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Path-independent integral for the sharp V-notch in longitudinal shear problem

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ABSTRACT

By applying Noether's theorem to the elastic energy density in longitudinal shear problem, it is shown that its symmetry-transformations of material space can be expressed by the real and imaginary parts of an analytic function. This kind of the symmetry-transformations leads to the existence of a conservation law in material space, which does not belong to trivial conservation laws and whose divergence-free expression gives a path-independent integral. It is found that by adjusting the analytic function, a finite value can be obtained from this path-independent integral calculated around the material point with any order singularity. For a sharp V-notch placed on the edge of homogenous materials and/or the interface of bi-materials, application shows that the finite value obtained from this path-independent integral is directly related to the notch stress intensity factor (NSIF) and does not depend on the location of integral endpoints chosen respectively along two traction-free surfaces of which form a notch opening angle. Usability is presented in an example to estimate the NSIF of a bi-material plate.

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1. Introduction

Real mechanical elements possess different kinds of elastic stress concentrations, and sharp V-notches are a source of them, to which there is a Williams' exact solution (Williams, 1952, 1957). The intensity of the stress field near the tip of a sharp V-notch is singular and quantified by means of the NSIF (Gross and Mendelson, 1972; Seweryn and Molski, 1996). It is well-known that Rice's J-integral is path-independent around the entire area occupied by an elastic plane, and a finite value from calculating this integral encircling the tip of a crack is an elastic fracture parameter, namely, the energy release rate (Rice, 1968a,b). However, when applying J-integral to the tip of a sharp V-notch with the unloaded notch surfaces, it vanishes with $r \to 0$ because there is a $\sigma \approx r^{-(1-\lambda)}$ singularity with $0 < 1-\lambda < \frac{1}{2}$ in stresses and strains at the tip of a sharp V-notch (Williams, 1952, 1957). Recently, some authors studied the J-integral analytically and numerically for sharp V-notch problems in detail (Livieri, 2003, 2008; Chen and Lu, 2004; Matvienko and Morozov, 2004; Berto and Lazzarin, 2007), and one of their contributions is to getting a finite value J^L based on J-integral calculated by encircling the sharp V-notch tip (Lazzarin et al., 2002).

It is well-known that the J-integral given independently by Eshelby (1951, 1970), Rice (1968a,b) and Cherepanov (1967, 1979) belongs to a conservation law of elastic materials in material space (Herrmann, 1981), which is a divergence-free expression and can be obtained by Noether's theorem (Noether, 1918; Fletcher, 1976; Shi et al., 2006). In this paper, we focus on the sharp V-notch in longitudinal shear problem, which possesses the significance in engineering (Noda and Takase, 2003; Zappalorto et al., 2008, 2009). In Section 2, it is shown that a conservation law in material space involves two functions which can be expressed by the real and imaginary parts of an analytic function. In Section 3, by adjusting this analytic function, a path-independent integral for a sharp V-notch placed on the edge of homogeneous materials and/or the interface of bi-materials is presented. Concluding remarks are given in Section 4.

2. Conservation integrals

2.1. Special symmetry-transformations

For the longitudinal shear problem of elastic materials, the elastic energy density is

$$W = \frac{1}{2}\sigma_{3i}\varepsilon_{3i} = \frac{1}{2}G\varepsilon_{3i}\varepsilon_{3i} = \frac{1}{2}Gu_{3,i}u_{3,i}, \tag{1}$$

where σ_{3i} is the stress, ε_{3i} the strain, u_3 the displacement, and G the shear modulus. Here, the summation convention for repeated indices is implied and the Latin indices run from 1 to 2. Making use of Hooke's law

$$\sigma_{3i} = G\varepsilon_{3i}, \tag{2}$$

when calculating the variation of an integral of the elastic energy density (1), we obtain the field equation

$$\sigma_{3i,i} = 0, \tag{3}$$

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and the related nature, homogenous boundary conditions. This result satisfies the requirement of Noether's theorem (Noether, 1918).

It is convenient for us to use Lie's infinitesimal invariance criterion to compute the symmetry groups of the elastic energy density (1). By following the work of Olver (1993), for a first order variational problem, all possible conserved quantities P_i and non-trivial conservation laws derived from the invariance of the elastic energy density (1) can be formulated as

$$P_i = \zeta_p T_{ip} + \eta\sigma_{3i}, \tag{4}$$

$$D_i P_i = 0, \tag{5}$$

where

$$D_i = \frac{\partial}{\partial x_i} + \frac{\partial}{\partial u_3} u_{3,i} + \frac{\partial}{\partial u_{3,p}} u_{3,pi} + \cdots, \quad \text{and}$$
$$T_{ip} = W\delta_{ip} - \sigma_{3i} u_{3,p} \tag{6}$$

is the energy–momentum tensor, δ_{ip} the Kronecker delta. Here, ζ_i and η represent the symmetry-transformations of space and the displacement u_3, respectively, in an infinitesimal form of one parameter transformation group

