustrate dependences of the elastic constants of microcracked Berea sandstone and wet bovine dentin, respectively, on the crack density assuming that all the cracks are circular and are randomly oriented.

An interesting observation is related to the effect of microcracks on the extent of the overall anisotropy. Generally, microcracking in an elastic anisotropic material weakens the anisotropy, provided the orientation distribution of microcracks is more or less random (see Mauge and Kachanov (1994) for orthotropic 2-D material with randomly oriented cracks). The underlying reason is that a crack normal to the "stiffer" matrix direction produces a larger contribution to the overall compliance than a crack of the same size that is normal to the "softer" direction; hence, a set of cracks of diverse orientations reduces the anisotropy. In 3-D case the situation is complicated by the fact that anisotropy on Young’s moduli and shear moduli may be different. Figure 7 illustrates variation of the extent of anisotropy with increasing crack density. For wet bovine dentin the anisotropy in Young’s moduli changed its directionality with growing crack density while anisotropy in shear moduli is practically not changed. For Berea sandstone, the anisotropy in Young’s moduli slightly increases with crack density. The explanation of this observation is not fully clear – on one side it reflects the fact that the character of anisotropy in 3-D is more complex than in 2-D; on the other side it may be result of the approximation error used in our approach.

Remark. We restrict our analysis by Non-Interaction Approximation since it reflects qualitatively all the features characterizing effect of microcracks on the overall elastic properties. Involvement of Maxwell scheme, differential scheme, or any else homogenization technique only complicates the procedure without increasing the quality of the approximation since the accuracy of these schemes cannot be evaluated a’priori (*i.e.* without knowing the exact solution).

5. Concluding remarks

We proposed an approximate micromechanical model to evaluate overall elastic properties of a transversely isotropic material containing randomly oriented circular cracks. The model is based on the observation of Guerrero et al. (2007) the crack opening displacement tensor, that relates average displacement discontinuity vector to the vector of uniform traction induced at the crack site, is approximately independent on the crack orientation if anisotropy of the material is of elliptic type (Sevostianov and Kachanov, 2008). This observation yields approximate explicit expressions for the components of the compliance contribution tensor for a crack arbitrarily oriented in a transversely isotropic material. Such a tensor serves as the basic building block for various micromechanical schemes (see, for example, discussion in section 3 of Sevostianov and Kachanov, 2014). We use this tensor to write explicit expressions for overall elastic compliances of a transversely isotropic material containing randomly oriented circular cracks (equation (4.4)) in the framework of Mori-Tanaka-Benveniste scheme. We illustrate our approach on two examples – Berea sandstone and wet bovine dentin. For both these materials we first verified independence of the crack opening displacement tensor on the crack orientation and then used the proposed methodology to calculate reduction of the elastic moduli in dependence on crack density.

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Appendix. Elliptically orthotropic materials and best fit elliptic orthotropy.

Elliptic transverse isotropy represents a special type of transverse isotropy discussed in detail by Sevostianov and Kachanov (2008). They call the orthotropy *elliptic* if the fourth-rank tensor of elastic constants $\boldsymbol{D}$ (that can represent either stiffnesses or compliances) can be expressed as tensor function $\boldsymbol{D} = \boldsymbol{D}(\boldsymbol{\omega})$ of certain symmetric second-rank tensor $\boldsymbol{\omega}$, this function being isotropic (any element of symmetry of $\boldsymbol{\omega}$ - a transformation $\boldsymbol{T}$ such that $\boldsymbol{\omega} = \boldsymbol{T} \cdot \boldsymbol{\omega}$ - is also an element of symmetry of $\boldsymbol{D}$; more precisely, subjecting $\boldsymbol{\omega}$ to any transformation described by orthogonal second-rank tensor $\boldsymbol{Q}$: $\omega_{ij} \to \omega_{mn} Q_{mi} Q_{nj}$ would imply subjecting $\boldsymbol{D}$ to the same transformation, $D_{ijkl} \to D_{mnpq} Q_{mi} Q_{nj} Q_{pk} Q_{ql}$) and, secondly, linear:

