

Numerical evaluation of the thermomechanical effective properties of a functionally graded material using the homogenization method

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Abstract

When the stresses of the functionally graded materials (FGMs) are discussed under thermal and/or mechanical loading conditions, the different thermomechanical effective properties are needed. For the steady state thermal analyses, these properties include the Young's modulus, Poisson's ratio, thermal expansion coefficient and thermal conductivity. For the transient analyses of the heat conduction problem, on the other hand, the density and heat capacity should be added to the aforementioned properties. The homogenization method (HM) based on the finite element method (FEM) is used as it has advantages, such as it is appropriate for estimating the effective properties of composites with a given periodic fiber distribution and complicated geometries. For a periodic composite structure, it is not necessary to study the whole structure but only a representative volume element (RVE) or a unit cell (UC). As the overall behavior of composites depends on the arrangement of the reinforcements, the corresponding UCs of two different arrangements of the fibers are analyzed; namely the square and hexagonal arrangements. It is found that the square arrangement predicts higher values of the Young's modulus than the hexagonal one but with small difference. In order to verify the computed values of the properties, the results are compared with previous experimental measurements and results of analytical and numerical methods, and good agreement is achieved.

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1. Introduction

Composite materials have different attractive properties such as high strength, high stiffness and high thermal stability at elevated temperatures. Therefore, aerospace, automotive and marine industries and biological aspects are taken the advantages of the special characteristics of these materials. Functionally graded material (FGM) is a composite material and it can be distinguished from a traditional composite by its spatial change

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of the reinforcement volume fraction as it is a function of the position. Therefore, the volume fraction of certain constituent, in a complete FGM, changes from 0% to 100%. This spatial change of the volume fraction results in changes of the properties from a position to another. This leads to a prestigious performance of the FGM over other composites. Eventually, the properties control the behavior of a material and also have a significant contribution to its superiority. FGMs are subjected to temperature gradients not only during the manufacturing process but also during their service lives. Hence, a clear understanding of the thermomechanical properties is an important step in assessing the performance of these materials in applications.

Many models and approaches are proposed in the literature for the evaluation of the different effective properties. The homogenization method (HM) based on the FEM has advantages as it is appropriate for estimating the effective properties of composites with a given periodic fiber distribution and more complicated geometries. Grujicic and Zhang (1998) used the voronoi cell finite element method to evaluate some of the FGM elastic properties of plane strain problems; mainly Young's modulus and Poisson's ratio. Theocaris and Stavroulakis (1998) used the HM to study the variation of Poisson's ratio in fiber composites. For the modulus of rigidity, on the other hand, the Chamis model (1983) gives predictions for a composite of anisotropic fiber and isotropic matrix. When thermal stresses are analyzed, the Schapery model (1968) may be used to predict the thermal expansion coefficients. It is more convenient sometimes to have one model that predicts most of the effective properties. Shabana (2003) proposed a three dimensional constitutive equation for the thermo-elasto-plastic-macroscopic-microscopic behavior of composites. This constitutive equation is based on the Eshelby (1957) and Mori and Tanaka (1973) models and can be used for a composite material to predict many different properties, in different directions, including the thermal expansion coefficients.

The thermal conductivity, which is needed to assess the temperature distribution, may receive little attention in the literature compared to other properties such as Young's modulus. Hesselman and Johnson (1987) derived a complicated expression that accounts for the conductivity of the fiber/matrix interface region that is less than perfect due to thermal mismatch of the fiber and the host matrix. Hatta and Taya (1986) proposed a model that takes advantage of the Eshelby's method that developed for elasticity problems. This model can be also applied to cases of complicated morphologies of reinforcement. Gurtman et al. (1981) used the multi-variable asymptotic expansion to propose a model that is based on a unit cell (UC) of binary unidirectional periodic composite of anisotropic constituents. When the ratio of the fiber conductivity to the matrix conductivity is small, the Gurtman et al. model and the Hatta and Taya model predict almost same values (Taya and Arsenault, 1989). When the unsteady state thermal load is investigated, the density and heat capacity are needed for the evaluation of the temperature distribution in the domain. Christian and Campbell (1972) measured the specific heat of various aluminum matrix composites and it is found that his measures are close to those predicted by the rule of mixture.

In this framework, the HM and the FEM are used to determine the full set of the aforementioned macroscopic effective properties which lead to the same thermomechanical behavior as the one of the material with the periodic microstructure. These effective properties can be used for FGM stress analyses under thermal and/or mechanical loading conditions. Analyzing a small UC, which represents a structure, is enough for a periodic composite material. Two UC models, which are suitable for the square and hexagonal arrangements of the fibers, with appropriate periodic boundary conditions and load cases, are discussed in order to study the effect of different fiber arrangements on the predicted effective properties. In order to verify the obtained effective properties by the numerical homogenization technique, the results are compared with previous experimental measurements and analytical and numerical methods, and good agreement is achieved.

