



Thermodynamically based fictitious crack/interface model for general normal and shear loading



Niels Saabye Ottosen*, Matti Ristinmaa

Division of Solid Mechanics, Lund University, SE-22100 Lund, Sweden

ARTICLE INFO

Article history:

Received 8 October 2012

Received in revised form 14 May 2013

Available online 6 July 2013

Keywords:

Fictitious crack model

Thermodynamics

Damage

Elasto-plasticity

Finite deformations

ABSTRACT

For small deformations a crack/interface model that considers general 3D normal and shear loading is proposed. It involves elasticity, plasticity and damage and it is thermodynamically based. An essential feature of the model is its consistency with the concepts behind the fictitious crack model. In particular, no crack deformation occurs before the crack is initiated and when a crack has just been initiated the proposed model provides an unloading stiffness that is infinitely large. For the same set of parameters, it is demonstrated that the proposed model is able to provide predictions that are in close agreement with experimental data for concrete for a wide range of loading situations.

© 2013 Elsevier Ltd. All rights reserved.

1. Introduction

In brittle materials, cracking is initiated when the maximum principal stress reaches the uniaxial tensile strength. A cohesive crack is then initiated and the direction of the crack is orthogonal to the corresponding principal stress direction. The well-known fictitious crack model suggested by [Hillerborg et al. \(1976\)](#) then provides a suitable description of the response in the post-crack regime. Thermodynamical arguments were presented by [Ottosen \(1986\)](#) which support the concepts behind the fictitious crack model.

The fictitious crack model was formulated for mode I conditions for which the response both during loading and unloading is straightforward to formulate. However, after crack initiation there are a number of situations where the crack subsequently is exposed also to shear loading and then the problem becomes much more complex. In that situation, the problem involves two aspects: the constitutive model for the crack and the numerical treatment that must reflect the discontinuity created by the crack. Here, we will just discuss the constitutive model for the crack/interface and only mention that the numerical finite element treatment can follow different lines, e.g. interface elements, smeared crack approach, embedded crack model and the extended finite element method; see, for instance, the reviews provided by [Jirásek and Belytschko \(2002\)](#) and [de Borst \(2003\)](#).

Modeling of cracks and interfaces exposed to combined normal and shear loading are closely related, but one aspect differs. An interface, a joint for instance, exists before any loading is applied whereas a certain loading is required before a crack is initiated; after that the crack can be considered as an interface.

Due to its importance and complexity, a number of crack/interface models have been presented in the literature and we will here confine ourselves to a discussion of certain main trends.

An interface model accounting for both normal and shear loading was proposed by [Needleman \(1987\)](#) and this model has served as a vehicle for a number of later developments. It has the property that the fracture energy obtained for complete separation is the same irrespective of the load path; this assumption is normally not fulfilled, cf. for instance the experimental data of [Hassanzadeh \(1990\)](#) for cracks in concrete and for interfaces, cf. for instance, the discussion provided by [van der Bosch et al. \(2006\)](#).

The model of [Needleman \(1987\)](#) was later generalized by [Xu and Needleman \(1993\)](#) to account for different fracture energies in separation and shear, but also this model is reversible and thereby history-independent; even the recent model of [Park et al. \(2009\)](#) fits into this framework. To account for different responses during loading and unloading, [Ortiz and Pandolfi \(1999\)](#) formulated a theory that, in principle, corresponds to a damage model. Since no plasticity is considered, on reversal of the load all deformations disappear and no residual deformations exist. In the well-known paper by [Carol et al. \(1997\)](#), a generalized crack model is proposed which reflects the elastic and plastic response of the crack; close agreement is obtained with the experimental data of [Hassanzadeh \(1990\)](#) for concrete. A hyperbolic yield criterion previously

* Corresponding author. Tel.: +46 46 2228001; fax: +46 46 2223115.