$$x_1' = x_1 + \varepsilon\zeta_1(x_1, x_2, u_3), \quad x_2' = x_2 + \varepsilon\zeta_2(x_1, x_2, u_3),$$
$$u_3' = u_3 + \varepsilon\eta(x_1, x_2, u_3), \tag{7}$$

where ε is an infinitesimal group parameter. In order to find functions ζ_i and η in expression (4), we expand Eq. (5) as follows

$$G\left[\frac{\partial\eta}{\partial x_i}u_{3,i} + \frac{1}{2}\left(2\frac{\partial\eta}{\partial u_3} + \frac{\partial\zeta_2}{\partial x_2} - \frac{\partial\zeta_1}{\partial x_1}\right)u_{3,1}^2 + \frac{1}{2}\left(2\frac{\partial\eta}{\partial u_3} + \frac{\partial\zeta_1}{\partial x_1} - \frac{\partial\zeta_2}{\partial x_2}\right)u_{3,2}^2 - \left(\frac{\partial\zeta_1}{\partial x_2} + \frac{\partial\zeta_2}{\partial x_1}\right)u_{3,1}u_{3,2} - \frac{1}{2}\frac{\partial\zeta_i}{\partial u_3}u_{3,i}u_{3,p}u_{3,p}\right] = 0. \tag{8}$$

Since Noether's theorem (Noether, 1918) demands that Eq. (8) vanish identically, the coefficients of all the independent linear, quadratic and cubic terms of $u_{3,i}$ must be equal to zero. This requirement gives the determining equations as follows

$$u_{3,i}: \quad \frac{\partial\eta}{\partial x_i} = 0, \quad (i = 1, 2), \tag{9}$$

$$u_{3,1}^2: \quad 2\frac{\partial\eta}{\partial u_3} + \frac{\partial\zeta_2}{\partial x_2} - \frac{\partial\zeta_1}{\partial x_1} = 0, \tag{10}$$

$$u_{3,2}^2: \quad 2\frac{\partial\eta}{\partial u_3} + \frac{\partial\zeta_1}{\partial x_1} - \frac{\partial\zeta_2}{\partial x_2} = 0, \tag{11}$$

$$u_{3,1}u_{3,2}: \quad \frac{\partial\zeta_1}{\partial x_2} + \frac{\partial\zeta_2}{\partial x_1} = 0, \tag{12}$$

$$u_{3,i}u_{3,p}u_{3,p}: \quad \frac{\partial\zeta_i}{\partial u_3} = 0, \quad (i = 1, 2). \tag{13}$$

Clearly, Eqs. (9) and (13) require that $\eta = \eta(u_3)$ and $\zeta_i = \zeta_i(x_1,x_2)$, from which we obtain with the help of Eqs. (10)–(12) as follows

$$\eta = C, \tag{14}$$

$$\frac{\partial\zeta_1}{\partial x_1} = \frac{\partial\zeta_2}{\partial x_2}, \quad \frac{\partial\zeta_1}{\partial x_2} = -\frac{\partial\zeta_2}{\partial x_1}, \tag{15}$$

where C is an independent arbitrary constant. Moreover, Eq. (15) indicate that functions $\zeta_i = \zeta_i(x_1,x_2)$ must satisfy the Cauchy–Riemann equations, from which they can be expressed by

$$\zeta_1 + i\zeta_2 = \phi(z), \tag{16}$$

where $\phi(z)$ is an analytic function. Clearly, expression (16) can further be expressed by a complex power function

$$\zeta_1 + i\zeta_2 = \phi(z) = z^\delta = r^\delta(\cos\delta\theta + i\sin\delta\theta). \tag{17}$$

Obviously, substituting (14) and (17) into expressions (4) and Eq. (5), because of the independence of C, we obtain the field Eq. (3). For the other conservation laws, by using expanded forms of the energy–momentum tensor (6)

$$T_{11} = -T_{22} = -\frac{1}{2}G\left(u_{3,1}^2 - u_{3,2}^2\right), \quad T_{12} = T_{21} = -Gu_{3,1}u_{3,2}, \tag{18}$$

the conserved quantities (4) become

$$P_i = \zeta_1 T_{i1} + \zeta_2 T_{i2} = r^\delta(T_{i1}\cos\delta\theta + T_{i2}\sin\delta\theta). \tag{19}$$

Hence, a path-independent integral is obtained by divergence-free expression (5)

$$SW = \int_{(x_{10},x_{20})}^{(x_1,x_2)} P_i n_i \, d\Gamma, \tag{20}$$

where $n_1 = \cos(n,x_1)$ and $n_2 = \cos(n,x_2)$. It should be mentioned that whenever the functions $\zeta_i = \zeta_i(x_1,x_2)$ are expressed by an analytic function (16), the path independency of (20) holds. On the other hand, we have to choose a power form (17) for the functions $\zeta_i = \zeta_i(x_1,x_2)$, so that a finite value can be obtained by adjusting δ in calculating SW-integral (20) encircling the tip of a sharp V-notch.