$$\begin{aligned} D_{ijkl} &= A_1 \delta_{ij} \delta_{kl} + A_2 \left(\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk} \right) \\ &\quad + A_3 \left(\omega_{il} \delta_{jk} + \omega_{jk} \delta_{il} + \omega_{jl} \delta_{ik} + \omega_{ik} \delta_{jl} \right) + A_4 \left(\delta_{ij} \omega_{kl} + \delta_{kl} \omega_{ij} \right) \end{aligned} \tag{A.1}$$

where coefficients A_i may depend on invariants of $\boldsymbol{\omega}$. Since $\boldsymbol{\omega}$ is a symmetric second-rank tensor, the elastic properties are orthotropic, and the orthotropy axes coincide with the principal axes of $\boldsymbol{\omega}$. The elliptic type of orthotropy is characterized by reduced number of independent constants – only five, due to the following four constraints:

$$\begin{aligned} &4D_{1212} - D_{1111} - D_{2222} + 2D_{1122} = 0, \\ &4D_{2323} - D_{2222} - D_{3333} + 2D_{2233} = 0, \\ &4D_{3131} - D_{3333} - D_{1111} + 2D_{3311} = 0, \\ &D_{1111}(D_{1122} - D_{1133}) + D_{2222}(D_{2233} - D_{2211}) + D_{3333}(D_{3311} - D_{3322}) = 0 \end{aligned} \tag{A.2}$$

For *transverse isotropy of the elliptic type*, only one of the relations (A.2) is non-trivial, thus reducing the number of independent constants from five to four.

Sevostianov and Kachanov (2008) also proposed the concept that any element of elastic symmetry is always present, with certain accuracy (measured by appropriately chosen norm) and therefore, the idea of approximate symmetry (widely used, for example in geomechanics, Fedorov, 2013) becomes relevant for many practical applications. This concept eliminates the symmetry “jumps” that may accompany small changes in elastic constants: they are replaced by small changes in accuracy in the above statement. For example, emergence of weak anisotropy means that the error of the statement "the material is isotropic" changes continuously from zero to a small value. Sevostianov and Kachanov (2008) also developed a procedure of best-fit approximation of general orthotropy by the elliptic orthotropy. For a transversely-isotropic material, imposing the three constraints (A.2) that define the elliptic orthotropy and minimizing the error by using the Lagrange multipliers technique, yields the following elliptically transversely isotropic constants (ETI)

$$C_{1111}^{ETI} = C_{1111} + (\lambda_1 + \lambda_2)/2, \quad C_{3333}^{ETI} = C_{3333} + \lambda_2, \quad C_{1122}^{TIO} = C_{1122} - \lambda_1/2,$$

$$C_{1133}^{TIO} = C_{1133} - \lambda_2/2, \; C_{1313}^{TIO} = C_{1313} - \lambda_2/2 \quad , \tag{A.4}$$

where the Lagrange multipliers are

$$\lambda_1 = \frac{9f_1 - 2f_2}{35}; \quad \lambda_2 = \frac{8f_2 - f_1}{35} \tag{A.5}$$

with $f_1 \equiv 4C_{1212} - C_{1111} - C_{2222} + 2C_{1122}$, $f_2 \equiv 4C_{2323} - C_{2222} - C_{3333} + 2C_{2323}$,

(f_i turn to zero if material possesses elliptic symmetry). Note that the paper of Sevostianov and Kachanov (2008) contains a typo in formulas for λ_i .

Tables.

Table 1. Stiffness constant of two transversely isotropic materials (Ding, et. al. 2006)

	C_{1111}	C_{3333}	C_{1122}	C_{1133}	C_{1313}
Wet Bovine Dentin (WBD)	37	39	16.6	8.7	5.7
Berea Sandstone (BS)	21.17	17.34	4.34	3.84	13.97

Table 2. Best fit elliptic transversely-isotropic (ETI) constants

	C_{1111}^{ETI}	C_{3333}^{ETI}	C_{1122}^{ETI}	C_{1133}^{ETI}	C_{1313}^{ETI}
Wet Bovine Dentin (WBD)	35.47	34.91	16.09	10.75	7.75
Berea Sandstone (BS)	23.32	23.07	5.06	0.98	11.11