2. Homogenization method

One of the most powerful tools for speeding up the modeling process, both for the composite discretization and the computer simulation of composites in realistic conditions, is the homogenization method. The main idea of the method is to find a globally homogeneous medium equivalent to the original composite. The formulation of the homogenization technique that exists in (Takano and Zako, 2000) can be used for the evaluation of the Young's modulus, Poisson's ratio, thermal expansion coefficient and thermal conductivity. Therefore brief steps will be shown here.

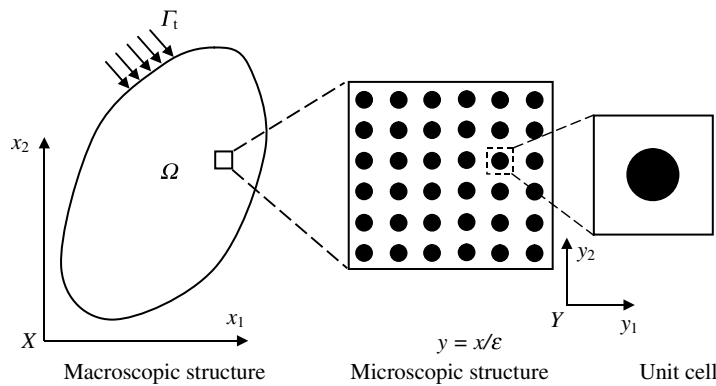


Fig. 1. Coordinate systems of homogenization analysis.

Fig. 1 shows the two distinct coordinate systems that the HM adopts. The macroscopic (global) scale x is defined for the macroscopic structure and the microscopic (local) scale y is defined in a unit cell for the micro-structure of periodic composites. The relation between x and y is as follows:

$$\varepsilon = x/y \quad (1)$$

where ε is a very small positive parameter that represents the ratio of the microscopic representative length to the macroscopic one.

Before proceeding with the HM relations, it may be better to introduce the volume average technique. The volume averages of different quantities such as stress can be evaluated using the FEM by,

$$\langle \sigma \rangle = \frac{1}{|Y|} \int_Y \sigma dY$$

where $|Y|$ denotes the volume of the unit cell and $\langle \rangle$ signifies the volume average in Y .

Intensive details of the boundary conditions and the admissible spaces of the temperatures and displacements can be found in [Cheng Cho-Hsien \(1992\)](#).

Considering the microstructure effects, the governing equations of the heat conduction and thermal stress problems can be described, respectively, by

$$\int_{\Omega} K_{ij} \frac{\partial T}{\partial x_j} \frac{\partial T}{\partial x_i} d\Omega = \int_{\Omega} f \delta T d\Omega + \int_{\Gamma_h} h \delta T d\Gamma \quad \forall \delta T \quad (2)$$

$$\int_{\Omega} E_{ijkl} \frac{\partial u_k}{\partial x_l} \frac{\partial \delta u_i}{\partial x_j} d\Omega = \int_{\Omega} b_i \delta u_i d\Omega + \int_{\Gamma_t} t_i \delta u_i d\Gamma + \int_{\Omega} E_{ijkl} \alpha_{kl} (T - T_0) \frac{\partial \delta u_i}{\partial x_j} d\Omega \quad \forall \delta u_i \quad (3)$$

where K_{ij} and E_{ijkl} are the thermal conductivity and the elastic tensors, α_{kl} are the coefficients of thermal expansions. Let f and h denote the internal heat sources and inflow of heat on Γ_h boundary, b_i and t_i denote the body force and the traction on Γ_t boundary and T_0 is the initial temperature.

The temperature and displacement fields are expressed using the asymptotic expansion method as follows:

$$T(x, y) = T^0(x) + \varepsilon T^1(x, y) \quad (4)$$

$$u_i(x, y) = u_i^0(x) + \varepsilon u_i^1(x, y) \quad (5)$$

Here, T^0 and u_i^0 are the macroscopic (homogenized) temperature and displacement, T^1 and u_i^1 are the microscopic temperature and displacement, respectively. The macroscopic quantities are functions of the macroscale (x) only as shown in Eqs. (4) and (5). Substituting Eqs. (4) and (5) into the governing Eqs. (2) and (3), and using the averaging technique for a periodic function ψ in the same way with the standard homogenization procedure, the micro-macro coupled problem can be resolved into macroscopic and microscopic problems:

$$\lim_{\varepsilon \rightarrow 0} \int_{\Omega} \psi\left(\frac{x}{\varepsilon}\right) d\Omega = \int_{\Omega} \frac{1}{|Y|} \int_Y \psi(y) dY d\Omega \quad (6)$$

For the heat conduction problem, the microscopic and macroscopic equations are derived, respectively, as

$$\int_{\Omega} K_{ip} \frac{\partial \phi^j}{\partial y_p} \frac{\partial \delta T^1}{\partial y_i} dY = \int_{\Omega} K_{ij} \frac{\partial \delta T^1}{\partial y_i} dY \quad \forall \delta T^1 \quad (7)$$

$$\int_{\Omega} K_{ij}^H \frac{\partial T^0}{\partial x_j} \frac{\partial \delta T^0}{\partial x_i} d\Omega = \int_{\Omega} f^H \delta T^0 d\Omega + \int_{\Omega} h \delta T^0 d\Gamma \quad \forall \delta T^0 \quad (8)$$

