International Journal of Solids and Structures 46 (2009) 2225–2238

Contents lists available at ScienceDirect

International Journal of Solids and Structures

journal homepage: www.elsevier.com/locate/ijsolstr

Localization properties of strain-softening gradient plasticity models. Part I: Strain-gradient theories

Milan Jirásek [a,*], Simon Rolshoven [b]

[a] Department of Mechanics, Faculty of Civil Engineering, Czech Technical University in Prague, Thákurova 7, 166 29 Prague, Czech Republic
[b] Bu Anchor, Hilti Aktiengesellschaft, 9494 Schaan, Liechtenstein

A R T I C L E I N F O

Article history:
Received 19 May 2008
Received in revised form 27 November 2008
Available online 9 January 2009

Keywords:
Plasticity
Strain-gradient models
Nonlocal models
Localization
Softening

A B S T R A C T

This paper compares and evaluates strain-gradient extensions of the conventional plasticity theory. Attention is focused on the ability of individual formulations to act as localization limiters, i.e., to regularize the boundary value problem in the presence of softening and to prevent localization of plastic strain increments into a set of zero measure. To keep the presentation simple and to highlight the essential properties of the investigated models, only the static, rate-independent response in the small-strain range and in the one-dimensional setting is considered. These restrictions permit an analytical or semi-analytical treatment of the problem, while the basic characteristics of the solutions remain valid in the general, multi-dimensional case. The onset of localization is characterized as a bifurcation from a uniform state. The subsequent evolution of the localized process zone and of the shape of the strain profile is studied numerically. It is shown that certain pathologies, e.g., expansion of the plastic region accompanied by stress locking, may arise at later stages of localization. A similar analysis of models with gradients of internal variables is presented in a companion paper.

© 2008 Elsevier Ltd. All rights reserved.

1. Introduction

Classical plasticity models, typically used in most engineering applications, rely on the assumption that the stress at a certain material point can be uniquely determined from the history of the strain (directly related to the first-order deformation gradient) at that point only and, in the thermoplastic extension, from the history of temperature and the current value of temperature gradient at that point only. Individual material points are supposed to interact only with their immediate neighbors, and the mechanical part of the interaction is fully described by the usual symmetric stress tensor. Force interaction at finite distance is excluded (except for externally applied body forces), same as the dependence on the second- and higher-order gradients of the displacement field. In the actual constitutive equations, the functional dependence on the strain history is usually replaced by dependence on the current values of strain and internal variables, but again, this dependence is strictly local, and gradients of the internal variable fields are not taken into account. The same holds for the evolution equations that specify the rates of internal variables.

Constitutive models formulated within the classical framework of simple nonpolar materials are not equipped with any length scale that would reflect the typical size and spacing of characteristic microstructural features; therefore, they are inherently incapable of describing size effects of the transitional type that are experimentally observed if the characteristic wave length of the deformation pattern becomes comparable with the intrinsic material length scale. Such models also fail to describe strain localization due to softening or nonassociated flow in an objective and mathematically consistent way, and their application in the post-critical regime leads to ill-posed boundary value problems.

The remedy has been sought in various enrichments that incorporate, at least in a simplified way, some information about the material heterogeneity at the mesoscopic or microscopic levels. Examples of such enrichments include additional kinematic variables (e.g., independent micropolar rotations in unconstrained Cosserat-type theories), weighted spatial averages, higher-order gradients, or rate-dependent terms. In the present study we focus on the broad class of gradient theories, which can be divided into two distinct groups:

- *Strain-gradient models*, which characterize the deformation at a material point not only by the conventional strain (related to the displacement gradient) but also by the strain gradient (related to the second gradient of displacement). In a general case, such models can also take into account second- or higher-order gradients of strain.
- *Models with gradients of internal variables*, some of which also incorporate the gradients of the dissipative forces conjugate to the internal variables.

* Corresponding author. Fax: +41 21 6936340.
E-mail address: Milan.Jirasek@fsv.cvut.cz (M. Jirásek).

0020-7683/$ - see front matter © 2008 Elsevier Ltd. All rights reserved.
doi:10.1016/j.ijsolstr.2008.12.016

The differences between these two groups of models will be explained in detail in Section 3.1.

This paper compares a number of gradient-type extensions of the conventional plasticity theory. Attention is focused on the suitability of individual formulations for problems with strain localization due to softening (decrease of yield stress with increasing plastic strain). To keep the presentation simple and to highlight the essential properties of the investigated models, we consider only the static, rate-independent response in the small-strain range and in the one-dimensional setting. These restrictions permit an analytical or semi-analytical treatment of the problem, while the basic characteristics of the solutions remain valid in the general, multi-dimensional case.

The entire study is divided into two parts. Part I starts with a brief summary of the basic equations of standard local plasticity in Section 2 and with general comments on the classification of gradient theories in Section 3. A quick overview of the basic concepts of strain-gradient elasticity is provided in Section 4. Experts in this field may wish to skip the preparatory considerations and proceed directly to Sections 5–7, devoted to localization analysis of the specific models that belong to the family of strain-gradient theories. Plasticity models with gradients of internal variables are covered in Part II.

2. Flow theory of plasticity in one dimension

In the one-dimensional setting, standard (local, gradient-independent) elastoplasticity with linear isotropic hardening is described by the elastic stress–strain law

$$\sigma = E(\varepsilon - \varepsilon_p) \tag{1}$$

hardening law

$$q = -H\kappa \tag{2}$$

and evolution laws

$$\dot{\varepsilon}_p = \dot{\lambda}\frac{\partial f(\sigma, q)}{\partial \sigma} \tag{3}$$

$$\dot{\kappa} = \dot{\lambda}\frac{\partial f(\sigma, q)}{\partial q} \tag{4}$$

with loading–unloading conditions

$$\dot{\lambda} \geqslant 0, \quad f(\sigma, q) \leqslant 0, \quad \dot{\lambda} f(\sigma, q) = 0 \tag{5}$$

In the foregoing relations, σ is the stress, ε is the (total) strain, ε_p is the plastic strain, E is the elastic modulus, H is the plastic modulus (positive for hardening and negative for softening), κ is the hardening variable, q is the dissipative thermodynamic force conjugate to κ, $\dot{\lambda}$ is the rate of the plastic multiplier, and

$$f(\sigma, q) = |\,\sigma\,| - \sigma_0 + q \tag{6}$$

is the yield function. Initially, the variables κ and q have zero values, and so the parameter σ_0 is the initial yield stress. As usual, a superimposed dot denotes the derivative with respect to time.

From the thermodynamic point of view, Eqs. (1) and (2) are the state laws that can be derived from the free-energy potential

$$\rho\psi(\varepsilon, \varepsilon_p, \kappa) = \frac{1}{2}E(\varepsilon - \varepsilon_p)^2 + \frac{1}{2}H\kappa^2 \tag{7}$$

Eqs. (3)–(5) are the complementary laws that can be derived from the dual dissipation potential $\phi^*(\sigma, q)$ defined as the indicator function of the set of plastically admissible states; for a detailed discussion, see e.g., Jirásek and Bažant (2002), Chapter 23.

The basic Eqs. (1)–(5) are written in a form that reflects the general structure of associated plasticity and reveals a certain symmetry in the state laws and complementary laws. Making use of the particular definition of the yield function (6), it is possible to replace $\partial f/\partial\sigma$ in (3) by $\operatorname{sgn}\sigma$ and $\partial f/\partial q$ in (4) by 1. According to the latter equation, rewritten as $\dot{\kappa} = \dot{\lambda}$, the plastic multiplier rate can be replaced by the rate of the hardening variable and eliminated from the basic equations. The flow rule (3), rewritten as $\dot{\varepsilon}_p = \dot{\kappa}\operatorname{sgn}\sigma$ and combined with the condition $\dot{\kappa} \geqslant 0$, then implies that $\dot{\kappa} = |\,\dot{\varepsilon}_p\,|$, which endows the hardening variable κ with the physical meaning of cumulative plastic strain. To give a clear physical meaning to the variable that controls the size of the elastic domain, we can introduce the current yield stress $\sigma_Y = \sigma_0 - q$ and rewrite the hardening law (2) as $\sigma_Y = \sigma_0 + H\kappa$. After all these adjustments, the basic equations reduce to

$$\sigma = E(\varepsilon - \varepsilon_p) \tag{8}$$

$$\sigma_Y = \sigma_0 + H\kappa \tag{9}$$

$$\dot{\varepsilon}_p = \dot{\kappa}\operatorname{sgn}\sigma \tag{10}$$

$$\dot{\kappa} \geqslant 0, \quad f(\sigma, \sigma_Y) \leqslant 0, \quad \dot{\kappa} f(\sigma, \sigma_Y) = 0 \tag{11}$$

where the yield function is now given by

$$f(\sigma, \sigma_Y) = |\,\sigma\,| - \sigma_Y \tag{12}$$

As an illustrative test problem, consider a bar of a constant cross section fixed at one end and loaded by an applied displacement at the opposite end (Fig. 1a). Alternatively, one could consider a semi-infinite layer of material between two parallel planes, one of which is fixed and the other displaced in the tangential direction and free in the normal direction, so that the material is under pure shear stress (Fig. 1b). In the elastic regime, the response is unique, and the distribution of strain remains uniform. The same holds in the elastoplastic regime, provided that the material is hardening. However, if the material is softening, the governing equations admit infinitely many solutions with nonuniform strain distributions. Due to the static equilibrium condition, the stress must remain uniform, but plastic yielding does not need to occur at all points of the body. The plastic zone I_p can become arbitrarily small and the bar or shear layer can fail at arbitrarily small dissipation. These physically inadmissible properties of the theoretical solutions are the source of a pathological sensitivity of the numerical results to the computational grid (e.g., to the finite element mesh), as amply documented in the literature.

From now on, we focus on the case of *softening*. To emphasize that, κ will be called the softening variable and Eq. (9) will be referred to as the softening law.

A linear softening law as stated in (9) has only a limited range of validity. Since the (tensile) yield stress cannot become negative, the law must be in fact considered as bilinear, with the softening

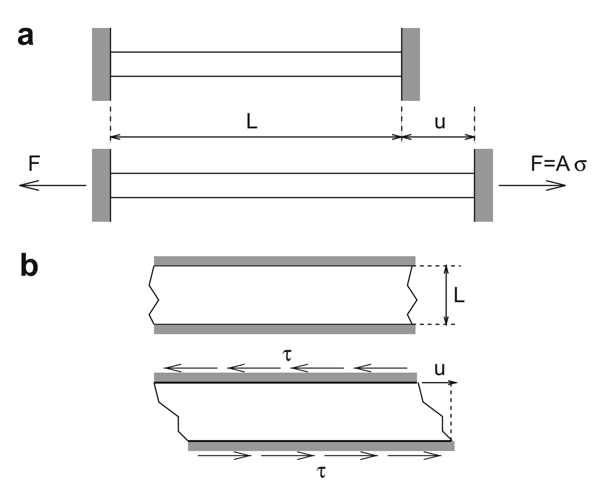

Fig. 1. (a) Bar under uniaxial tension, (b) shear layer.