Table 3. Result of the compliance contribution tensor for randomly oriented circular cracks

	$\left(\sum H\right)_{1111}$	$\left(\sum H\right)_{3333}$	$\left(\sum H\right)_{1313}$	$\left(\sum H\right)_{1212}$
Wet Bovine Dentin (WBD)	0.1593	0.1313	0.0726	0.0797
Berea Sandstone (BS)	0.1593	0.1582	0.0807	0.0822

Figure captions

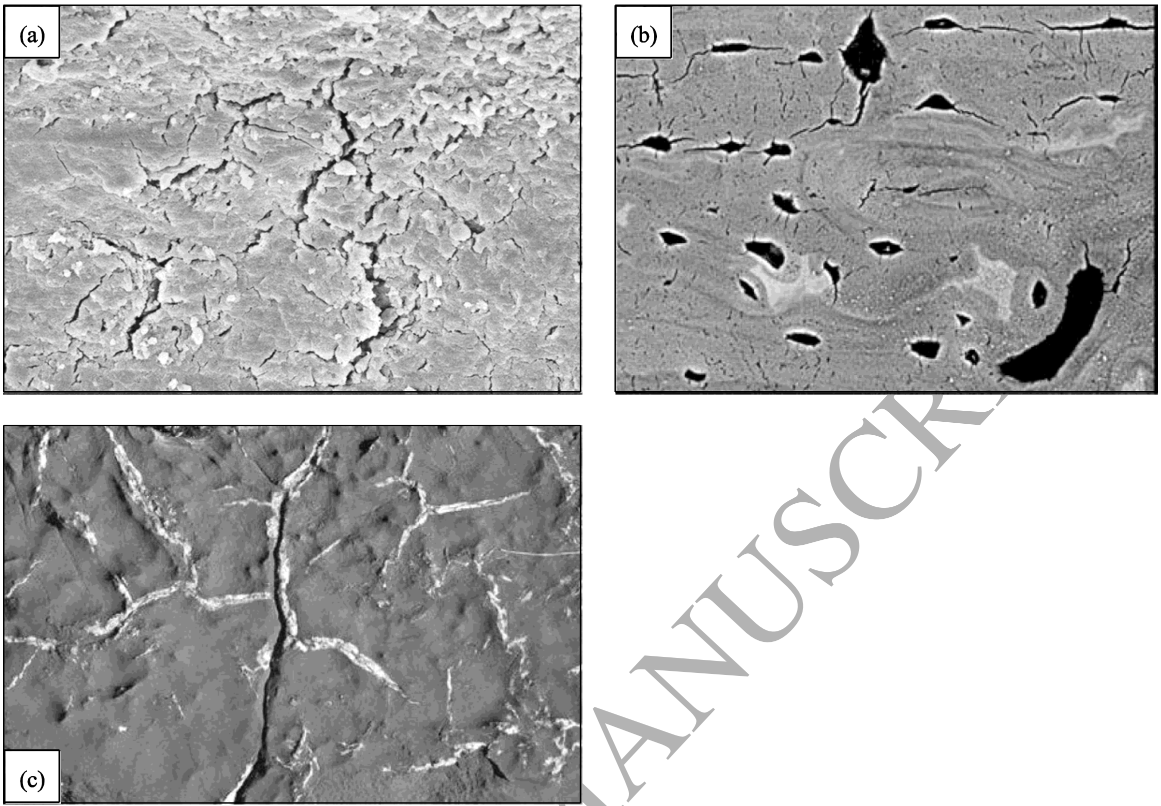

Figure 1. Examples of microstructures of various transversely isotropic materials containing multiple randomly oriented cracks: (a) cracks in human dentin subjected to diode laser irradiation (from Faria et al., 2011), (b) cracks accumulated in cortical bone of mice due to high-fat diet (from Ionova-Martin et al., 2011), and (c) cracks in sedimentary rock from the Neoarchaean Gamohaan Formation, South Africa (from Mahon et al., 2016).