E-mail addresses: Niels.Saabye.Ottosen@solid.lth.se (N.S. Ottosen), Matti.Ristinmaa@solid.lth.se (M. Ristinmaa).

proposed by Carol and Prat (1990) and also adopted by Červenka (1994) is used in combination with nonassociated plasticity. Indeed, the models of Carol et al. (1997) and Červenka (1994) have been used not only in a number applications, but several papers are based on certain variations of these models, see, for instance, Gálvez et al. (2002a,b). Since damage is not considered in the models of Carol et al. (1997) and Červenka (1994), the unloading stiffness is constant, irrespective of the load history. This drawback was remedied in the model of Červenka et al. (1998). A somewhat different model, splitted into a cohesive part and a frictional part, that considers elasticity, plasticity and damage of the crack was recently proposed by Nielsen et al. (2010). Also the predictions of this model was compared with the experimental data of Hassanzadeh (1990) for concrete and good agreement was achieved.

A consistent formulation of a crack/interface model applicable to finite deformations is far from being trivial and we may, for instance, refer to the works of Allix and Corigliano (1999); Larsson and Jansson (2002); Fagerström and Larsson (2006) and Mergheim et al. (2007) as well as Mosler and Scheider (2011) for important contributions and discussions. Apart from being kinematically consistent, one also wants a large displacement crack/interface model to fulfil thermodynamic requirements as well as objectivity and this complicates the picture considerably and a profound discussion on these matters is provided by Mosler and Scheider (2011). Focus in this contribution will be on deriving a conceptually simple fictitious crack model and therefore at the present stage only small deformations will be considered.

With this survey, let us return to a discussion of some basic features. In the fictitious crack model of Hillerborg et al. (1976) all crack deformation is localized to the crack itself and the response of the material outside the crack is not affected by its presence. The crack deformation is then described by a relation $\sigma = \sigma(w)$ where σ is the normal stress and w is the crack opening. We then get the response shown in Fig. 1(a) where σ_t is the uniaxial strength and we note, in particular, that when the crack has just been initiated, the unloading stiffness is infinitely large. This feature is easily achieved if a model based entirely on damage is adopted; this is the case for the proposal of Ortiz and Pandolfi (1999). However, as soon as an plastic response is introduced in the model, all existing proposals exhibit the response shown in Fig. 1(b). This response is certainly not in accordance with the fictitious crack model and even though the initial stiffness can be made arbitrarily large, it implies that crack opening occurs before the uniaxial strength σ_t is achieved. The reason that this drawback traditionally is accepted is that it opens for a direct application of traditional elasto-plasticity and damage mechanics. However, this deficiency was also observed by Jirásek and Zimmermann (2001) who discussed the introduction of a so-called compliance parameter to avoid this inconsistency.

We shall here present a model that is in full accordance with the response of the fictitious crack model, i.e. a model which exhibits the properties shown in Fig. 1(a). The model is formulated for the general 3D case. Moreover, the model is thermodynamically based, it includes elasticity, plasticity and damage

and it will turn out to be conceptionally very simple. The model parameters will then, as an example, be calibrated to experimental data for cracking of concrete and it will be demonstrated that for the same set of parameters, close agreement is achieved not only with the experimental data of Hassanzadeh (1990) for combined normal and shear loading, but also with experimental data for uniaxial tests that include the unloading response and, finally, with experimental data for the initial shear stiffness as function of the crack opening.

2. Dissipation inequality

In a body, consider the crack (interface) in the reference configuration defined by the surface Γ which in the current configuration separates into surfaces Γ^+ and Γ^- . From linear momentum of the crack (interface), it follows that $\mathbf{t}^+ = -\mathbf{t}^-$ where $\mathbf{t}$ denotes the Cauchy traction vector (recall that small deformations are assumed). With the displacement jump $\mathbf{w} = \mathbf{u}^+ - \mathbf{u}^-$, the dissipation inequality for isothermal conditions becomes

$$D = \mathbf{t} \cdot \dot{\mathbf{w}} - \dot{\psi} \geq 0 \quad (1)$$

where a dot denotes the time derivative. Here, $\mathbf{t} = \mathbf{t}^+$ and Helmholtz' free energy ψ is the energy per unit surface.