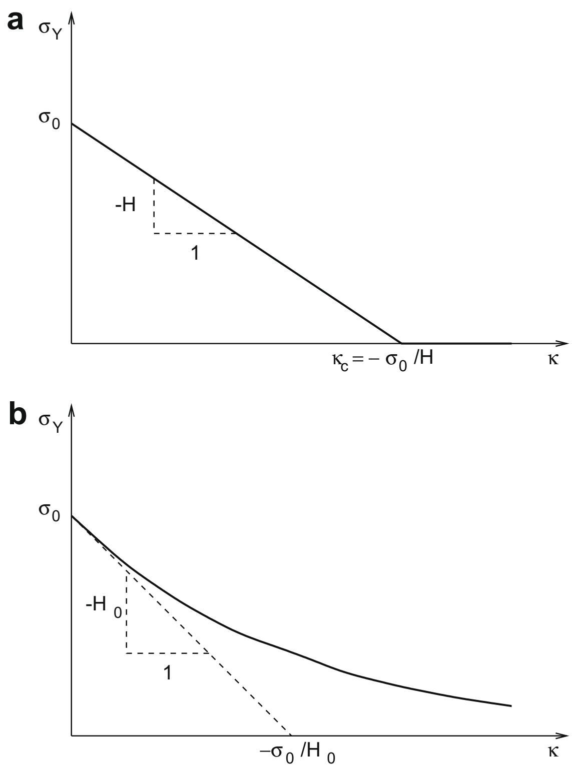

Fig. 2. (a) "Linear" softening curve consisting of two straight segments, (b) exponential softening curve.

curve consisting of a descending segment and a horizontal segment at zero stress level; see Fig. 2a. The corresponding generalized form of (9) reads

$$\sigma_Y = \langle \sigma_0 + H\kappa \rangle = \begin{cases} \sigma_0 + H\kappa & \text{if } \kappa \leqslant \kappa_c \\ 0 & \text{if } \kappa \geqslant \kappa_c \end{cases} \tag{13}$$

where $\langle \ldots \rangle$ are the Macauley brackets denoting the positive part, and $\kappa_c = -\sigma_0/H$ is the plastic strain at which the yield stress first vanishes.

We will also consider nonlinear softening laws, for instance in the exponential form

$$\sigma_Y = \sigma_0 \exp\left(-\frac{H_0\kappa}{\sigma_0}\right) \tag{14}$$

The meaning of parameter H_0 (the initial softening modulus) is clear from Fig. 2b.

3. Gradient-enriched theories

3.1. Classification

In the context of the standard continuum theory with simple nonpolar material models in the sense of Noll (1972), softening or nonassociated plastic flow may lead to the loss of ellipticity of the governing differential equations and to an ill-posed boundary value problem. In order to arrive at objective numerical solutions that are not pathologically sensitive to the discretization, an appropriate regularization technique must be used. Such techniques usually enrich the model equations so as to prevent localization of strain into a set of zero measure, and therefore serve as localization limiters. As already alluded to in the Introduction, in the present study we focus on the broad class of gradient theories, which can be divided into

- strain-gradient models, and
- models with gradients of internal variables.

The fundamental difference between these two groups of models is that strain gradients considered as additional observable state variables are conjugate to higher-order stresses that enter the equilibrium equations, while gradients of internal variables are conjugate to certain dissipative thermodynamic forces that can enter the evolution equations for internal variables but do not appear in the equilibrium (momentum balance) equations. Thus the latter group of theories modifies only the constitutive description while the kinematic and equilibrium equations remain standard. In thermodynamic terms, one could say that the theories with gradients of internal variables enrich only the free-energy potential and the dissipation potential, while the strain-gradient theories require also generalizations of the external and internal work expressions.

Of course, certain models may have a mixed character. For instance, Zervos et al. (2001) proposed a model that can be interpreted as a strain-gradient theory with softening law enriched by the second gradient of an internal variable. This so-called gradient elastoplasticity theory will be described and analyzed in Part II, while in Part I we restrict our attention to the "pure" strain-gradient elastoplastic models.

In the elastic regime, the internal variables do not evolve and remain equal to their initial values (usually zero), and so their gradients vanish. Consequently, the initial response of a model with gradients of internal variables is governed by standard elasticity, and gradient effects are activated only by inelastic processes. In contrast to that, the response of strain-gradient models deviates from standard models already in the elastic regime, unless the strain remains uniform. The elastic response of strain-gradient models is described by the nowadays classical strain-gradient elasticity theory pioneered by Toupin (1962) and Mindlin (1964, 1965).

The terminology used in the literature on gradient-enriched material models is not unified, which presents a potential source of misunderstanding. Gradient theories are often classified according to the order of the enrichment terms and are called first-gradient theories, second-gradient theories, etc. But this terminology should be used only when the context is clear, because it can lead to confusion. First of all, it is important to know on which field the gradient operators act. For instance, the basic version of strain-gradient elasticity enriches the free-energy potential by dependence on the first gradient of strain. Since the strain itself is the symmetric part of the displacement gradient, the strain gradient is directly related to the second gradient of the displacement field. So one can consider this theory as a first- or second-gradient theory, depending on the interpretation.

Another potential source of misunderstanding is hidden in the fact that enrichments of the observable state variables inevitably lead to modifications of the corresponding balance laws, and the same is true for the internal variables and corresponding evolution equations if the theory is formulated within a thermodynamic framework. For instance, in strain-gradient elasticity the strain gradient is work-conjugate to the so-called double stress, and the second gradient of the double stress appears in the momentum balance equation. When the basic equations are combined and the momentum balance is written in terms of displacements, it turns out to be a fourth-order differential equation, i.e., its order increases by 2 as compared to the standard theory. So even though the enrichment of the kinematic part of the model is of the first order, duality induces another first-order enrichment of the equilibrium equation and the resulting effect is of the second order. On the other hand, some theories postulate the enrichment of the constitutive equations directly, without using the thermodynamic framework, and then two dual first-order enrichments are presented as one second-order enrichment. A typical example is the phenomenological gradient plasticity of Aifantis (1984), which incorporates in the hardening or softening law the dependence on the second gradient of cumulative plastic strain. It is therefore considered as a second-gradient theory, but the same constitutive model could be constructed from a thermodynamically based theory with free energy dependent on the first gradient of cumulative plastic strain (Svedberg, 1996; Svedberg and Runesson, 1997; Liebe and Steinmann, 2001).

3.2. Size effects and internal length scale

Since gradients are sensitive to the spatial scale (for geometrically similar bodies with homothetic distributions of state variables they are inversely proportional to the characteristic size of the body), gradient enrichments are perfectly suited for incorporating information about the characteristic length scale(s) dictated by the internal structure of the material. This is important, for instance, for the description of size effects of a transitional type. The standard continuum theory does not possess any characteristic length, and it can be shown that in such a case all scaling laws must be of a power type (e.g., Bažant, 2002). However, in many cases a certain property scales differently for "large" sizes than for "small" sizes. A typical example is the nominal strength of notched quasibrittle structures, which is almost constant for small structures but decreases in inverse proportion to the square root of the structure size for large structures. The small-size range can be described by standard plasticity and the large-size range by linear elastic fracture mechanics, both being theories with no characteristic length. To describe the entire range of sizes by one single model, a length parameter must be incorporated into the theory, otherwise it is impossible for the model to distinguish between a small structure and a large one.

Quasibrittle failure is closely related to softening material modeQuasibrittle failure is closely related to softening material models, but similar examples of transitional size effects can be found for hardening materials or even for the elastic response; they include the dispersion of short elastic waves, scaling of the torsional elastic stiffness of thin wires or bones (Morrison, 1939; Lakes, 1986; Fleck et al., 1994) and the bending stiffness of thin beams (metallic films) (Richards, 1958; Stolken and Evans, 1998), or size-dependence of the apparent hardening curve evaluated from microindentation tests (Nix, 1989; Ma and Clarke, 1995; Poole et al., 1996). The main purpose of certain gradient theories is an accurate modeling of such size effects, and not the objective description of strain localization due to softening. However, it seems useful to include even such models in the present comparative study, which scrutinizes their possible exploitation as localization limiters. Localization properties of several gradient plasticity models have recently been investigated by Engelen et al. (2006), who focused on harmonic incremental solutions from a uniform reference state. In the present study we look at incremental solutions with plastic yielding localized into a single interval.

One should bear in mind that the choice of an appropriate model and the need for gradient-type or other enrichments depends on the particular problem type, including the size and shape of the body of interest, properties of the material and type of loading. If the strain distribution is smooth and the strain gradients are small (with respect to the strain amplitude divided by the characteristic length of the material), as is often the case in the elastic regime, the standard theory provides a good approximation and no important deviations from the actual behavior can be observed. However, gradient effects may play an important role already in the elastic regime, e.g., in regions of high stress and strain concentrations around notches and cracks, or if the wave length of elastic waves is comparable to the size of characteristic heterogeneities in the material. For metals, this scale is in the order of microns, but for concrete and other highly heterogeneous composite materials, it is substantially larger.

Still, in many situations it is perfectly legitimate to describe the elastic response by the conventional theory. Only in the inelastic regime, especially after strain localization, the characteristic "wave length" of the deformation field decreases, and the influence of material heterogeneity at the meso- or microscale becomes important. For this reason, gradient theories constructed as localization limiters usually do not modify the elastic part of the response. Gradient enhancements are then applied only to an internal variable (or thermodynamic force) linked to the dissipative processes. In plasticity, this is naturally the softening variable (cumulative plastic strain), or the plastic strain itself. The elastic behavior of the material remains unchanged. In contrast to that, strain-gradient theories reflect the influence of the length scale on the material properties already in the elastic range. This is useful, e.g., for modeling of dispersion of elastic waves in heterogeneous materials or in crystal lattices by homogenized continuum models.

3.3. Strong and weak nonlocality

From another viewpoint, gradient models can be divided into explicit and implicit ones.

Explicit models enrich the governing equations directly by gradients of the local state variables or thermodynamic forces. The dependence on the gradients makes the stress response of one material point depend on the behavior of a neighborhood of that point, but that neighborhood can be arbitrarily small. This is why such models are called *weakly nonlocal*. Precise definitions of weak and strong nonlocality are given in Rogula (1982).

Implicit models work with higher-order gradients, too, but do not insert them directly into the constitutive equations. The differential operators are not applied to the local internal variable field, f, but they implicitly define a nonlocal field, $\bar{f}$, constructed, e.g., as the solution of the Helmholtz-type differential equation

$$\bar{f} - c\,\nabla^2\bar{f} = f \quad \text{in } V \tag{15}$$

where ∇^2 is the Laplace operator, V is the domain occupied by the body of interest, and c is a material parameter with the dimension of length squared. To uniquely specify $\bar{f}$, Eq. (15) must be supplemented by appropriate boundary conditions. The precise form of these conditions is not obvious, but it seems reasonable to require that the transformation of f into $\bar{f}$ should not alter a constant field. If $f(\boldsymbol{x}) = f_0 = \text{const.}$, then $\bar{f}(\boldsymbol{x}) = f_0$ satisfies the differential Eq. (15), and it should also satisfy the boundary conditions, independently of the value of f_0. Clearly, it is not possible to use the homogeneous Dirichlet boundary conditions, but every constant field satisfies the homogeneous Neumann boundary conditions

$$\boldsymbol{n}\cdot\nabla\bar{f} = 0 \quad \text{on } \partial V \tag{16}$$

where ∇ is the gradient operator, ∂V is the boundary of the domain V, and $\boldsymbol{n}$ is the unit vector normal to the boundary. For an infinite domain, the boundary conditions are replaced by the requirement that the solution must remain bounded.