By solving the microscopic Eq. (7) using the periodic boundary conditions, we obtain the characteristic function associated with heat conduction in the microstructure due to the mismatch of the thermal conductivities of the constituents. In Eq. (8), the homogenized thermal conductivity tensor K_{ij}^H and the homogenized internal heat source f^H are calculated by the integration over the whole domain and rigorously defined as follows:

$$K_{ij}^H = \frac{1}{|Y|} \int_Y \left(K_{ij} - K_{ip} \frac{\partial \phi^j}{\partial y_p} \right) dY \quad (9)$$

$$f^H = \frac{1}{|Y|} \int_Y f dY \quad (10)$$

For a thermal stress problem, macroscopic and microscopic equations are derived in the same way as;

$$\int_Y E_{ijmn} \frac{\partial \chi_m^{kl}}{\partial y_n} \frac{\partial \delta u_i^1}{\partial y_j} dY = \int_Y E_{ijkl} \frac{\partial \delta u_i^1}{\partial y_j} dY \quad \forall \delta u_i^1 \quad (11)$$

$$\int_Y E_{ijkl} \frac{\partial \phi_k}{\partial y_l} \frac{\partial \delta u_i^1}{\partial y_j} dY = \int_Y E_{ijkl} \alpha_{kl} \frac{\partial \delta u_i^1}{\partial y_j} dY \quad \forall \delta u_i^1 \quad (12)$$

$$\int_{\Omega} E_{ijkl}^H \frac{\partial u_k^0}{\partial x_l} \frac{\partial \delta u_i^0}{\partial x_j} d\Omega = \int_{\Omega} b_i^H \delta u_i^0 d\Omega + \int_{\Gamma_t} t_i \delta u_i^0 d\Gamma + \int_{\Omega} E_{ijkl}^H \alpha_{kl}^H (T^0 - T_0) \frac{\partial \delta u_i^0}{\partial x_j} d\Omega \quad \forall \delta u_i^0 \quad (13)$$

Microscopic Eq. (11) provides the characteristic displacement due to the mismatch of the elastic tensor of the constituents, and microscopic Eq. (12) provides the characteristic displacement due to the mismatch of the thermal expansion coefficient. The homogenized elastic tensor E_{ijkl}^H , thermal expansion coefficient α_{ij}^H and the body force b_i^H are derived as follows:

$$E_{ijkl}^H = \frac{1}{|Y|} \int_Y \left(E_{ijkl} - E_{ijmn} \frac{\partial \chi_m^{kl}}{\partial y_n} \right) dY \quad (14)$$

$$\alpha_{ij}^H = C_{ijpq}^H \frac{1}{|Y|} \int_Y E_{pqkl} \left(\alpha_{kl} - \frac{\partial \phi_k}{\partial y_l} \right) dY \quad (15)$$

$$b_i^H = \frac{1}{|Y|} \int_Y b_i dY \quad (16)$$

where C_{ijpq}^H denotes the homogenized compliance tensor, which is the inverse of the homogenized elastic tensor. The above partial differential equations can be solved numerically by the finite element method. Microscopic equations are solved with the periodic boundary condition that is put on the unit cell.

For the transient (quasi-static) thermal analyses, the product of the density and heat capacity is needed to assess the temperature distribution in the domain. This product is also called effective volumetric heat capacity and it can be evaluated using the average technique as follows (Auriault, 1983):

$$(\rho c)^H = \frac{1}{|Y|} \int_Y (\rho c) dY \quad (17)$$

This relation can be used for composites with periodic, quasi-periodic or even non-periodic structures (Kaminski, 2003).

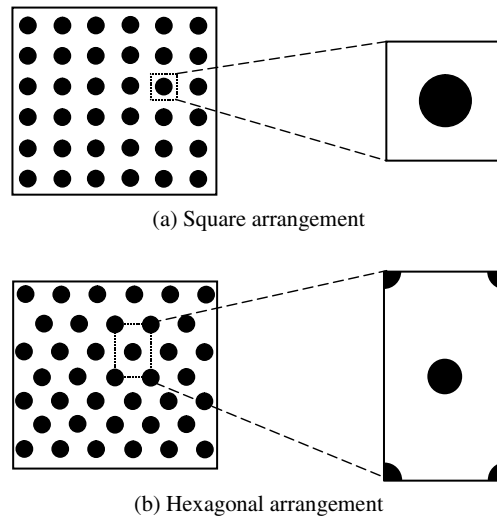


Fig. 2. Square and hexagonal arrangements of the fibers and their unit cells.

2.1. Unit cell models and periodic boundary conditions

The common approach for modeling the macroscopic properties of fiber composites is creating a UC that captures the major features of the underlying microstructure. The mechanical, thermal and physical properties of the constituent materials are always regarded as a small-scale/microscopic properties.