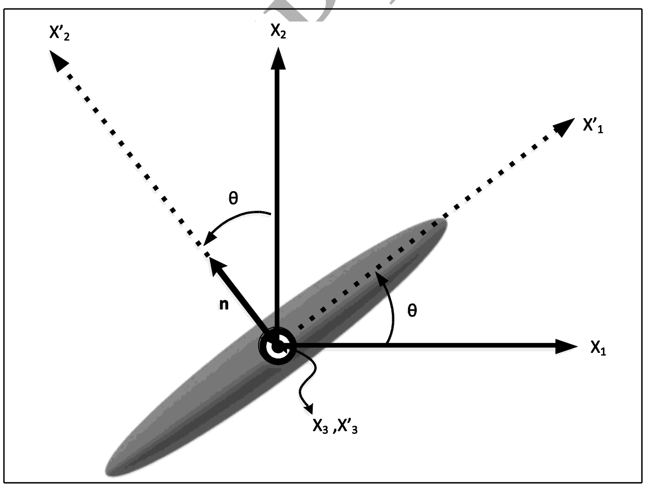

Figure 2. Local and global coordinates associated with arbitrarily oriented crack.

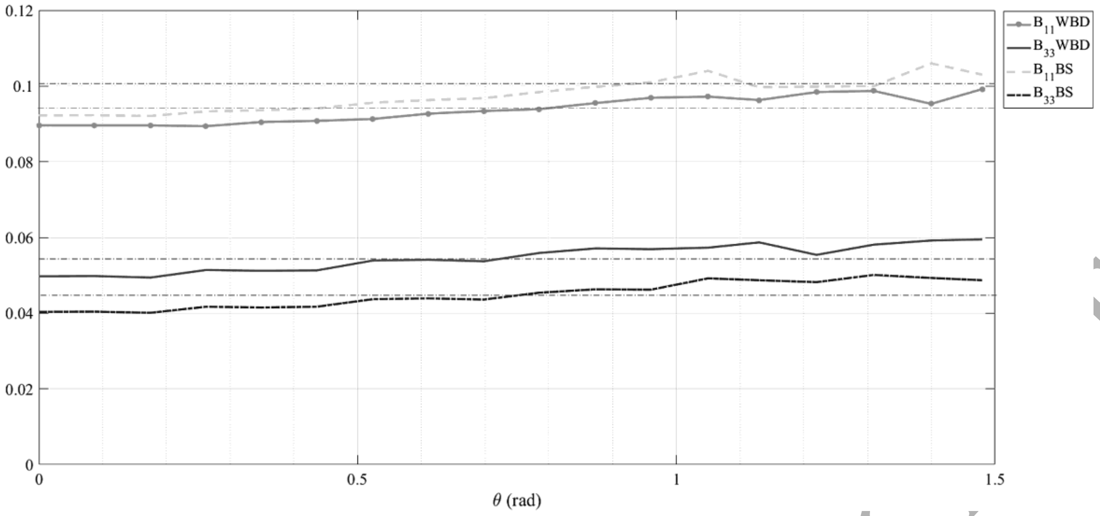

Figure 3. Dependence of the components of tensor ***B*** on the angle of crack inclination for wet bovine dentine and Berea sandstone.

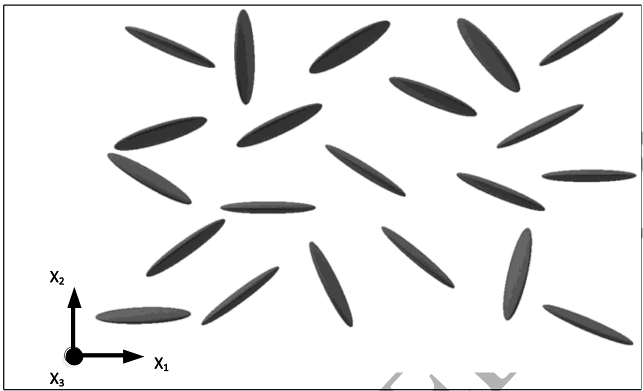

Figure 4. Multiple penny shaped randomly oriented cracks in a transversely isotropic (TI) media.