Implicit gradient models that incorporate a transformed field defined as the solution of a boundary value problem such as (15) and (16) have been developed, e.g., by Peerlings et al. (1996, 1998), Engelen et al. (2003) and Geers (2004) and applied by Peerlings et al. (2004), César de Sá et al. (2006) and Tovo and Livieri (2008). In contrast to explicit gradient models, they are *strongly nonlocal*, because the value of the solution $\bar{f}$ at a given point $\boldsymbol{x}$ depends on the values of the right-hand side f in the entire body, and so the stress at $\boldsymbol{x}$ depends on the state of the entire body. In fact, the implicit gradient models can equivalently be written in an integral nonlocal format (Peerlings et al., 1996). To show that, let us introduce the Green function of the boundary value problem (15) and (16). Taking the one-dimensional case as an example, the domain V reduces to an interval $\mathscr{L}$, and the Green function is a function $G(x,\xi)$ satisfying for every $\xi \in \mathscr{L}$ the differential equation

$$G(x,\xi) - c\frac{\partial^2 G(x,\xi)}{\partial x^2} = \delta(x-\xi) \quad \forall x \in \mathscr{L} \tag{17}$$

and the boundary conditions

$$\frac{\partial G(x,\xi)}{\partial x} = 0 \quad \forall x \in \partial\mathscr{L} \tag{18}$$

The symbol δ in (17) denotes the Dirac distribution. Since $\int_{\mathscr{L}} \delta(x-\xi) f(\xi)\,\mathrm{d}\xi = f(x)$, the solution of (15) and (16) can formally be written as

$$\bar{f}(x) = \int_{\mathscr{L}} G(x,\xi) f(\xi)\mathrm{d}\xi \tag{19}$$

Consequently, the transformed field $\bar{f}$, implicitly defined as the solution of the boundary value problem, is a weighted spatial average of the local field f, with the Green function playing the role of the weight function. So, the implicit gradient models are equivalent to *integral-type nonlocal models* with special weight functions; see Bažant and Jirásek (2002) for a general review of integral nonlocal models and Jirásek and Rolshoven (2003) for a comparative study on localization properties of integral nonlocal plasticity models.

3.4. Test problem: localization in one dimension

To demonstrate the differences among various formulations of gradient plasticity and to assess their regularizing capabilities, we will analyze a simple one-dimensional localization problem. In the course of analysis, this problem will be interpreted as a uniaxial tensile test (Fig. 1a), but if the yield condition is pressure-insensitive and the plastic flow is isochoric, the results are also valid for localization in a semi-infinite shear layer (Fig. 1b). It is sufficient to replace the normal stress by shear stress, normal strain by engineering shear strain, Young's modulus by shear modulus, etc. A truly two-dimensional analysis of the shear layer problem for a strain-gradient plasticity model with a pressure-sensitive yield condition was presented by Chambon et al. (2001).

For models with gradients of internal variables only, the kinematic equation

$$\varepsilon(x) = u'(x) \tag{20}$$

and the equilibrium equation

$$\sigma'(x) + b(x) = 0 \tag{21}$$

remain standard. Here, u is the displacement, ε is the strain, σ is the stress and b are the body forces. We consider a straight bar of length L and constant cross section A, loaded by an increasing imposed displacement $\bar{u}$ at the right end while the left end remains fixed. This type of loading is described by the kinematic boundary conditions

$$u(0) = 0, \quad u(L) = \bar{u} \tag{22}$$

where $\bar{u}$ is a monotonically increasing parameter that controls the deformation process. The reaction at the right support, generated by the applied displacement, can be evaluated as $F = A\sigma(L)$. A generalized form of these equations valid for the strain-gradient theories will be presented in the next section.

Despite the apparent simplicity of the one-dimensional localization problem, there are many interesting aspects to be explored. We will look not only at the first bifurcation from a uniform strain state, which is captured by most of the models in a proper way, but also at the subsequent evolution of the localized plastic zone up to complete failure (full softening to zero residual strength). It will be shown that the model response strongly depends on the particular formulation and that some formulations lead to very unrealistic results. Another issue to be addressed is the influence of boundaries, i.e., the difference between localization in an infinite bar (or sufficiently far from the boundary) and in the proximity of the bar end. This issue is closely related to the choice of boundary conditions, which are needed by gradient models but whose physical meaning is not always clear.

4. Strain-gradient elasticity

The elastic strain-gradient theory has its roots in the pioneering work of Toupin (1962) and Mindlin (1964). The linear version of strain-gradient elasticity is derived from a quadratic free-energy potential that depends not only on the strain but also on its gradient. In one dimension, we can write

$$\rho\psi(\varepsilon,\eta) = \frac{1}{2}E\varepsilon^2 + \frac{1}{2}El^2\eta^2 \tag{23}$$

where ρ is the mass density, ψ is the free energy per unit mass, $\eta = \varepsilon'$ is the spatial derivative of strain and l is a material parameter with the dimension of length. For an elastic material, the dissipation density $\mathscr{D} = \mathscr{P}_{int} - \rho\dot{\psi}$ must vanish. Since the rate of free energy density depends not only on the strain rate but also on the strain-gradient rate (which is *locally* independent of the strain rate), the standard expression for the internal power $\mathscr{P}_{int} = \sigma\dot{\varepsilon}$ must be enriched by a term $\chi\dot{\eta}$ where χ, called the double stress, is the thermodynamic force conjugate to the state variable η. From the condition that $\mathscr{D} = 0$ for an arbitrary combination of rates $\dot{\varepsilon}$ and $\dot{\eta}$, we obtain the state equations

$$\sigma = \rho\frac{\partial\psi}{\partial\varepsilon} = E\varepsilon \tag{24}$$

$$\chi = \rho\frac{\partial\psi}{\partial\eta} = El^2\eta \tag{25}$$

consisting of the standard stress–strain law of linear elasticity (24) and another linear elastic law (25) that links the double stress to the strain gradient. The higher-order elastic modulus El^2 has a physical dimension different from Young's modulus E and their ratio is controlled by the square of the intrinsic length parameter l. For $l = 0$, standard elasticity is recovered as a special case.

The state variables ε and η are linked to the displacement field by the kinematic equations

$$\varepsilon = u' \tag{26}$$

$$\eta = u'' \tag{27}$$

and the corresponding static equations follow from duality. Integrating by parts the internal power for the entire bar (taken per unit cross-sectional area), we obtain

$$\begin{aligned} P_{\mathrm{int}} &= \int_{\mathscr{L}} \mathscr{P}_{int}\mathrm{d}x = \int_{\mathscr{L}} (\sigma\dot{\varepsilon} + \chi\dot{\eta})\mathrm{d}x = \int_{\mathscr{L}} (\sigma\dot{u}' + \chi\dot{u}'')\mathrm{d}x \\ &= \int_{\partial\mathscr{L}} n[(\sigma - \chi')\dot{u} + \chi\dot{u}']\mathrm{d}(\partial\mathscr{L}) - \int_{\mathscr{L}} (\sigma' - \chi'')\dot{u}\mathrm{d}x \end{aligned} \tag{28}$$

where $\partial\mathscr{L}$ is the boundary of the interval $\mathscr{L} = [0, L]$ and n is the "unit outward normal" to this boundary. Of course, in one dimension $\partial\mathscr{L}$ consists of two points, 0 and L, and n is a scalar equal to -1 on the left boundary (at $x = 0$) and to 1 on the right boundary (at $x = L$). The formal presentation with integrals over the boundary (which in one dimension reduce to sums over the boundary points) emphasizes the general structure of the theory and facilitates its extension to multiple dimensions.

The standard expression for the external power (per unit cross-sectional area) supplied by body forces b and surface tractions t is

$$P_{\mathrm{ext}} = \int_{\mathscr{L}} b\dot{u}\,\mathrm{d}x + \int_{\partial\mathscr{L}} t\dot{u}\,\mathrm{d}(\partial\mathscr{L}) \tag{29}$$

Substituting this into the power equality $P_{\mathrm{ext}} = P_{\mathrm{int}}$ and considering that the displacement rate must satisfy the essential (kinematic) boundary conditions but otherwise is arbitrary, we obtain the general form of the equilibrium equation

$$\sigma' - \chi'' + b = 0 \tag{30}$$

and the boundary conditions

$$u = \bar{u} \text{ or } n(\sigma - \chi') = \bar{t} \quad \text{on } \partial\mathscr{L} \tag{31}$$

$$u' = \bar{\varepsilon} \text{ or } \chi = 0 \quad \text{on } \partial\mathscr{L} \tag{32}$$

where $\bar{u}$ is the prescribed displacement, $\bar{\varepsilon}$ is the prescribed strain, and $\bar{t}$ is the prescribed traction. The expression for the external power could contain additional nonstandard boundary forces acting on the displacement gradients, but such higher-order tractions are usually assumed to vanish (if not, they replace zero on the right-hand side of the second condition in (32)).

In the absence of body forces, the equilibrium equation (30) can be written as

$$(\sigma - \chi')' = 0 \tag{33}$$

Due to the contribution of the double stress χ, the distribution of stress σ is not necessarily uniform. The quantity that remains uniform in space is $\sigma - \chi'$, sometimes called the *total* stress (while σ is more specifically called the *Cauchy* stress). From the boundary condition (31b) it follows that $\sigma - \chi'$ is equal to the traction $t = F/A$ applied at the right boundary (where $n = 1$). So the reaction force F at the support is related to the total stress and not to the Cauchy stress.

Probably the first extension of strain-gradient theory to plasticity, proposed by Dillon and Kratochvil (1970), was based on the elastic theory with second gradients of strain (Mindlin, 1965). Higher-order effects were loosely motivated by dislocation interactions and activated only after the onset of plastic flow. The main purpose of the Dillon–Kratochvil model was to reflect the formation of nonuniform deformation patterns on the microscopic level in hardening metals. In the subsequent sections we will present and analyze more recent formulations of strain-gradient plasticity that can serve as localization limiters.

5. Strain-gradient plasticity model of Chambon et al.

5.1. Model equations

Chambon et al. (1998) proposed a localization limiter based on the theory of continua with microstructure. They discussed a rather general framework, but the specific formulation they finally used can be considered as an elastoplastic extension of strain-gradient elasticity (Toupin, 1962; Mindlin, 1964). This extension is based on the assumption that the yield function depends only on σ, and not on χ. Thus, the yield function as well as loading–unloading conditions and softening law remain the same as in standard plasticity. Under the assumption of associated flow, plastic deformation is described only by the classical plastic strain. Consequently, the constitutive part of the model consists of standard plasticity Eqs. (8)–(12) and the higher-order elastic law (25). Note that, on the constitutive level, the stress–strain relation is fully decoupled from the relation between the double stress and the strain gradient. The elastic constitutive equation for the double stress (25) remains valid without any plastic correction. The double stress is linked to the stress by the equilibrium Eq. (30) and the strain gradient is linked to the strain by the compatibility equation $\eta = \varepsilon'$ that follows from the kinematic Eqs. (26) and (27).

5.2. Bifurcation from a uniform state

Chambon et al. (1998) outlined the general approach to analytical treatment of their model in one dimension. For comparison with other gradient-type formulations, it is useful to provide here the specific solution of the one-dimensional localization problem and discuss the main localization properties of Chambon's model.

To analyze the bifurcation from a uniform state, the equilibrium Eq. (33) is first integrated in space and then rewritten in the rate form

$$\dot{\sigma}(x) - \dot{\chi}'(x) = \dot{t} \tag{34}$$

Since the stress–strain law is given by the standard equations of plasticity (8)–(12), the stress rate can be expressed as $\dot{\sigma} = E\dot{\varepsilon}$ in the elastic region and $\dot{\sigma} = E_t\,\dot{\varepsilon}$ in the plastic region, with $E_t = EH/(E+H) < 0$ being the tangent elastoplastic modulus. The rate of the double stress is easily expressed from (25). Substituting all this into (34), we obtain

$$E\dot{\varepsilon}(x) - El^2\dot{\varepsilon}''(x) = \dot{t} \quad \text{for } x \in \mathscr{I}_{\mathrm{e}} \tag{35}$$

$$E_t\,\dot{\varepsilon}(x) - El^2\dot{\varepsilon}''(x) = \dot{t} \quad \text{for } x \in \mathscr{I}_{\mathrm{p}} \tag{36}$$

where the elastic region $\mathscr{I}_{\mathrm{e}}$ is characterized by $\dot{\varepsilon} \leqslant 0$ and the plastic region $\mathscr{I}_{\mathrm{p}}$ is characterized by $\dot{\varepsilon} \geqslant 0$. The above differential equation has the general solution

$$\dot{\varepsilon}(x) = \frac{\dot{t}}{E} + C_1\cosh\frac{x}{l} + C_2\sinh\frac{x}{l} \quad \text{for } x \in \mathscr{I}_{\mathrm{e}} \tag{37}$$

$$\dot{\varepsilon}(x) = \frac{\dot{t}}{E_t} + C_3\cos\frac{\alpha x}{l} + C_4\sin\frac{\alpha x}{l} \quad \text{for } x \in \mathscr{I}_{\mathrm{p}} \tag{38}$$

where

$$\alpha = \sqrt{-\frac{E_t}{E}} = \sqrt{-\frac{H}{E+H}} \tag{39}$$

is a positive parameter that depends only on the ratio H/E between the softening modulus and the modulus of elasticity, and C_i, $i = 1,2,3,4$, are integration constants. If the elastic region or plastic region consists of several disjoint intervals, the integration constants are of course different in each contiguous part.