It seems natural to decompose the various vectors into a part that is directed normal to the surface Γ and a part tangential to Γ . Therefore, denoting the normal to the surface Γ as $\mathbf{n}$, the traction vector is decomposed as

$$\mathbf{t} = \sigma \mathbf{n} + \boldsymbol{\tau} \quad (2)$$

where

$$\sigma = \mathbf{n} \cdot \mathbf{t}, \quad \boldsymbol{\tau} = (\mathbf{I} - \mathbf{n} \otimes \mathbf{n}) \cdot \mathbf{t} \quad (3)$$

Here σ is the normal stress component and the $\boldsymbol{\tau}$ is the shear stress vector. Similarly it follows that the displacement jump can be decomposed as

$$\mathbf{w} = w_n \mathbf{n} + \mathbf{w}_s \quad (4)$$

where the crack/interface opening w_n and slip displacement vector $\mathbf{w}_s$ are defined as

$$w_n = \mathbf{n} \cdot \mathbf{w} \quad \mathbf{w}_s = (\mathbf{I} - \mathbf{n} \otimes \mathbf{n}) \cdot \mathbf{w} \quad (5)$$

Returning to the dissipation inequality (1) and taking advantage of (2) and (4) it follows that

$$D = \sigma \dot{w}_n + \boldsymbol{\tau} \cdot \dot{\mathbf{w}}_s - \dot{\psi} \geq 0 \quad (6)$$

To introduce plasticity, the plastic part of the jump is decomposed in analogy with the considerations above, i.e

$$\mathbf{w}^p = w_n^p \mathbf{n} + \mathbf{w}_s^p \quad (7)$$

where

$$w_n^p = \mathbf{n} \cdot \mathbf{w}^p \quad \mathbf{w}_s^p = (\mathbf{I} - \mathbf{n} \otimes \mathbf{n}) \cdot \mathbf{w}^p \quad (8)$$

With these consideration, a specific form of Helmholtz' free energy will now be considered where, in addition to elasticity and plasticity, also damage is considered.

3. Helmholtz' free energy

In continua, thermodynamics involves certain quantities that are expressed per unit volume. Now we are dealing with a surface and the corresponding quantities are then expressed per unit surface. Apart from that, the thermodynamic framework is similar to the standard formulation and we refer, for instance, to the survey given by Ottosen and Ristinmaa (2005).

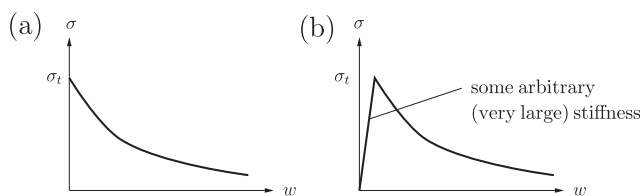


Fig. 1. (a) Response according to the fictitious crack model with an unloading stiffness that initially is infinitely large; (b) Response for previous models that include an elastic part.

Assume Helmholtz' free energy per unit surface is given by

$$\psi = (1 - \kappa)^m \frac{E_c}{2\kappa^m} (w_n - w_n^p)^2 + (1 - \kappa)^m \frac{G_c}{2\kappa^m} (\mathbf{w}_s - \mathbf{w}_s^p) \cdot (\mathbf{w}_s - \mathbf{w}_s^p) \quad (9)$$

where κ is a damage variable; $0 \leq \kappa \leq 1$. Moreover, m = dimensionless parameter ($m > 0$) whereas E_c and G_c are elastic stiffnesses of the crack/interface of dimension $[N/m^3]$ (subscript c refers to crack/interface). It appears that the first term in (9) refers to opening of the crack/interface whereas the second term refers to slip.

Expression (9) is now inserted in (6) resulting in

$$\begin{aligned} & \left[\sigma - (1 - \kappa)^m \frac{E_c}{\kappa^m} (w_n - w_n^p) \right] \dot{w}_n + \left[\tau - (1 - \kappa)^m \frac{G_c}{\kappa^m} (\mathbf{w}_s - \mathbf{w}_s^p) \right] \cdot \dot{\mathbf{w}}_s \\ & + (1 - \kappa)^m \frac{E_c}{\kappa^m} (w_n - w_n^p) \dot{w}_n^p + (1 - \kappa)^m \frac{G_c}{\kappa^m} (\mathbf{w}_s - \mathbf{w}_s^p) \cdot \dot{\mathbf{w}}_s^p \\ & + \frac{m(1 - \kappa)^{m-1}}{2\kappa^{m+1}} [E_c (w_n - w_n^p)^2 + G_c (\mathbf{w}_s - \mathbf{w}_s^p) \cdot (\mathbf{w}_s - \mathbf{w}_s^p)] \dot{\kappa} \geq 0 \end{aligned} \quad (10)$$