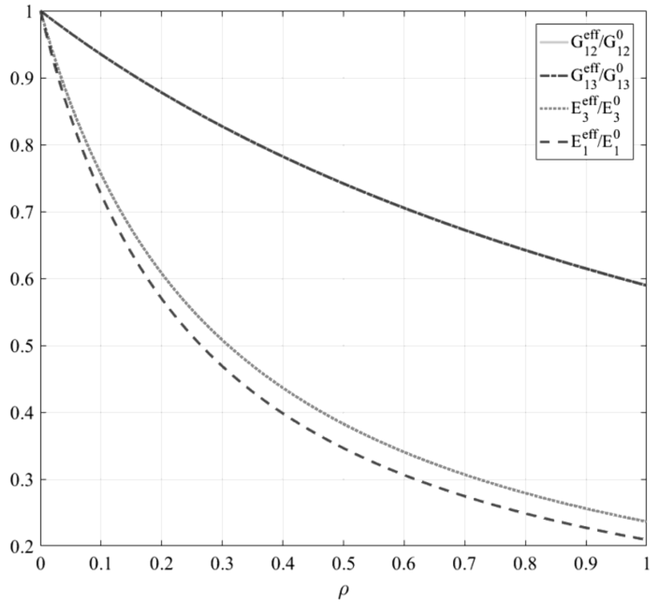

Figure 5. Effective Young's and shear moduli of wet bovine dentin as functions of the crack density.

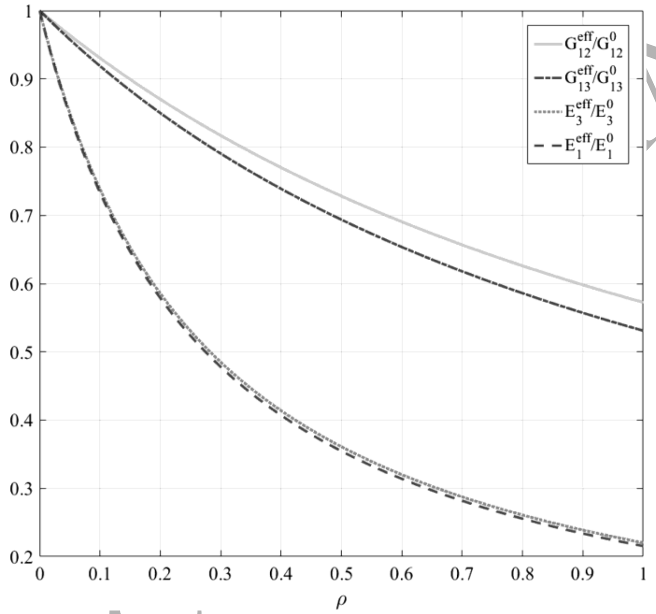

Figure 6. Effective Young's and shear moduli of Berea sandstone as functions of the crack density

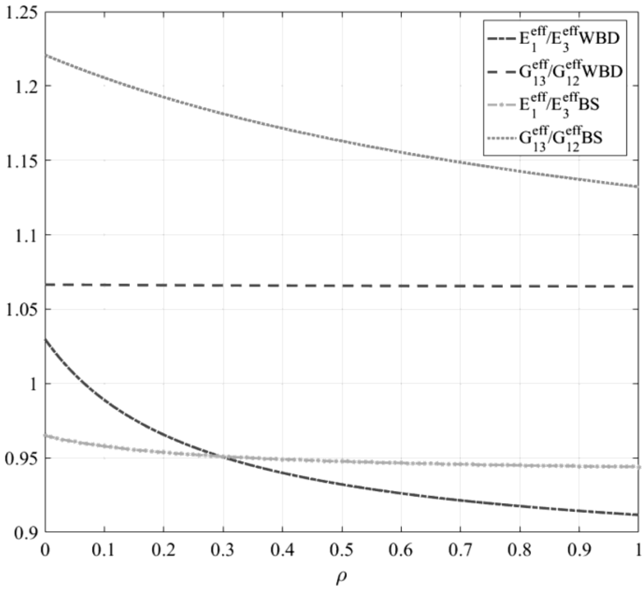

Figure 7. Variation of the extent of anisotropy, as measured by the ration of Young's moduli E_1 and E_3 and shear moduli G_{13} and G_{12} with crack density growth for wet bovine dentin (WBD) and Berea Sandstone (BS).