To obtain a valid solution, it is necessary to find the regions $\mathscr{I}_{\mathrm{e}}$ and $\mathscr{I}_{\mathrm{p}}$ and determine the integration constants such that the solution satisfies the boundary conditions at both end sections of the bar and the appropriate continuity conditions at the internal elastoplastic boundaries. In general, one should enforce continuity of u, u', χ and $\sigma - \chi'$. In the present simple case, continuity of $\sigma - \chi'$ is satisfied automatically (since the same $\dot{t}$ is used in (35) and (36)), and continuity of u would be used to obtain the displacement field by integration of the strain field but does not need to be considered when solving for the strains only. So it is sufficient to enforce continuity of u' and χ, which is equivalent to continuity of ε and ε'. On each external boundary, only one of conditions (32) is enforced. Since it would be physically unrealistic to prescribe the value of strain on the boundary, the condition to be used is $\chi = 0$, which is equivalent to $\varepsilon' = 0$. As already mentioned in Section 4, a nonzero higher-order traction could be prescribed on the boundary. This could be useful, e.g., in shear localization problems, where the higher-order strain corresponds to a micro-curvature and the higher-order traction is then a micro-stress couple acting on the boundary. Nevertheless, in the subsequent analysis we restrict attention to the basic case with vanishing higher-order traction on the boundary.

The size and location of the elastic and plastic regions are not given in advance; they must be determined such that the solution is admissible. Recall that, according to the loading–unloading conditions, the strain rate is nonnegative in $\mathscr{I}_{\mathrm{p}}$ and nonpositive in $\mathscr{I}_{\mathrm{e}}$. Since, for the present model, the strain must be continuous, these conditions imply that each internal elastoplastic boundary is in the regime of neutral loading, with a vanishing strain rate. This provides a sufficient number of additional conditions that make it possible to determine the exact location of the elastoplastic boundaries. However, the number of elastic and plastic intervals must be selected first. This is a typical situation in one-dimensional localization analysis. The choice is not unique, but the most interesting solutions are those for which the plastic strain localizes into one single interval, which can either be adjacent to one bar end, or form in the interior of the bar. The number of elastic intervals is then 1 or 2, resp.

Consider first the case of localization near the left end. The bar domain $\mathscr{L} = [0, L]$ is divided into the plastic region $\mathscr{I}_p = [0, L_p]$ and the elastic region $\mathscr{I}_e = [L_p, L]$, where L_p is the size of the plastic region, still to be determined. Imposing conditions $\dot{\varepsilon}'(0) = 0$ and $\dot{\varepsilon}(L_p) = 0$ on the general solution (38) and conditions $\dot{\varepsilon}(L_p) = 0$ and $\dot{\varepsilon}'(L) = 0$ on the general solution (37), we obtain a family of particular solutions described by

$$\dot{\varepsilon}(x) = \begin{cases} \dfrac{\dot{t}}{E_t}\left(1 - \dfrac{\cos\frac{\alpha x}{l}}{\cos\frac{\alpha L_p}{l}}\right) & \text{for } x \in I_p \equiv [0, L_p] \\ \dfrac{\dot{t}}{E}\left(1 - \dfrac{\cosh\frac{L-x}{l}}{\cosh\frac{L-L_p}{l}}\right) & \text{for } x \in I_e \equiv [L_p, L] \end{cases} \tag{40}$$

This family is parameterized by the unknown length L_p, which can be determined from the condition of continuity of $\dot{\varepsilon}'$ at $x = L_p$. The resulting equation has the form

$$\tan\frac{\alpha L_p}{l} + \alpha \tanh\frac{L - L_p}{l} = 0 \tag{41}$$

Since an equation of this type plays an important role for this and several other models (to be discussed later), it is useful to introduce a function $\lambda_p(\alpha, \lambda)$, defined implicitly as the smallest positive solution of the transcendental equation

$$\tan \alpha\lambda_p + \alpha \tanh(\lambda - \lambda_p) = 0 \tag{42}$$

The graph of λ_p as a function of α is plotted in Fig. 3 for several values of λ. It is easy to show that, for any fixed α, λ_p is a decreasing function of λ, for $\lambda \to \infty$ approaching its minimum possible value,

$$\lim_{\lambda\to\infty} \lambda_p(\alpha, \lambda) \equiv \lambda_{p,\infty}(\alpha) = \frac{\pi - \arctan\alpha}{\alpha} \tag{43}$$

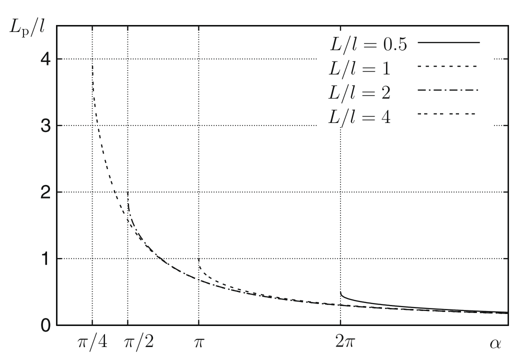

Fig. 3. Normalized plastic zone size $\lambda_p = L_p/l$ as a function of parameter α, for different values of the normalized bar length $\lambda = L/l$.

In terms of function λ_p, the size of the plastic region solved from (41) can be written as $L_p = l\lambda_p(\alpha, L/l)$. For very long bars, L_p tends to its limit value $L_{p,\infty} = l\lambda_{p,\infty}(\alpha) = (\pi - \arctan\alpha)l/\alpha$. This minimum size of the plastic region is proportional to the characteristic length l, but it also depends on the ratio H/E. For very steep softening, when H is close to its minimum admissible value, $-E$, parameter α is very large and the plastic region can become arbitrarily small. For mild softening, when the absolute value of H is small, parameter α is close to zero and the plastic region can become arbitrarily large. For a finite bar, the size of the plastic region is increasing with decreasing bar length, and if the theoretical size of the plastic region solved from (41) is larger than the actual bar length, localization is impossible. The critical bar length below which bifurcation cannot take place,

$$L_{min} = \frac{\pi l}{\alpha} \tag{44}$$

can be obtained from (41) by setting $L_p = L = L_{min}$ and looking for the smallest positive solution.

The foregoing derivations were based on the assumption that the plastic region is adjacent to one of the bar ends. Another possible localization pattern is a plastic region inside the bar, separated from the bar ends by two elastic intervals. It turns out that solutions of this type are symmetric with respect to the middle section and that each of them can be constructed by combining two solutions describing localization near the boundary for a bar of length $L/2$; see Fig. 4. Such solutions exist only if the bar length exceeds $2L_{min} = 2\pi l/\alpha$. The size of the plastic region is then given by $L_p = 2l\lambda_p(\alpha, L/2l)$.

For bar lengths between L_{min} and $2L_{min}$, the plastic strain localizes into a region adjacent to one of the bar ends (which end will attract the plastic region is decided by random imperfections). For bar lengths that exceed $2L_{min}$, there are multiple localization patterns and their number increases with increasing bar length but always remains finite. According to the criterion proposed by Bažant (1988) and based on thermodynamic arguments, the actual solution (stable under displacement control) is that which leads to the steepest slope of the resulting load–displacement diagram or, equivalently, to the algebraically largest tangent compliance. In the present context, the load is the traction t and the displacement is the total bar elongation $\bar{u} = u(L)$. Integrating the strain rate along the bar, we obtain the elongation rate

$$\dot{u}^s(L) = \dot{t}\left\{\frac{L}{E} + \frac{2l}{\alpha H}\left[\alpha\lambda_p\left(\alpha, \frac{L}{2l}\right) - \tan\left(\alpha\lambda_p\left(\alpha, \frac{L}{2l}\right)\right)\right]\right\} \tag{45}$$

for the symmetric solution and

$$\dot{u}^b(L) = \dot{t}\left\{\frac{L}{E} + \frac{l}{\alpha H}\left[\alpha\lambda_p\left(\alpha, \frac{L}{l}\right) - \tan\left(\alpha\lambda_p\left(\alpha, \frac{L}{l}\right)\right)\right]\right\} \tag{46}$$

for the solution localized near the boundary. Inspection of (45) and (46) reveals that $\dot{u}^s(L)$ can be obtained from the expression for $\dot{u}^b(L)$ simply by doubling the characteristic length l. So, to decide which solution will actually occur, we need to study the dependence of the compliance on the characteristic length, with all other parameters kept fixed. The structural compliance $C = \dot{u}(L)/\dot{t}$ can be expressed as a sum of the elastic compliance, $C_e = L/E$, and the

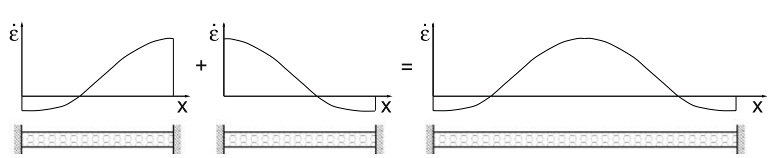

Fig. 4. Symmetric solution obtained by combining two solutions localized at the boundary.

plastic compliance, C_p. The relative plastic compliance $\Gamma_p = C_p/C_e$ depends only on the parameters α and $\lambda = L/l$:

$$\Gamma_p(\alpha,\lambda) = \frac{1+\alpha^2}{\lambda\alpha^3}\left[\tan(\alpha\lambda_p(\alpha,\lambda)) - \alpha\lambda_p(\alpha,\lambda)\right] \quad (47)$$

In Fig. 5, Γ_p is plotted as a function of α for several fixed values of λ. The graph shows that the plastic compliance is an increasing function of λ, which can be proven rigorously, based on the fact that λ_p is a decreasing function of λ and that $\tan(\alpha\lambda_p) - \alpha\lambda_p$ is a negative increasing function of λ_p in the range of interest, $\pi - \arctan\alpha < \alpha\lambda_p \leqslant \pi$. Consequently, for any $L > 2L_{min}$, the plastic compliance corresponding to the solution localized at the boundary is always larger than the plastic compliance corresponding to the symmetric solution, and the plastic strain tends to localize at the boundary. Of course, this conclusion refers to the properties of the specific model considered here and to the specific criterion advocated by Bažant (1988). For a real specimen, it is hard to reproduce the idealized conditions of a perfectly uniform cross section and material properties. So the question where the localized zone would evolve in a perfectly uniform bar is rather academic. In reality, the localization process would be strongly affected by random imperfections, and also by the exact nature of the connection between the specimen boundary and the loading device.