Since $\dot{w}_n$ and $\dot{\mathbf{w}}_s$ are arbitrary, we conclude that

$$\sigma = E_{ceff} (w_n - w_n^p) \quad \text{and} \quad \tau = G_{ceff} (\mathbf{w}_s - \mathbf{w}_s^p) \quad (11)$$

where the effective elastic stiffnesses E_{ceff} and G_{ceff} are defined by

$$E_{ceff} = (1 - \kappa)^m \frac{E_c}{\kappa^m} \quad G_{ceff} = (1 - \kappa)^m \frac{G_c}{\kappa^m} \quad (12)$$

With expressions (11), the dissipation inequality (10) reduces to

$$\sigma \dot{w}_n^p + \tau \cdot \dot{\mathbf{w}}_s^p + K \dot{\kappa} \geq 0 \quad (13)$$

where the conjugated force K is defined by

$$K = \frac{m\kappa^{m-1}}{2(1 - \kappa)^{m+1}} \left(\frac{\sigma^2}{E_c} + \frac{\tau^2}{G_c} \right) \quad (14)$$

and

$$\tau^2 = \tau \cdot \tau \quad (15)$$

For elastic behavior, i.e. $\dot{w}_n^p = \dot{\kappa} = 0$ and $\dot{\mathbf{w}}_s^p = \mathbf{0}$, it follows that the process is nondissipative. For elastic unloading, we get from (11) that

$$\dot{\sigma} = E_{ceff} \dot{w}_n \quad \dot{\tau} = G_{ceff} \dot{\mathbf{w}}_s \quad (16)$$

When the crack just has been initiated, κ is, in principle, zero and the stiffnesses E_{ceff} and G_{ceff} then become infinitely large. This implies an infinitely large unloading stiffness precisely in accordance with the fictitious crack model and as discussed in relation to Fig. 1(a).

4. Loading surface

Let us next introduce the loading surface, i.e. the activation of which implies evolution of both plasticity and damage. In the first place, we consider the hyperbolic yield function used by Carol et al. (1997) and Červenka (1994), but written a slightly different form that gives a clear interpretation of the parameters. This hyperbola is given by $f = 0$ where

$$f = \tau^2 - \mu^2 \{ [(1 + k)\sigma_t - \sigma]^2 - k^2 \sigma_t^2 \} \quad (17)$$

where $\tau^2 = \tau \cdot \tau$, σ_t = uniaxial strength, μ = friction coefficient and k = dimensionless parameter (≥ 0). As apparent from Fig. 2(a), the larger the k -value, the larger the cusp of the hyperbola.

So far, no hardening or softening effects are present. To introduce softening effects, we take the loading function as

$$f = \tau^2 - \mu^2 \{ [(1 + k)(1 - \kappa)^n \sigma_t - \sigma]^2 - k^2 (1 - \kappa)^{2n} \sigma_t^2 \} \quad (18)$$

where $n > 0$ is a dimensionless parameter; this gives the evolution of the loading surface as shown in Fig. 2(b).

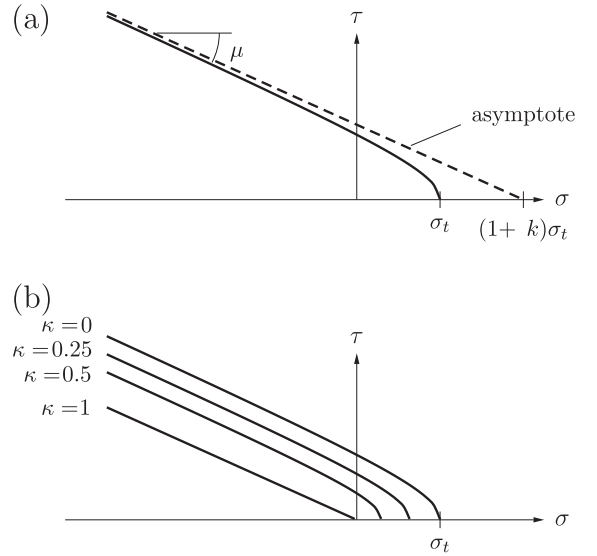


Fig. 2. (a) Interpretation of $f = 0$ where f is given by (17); (b) Loading criterion $f = 0$ where f is given by (18) showing softening caused by the damage parameter κ .