For square arrangements of fibers, the UC has a square geometry as shown in Fig. 2a. In order to provide the periodicity for hexagonal arrangements, the UC has rectangular geometry like that shown in Fig. 2b. Other geometrical forms are also available for the hexagonal arrangements, but the rectangular cell shown enables all the necessary periodic boundary conditions to be easily described in a Cartesian coordinate system.

Since composite materials can be represented as a periodical array of UCs, periodic boundary conditions should be applied to the UC models. This implies that, each UC in the composite has the same deformation mode and there is no separation or overlap between the neighboring UCs after deformation. This means all boundary nodes separated by the periods Y_1 and Y_2 along the coordinate directions, will follow the following displacement constraints:

$$\mathbf{u}_i(x_1; x_2) = \mathbf{u}_i(x_1 \pm k_1 Y_1; x_2 \pm k_2 Y_2) \quad (18)$$

where \mathbf{u}_i is the displacement vector at node i and k_1 and k_2 are integers.

In addition to the microstructural periodicity, meshes are supposed to be the same on opposite faces of the UC, so that after deformation, for each master node on the left face, there is a slave node, which exists on the same level, on the right face. The displacement vector of the tied (master and slave) nodes should be the same, see Fig. 3. The total number of master nodes is less than the number of nodes on both the bottom and left faces by two. Because, if node 1 is chosen as a master node, its corresponding slave nodes will be nodes 2, 3 and 4. Therefore, the total number of slave nodes, on the other hand, equals the total number of nodes on both the top and right faces. The three periodic boundary conditions related to the master node 1 may be rewritten as:

$$\mathbf{u}_2 = \mathbf{u}_1, \quad \mathbf{u}_3 = \mathbf{u}_1, \quad \text{and} \quad \mathbf{u}_4 = \mathbf{u}_1$$

Other master nodes periodic boundary conditions can be extracted easily.

3. Results and discussion

The plane strain conditions are considered to study the effective material properties for the uncoupled thermomechanical FGM problems. Since in most often cases FGM consists of a ceramic and a metal, the consid-

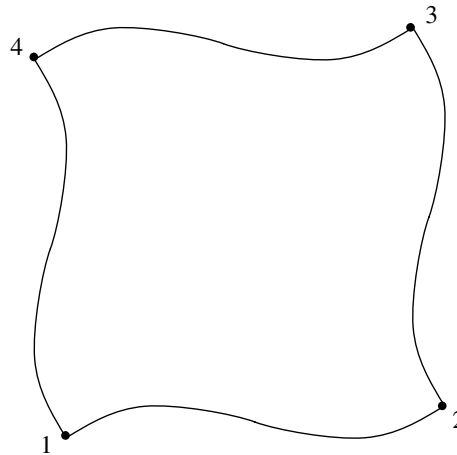


Fig. 3. Definition of the boundaries, master node (node 1) and slave nodes (nodes 2, 3 and 4).

ered two constituent isotropic materials of the main analyzed FGM are ZrO_2 and Ti-6Al-4V. The different thermomechanical and physical properties of these two materials are listed in Table 1. For the finite element analyses, the generated meshes of the unit cells have the same adaptivity method as that exists in Kari (2006). Also, the four-node element is used here and the minimum numbers of elements and nodes are 10000 and 11200, respectively.

All simulations of this work are performed by developing finite element FORTRAN programs in order to evaluate all of the thermally independent effective properties for each ceramic volume fraction. However, the meshes are generated by the commercial finite element package ABAQUS.

The stiffness matrix of 2D composite analyses may be written in its general form as:

$$[E] = \begin{bmatrix} E_{11} & E_{12} & E_{13} \\ E_{21} & E_{22} & E_{23} \\ E_{31} & E_{32} & E_{33} \end{bmatrix}$$

As the constituent materials are isotropic and the generated finite element meshes are symmetric, it is found that the obtained results of the stiffness coefficients satisfy the well known conditions of an isotropic material. These are $E_{11} = E_{22}$, $E_{12} = E_{21}$, $E_{33} = \frac{E_{11} - E_{12}}{2}$ and E_{13} , E_{23} , E_{31} and E_{32} are very small and tend to be zero compared to other terms. This means that the material satisfies the hypothesis of isotropic elasticity under the current analyses.

After evaluating the stiffness matrix coefficients under the plane strain conditions, the Poisson's ratio, Young's modulus, and shear modulus can be evaluated, respectively, by

$$\nu = R/(1 + R)$$

$$E = E_{11}(1 + \nu)(1 - 2\nu)/(1 - \nu)$$

$$G = E_{33}$$

where $R = E_{12}/E_{11}$

Table 1
Properties of the FGM constituent materials

Property	ZrO ₂	Ti-6Al-4V
Thermal expansion coefficient (1/K)	7.11×10^{-6}	10.3×10^{-6}
Thermal conductivity (W/mK)	2.036	18.1
Young's modulus (GPa)	117	66.2
Poisson's ratio	0.333	0.321

3.1. Verification of the results

Before using the HM to compute the effective macroscopic properties at each ceramic volume fraction of the FGM, one problem for which the experimental measurements and analytical and numerical solutions are available is investigated using the HM.