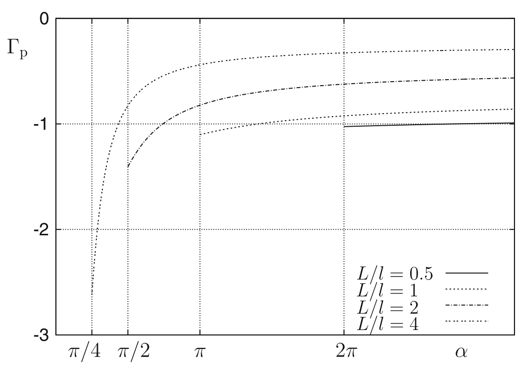

Fig. 5. Dimensionless plastic compliance $\Gamma_p = C_p/C_e$ as a function of parameter α for several fixed values of dimensionless bar length $\lambda = L/l$.

5.3. Evolution of plastic region

The foregoing analysis referred to the rates at the onset of localization, i.e., at the first bifurcation from a uniform state. Since the solution depends only on the dimensionless ratios L/l and H/E, it remains valid as long as these ratios do not change. Clearly, L and E are constant parameters, and the internal length l is normally considered as a constant as well. The softening modulus H is the derivative of the yield stress with respect to the cumulative plastic strain, and it is constant if the softening law is linear. In this case, the solution remains valid if the rates are replaced by finite increments with respect to the state at bifurcation. The size of the plastic zone and the shape of the plastic strain profile therefore remain constant, and the plastic strains at all points of the plastic region grow proportionally to the applied increment of bar elongation (equal to the displacement of the right end if the left end is fixed); see Fig. 6a.

For nonlinear softening laws, the solution of the rate problem changes during the localization process and it cannot be constructed analytically. However, if the solution was artificially kept uniform until a certain state in the post-peak range and only then the bifurcation was allowed, the plastic region would localize into an interval whose length can be obtained from the foregoing formulae with $H = d\sigma_Y/d\kappa$ being the current (tangent) softening modulus. This indicates that if the magnitude of the softening

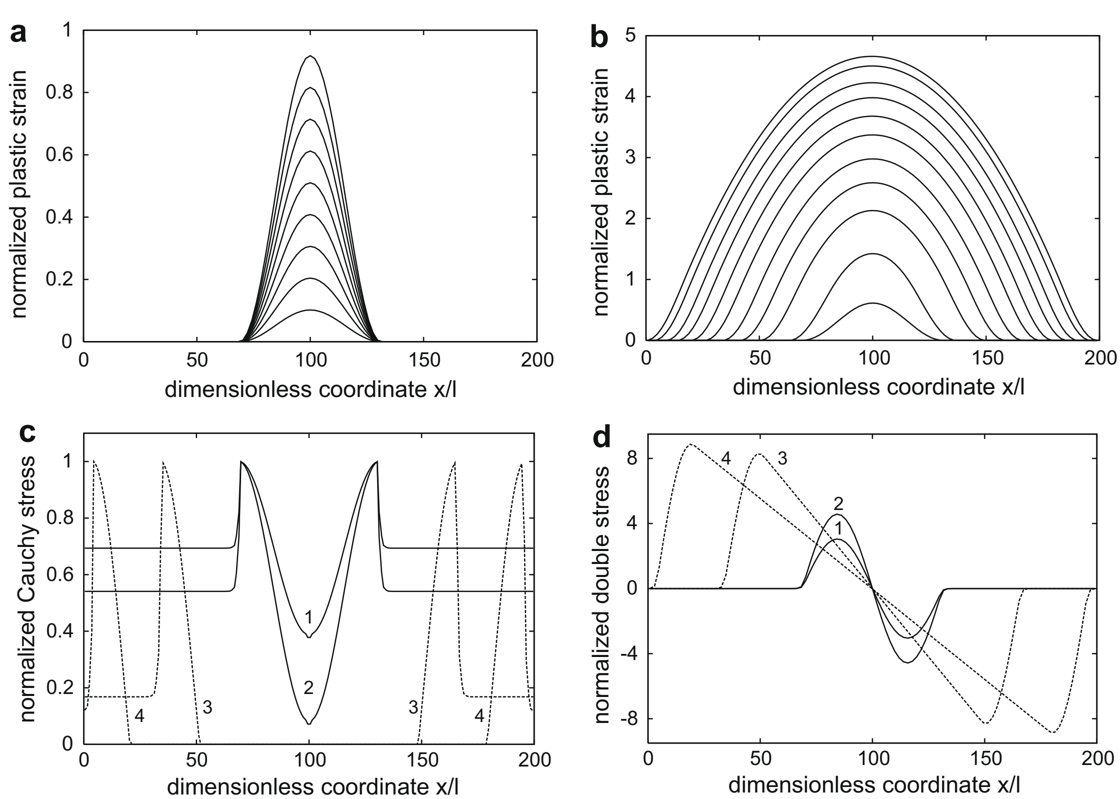

Fig. 6. Chambon's model with linear softening: evolution of (a) plastic strain profiles at early stages of softening, (b) plastic strain profiles at late stages of softening, (c) Cauchy stress profiles, (d) double stress profiles.

modulus decreases during the softening process, the plastic region is expected to expand. Some limited expansion of the plastic region would be acceptable, but the problem is that as the current softening modulus tends to zero, the size of the plastic region grows without any bounds (Fig. 8a). The bar then cannot fail by yielding of its limited segment, and all sections must sooner or later start yielding. This is a spurious, nonphysical effect, which is typically accompanied by stress locking, manifested by a nonvanishing residual resistance of the structure even at very large applied elongations. A model exhibiting this kind of behavior cannot correctly predict complete failure.

Similar pathological effects can be expected for the full form of the linear softening law (13) with a horizontal branch at zero residual yield stress. The analytical solution with a constant size of the plastic region remains valid only if the softening modulus is constant across the entire plastic region, which is true as long as the maximum plastic strain remains below the critical level κ_c. The sudden jump of the tangent softening modulus from a constant negative value to zero at the critical plastic strain level can be expected to produce qualitatively similar locking effects as the gradual decrease of the tangent softening modulus characteristic of nonlinear softening laws.

Numerical calculations confirm the foregoing simplified and partially intuitive analysis. In the case of a linear softening law with cut-off at zero stress, the plastic zone is indeed initially of constant size; see the evolution of plastic strain profiles in Fig. 6a and the profiles of Cauchy stress and double stress marked as 1 and 2 in Fig. 6c and d. The Cauchy stress distribution (Fig. 6c) is nonuniform, with two maxima at the elastoplastic boundaries, where σ remains at the initial yield stress level σ_0, and with a minimum at the center of the plastic zone. In the elastic zones, the stress decreases with increasing distance from the elastoplastic boundary and asymptotically approaches a constant level. The size of the transition layers is in the order of the characteristic length l. The reason why these layers in Fig. 6c appear so small is that, for the selected model parameters, the initial plastic zone size L_p is about 60 times larger than l. The example has been computed with $H/E = -0.01$ and $L/l = 200$, which gives $\alpha = \sqrt{-H/(E+H)} = 0.1005$ and $L_p = 2l\lambda_p(\alpha, L/2l) = 2l\lambda_p(0.1005, 100) = 60.52\,l$.

When the maximum plastic strain at the center of the plastic zone attains its critical value $\kappa_c = -\sigma_0/H$, the yield stress at that point vanishes. Upon further loading, an initially small interval with zero yield stress forms at the center of the plastic zone, and the plastic zone starts expanding; see Fig. 6b. Typical profiles of Cauchy stress and double stress corresponding to this loading stage are shown in Fig. 6c and d and marked by labels 3 and 4. The maximum Cauchy stress at the moving elastoplastic boundaries is still equal to the initial yield stress. The reaction at the bar end (equal to the total stress $\sigma - \chi'$ multiplied by the sectional area) decreases only slowly with increasing bar elongation. The load–displacement curve, shown in Fig. 7a, crosses the dashed curve that would correspond to the (unstable) uniform solution. The total work needed to completely break the bar (given by the area under the load–displacement diagram) is larger than in the absence of localization. This is clearly a pathological locking effect.

For a nonlinear softening law with a decreasing magnitude of the tangent softening modulus, the expansion of the plastic zone starts immediately after localization. This is documented for the exponential softening law (14) in Fig. 8. At later stages of the softening process, high residual reactions are obtained even for large bar elongations, and the load–displacement diagram eventually crosses that corresponding to the unstable uniform solution; see Fig. 7b.

Excessive residual forces transmitted at late stages of softening are certainly inadmissible for quasi-brittle materials in tension-

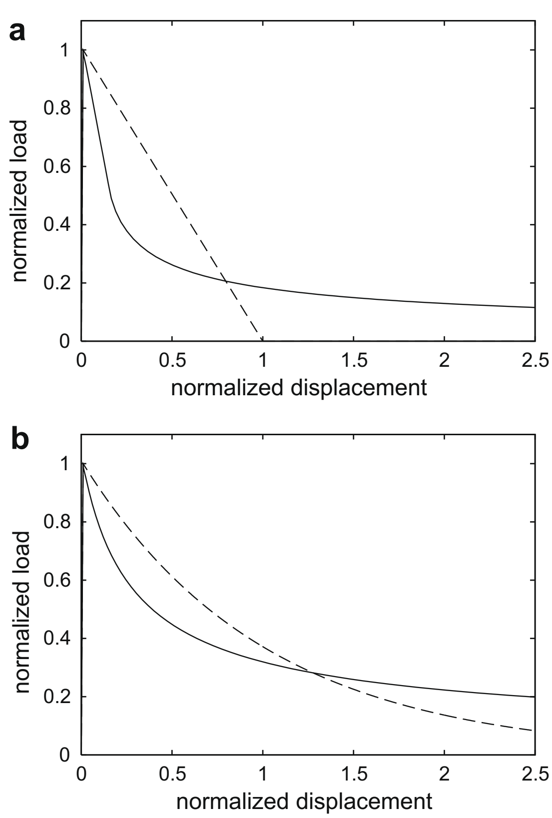

Fig. 7. Load–displacement diagrams for Chambon's model with (a) linear softening, (b) exponential softening; the solid curves correspond to the actual (localized) solutions while the dashed curves correspond to the unstable uniform solutions.

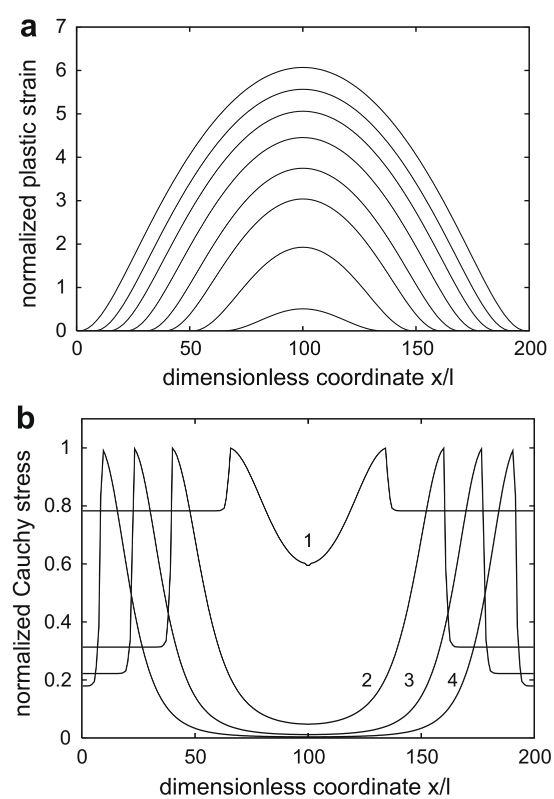

Fig. 8. Chambon's model with exponential softening: evolution of (a) plastic strain profiles, (b) Cauchy stress profiles.

dominated failure modes. It may be argued that for other types of failure (e.g., of ductile materials, or for frictional cohesive materials under shear and compression) the residual resistance is desirable and reflects their real physical behavior. In the authors' opinion, if a material exhibits some residual strength, this should be incorporated into the model at the level of the local stress–strain law and not obtained as a by-product of the regularization technique. Otherwise it is hard to control such phenomena and calibrate the model parameters.