2.2. Non-triviality of conservation law (5) with (19)

From expressions (18) and (19), the conserved quantities P_i can also be expressed by

$$P_1 = \zeta_1 T_{11} + \zeta_2 T_{12} = -\frac{1}{2}G\left[\zeta_1\left(u_{3,1}^2 - u_{3,2}^2\right) + 2\zeta_2 u_{3,1}u_{3,2}\right],$$
$$P_2 = \zeta_1 T_{21} + \zeta_2 T_{22} = \frac{1}{2}G\left[\zeta_2\left(u_{3,1}^2 - u_{3,2}^2\right) - 2\zeta_1 u_{3,1}u_{3,2}\right]. \tag{21}$$

There are two kinds of trivial conservation laws (Olver, 1993). The first kind is that P_i themselves in (5) vanish, which is not true for expressions (21). The second kind requires that the conserved quantities P_i be expressed as follows

$$P_1 = D_2 Q_{12}, \quad P_2 = D_1 Q_{21}, \quad Q_{12} + Q_{21} = 0, \quad Q_{11} = Q_{22} = 0. \tag{22}$$

Actually, expanding the first and second in (22) gives

$$P_1 = \frac{\partial Q_{12}}{\partial x_2} + \frac{\partial Q_{12}}{\partial u_3}u_{3,2} + \frac{\partial Q_{12}}{\partial u_{3,1}}u_{3,12} + \frac{\partial Q_{12}}{\partial u_{3,2}}u_{3,22} + \cdots,$$
$$P_2 = \frac{\partial Q_{21}}{\partial x_1} + \frac{\partial Q_{21}}{\partial u_3}u_{3,1} + \frac{\partial Q_{21}}{\partial u_{3,1}}u_{3,11} + \frac{\partial Q_{21}}{\partial u_{3,2}}u_{3,21} + \cdots \tag{23}$$

Clearly, it is impossible that the conserved quantities in (21) accord with two expressions in (23). Therefore, the obtained SW-integral (20) is non-trivial.

2.3. Remarks

According to the viewpoints of Newtonian mechanics, the translation, rotation and scale change of coordinates are the symmetry-transformations of space. The existence and existent forms of J-integral, L-integral and M-integral depend on whether and how the elastic energy density accords with the symmetry-transformations of space when Noether's theorem is applied. As a physical system, its elastic energy density constructed by a linear and/or non-linear theory describing an elastic body can not be changed when we make a translation of coordinates, so that J-integral always exists (Knowles and Sternberg, 1972; Fletcher, 1976; Olver, 1984a,b; Caviglia and Morro, 1988; Honein and Herrmann, 1997; Shi, 2005; Shi et al., 2006). Apart from the isotropic and transversely isotropic materials, there is no L-integral for anisotropic materials because their elastic energy densities do not accord with the rotational symmetry of space (Caviglia and Morro, 1988). For a non-linear theory different from a linear theory, it is usually impossible that the value of physical quantities remains unchanged when the scale of both coordinates and field variables is considered changed. Therefore, there is no M-integral from the non-linear theory of an elastic body (Knowles and Sternberg, 1972; Shi et al., 2006).

The fact mentioned above is right for the two- and three-dimensional elastic problems. However, the longitudinal shear problem is special because there is no projection of the displacement u_3 onto the physical (x_1,x_2)-plane with the translation, rotation and scale change of coordinates. This leads to that the symmetry-transformation of the elastic energy density (1) in material space can be expressed by the real and imaginary parts of an analytic function (16). Actually, the two families of curves $\zeta_1(x_1,x_2) = c_1$ and $\zeta_2(x_1,x_2) = c_2$ in (16) are mutually perpendicular, and the analytic function $\phi(z)$ represents a conformal mapping. Therefore, the translation, rotation and scale change of coordinates are some particular cases involved in the analytic function $\phi(z)$. For example, consider the following transformations

$$\zeta_i = Ax_i + e_{i3k}\Omega_3 x_k + B_i, \tag{R.1}$$

where A, Ω_3 and B_i are the independent arbitrary constants. Obviously, expressions (R.1) satisfy the Cauchy–Riemann Eq. (15). Substituting (14) and (R.1) into expressions (4) and Eq. (5), because of the independence of C, A, Ω_3 and B_i, respectively, we obtain the field Eq. (3) and the following conservation integrals:

(i) J-integral from translation ($B_k \neq 0$)

$$J_k = \int T_{ik} n_i \, d\Gamma = \int (W\delta_{ik} - \sigma_{3i} u_{3,k}) n_i \, d\Gamma; \tag{R.2}$$

(ii) L-integral from rotation ($\Omega_3 \neq 0$)

$$L = \int e_{k3j} x_j T_{ik} n_i \, d\Gamma = \int e_{k3j} x_j (W\delta_{ik} - \sigma_{3i} u_{3,k}) n_i \, d\Gamma; \tag{R.3}$$

(iii) M-integral from scaling ($A \neq 0$)

$$M = \int x_k T_{ik} n_i \, d\Gamma = \int x_k (W\delta_{ik} - \sigma_{3i} u_{3,k}) n_i \, d\Gamma. \tag{R.4}$$

Clearly, when we set $\delta = 0$ in (19), the path-independent integral (20) coincides with J_1 in (R.2). That is, J-integral is a particular case of SW-integral in longitudinal shear problem.