It is recalled that when the loading surface is activated, both plasticity and damage evolve. This simple assumption is in good agreement with experimental facts from uniaxial tension of, for instance, concrete showing that evolution of plasticity also means evolution of damage and *vice versa*. This observation is in contrast to the situation in, say, uniaxial compression where significant plasticity is required before damage develops. Apart from that, it is recalled that for cracking the entire process starts when the maximum principal stress reaches the uniaxial strength, i.e. at the point $(\sigma, \tau) = (\sigma_t, 0)$ in Fig. 2(b). For a general load history leading to evolution of damage and plasticity, the stress point moves along the loading curve, but – if the model is used to simulate cracking – no further crack initiation will occur; this means that we ignore the possible existence of secondary cracking.

For later calibration and simulation purposes rate equations will be derived. To obtain these equations the consistency equation is considered $\dot{f}(\sigma, \tau, \kappa) = 0$ which takes the format

$$C_1 \dot{\sigma} + C_2 \cdot \dot{\tau} + C_3 \dot{\kappa} = 0 \quad (19)$$

where

$$C_1 = \frac{\partial f}{\partial \sigma} \quad C_2 = \frac{\partial f}{\partial \tau} \quad C_3 = \frac{\partial f}{\partial \kappa} \quad (20)$$

i.e.

$$\begin{aligned} C_1 &= 2\mu^2 [(1 + k)(1 - \kappa)^n \sigma_t - \sigma], \quad C_2 = 2\tau \\ C_3 &= 2\mu^2 n \sigma_t [(1 + 2k)(1 - \kappa)^{2n-1} \sigma_t - (1 + k)(1 - \kappa)^{n-1} \sigma] \end{aligned} \quad (21)$$

From the stresses given by (11) we obtain

$$\begin{aligned} \dot{\sigma} &= -\frac{m}{\kappa(1 - \kappa)} \sigma \dot{\kappa} + E_{ceff} (\dot{w}_n - \dot{w}_n^p) \\ \dot{\tau} &= -\frac{m}{\kappa(1 - \kappa)} \tau \dot{\kappa} + G_{ceff} (\dot{\mathbf{w}}_s - \dot{\mathbf{w}}_s^p) \end{aligned} \quad (22)$$

By inserting these expressions in the consistency relation (19) we obtain

$$-B\dot{\kappa} - E_{ceff} C_1 \dot{w}_n^p - G_{ceff} C_2 \cdot \dot{\mathbf{w}}_s^p + E_{ceff} C_1 \dot{w}_n + G_{ceff} C_2 \cdot \dot{\mathbf{w}}_s = 0 \quad (23)$$

where

$$B = \frac{m}{\kappa(1 - \kappa)} (\sigma C_1 + 2\tau^2) - C_3 \quad (24)$$

where it was used that $C_2 \cdot \tau = 2\tau^2$.

5. Evolution laws

The next step is to choose the evolution laws for $\dot{w}_n^p$, $\dot{w}_s^p$ and $\dot{\kappa}$ such that the dissipation inequality (13) is fulfilled. For that purpose, we introduce two potential functions, the plasticity potential function $g_p(\sigma, \tau, \kappa)$ and the damage potential function $g_d(K, \sigma, \tau, \kappa)$. These potential functions are required to be convex and to possess the properties: $g_p(\sigma, \tau, \kappa) - g_p(0, 0, \kappa) \geq 0$ and $g_d(K, \sigma, \tau, \kappa) - g_d(0, \sigma, \tau, \kappa) \geq 0$. These requirements ensure that the dissipation inequality (13) is fulfilled by the following evolution laws

$$\dot{w}_n^p = \dot{\lambda} \frac{\partial g_p}{\partial \sigma}, \quad \dot{w}_s^p = \dot{\lambda} \frac{\partial g_p}{\partial \tau}, \quad \dot{\kappa} = \dot{\lambda} \frac{\partial g_d}{\partial K} \quad \text{and} \quad \dot{\lambda} \geq 0, \quad \dot{\lambda} f = 0 \quad (25)$$