The values pertaining to the variation of the Young's modulus with the volume fraction of TiC in the $\text{Ni}_3\text{Al}/\text{TiC}$ FGM composite taken from the work of [Grujicic and Zhang \(1998\)](#) are shown in [Fig. 4](#). These values are related to the computations using the analytical methods, which are the equivalent inclusion method (EIM) and the self consistent method (SCM), numerical method, which done by the voronoi cell finite element method (VCFEM), and the experimental measurements. Following the aforementioned homogenization procedure for the square arrangement UC at different volume fractions of TiC, the effective Young's modulus is computed. In these calculations the following elastic properties of fully dense Ni_3Al and TiC are used ([Grujicic and Zhang, 1998](#)): $E_{\text{Ni}_3\text{Al}} = 217 \text{ GPa}$, $\nu_{\text{Ni}_3\text{Al}} = 0.30$, $E_{\text{TiC}} = 440 \text{ GPa}$ and $\nu_{\text{TiC}} = 0.19$. Also, the porosity is assumed to reduce the fully dense results of the homogenization method by the same percentage. It can be seen from [Fig. 4](#) that, although the UC model used in the HM considers the periodicity of the fiber arrangements, the results obtained by the two analytical methods and by the HM are quite comparable with each other. The agreement between the computed analytical and the experimental results for E is, however, only fair. It can be seen from [Fig. 4](#) that the HM results are, in general, in better agreement with the experimental and analytical results than the VCFEM results. Since the rule of mixture (RM) is intensively used in the literature, see for example [Yang et al. \(2006\)](#), it is valuable to have some insight about its predictions relative to the other models including the HM. It can be seen from [Fig. 4](#) that the RM predicts the highest values of Young's modulus and therefore it represents the upper bound as it is well known.

3.2. Influence of the arrangements of the fibers

The overall behavior of composite materials depends on the arrangement of the reinforcements. Therefore, in order to study how much effect the fiber arrangement has on the effective properties, the effective Young's modulus is evaluated for two UCs of composites with periodic square ([Fig. 2a](#)) and periodic hexagonal ([Fig. 2b](#)) fiber arrangements. It can be seen from [Fig. 5](#) that the square arrangement has higher values than

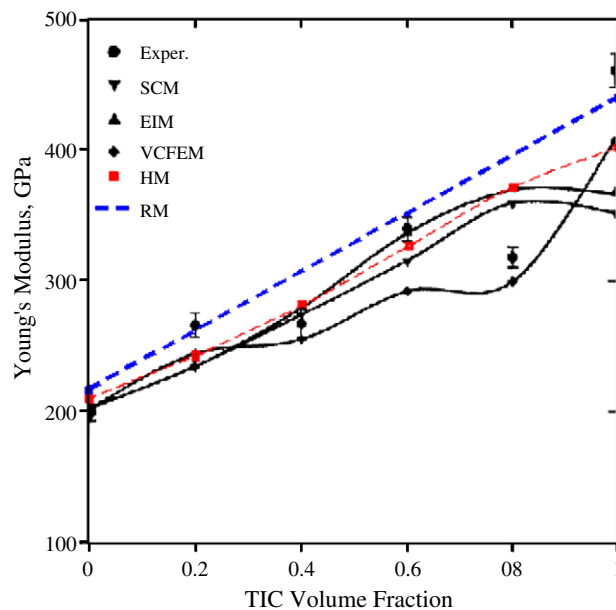


Fig. 4. Variation of Young's modulus with the ceramic volume fraction.

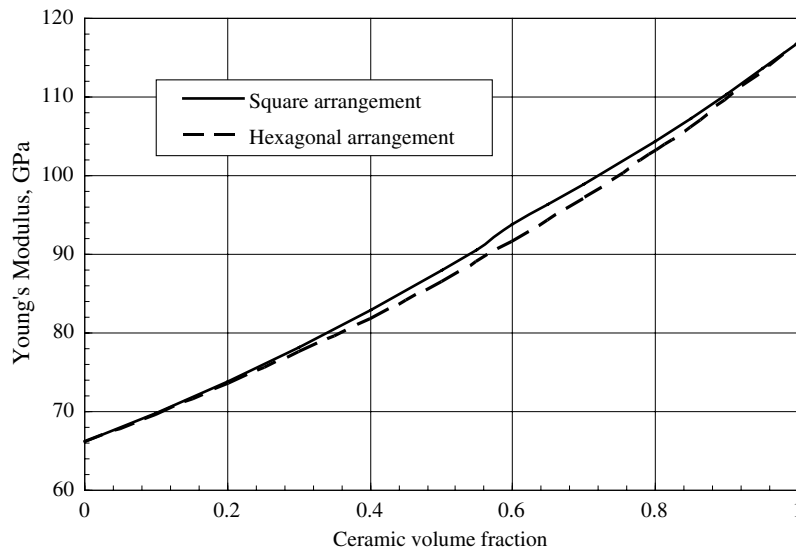


Fig. 5. Young's modulus as a function of the ceramic volume fraction for the square and hexagonal arrangements of the fibers.

the hexagonal one. However, the difference is not that large and this behavior is consistent with the results by Kari (2006). Since mostly higher Young's modulus results in higher induced stresses, the square arrangement will be used for the following analyses in order for a safe structural design.