3. Application to sharp V-notches

There are several articles concerning the problem of sharp V-notches (Chen and Nisitani, 1992; Qian and Hasebe, 1997; Dunn et al., 1997), as shown in Fig. 1, which is most conveniently analyzed in terms of polar coordinates (r,θ) with $r = 0$ at the tip of sharp V-notches. The transformation between the strains described in polar and Cartesian coordinates, and the Hooke's law are

$$\varepsilon_{3r} = \frac{\partial u_3}{\partial r} = \varepsilon_{31}\cos\theta + \varepsilon_{32}\sin\theta,$$
$$\varepsilon_{3\theta} = \frac{1}{r}\frac{\partial u_3}{\partial \theta} = -\varepsilon_{31}\sin\theta + \varepsilon_{32}\cos\theta, \tag{24}$$

$$\varepsilon_{3r} = \frac{1}{G}\sigma_{3r}, \quad \varepsilon_{3\theta} = \frac{1}{G}\sigma_{3\theta}. \tag{25}$$

By introducing a stress function Φ such that

$$\sigma_{3r} = -\frac{1}{r}\frac{\partial \Phi}{\partial \theta}, \quad \sigma_{3\theta} = \frac{\partial \Phi}{\partial r}, \tag{26}$$

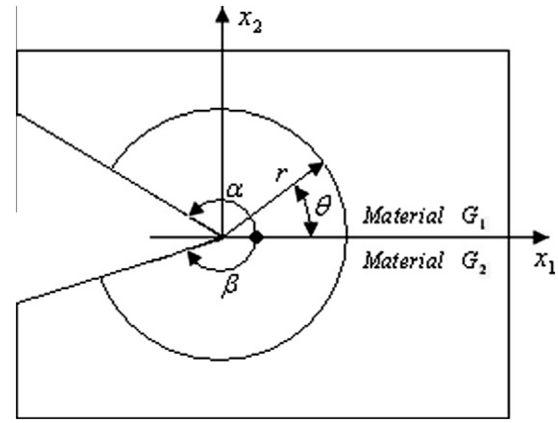

Fig. 1. Configuration of a sharp V-notch. $G_1 = G_2$ for homogenous material and $G_1 \neq G_2$ for bi-materials.

the polar coordinate form of equilibrium Eq. (3) is identically satisfied and the compatibility equation can be solved by the standard technique of separation variables, resulting in

$$\Phi = r^{\lambda}(A\cos\lambda\theta + B\sin\lambda\theta), \tag{27}$$

where λ, A and B are the constants.

3.1. A sharp V-notch placed on the edge of homogeneous material

From the boundary conditions along the unloaded notch surfaces, that is,

$$\sigma_{3\theta} = \frac{\partial \Phi}{\partial r} = 0, \quad \theta = \alpha, \quad \theta = -\beta, \tag{28}$$

the eigen equations can be obtained from (27), and the vanishing of the determinant of their coefficient matrix gives

$$\sin\lambda(\alpha+\beta) = 0, \quad (\pi < \alpha+\beta \leqslant 2\pi), \tag{29}$$

$$\Phi = \sum_{n=1}^{+\infty} r^{\frac{n\pi}{\alpha+\beta}} \left(A_n \cos\frac{n\pi\theta}{\alpha+\beta} + B_n \sin\frac{n\pi\theta}{\alpha+\beta} \right), \quad \lambda_n = \frac{n\pi}{\alpha+\beta}, \quad (n = 1, 2, \ldots), \tag{30}$$

$$B_n = -A_n ctg\frac{n\pi\alpha}{\alpha+\beta} = A_n ctg\frac{n\pi\beta}{\alpha+\beta}. \tag{31}$$

Here, the terms with $n = -1, -2, \ldots$ have been excluded in order that the displacement remains finite.

Using (26), (31) and the first in (30), according to the definition of NSIF (Seweryn and Molski, 1996)

$$K_{III}^{N} = \sqrt{2\pi} \lim_{r \to 0} r^{1-\frac{\pi}{\alpha+\beta}} \sigma_{3\theta}(r, \theta = 0), \tag{32}$$

we get

$$\sigma_{3r} = -\frac{K_{III}^{N}}{\sqrt{2\pi}\sin\frac{\pi\beta}{\alpha+\beta}} r^{\frac{\pi}{\alpha+\beta}-1} \cos\frac{\pi(\theta+\beta)}{\alpha+\beta} + O\left(r^{\frac{2\pi}{\alpha+\beta}-1}\right),$$
$$\sigma_{3\theta} = \frac{K_{III}^{N}}{\sqrt{2\pi}\sin\frac{\pi\beta}{\alpha+\beta}} r^{\frac{\pi}{\alpha+\beta}-1} \sin\frac{\pi(\theta+\beta)}{\alpha+\beta} + O\left(r^{\frac{2\pi}{\alpha+\beta}-1}\right), \tag{33}$$

where

$$A_1 = \frac{\alpha+\beta}{\pi}\frac{K_{III}^{N}}{\sqrt{2\pi}},$$
$$B_1 = -\frac{\alpha+\beta}{\pi}\frac{K_{III}^{N}}{\sqrt{2\pi}} ctg\frac{\pi\alpha}{\alpha+\beta} = \frac{\alpha+\beta}{\pi}\frac{K_{III}^{N}}{\sqrt{2\pi}} ctg\frac{\pi\beta}{\alpha+\beta}. \tag{34}$$