6. Strain-gradient plasticity model of Fleck and Hutchinson (1997)

6.1. Model equations

Fleck et al. (1994) proposed an extension of the couple-stress theory to the nonlinear range and interpreted it as a deformation theory of plasticity. They defined the free energy as a power function of the combined strain, which is a scalar measure of the strain and the local curvatures (gradients of the rotational part of the displacement gradient). Once this measure was defined, the stress–strain relations were easily obtained as the state laws. No internal variables were used and the dissipation was not considered.

The deformation theory of plasticity cannot realistically describe unloading, and so it would not be appropriate for the description of localization phenomena. A more realistic model is provided by the flow theory of strain-gradient plasticity, which was first proposed by Fleck and Hutchinson (1993) in the context of the couple-stress theory, and later reformulated by Fleck and Hutchinson (1997) in the context of the general strain-gradient theory, which takes into account not only the microcurvatures (rotation gradients) but also the stretch gradients. This is the model to be analyzed next.

In contrast to Chambon et al. (1998), the yield condition is assumed to depend not only on the standard (Cauchy) stress but also on the double stress. The yield function is defined as

$$f(\sigma, \chi, \sigma_Y) = \Sigma(\sigma, \chi) - \sigma_Y \tag{48}$$

where

$$\Sigma(\sigma, \chi) = \sqrt{\sigma^2 + \frac{\chi^2}{l_p^2}} \tag{49}$$

is the overall effective stress. Since σ and χ have different physical dimensions, the definition of Σ must contain a scaling parameter l_p related to the internal structure of the material. This plastic characteristic length is in general different from the characteristic length used by the elastic strain-gradient theory, which will be from now on denoted as l_e. The elastic constitutive laws read

$$\sigma = E(\varepsilon - \varepsilon_p) \tag{50}$$

$$\chi = El_e^2(\eta - \eta_p) \tag{51}$$

where ε_p is the usual plastic strain and η_p is the plastic part of the strain gradient, considered as an independent internal variable. Note that $\eta = \varepsilon'$ is the strain gradient, but η_p has no direct relation to the derivative of the plastic strain, ε_p'.

Within an associated framework, the evolution laws for plastic strain ε_p, plastic strain gradient η_p and softening variable κ are

$$\dot{\varepsilon}_p = \dot{\lambda}\frac{\partial f}{\partial \sigma} = \dot{\lambda}\frac{\sigma}{\Sigma} \tag{52}$$

$$\dot{\eta}_p = \dot{\lambda}\frac{\partial f}{\partial \chi} = \dot{\lambda}\frac{\chi}{l_p^2 \Sigma} \tag{53}$$

$$\dot{\kappa} = \dot{\lambda}\frac{\partial f}{\partial(-\sigma_Y)} = \dot{\lambda} \tag{54}$$

Under plastic loading, the rate of the plastic multiplier $\dot{\lambda}$ can be determined from the consistency condition $\dot{f} = 0$. After substitution of (50)–(54) and $\dot{\sigma}_Y = H\dot{\kappa}$ into the consistency condition and algebraic manipulations, the resulting expression reads

$$\dot{\lambda} = E\Sigma\frac{\sigma\dot{\varepsilon} + \chi\dot{\eta}l_e^2/l_p^2}{H\Sigma^2 + E(\sigma^2 + \chi^2 l_e^2/l_p^4)} \tag{55}$$

where $H = \mathrm{d}\sigma_Y/\mathrm{d}\kappa$ is the tangent plastic modulus.

6.2. Localization analysis

At the first bifurcation from a uniform state, the Cauchy stress σ is constant along the bar and the double stress χ vanishes (this is the unique elastic solution satisfying the appropriate boundary conditions). Consequently, (55) reduces to $\dot{\lambda} = E\dot{\varepsilon}/(E + H)$, the evolution laws give $\dot{\varepsilon}_p = \dot{\lambda}$ and $\dot{\eta}_p = 0$, and the stress rates are $\dot{\sigma} = E_t\dot{\varepsilon}$ and $\dot{\chi} = El_e^2\dot{\eta}$ where $E_t = EH/(E + H)$ is the tangent modulus of standard elastoplasticity. The problem is completely identical with the bifurcation problem for the model of Chambon et al., and its solution is therefore exactly the same as discussed in Section 5.2. However, this is true only at the onset of localization. As a nonuniform strain profile develops, double stresses build up and the rate of the plastic strain gradient becomes nonzero. The subsequent evolution is thus different from that predicted by Chambon's model, except for the limit case when $l_p \to \infty$. In this special case, the Fleck–Hutchinson model reduces to the model of Chambon et al. (1998), with $l = l_e$.

6.3. Evolution of plastic region

As follows from the foregoing analysis, the initial size of the plastic region is controlled by the ratio H/E and by the elastic characteristic length l_e. The subsequent evolution is not amenable to an analytical solution but it can be studied numerically. Fig. 9 shows the evolution of the strain profile for two different ratios l_p/l_e equal to 2 and 5 (with $H/E = -0.01$ and $L/l_e = 200$). The initial size of the plastic zone $L_p = 60.52l_e$ is in both cases the same and the strain profiles during the initial stage of localization are very similar (Fig. 9 top). At late stages of the softening process, the plastic zone is wider for larger l_p (Fig. 9 bottom). In contrast to Chambon's model, the active part of the plastic zone shrinks during the softening process, and the load–displacement diagrams become steeper; see Fig. 10. At late stages of softening, the shrinking is accelerated and an instability develops. Detailed analysis of this intriguing phenomenon is out of scope of the present paper and will be addressed separately.

7. Mechanism-based strain-gradient plasticity

7.1. Model equations

Fleck–Hutchinson strain-gradient theories of plasticity are physically motivated by the concept of statistically stored and geometrically necessary dislocations, but their essence remains phenomenological. A micromechanically based strain-gradient law for the flow strength of materials, derived from the Taylor hardening model by Nix and Gao (1998), serves as the basis of a group of theories called the mechanism-based strain-gradient (MSG) plasticity.

The original MSG model (Gao et al., 1999; Huang et al., 2000) was a nonlinear extension of the Toupin–Mindlin theory, interpreted as a deformation theory of plasticity. The Cauchy stress and double stress were considered as functions of the strain and strain gradient, without any internal variables that could describe irreversible processes. Localization analysis of the shear layer problem using the deformation theory of MSG plasticity was presented by Shi et al. (2000).

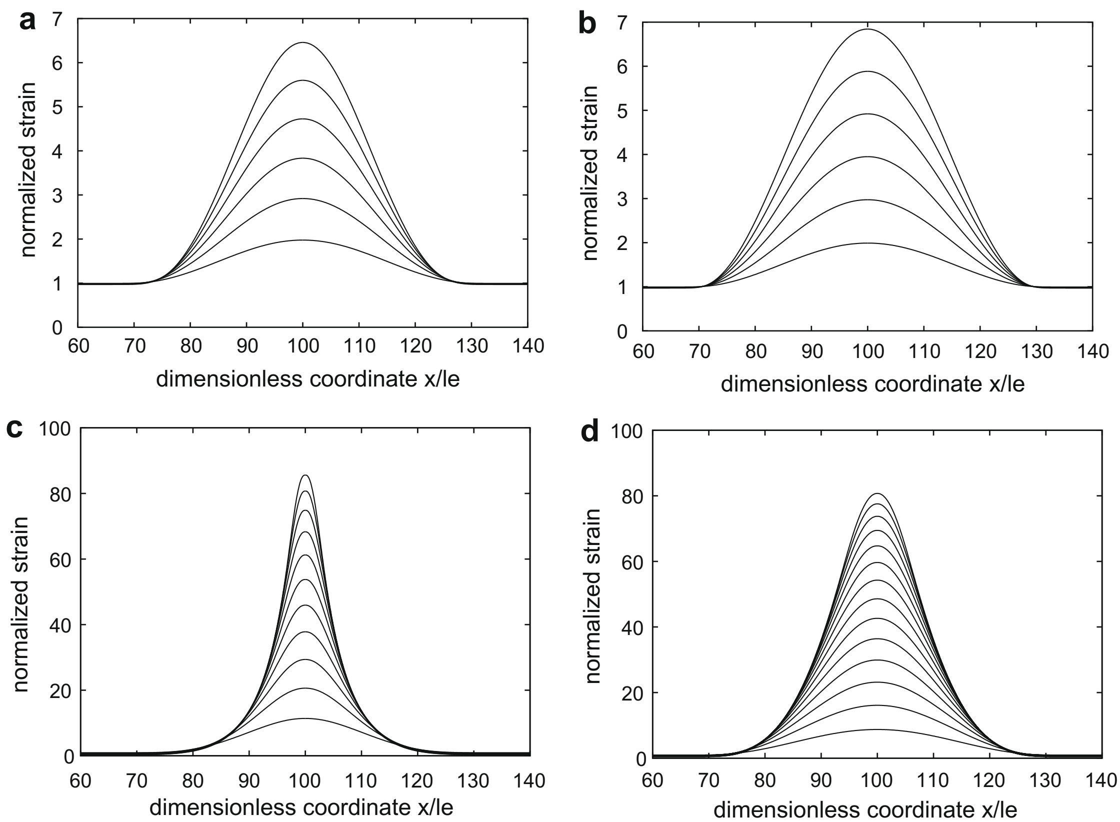

Fig. 9. Fleck–Hutchinson model: evolution of the strain profile for (a,c) $l_p = 2l_e$, (b,d) $l_p = 5l_e$.

Since deformation theories are usually inappropriate for loading programs that deviate from proportionality, we focus on the flow theory of MSG plasticity (Qiu et al., 2003). The hardening law is considered in the form

$$\sigma_Y = \sqrt{h^2(\kappa) + 3\alpha^2 bE^2 \mid \eta \mid} \tag{56}$$

where h is a function describing the hardening law $\sigma_Y = h(\kappa)$ in the absence of strain gradients, α is the Taylor factor with values between 0.1 and 0.5, and b is the norm of the Burgers vector (spacing between neighboring atoms in the crystal lattice). This and all subsequent equations refer to the shear layer problem (Fig. 1b), but for easier comparison with other models we keep using the notation that corresponds to the tensile test. In the present context, E should be interpreted as the shear modulus of elasticity, κ as the cumulative value of plastic engineering shear strain, σ as the shear stress, σ_Y as the yield stress in shear, ε as the engineering shear strain, and η as its derivative in the direction perpendicular to the boundaries of the layer.