3.3. Stiffness matrix coefficients

The stiffness matrix coefficients are evaluated for different volume fractions of ZrO_2 . For the current analyses, there are basically three important coefficients. These are E_{11} , E_{12} , and E_{33} . Fig. 6 depicts the variation of E_{11} with the ceramic volume fraction. As the linear relation (rule of mixture) is expected for the axial direction of the fiber and the current analyses is devoted to the transverse directions of the fibers, it can be seen from Fig. 6 that E_{11} increases nonlinearly with the ceramic volume fraction and its values are usually less than the

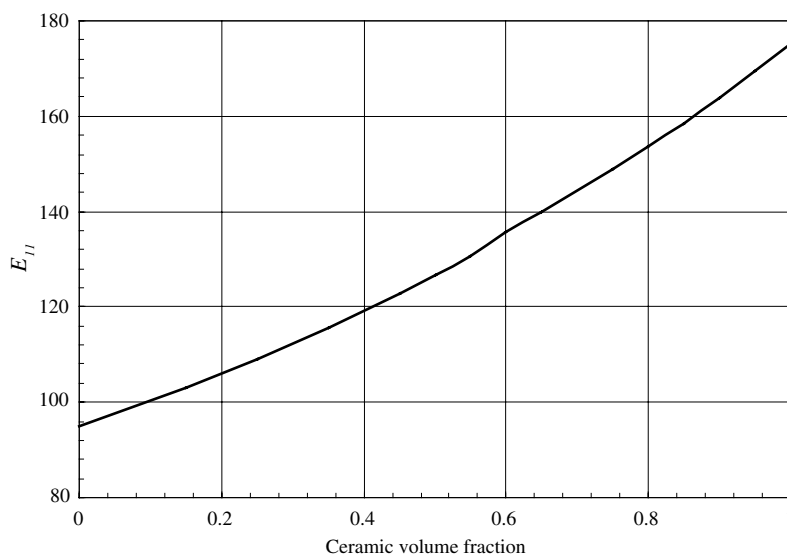


Fig. 6. Variation of E_{11} with the ceramic volume fraction.

linear relation. The behavior of E_{12} is almost same as that of E_{11} but with lower values as shown in Fig. 7. In the current analyses, the modulus of rigidity is represented by E_{33} . Fig. 8 shows that the variations of E_{33} agree well with Chamis model (1983) that can be written as:

$$G_c = G_m / [1 - V_f(1 - G_m/G_f)]$$

where, G is the modulus of rigidity, V is the volume fraction and the subscripts c , m and f stand for composite, matrix and fiber, respectively.

The above investigations clarify the consistency of the obtained results by the HM and other models predictions.

For the coefficient of thermal expansion (CTE), on the other hand, fibers have lower expansion coefficient than the matrix. Thus, fibers are mechanically constraining the matrix. Therefore, it is expected that the transverse expansion of the considered composite material is greater than the matrix in isolation. The long stiff

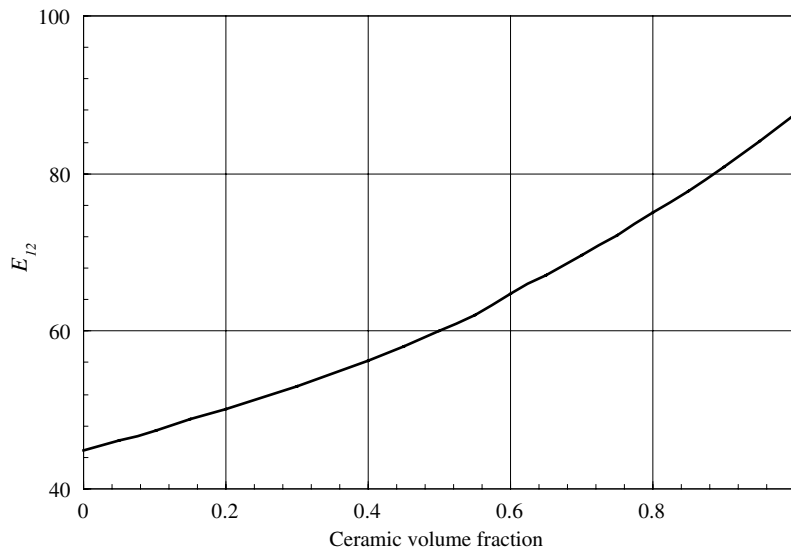


Fig. 7. Variation of E_{12} with the ceramic volume fraction.