By using (18), (24) and (25), the conserved quantities (19) can be written as

$$P_1 = -\frac{K_{III}^{N^2}}{4\pi G \sin^2 \frac{\pi\beta}{\alpha+\beta}} r^{\frac{2\pi}{\alpha+\beta}-2+\delta} \cos\left[\frac{2(\pi-\alpha-\beta)\theta+2\pi\beta}{\alpha+\beta}+\delta\theta\right] + O\left(r^{\frac{3\pi}{\alpha+\beta}-2+\delta}\right),$$
$$P_2 = \frac{K_{III}^{N^2}}{4\pi G \sin^2 \frac{\pi\beta}{\alpha+\beta}} r^{\frac{2\pi}{\alpha+\beta}-2+\delta} \sin\left[\frac{2(\pi-\alpha-\beta)\theta+2\pi\beta}{\alpha+\beta}+\delta\theta\right] + O\left(r^{\frac{3\pi}{\alpha+\beta}-2+\delta}\right). \tag{35}$$

As mentioned above, the path independency of (20) holds for the conserved quantities (35) with any value δ. Therefore, when we set

$$\delta = 1 - \frac{2\pi}{\alpha+\beta}, \tag{36}$$

the conserved quantities (35) become

$$P_1 = -\frac{K_{III}^{N^2}}{4\pi G \sin^2 \frac{\pi\beta}{\alpha+\beta}} r^{-1} \cos\frac{2\pi\beta-(\alpha+\beta)\theta}{\alpha+\beta} + O\left(r^{\frac{\pi}{\alpha+\beta}-1}\right),$$
$$P_2 = \frac{K_{III}^{N^2}}{4\pi G \sin^2 \frac{\pi\beta}{\alpha+\beta}} r^{-1} \sin\frac{2\pi\beta-(\alpha+\beta)\theta}{\alpha+\beta} + O\left(r^{\frac{\pi}{\alpha+\beta}-1}\right) \tag{37}$$

and the path-independent integral (20) with the help of expressions (18) and (19) can be written for the sharp V-notch problem as

$$SW = \int_{(x_{10},x_{20})}^{(x_1,x_2)} P_i n_i d\Gamma$$
$$= \int_{x_{10},x_{20}}^{x_1,x_2} r^{1-\frac{2\pi}{\alpha+\beta}}\left[T_{i1}\cos\left(1-\frac{2\pi}{\alpha+\beta}\right)\theta + T_{i2}\sin\left(1-\frac{2\pi}{\alpha+\beta}\right)\theta\right] n_i d\Gamma. \tag{38}$$

This path-independent integral is valid for any path in entire plane. Choosing a circle path $d\Gamma = rd\theta$ with $r \to 0$ encircling the tip of a sharp V-notch as shown in Fig. 1 and making use of (37), we obtain

$$SW = \int_{-\beta}^{\alpha}(P_1\cos\theta + P_2\sin\theta)r\,d\theta$$
$$= \frac{K_{III}^{N^2}}{2G}\frac{\alpha+\beta}{\pi}\frac{\cos\frac{(\alpha-\beta)\pi}{\alpha+\beta}}{1+\cos\frac{(\alpha-\beta)\pi}{\alpha+\beta}}, \quad (\pi < \alpha+\beta \leqslant 2\pi). \tag{39}$$

One can check that when $\alpha = \beta$,

$$SW = \frac{K_{III}^{N^2}}{2G}\frac{\alpha}{\pi}, \quad \left(\frac{\pi}{2} < \alpha \leqslant \pi\right), \tag{40}$$

and when $\alpha = \beta = \pi$,

$$SW = \frac{K_{III}^2}{2G}, \tag{41}$$

which is the energy release rate of a Mode III crack.