The dependence of the yield stress on the magnitude of the strain gradient η (in multiple dimensions replaced by a suitable scalar measure of the strain-gradient tensor) is supposed to reflect the influence of geometrically necessary dislocations. The yield function preserves its classical form (12), i.e., it takes into account only the Cauchy stress σ but not the double stress χ. The consistency condition $\dot{\sigma} = \dot{\sigma}_Y$ combined in the usual way with the elastic stress–strain law (8) leads to the following expressions for the rates of cumulative plastic strain and of Cauchy stress:

$$\dot{\kappa} = \frac{\sigma_Y \dot{\varepsilon} - 1.5\alpha^2 bEn\dot{\eta}}{\sigma_Y + hH/E} \tag{57}$$

$$\dot{\sigma} = \frac{hH\dot{\varepsilon} + 1.5\alpha^2 bE^2 n\dot{\eta}}{\sigma_Y + hH/E} \tag{58}$$

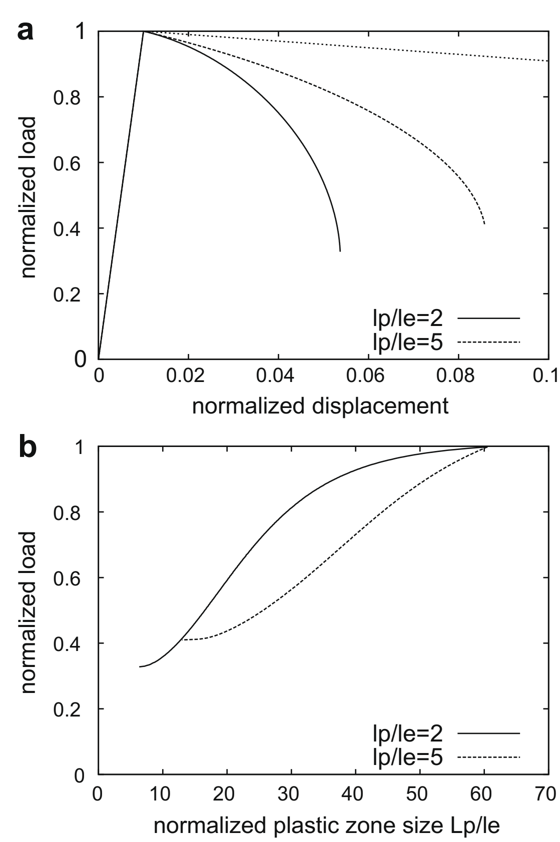

Fig. 10. Fleck–Hutchinson model: (a) normalized load–displacement diagrams (the dashed curve corresponds to the unstable uniform solution), (b) evolution of the plastic zone size.

Here, $H = \mathrm{d}h/\mathrm{d}\kappa$ is the (standard) plastic modulus, and n is a signed unity defined by $n = \mathrm{sgn}(\eta)$ if $\eta \neq 0$, $n = \mathrm{sgn}(\dot{\eta})$ if $\eta = 0$.

In MSG plasticity, the rate of double stress is derived by first-order averaging over the mesocell of size l_ε. After simplifications that correspond to the one-dimensional case, the complicated tensorial expression given in Qiu et al. (2003) reduces to

$$\dot{\chi} = \frac{El_\varepsilon^2}{12}\frac{hH\dot{\eta}}{E\sigma_Y + hH} + \frac{\chi(e\sigma_Y\dot{\kappa} - \dot{\sigma})}{\sigma_Y + hH/E} \tag{59}$$

where

$$e = \frac{H}{h} + \frac{1}{H}\frac{\mathrm{d}H}{\mathrm{d}\kappa} \tag{60}$$

Eqs. (57)–(59) apply to plastic loading. Elastic unloading is characterized by $\dot{\kappa} = 0$, $\dot{\sigma} = E\dot{\varepsilon}$ and $\dot{\chi} = E(l_\varepsilon^2/12)\dot{\eta}$.

7.2. Localization analysis

At bifurcation from a uniform state, the double stress χ vanishes and therefore the second term in (59) drops out. The strain gradient η also vanishes, and the yield stress is $\sigma_Y = h(\kappa)$. Taking all this into account, and substituting (58) and (59) and $\dot{\eta} = \dot{\varepsilon}'$ into the rate form of the equilibrium Eq. (34), we obtain the second-order differential equation

$$\dot{\varepsilon} - \frac{\beta l_\varepsilon}{\sqrt{12}} \mid \dot{\varepsilon}' \mid - \frac{l_\varepsilon^2}{12}\dot{\varepsilon}'' = \frac{\dot{t}}{E_t} \tag{61}$$

in which $E_t = EH/(E + H)$ is the elastoplastic modulus and

$$\beta = -\frac{3\sqrt{3}\alpha^2 bE^2}{H\sigma_Y l_\varepsilon} \tag{62}$$

is a dimensionless parameter. The negative sign in the definition (62) has been selected such that β is positive for softening ($H < 0$), which is the case of our interest. However, Eq. (61) is valid even for hardening, characterized by a negative value of β. In the special case of perfect plasticity ($H = 0$), the elastoplastic modulus E_t vanishes and Eq. (61) must be written in a different way.

Independently of the sign of plastic modulus H, the characteristic equation of (61) has two real roots, one positive and one negative:

$$\lambda_1 = \frac{-\sqrt{3}n\beta + \sqrt{3\beta^2 + 12}}{l_\varepsilon}, \quad \lambda_2 = \frac{-\sqrt{3}n\beta - \sqrt{3\beta^2 + 12}}{l_\varepsilon} \tag{63}$$

Therefore, the general solution of (61) has the form

$$\dot{\varepsilon}(x) = \frac{\dot{t}}{E_t}[1 + C_1\exp(\lambda_1 x) + C_2\exp(\lambda_2 x)] \tag{64}$$

on each interval with a constant value of $n = \mathrm{sgn}(\dot{\varepsilon}')$, i.e., on each interval where the rate of the strain gradient does not change its sign.

The foregoing derivation applies to the plastic region. In the elastic region, the constitutive equations of linear strain-gradient elasticity $\dot{\sigma} = E\dot{\varepsilon}$ and $\dot{\chi} = E(l_\varepsilon^2/12)\dot{\eta}$ substituted into (34) lead to

$$\dot{\varepsilon} - \frac{l_\varepsilon^2}{12}\dot{\varepsilon}'' = \frac{\dot{t}}{E} \tag{65}$$

and the general solution is

$$\dot{\varepsilon}(x) = \frac{\dot{t}}{E}[1 + C_3\exp(\lambda_3 x) + C_4\exp(-\lambda_3 x)] \tag{66}$$

where $\lambda_3 = -2\sqrt{3}/l_\varepsilon$.

Let us now look for a solution with plastic strain increments localized in an interval $\mathscr{I}_p = [0, L_p]$, adjacent to the boundary of the shear layer at $x = 0$. To simplify the problem, we assume that the layer is sufficiently thick compared to the thickness of the localized zone, so that we can perform the analysis on a semi-infinite interval $[0, \infty)$ and replace the boundary condition at the elastic boundary by the requirement that the strain must remain bounded as $x \to \infty$, which implies that the integration constant C_4 must vanish. On the plastic boundary (at $x = 0$) we set $\chi = 0$, and on the elastoplastic interface (at $x = L_p$) we enforce continuity of strain and double stress. Note that continuity of strain gradient is not required and that continuity of total stress is satisfied automatically. The boundary conditions specified above correspond at the same time to a localized shear band of thickness $2L_p$ with symmetric strain distribution, placed in an infinite (or sufficiently large) body such that $x = 0$ corresponds to the center of the band.

Detailed analysis of this problem is relatively tedious and therefore is relegated to the Appendix. It reveals that, for the present model, the bifurcation problem does not possess any solution with plastic yielding localized into a layer of finite thickness. Even for softening, localization cannot take place and the solution remains uniform. So the enhancement of the softening law by the strain-gradient term cannot be used as a localization limiter in the usual sense, because it would completely exclude localization, even into a finite interval.

8. Conclusions

Three specific enhancements of the standard flow theory of plasticity have been considered here, all of them incorporating the influence of the gradient of total strain:

- The model initially proposed by Chambon et al. (1998), which combines the standard elastoplastic stress–strain law with a linear elastic higher-order law linking the strain gradient to the double stress.
- The 1997 version of the Fleck–Hutchinson (F & H) model, for which the yield function depends on the Cauchy stress and on the double stress, and the strain and strain gradient both have a plastic part.
- The mechanism-based strain-gradient (MSG) plasticity model (Qiu et al., 2003), for which the strain gradient affects the hardening law.

Detailed analysis of the onset of localization, considered as a bifurcation from a uniform state and investigated in the one-dimensional setting, has shown that the first two models act as proper localization limiters, preventing localization of plastic yielding into a set of zero measure and at the same time allowing localization into a process zone of a finite thickness. The MSG model tends to suppress localization completely and thus should not be used in problems with expected formation of localized plastic bands.

The models of Chambon et al. (1998) and of Fleck and Hutchinson (1997) lead to the same incremental solution at the onset of localization but differ by the subsequent evolution. For the former model, locking phenomena arise at later stages of softening and the plastic zone tends to expand over the entire specimen. For the latter model, the evolution of the plastic zone is controlled by an extra internal length parameter, which can be chosen such that the active part of the plastic zone shrinks and no locking effects appear. However, another problem arises. At late stages of softening the shrinking accelerates and an instability develops.

Other enhancements of the plasticity theory, dealing with gradients of internal variables, are studied in the second part of this paper. They include among others the 2001 version of the Fleck–Hutchinson model (Fleck and Hutchinson, 2001) and the model by Zervos et al. (2001), which combines strain gradients with gradients of internal variables.

For a quick overview, the main ingredients of the strain-gradient plasticity models that have been investigated are summarized in Table 1. The Zervos et al. (2001) gradient elastoplasticity (GEP)

Table 1
Overview of strain-gradient plasticity models.

Model	Sec.	Stress–strain laws		Yield function
Chambon	5	$\sigma = E(\varepsilon - \varepsilon_p)$	$\chi = El^2\eta$	$f(\sigma,\kappa) = \sigma - \sigma_Y(\kappa)$
F&H	6	$\sigma = E(\varepsilon - \varepsilon_p)$	$\chi = El^2(\eta - \eta_p)$	$f(\sigma,\chi,\kappa) = \Sigma(\sigma,\chi) - \sigma_Y(\kappa)$
MSG	7	$\sigma = E(\varepsilon - \varepsilon_p)$	(59)	$f(\sigma,\kappa,\eta) = \sigma - \sigma_Y(\kappa,\eta)$
GEP	II.4	$\sigma = E(\varepsilon - \varepsilon_p)$	$\chi = El^2(\eta - \varepsilon_p')$	$f(\sigma,\chi,\kappa) = \mid \sigma - \chi' \mid - \sigma_Y(\kappa,\kappa'')$

model is also included in the table for comparison, even though it will be described only later, in Section 4 of Part II.

Acknowledgements

This work has been financially supported by the Ministry of Education of the Czech Republic under the Research Plan MSM 6840770003 and by the Swiss National Science Foundation under project 2100-057062.99/1.

Appendix

This appendix contains the detailed proof that the MSG model does not admit a localized solution of the one-dimensional problem. The governing equations are outlined in Section 7.2. The general form of the solution is given by (64) in the plastic region and by (66) in the elastic region. We look for a solution with plastic strain increments localized in an interval $\mathscr{I}_p = [0, L_p]$, adjacent to the boundary of the shear layer at $x = 0$, and we further assume that the layer is sufficiently thick compared to the thickness of the localized zone, so that the analysis can be performed on a semi-infinite interval $[0, \infty)$. The boundary condition at the elastic boundary is then replaced by the requirement that the strain must remain bounded as $x \to \infty$, which implies that the integration constant C_4 in (66) must vanish.

Boundary condition $\chi(0) = 0$ and continuity conditions for $\varepsilon(x)$ and $\chi(x)$ at $x = L_p$ lead to a set of three linear equations, from which it is possible to determine the three remaining integration constants. To simplify the subsequent derivations, we introduce symbols $\mu = H/E$, $e_i = \exp(\lambda_i L_p)$, $i = 1, 2, 3$, and

$$\Lambda_{12} = \lambda_1\lambda_2(e_1 - e_2) \tag{67}$$

$$\Lambda_3 = \lambda_3(\lambda_1 e_2 - \lambda_2 e_1) \tag{68}$$

$$\Lambda = \mu\Lambda_{12} + (1+\mu)\Lambda_3 \tag{69}$$

The integration constants can then be expressed in the simple form

$$C_1 = \frac{\lambda_2\lambda_3}{\Lambda} \tag{70}$$

$$C_2 = -\frac{\lambda_1\lambda_3}{\Lambda} \tag{71}$$

$$C_3 = \frac{\Lambda_{12}}{e_3\Lambda} \tag{72}$$

Note that the size of the plastic region, L_p, is still undetermined. For an arbitrary $L_p > 0$, we can construct a formal solution that satisfies the equilibrium equation and the boundary and continuity conditions. However, the solution is admissible only if it also satisfies the loading–unloading conditions $\dot\kappa(x) \geqslant 0$ for $0 \leqslant x \leqslant L_p$ and $\dot\sigma(x) - \dot\sigma_Y(x) \leqslant 0$ for $L_p \leqslant x$.