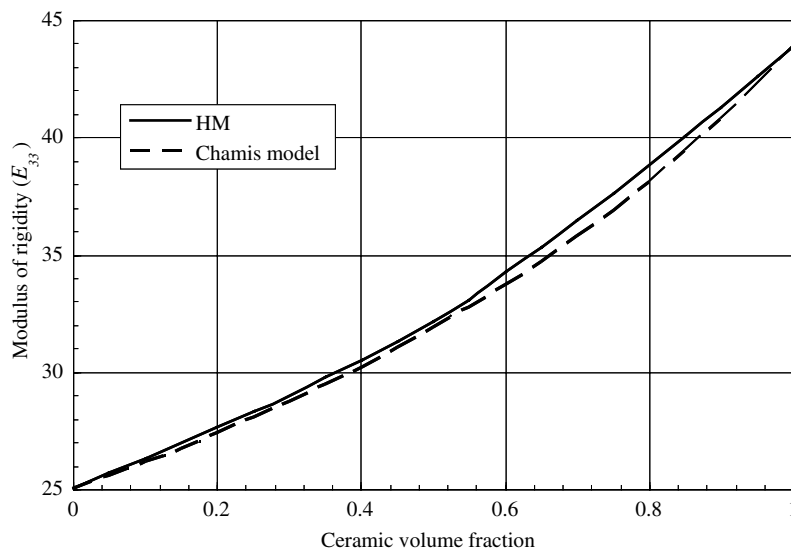


Fig. 8. Variation of E_{33} with the ceramic volume fraction.

fibers prevent the matrix from expanding in the longitudinal direction of the fiber, and as a result the matrix is forced to expand more than usual in the transverse directions. This results in the CTE in the longitudinal direction, which is represented by [Turner model \(1946\)](#) that accounts for the case of hydrostatic stresses, [Fig. 9](#), is lower than that in the transverse directions, which is represented by the current analyses results and the [Schapery model \(1968\)](#) predictions. Also, [Fig. 9](#) depicts that there is a good agreement between the HM results and the analytical results by Schapery model as the difference between them is small.

There are attempts in the literature for modelling the thermal conductivity (k) of composites. [Hesselman and Johnson \(1987\)](#) derived a complicated expression that accounts for the conductivity of the interface region. Since a perfect bonding between the matrix and fiber is considered here, omitting the interface region terms results in the following relation:

$$k_c = k_m [(k_f/k_m - 1)V_f + (1 + k_f/k_m)] / [(1 - k_f/k_m)V_f + (1 + k_f/k_m)]$$

Also, the [Hatta and Taya model \(1986\)](#) takes advantage of the Eshelby's method developed for elasticity problems. This model can be also applied to cases of more complicated morphologies of reinforcement. For uni-directional composite systems, this model is expressed as

$$k_c = k_m + [k_m(k_f - k_m)V_f] / 0.5[(1 - V_f) + (k_f - k_m) + k_m]$$

Moreover, the [Gurtman et al. model \(1981\)](#) predicts the thermal conductivity as follows:

$$k_c = V_f k_f (1 + (1 - V_f)A) + (1 - V_f)k_m (1 - V_f A)$$

$$A = [(k_m - k_f)] / [k_f + k_m + V_f(k_f + k_m)]$$

It can be seen from [Fig. 10](#) that when the ratio of the fiber conductivity to the matrix conductivity is small as in the current analyses, the Gurtman et al., Hatta and Taya and Hesselman and Johnson models predict almost same values. Also, the results of the current analyses are very consistent with these models. The Turner model, on the other hand, is shown in [Fig. 10](#) in order for comparison between the thermal conductivity values along the fiber with those normal to the fiber. [Fig. 11](#) shows the linear variation of effective volumetric heat capacity and it is consistent with the rule of mixture ([Auriault, 1983](#)).

4. Conclusion

In this study, a plane strain homogenization method based on the FE modeling procedure is investigated. This models inhomogeneous fiber-reinforced composite materials at a microscopic scale for predict-

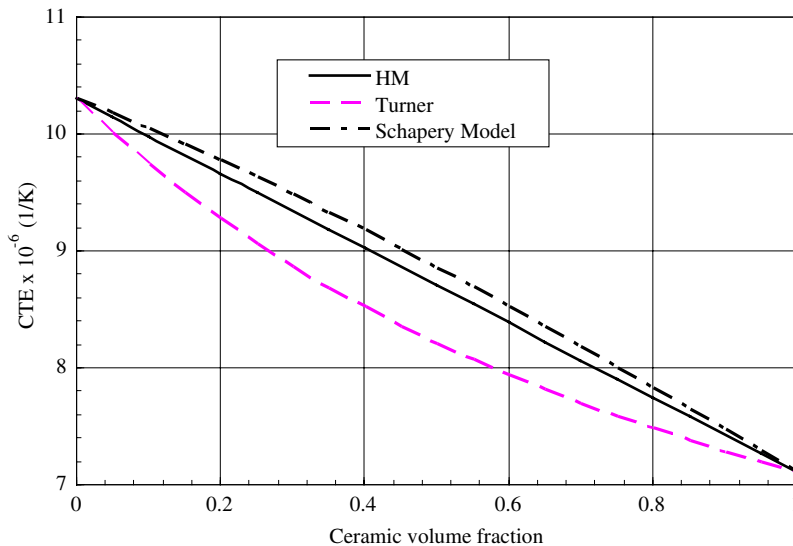


Fig. 9. Variation of CTE with the ceramic volume fraction.