Apart from the path independency of SW-integral from divergence-free expression (5) in elastic field, the vanishing of its integrand in (20)

$$P_i n_i = r^{\delta}(T_{i1}\cos\delta\theta + T_{i2}\sin\delta\theta)n_i, \tag{42}$$

respectively along the lower and upper surfaces with the traction-free condition is also important for usability, as shown in Fig. 2. Clearly, the unit normal vectors respectively on the lower and upper surfaces are

$$n_1^{LS} = -\sin\alpha, \quad n_2^{LS} = -\cos\alpha, \tag{43}$$

$$n_1^{US} = -\sin\alpha, \quad n_2^{US} = \cos\alpha. \tag{44}$$

With the help of transformation (24) and traction-free condition (28), the energy–momentum tensor (18) becomes

$$T_{11}^{LS} = -T_{22}^{LS} = -\frac{1}{2}G\varepsilon_{3r}^2\cos 2\alpha, \quad T_{12}^{LS} = T_{21}^{LS} = \frac{1}{2}G\varepsilon_{3r}^2\sin 2\alpha, \tag{45}$$

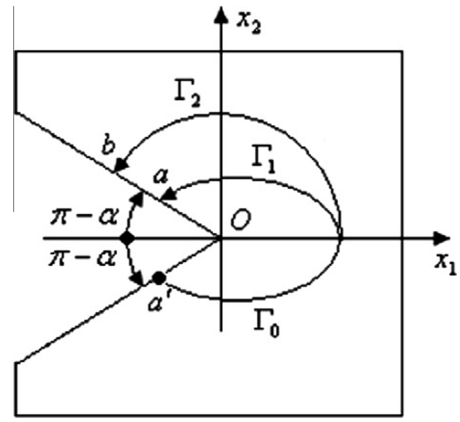

Fig. 2. Integral paths for sharp V-notch problem.

$$T_{11}^{US} = -T_{22}^{US} = -\frac{1}{2}G\varepsilon_{3r}^2\cos 2\alpha, \quad T_{12}^{US} = T_{21}^{US} = -\frac{1}{2}G\varepsilon_{3r}^2\sin 2\alpha. \tag{46}$$

Substituting (43)–(46) into (42) with $\theta = \alpha$ and/or $-\alpha$, we obtain the same integrand, respectively along the lower and upper surfaces

$$(P_i n_i)^{LS} = (P_i n_i)^{US} = r^{\delta}\frac{1}{2}G\varepsilon_{3r}^2\sin(\delta-1)\alpha. \tag{47}$$

Clearly, substitution of (36) into (47) with letting $\alpha = \beta$ gives

$$(P_i n_i)^{LS} = (P_i n_i)^{US} = 0. \tag{48}$$

This means that the value of SW-integral in (20) or (40) does not depend on the location of integral endpoints a' and b, as shown in Fig. 2, which can be arbitrarily chosen along the traction-free surfaces of a sharp V-notch. As mentioned above, when setting $\delta = 0$, SW-integral is reduced to J-integral in longitudinal shear problem. By doing so, the integrand (47) becomes

$$(P_i n_i)^{LS} = (P_i n_i)^{US} = -\frac{1}{2}G\varepsilon_{3r}^2\sin\alpha. \tag{49}$$

This means and is also emphasized by Lazzarin and Zappalorto (2008) that the starting and finishing points of J-integral must be a' and a with $|Oa'| = |Oa| = R$, as shown in Fig. 2.

3.2. A Sharp V-notch placed on the interface of bi-materials

A closed-form solution can be obtained for the case $\alpha = \beta$ shown in Fig. 1. By using the boundary and continuity conditions

$$\sigma_{3\theta}^{(1)} = 0, \quad \theta = \alpha;$$
$$\sigma_{3\theta}^{(2)} = 0, \quad \theta = -\alpha;$$
$$\sigma_{3\theta}^{(1)} = \sigma_{3\theta}^{(2)}, \quad \varepsilon_{3r}^{(1)} = \varepsilon_{3r}^{(2)}, \quad \theta = 0, \tag{50}$$

the eigen equations are derived from (26) and (27), and their solutions can be expressed as

$$\Phi^{(k)} = \sum_{n=1,3,5,\ldots}^{+\infty} A_n^{(k)} r^{\frac{n\pi}{2\alpha}}\cos\frac{n\pi\theta}{2\alpha} + \sum_{n=2,4,6,\ldots}^{+\infty} B_n^{(k)} r^{\frac{n\pi}{2\alpha}}\sin\frac{n\pi\theta}{2\alpha},$$
$$\lambda_n = \frac{n\pi}{2\alpha}, \quad (n = 1,2,\ldots), \quad A_n^{(2)} = A_n^{(1)}, \quad B_n^{(2)} = \frac{G_2}{G_1}B_n^{(1)}, \tag{51}$$

where $k = 1$ and 2 denote, as shown in Fig. 1, the upper and lower materials, respectively. Also, the terms with $n = -1, -2, \ldots$ have been excluded in order that the displacements remain finite. Introducing the definition of NSIF (Seweryn and Molski, 1996)

$$K_{III}^N = \sqrt{2\pi}\lim_{r\to 0} r^{1-\frac{\pi}{2\alpha}}\sigma_{3\theta}^{(k)}(r,\theta=0), \quad (k = 1,2), \tag{52}$$

with the help of (26) and (27), we obtain

$$A_1^{(2)} = A_1^{(1)} = \frac{2\alpha}{\pi}\frac{K_{III}^N}{\sqrt{2\pi}}, \quad \left(\frac{\pi}{2} < \alpha \leqslant \pi\right),$$
$$\sigma_{3r}^{(k)} = \frac{K_{III}^N}{\sqrt{2\pi}} r^{\frac{\pi}{2\alpha}-1} \sin\frac{\pi\theta}{2\alpha} + O\left(r^{\frac{\pi}{\alpha}-1}\right), \quad (k = 1, 2), \tag{53}$$
$$\sigma_{3\theta}^{(k)} = \frac{K_{III}^N}{\sqrt{2\pi}} r^{\frac{\pi}{2\alpha}-1} \cos\frac{\pi\theta}{2\alpha} + O\left(r^{\frac{\pi}{\alpha}-1}\right), \quad (k = 1, 2).$$