Since

$$\dot\sigma(x) - \dot\sigma_Y(x) = \dot t[1 + (1 - n\beta\mu)C_3\exp(\lambda_3 x)] \tag{73}$$

and λ_3 is negative, the condition $\dot\sigma(x) - \dot\sigma_Y(x) \leqslant 0$ written for $x \to \infty$ implies that the traction rate $\dot t$ must not be positive. Provided that $\dot t < 0$, condition $\dot\sigma(x) - \dot\sigma_Y(x) \leqslant 0$ is satisfied for all $x \geqslant L_p$ if it is satisfied at $x = L_p$, which is the case if

$$1 + (1 - n\beta\mu)C_3 e_3 \geqslant 0 \tag{74}$$

This is equivalent to

$$\frac{(1+\mu)(\Lambda_{12}+\Lambda_3) - n\beta\mu\Lambda_{12}}{\Lambda} \geqslant 0 \tag{75}$$

Nonnegative value of the plastic strain rate

$$\dot\kappa(x) = \frac{\dot t}{H}\left[1 + \left(1 - \frac{\beta n\mu\lambda_1}{\lambda_3}\right)C_1\exp(\lambda_1 x) + \left(1 - \frac{\beta n\mu\lambda_2}{\lambda_3}\right)C_2\exp(\lambda_2 x)\right] \tag{76}$$

at $x = L_p$ is guaranteed if

$$\frac{(1-n\beta)\Lambda_{12} + \Lambda_3}{\Lambda} \leqslant 0 \tag{77}$$

The signed unity n is defined as the sign of the strain-gradient rate in the plastic region, and can be evaluated as

$$n = \operatorname{sgn}\dot\varepsilon'(L_p) = \operatorname{sgn}\left(\frac{\dot t}{E_t}(C_1e_1\lambda_1 + C_2e_2\lambda_2)\right) = \operatorname{sgn}\left(\frac{\dot t}{\Lambda E_t}\lambda_1\lambda_2\lambda_3(e_1 - e_2)\right) \tag{78}$$

Since $\dot t < 0$, $\lambda_1 > 0$, $\lambda_2 < 0$, $\lambda_3 < 0$, $e_1 = \exp(\lambda_1 L_p) > \exp(\lambda_2 L_p) = e_2$, and $\operatorname{sgn} E_t = \operatorname{sgn}\mu$ (here we tacitly assume that $1 + \mu > 0$, i.e., $E + H > 0$, which is the usual assumption that excludes snapback in the stress–strain diagram under uniform strain), we get

$$n = -\operatorname{sgn}\mu\ \operatorname{sgn}\Lambda \tag{79}$$

According to the definition of β, Eq. (62), we have $\operatorname{sgn}\beta = -\operatorname{sgn}\mu$, and so $\operatorname{sgn}(\beta n) = \operatorname{sgn}\Lambda$.

Now we are ready to discuss conditions (75) and (77). Recall that Λ_{12} and Λ_3 are negative. Λ must not be zero, otherwise the integration constants would tend to infinity.

(1) If Λ is negative, then βn is negative and the numerator in (77) is negative; consequently, this condition is violated.
(2) If Λ is positive, then βn is positive. Inequalities (75) and (77) rewritten for the numerators only can be combined to get

$$\mu \leqslant -\frac{\Lambda_{12} + \Lambda_3}{(1-n\beta)\Lambda_{12} + \Lambda_3} \tag{80}$$

However, since βn is now positive, the right hand side in (80) is smaller than -1. Values of $\mu = H/E$ smaller than -1 are not allowed because the material model would exhibit snapback in the stress–strain diagram under uniform strain.

In summary, with $\mu > -1$ it is impossible to satisfy conditions (75) and (77) simultaneously, and the localized solution is never admissible.

References

Aifantis, E.C., 1984. On the microstructural origin of certain inelastic models. Journal of Engineering Materials and Technology, ASME 106, 326–330.
Bažant, Z.P., 1988. Stable states and paths of structures with plasticity or damage. Journal of Engineering Mechanics, ASCE 114, 2013–2034.
Bažant, Z.P., 2002. Scaling of Structural Strength, Hermes-Penton, London.
Bažant, Z.P., Jirásek, M., 2002. Nonlocal integral formulations of plasticity and damage: survey of progress, Journal of Engineering Mechanics, ASCE 128, 1119–1149.

César de Sá, J., Areias, P., Zheng, C., 2006. Damage modelling in metal forming problems using an implicit non-local gradient model. Computer Methods in Applied Mechanics and Engineering 195 (48–49), 6646–6660 (Computational Metal Forming).

Chambon, R., Caillerie, D., Hassan, N.E., 1998. One-dimensional localisation studied with a second grade model. European Journal of Mechanics/A: Solids 17, 637–656.

Chambon, R., Caillerie, D., Matsuchima, T., 2001. Plastic continuum with microstructure, local second gradient theories for geomaterials: localization studies. International Journal of Solids and Structures 38, 8503–8527.

Dillon, O.W., Kratochvil, J., 1970. A strain gradient theory of plasticity. International Journal of Solids and Structures 6, 1513–1533.

Engelen, R.A.B., Geers, M.G.D., Baaijens, F.P.T., 2003. Nonlocal implicit gradient-enhanced elasto-plasticity for the modelling of softening behaviour. International Journal of Plasticity 19, 403–433.

Engelen, R.A.B., Fleck, N.A., Peerlings, R.H.J., Geers, M.G.D., 2006. An evaluation of higher-order plasticity theories for predicting size effects and localization. International Journal of Solids and Structures 43, 1857–1877.

Fleck, N.A., Hutchinson, J.W., 1993. A phenomenological theory for strain gradient effects in plasticity. Journal of the Mechanics and Physics of Solids 41, 1825–1857.

Fleck, N.A., Hutchinson, J.W., 1997. Strain gradient plasticity. In: Hutchinson, J.W., Wu, T.Y. (Eds.), Advances in Applied Mechanics, vol. 33. Academic Press, New York, pp. 295–361.

Fleck, N.A., Hutchinson, J.W., 2001. A reformulation of strain gradient plasticity. Journal of the Mechanics and Physics of Solids 49, 2245–2271.

Fleck, N.A., Muller, G.M., Ashby, M.F., Hutchinson, J.W., 1994. Strain gradient plasticity: theory and experiment. Acta Metallurgica et Materialia 42, 475–487.

Gao, H., Huang, Y., Nix, W.D., Hutchinson, J.W., 1999. Mechanism-based strain gradient plasticity – I. Theory. Journal of the Mechanics and Physics of Solids 47, 1239–1263.

Geers, M.G.D., 2004. Finite strain logarithmic hyperelasto-plasticity with softening: a strongly non-local implicit gradient framework. Computer Methods in Applied Mechanics and Engineering 193 (30–32), 3377–3401 (Computational Failure Mechanics).

Huang, Y., Gao, H., Nix, W.D., Hutchinson, J.W., 2000. Mechanism-based strain gradient plasticity – II. Analysis. Journal of the Mechanics and Physics of Solids 48, 99–128.

Jirásek, M., Bažant, Z.P., 2002. Inelastic Analysis of Structures. John Wiley and Sons, Chichester.

Jirásek, M., Rolshoven, S., 2003. Comparison of integral-type nonlocal plasticity models for strain-softening materials. International Journal of Engineering Science 41, 1553–1602.

Lakes, R.S., 1986. Experimental microelasticity of two porous solids. International Journal of Solids and Structures 22, 55–63.

Liebe, T., Steinmann, P., 2001. Theory and numerics of a thermodynamically consistent framework for geometrically linear gradient plasticity. International Journal for Numerical Methods in Engineering 51, 1437–1467.

Ma, Q., Clarke, D.R., 1995. Size dependent hardness in silver single crystals. Journal of Materials Research 10, 853–863.

Mindlin, R.D., 1964. Micro-structure in linear elasticity. Archive for Rational Mechanics and Analysis 16, 51–78.

Mindlin, R.D., 1965. Second gradient of strain and surface tension in linear elasticity. International Journal of Solids and Structures 1, 417–438.

Morrison, J.L.M., 1939. The yield of mild steel with particular reference to the effect of size of specimen. Proceedings of the Institute of Mechanical Engineers 142, 193–223.

Nix, W.D., 1989. Mechanical properties of thin films. Metallurgical Transactions 20A, 2217–2245.

Nix, W.D., Gao, H., 1998. Indentation size effects in crystalline materials: a law for strain gradient plasticity. Journal of the Mechanics and Physics of Solids 46, 411–425.

Noll, W., 1972. A new mathematical theory of simple materials. Archive for Rational Mechanics and Analysis 48, 1–50.

Peerlings, R.H.J., de Borst, R., Brekelmans, W.A.M., de Vree, J.H.P., 1996. Gradient-enhanced damage for quasi-brittle materials. International Journal for Numerical Methods in Engineering 39, 3391–3403.

Peerlings, R.H.J., de Borst, R., Brekelmans, W.A.M., Geers, M.G.D., 1998. Gradient enhanced damage modelling of concrete fracture. Mechanics of Cohesive-Frictional Materials 3, 323–342.

Peerlings, R.H.J., Massart, T.J., Geers, M.G.D., 2004. A thermodynamically motivated implicit gradient damage framework and its application to brick masonry cracking. Computer Methods in Applied Mechanics and Engineering 193 (30–32), 3403–3417 (Computational Failure Mechanics).

Poole, W.J., Ashby, M.F., Fleck, N.A., 1996. Microhardness of annealed and work-hardened copper polycrystals. Scripta Metallurgica et Materialia 34, 559–564.

Qiu, X., Huang, Y., Wei, Y., Gao, H., Hwang, K.C., 2003. The flow theory of mechanism-based strain gradient plasticity. Mechanics of Materials, 245–258.

Richards, C.W., 1958. Effects of size on the yielding of mild steel beams. Proceedings of the American Society for Testing and Materials 58, 955–970.

Rogula, D., 1982. Introduction to nonlocal theory of material media. In: Rogula, D. (Ed.), Nonlocal Theory of Material Media, CISM Courses and Lectures, no. 268. Springer Verlag, Wien and New York, pp. 125–222.

Shi, M.X., Huang, Y., Hwang, K.C., 2000. Plastic flow localization in mechanism-based strain gradient plasticity. International Journal of Mechanical Sciences 42, 2115–2131.

Stolken, J.S., Evans, A.G., 1998. A microbend test method for measuring the plasticity length scale. Acta Metallurgica et Materialia 46, 5109–5115.

Svedberg, T., 1996. A thermodynamically consistent theory of gradient-regularized plasticity coupled to damage. Licentiate thesis, Chalmers University of Technology.

Svedberg, T., Runesson, K., 1997. A thermodynamically consistent theory of gradient-regularized plasticity coupled to damage. International Journal of Plasticity 13, 669–696.

Toupin, R.A., 1962. Elastic materials with couple-stresses. Archive for Rational Mechanics and Analysis 11, 385–414.

Tovo, R., Livieri, P., 2008. An implicit gradient application to fatigue of complex structures. Engineering Fracture Mechanics 75 (7), 1804–1814 (Critical Distance Theories of Fracture).

Zervos, A., Papanastasiou, P., Vardoulakis, I., 2001. A finite element displacement formulation for gradient elastoplasticity. International Journal for Numerical Methods in Engineering 50, 1369–1388.