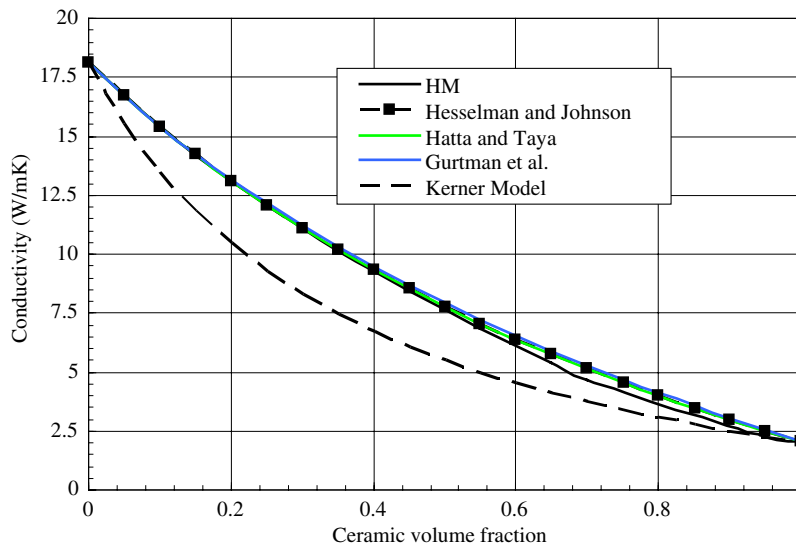


Fig. 10. Variation of thermal conductivity with the ceramic volume fraction.

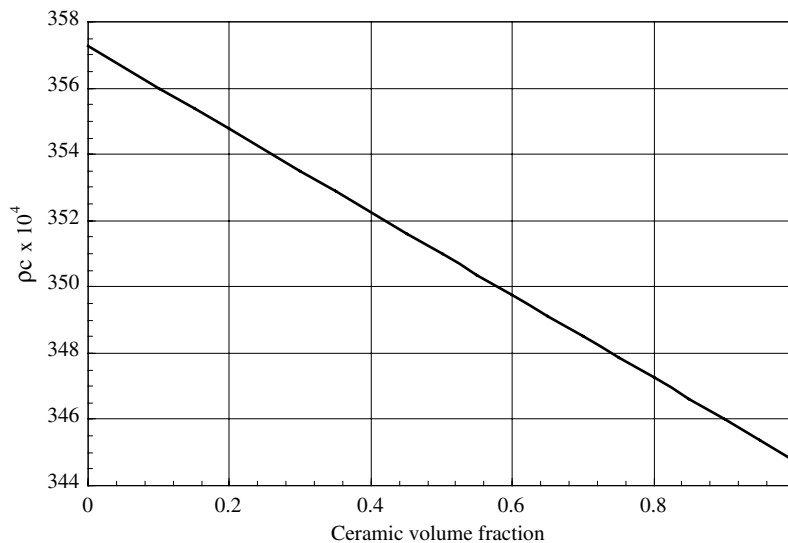


Fig. 11. Variation of ρc with the ceramic volume fraction.

ing the macroscopic thermomechanical and physical effective properties of these composites in order to be used for evaluating thermal stresses in uncoupled thermomechanical FGM problems by the homogenization method.

The effective Young's modulus is computed for square and hexagonal arrangements of the fibers and compared with three different kinds of results that found in the literature. These are based on the analytical methods, numerical method based on the VCFEM and the experimental measurements. This enables us to point out the validity of the current analyses and to quantify the influence of microstructural parameters such as the ceramic volume fraction on the effective macroscopic properties.

The following concluding remarks can be made referring to the computed obtained results of ZrO_2 and Ti-6Al-4V FGM:

- (1) The homogenization method results are, in general, in better agreement with the experimental and analytical results than the VCFEM results. Since different arrangements of the fibers are carried out, it may be worthy to mention that the difference between the results of the square and hexagonal arrangements of the fibers is small and the square arrangement predicts the upper level.
- (2) All of the stiffness matrix coefficients are logically increasing with the ceramic volume fraction and the computed shear modulus agrees well with Chamis model.
- (3) The coefficient of thermal expansion logically decreases with the ceramic volume fraction and it has good agreement with the results by Schapery model.
- (4) In order to evaluate the temperature distribution under steady state thermal load, the thermal conductivity is computed. The obtained results of thermal conductivity have an excellent agreement with three different analytical models. For unsteady state thermal load, on the other hand, the density and specific heat should be given. It is found from the computed results that the product of these two properties agrees well with previous estimations as it follows the linear relation (rule of mixture).

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