Similarly to the above, after using (18), (24) and (25), the conserved quantities (19) for the upper and lower materials, as shown in Fig. 1, can be written as

$$P_1^{(k)} = \frac{K_{III}^{N^2}}{4\pi G_k} r^{\frac{\pi}{\alpha}-2+\delta} \cos\left(\frac{\pi}{\alpha} - 2 + \delta\right)\theta + O\left(r^{\frac{3\pi}{2\alpha}-2+\delta}\right), \quad (k = 1, 2),$$
$$P_2^{(k)} = -\frac{K_{III}^{N^2}}{4\pi G_k} r^{\frac{\pi}{\alpha}-2+\delta} \sin\left(\frac{\pi}{\alpha} - 2 + \delta\right)\theta + O\left(r^{\frac{3\pi}{2\alpha}-2+\delta}\right), \quad (k = 1, 2). \tag{54}$$

Also, when we set

$$\delta = 1 - \frac{\pi}{\alpha}, \tag{55}$$

the conserved quantities (54) become

$$P_1^{(k)} = \frac{K_{III}^{N^2}}{4\pi G_k} r^{-1} \cos\theta + O\left(r^{\frac{\pi}{2\alpha}-1}\right), \quad (k = 1, 2),$$
$$P_2^{(k)} = \frac{K_{III}^{N^2}}{4\pi G_k} r^{-1} \sin\theta + O\left(r^{\frac{\pi}{2\alpha}-1}\right), \quad (k = 1, 2). \tag{56}$$

Hence, the path-independent integral (20) with the help of expressions (18) and (19) for bi-materials is

$$SW = \int_{(x_{10},x_{20})}^{(x_1,x_2)} P_i^{(k)} n_i \, d\Gamma$$
$$= \int_{(x_{10},x_{20})}^{(x_1,x_2)} r^{1-\frac{\pi}{\alpha}} \left[T_{i1}^{(k)} \cos\left(1 - \frac{\pi}{\alpha}\right)\theta + T_{i2}^{(k)} \sin\left(1 - \frac{\pi}{\alpha}\right)\theta\right] n_i \, d\Gamma, \tag{57}$$

where $k = 1$ for the upper material with $x_{20} > 0$ and $x_2 > 0$, and $k = 2$ for the lower material with $x_{20} < 0$ and $x_2 < 0$, as shown in Fig. 1. By selecting a circle path $d\Gamma = rd\theta$ with $r \to 0$ encircling the tip of a sharp V-notch and using the conserved quantities (56), the path-independent integral (57) can be calculated as follows

$$SW = \int_{-\alpha}^{0} \left(P_1^{(2)} \cos\theta + P_2^{(2)} \sin\theta\right) r \, d\theta + \int_{0}^{\alpha} \left(P_1^{(1)} \cos\theta + P_2^{(1)} \sin\theta\right) r \, d\theta$$
$$= \frac{\alpha K_{III}^{N^2}}{4\pi}\left(\frac{1}{G_1} + \frac{1}{G_2}\right), \quad \left(\frac{\pi}{2} < \alpha \leqslant \pi\right). \tag{58}$$

Clearly, expressions (40) and (58) will be the same when $G_1 = G_2$, and SW-integral (58) is also related to the NSIF of a sharp V-notch placed on the interface of bi-materials.

3.3. Example

Usability of the path-independent integrals (38) and (57) is illustrated below by using a bi-material plate with a built-in edge at right-hand side, as shown in Fig. 3, where the applied load F upward equals the load F downward at left-hand side. This kind of method to estimate the SIF had been discussed by Irwin (1960). The path-independent integral (57) with the result (58) implies

$$SW = \frac{\alpha K_{III}^{N^2}}{4\pi}\left(\frac{1}{G_1} + \frac{1}{G_2}\right) = \int_{\Gamma_1+\Gamma_2+\Gamma_3+\Gamma_4+\Gamma_5} P_i^{(k)} n_i \, d\Gamma. \tag{59}$$

Simple beam theory gives

$$u_{3,1}^{(k)}\Big|_{x_1=-l} = \frac{6Fl}{E_k h^3}\left[ctg\alpha + \frac{b}{2l} ctg^2\alpha \ln\left(1 - \frac{2l}{b} tg\alpha\right)\right], \quad (k = 1, 2), \tag{60}$$

for the tearing of the beam arms of two different materials when the ends at the bi-material sharp V-notch tip are considered clamped, where b is the width of two beams at left-hand side, h the thickness, l the length and E_k the Young's modulus, as shown in Fig. 3. Since Γ_3 is along the built-in edge, letting $u_{3,2}^{(k)} = 0$ in (18) and (19) with the help of that $\zeta_1 = r^\delta \cos\delta\theta \approx l^\delta$ when $l \gg b$, we know from (59) that

$$SW = \frac{\alpha K_{III}^{N^2}}{4\pi}\left(\frac{1}{G_1} + \frac{1}{G_2}\right) = \frac{1}{2} b l^\delta \left(G_1 u_{3,1}^{(1)^2} + G_2 u_{3,1}^{(2)^2}\right). \tag{61}$$

Here, it is implied that the load F is uniformly distributed on the area hb. Substituting (60) into (61), we obtain an estimation value of NSIF

$$K_{III}^N = \sqrt{\frac{G_1(1+\upsilon_1)^2 + G_2(1+\upsilon_2)^2}{(G_1+G_2)(1+\upsilon_1)^2(1+\upsilon_2)^2}} \frac{3Fl^{\frac{3}{2}-\frac{\pi}{2\alpha}}\sqrt{b}}{h^3} f\left(\frac{b}{l}, \alpha\right), \tag{62}$$

where $E_k = 2(1 + \upsilon_k)G_k$ has been used and

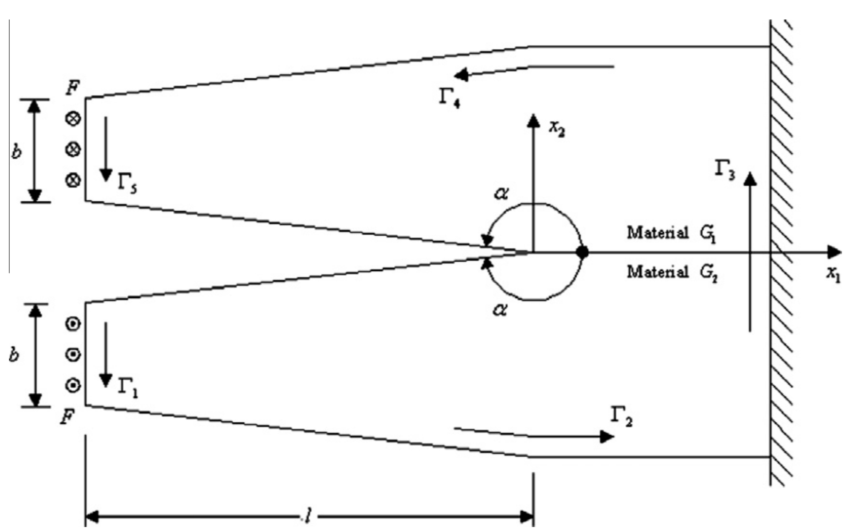

Fig. 3. A sharp V-notch placed on the interface of a bi-material plate.

$$f\left(\frac{b}{l},\alpha\right)=\sqrt{\frac{2\pi}{\alpha}}\left|ctg\alpha+\frac{b}{2l}ctg^2\alpha\ln\left(1-\frac{2l}{b}tg\alpha\right)\right|. \tag{63}$$

Clearly, when $G_1 = G_2$ and $\upsilon_1 = \upsilon_2$, expression (62) becomes a NSIF of homogeneous material

$$K_{III}^N=\frac{3Fl^{\frac{3}{2}-\frac{\pi}{2\alpha}}\sqrt{b}}{(1+\upsilon)h^3}f\left(\frac{b}{l},\alpha\right). \tag{64}$$

When $\alpha \to \pi$, $f(b/l,\alpha) \to l/b$, we get a stress intensity factor for crack problem

$$K_{III}=\frac{3Fl^2}{(1+\upsilon)h^3\sqrt{b}}. \tag{65}$$

4. Concluding remarks

Based on Noether's theorem, a conservation law and/or path-independent integral (20) in material space is obtained, which does not belong to trivial conservation law. Especially, the conserved quantities (19) can be expressed in terms of the components of energy–momentum tensor as well as the real and imaginary parts of an analytic function in a power form. It is shown that a finite value can be obtained from this path-independent integral calculated around the material point with any order singularity by adjusting the analytic function (38). Actually, *J*-integral for longitudinal shear problem is a particular case of *SW*-integral (20) and (38).

For a sharp V-notch placed on the edge of homogenous materials and/or the interface of bi-materials, application of this path-independent integral (38) calculated around the sharp V-notch's tip shows the validity, and the obtained finite value is irrelevant to the location of integral endpoints chosen respectively along two traction-free surfaces of which form an notch opening angle. Both the arbitrariness of integral endpoints and path independency of *SW*-integral (38) resulting in (40) and (58) represent a kind of physical invariance under the condition of a given fixed notch opening angle, and expressions (40) and (58) are directly related to the NSIF.

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