In analogy with the loading function (18), we choose the plastic potential function as

$$g_p = \tau^2 - \bar{\mu}^2 \{ [(1 + \bar{k})(1 - \kappa)^n \sigma_t - \sigma]^2 - \bar{k}^2 (1 - \kappa)^{2n} \sigma_t^2 \} \quad (26)$$

where $\bar{\mu}$ and $\bar{k}$ are dimensionless parameters; for simplicity, the exponent n is the same as in the loading function. According to (25) the damage potential function g_d must depend on the conjugated force K and the simplest possible expression is given by

$$g_d = \frac{K}{N(\sigma, \tau, \kappa)} \quad (27)$$

where the positive function $N(\sigma, \tau, \kappa)$ will be specified later when the model is calibrated. With these expressions for the potential functions, the evolution laws (25) become

$$\dot{w}_n^p = \dot{\lambda} \bar{C}_1, \quad \dot{w}_s^p = \dot{\lambda} \bar{C}_2, \quad \dot{\kappa} = \frac{\dot{\lambda}}{N} \quad (28)$$

where

$$\bar{C}_1 = \frac{\partial g_p}{\partial \sigma} = 2\bar{\mu}^2 [(1 + \bar{k})(1 - \kappa)^n \sigma_t - \sigma], \quad \bar{C}_2 = \frac{\partial g_p}{\partial \tau} = 2\tau \quad (29)$$

Insertion of the evolution laws (28) in the consistency relation (23) determines the multiplier $\dot{\lambda}$ and we obtain

$$\dot{\lambda} = \frac{N}{A} (C_1 E_{\text{ceff}} \dot{w}_n + G_{\text{ceff}} \mathbf{C}_2 \cdot \dot{\mathbf{w}}_s) \quad (30)$$

where

$$A = B + N(C_1 \bar{C}_1 E_{\text{ceff}} + \mathbf{C}_2 \cdot \bar{\mathbf{C}}_2 G_{\text{ceff}}) \quad (31)$$

With the multiplier $\dot{\lambda}$ given by (30) the damage evolution takes the form

$$\dot{\kappa} = \frac{1}{A} (C_1 E_{\text{ceff}} \dot{w}_n + G_{\text{ceff}} \mathbf{C}_2 \cdot \dot{\mathbf{w}}_s) \quad (32)$$

In these expressions, all terms are known except for the positive function $N(\sigma, \tau, \kappa)$. It appears from (30) that the “elastic trial stress” rate $C_1 E_{\text{ceff}} \dot{w}_n + G_{\text{ceff}} \mathbf{C}_2 \cdot \dot{\mathbf{w}}_s$ determines whether loading occurs or not; if this quantity is positive, zero or negative then we have loading, neutral loading and unloading, respectively.

As already mentioned, the only issue that remains to be specified is the function N entering the damage evolution law (28).

6. Identification of function N

To identify the, as yet, unknown function N present in the damage evolution law (28), the following simple strategy is adopted. Considering uniaxial tension, an analytic expression for the response $\sigma = \sigma(w_n)$ is chosen and it is required that the general constitutive model should reproduce this behavior in uniaxial tension exactly; this will identify the function N .

For uniaxial tension, we will here choose the following simple exponential relation

$$\sigma = \sigma_t e^{-b w_n} \quad (33)$$

where b is some parameter with dimension [1/m]; this format was suggested by Gopalaratnam and Shah (1985) and there shown to be in close agreement with experimental data for concrete. Indeed, this exponential format is often adopted in finite deformation analysis of cracks and interfaces, see e.g. Mergheim et al. (2007). The fracture energy G_f [N/m] is defined by $G_f = \int_0^\infty \sigma dw_n$ and with the expression above, we conclude that the parameter b is given by

$$b = \frac{\sigma_t}{G_f} \quad \text{i.e.} \quad \sigma = \sigma_t e^{-\frac{\sigma_t}{G_f} w_n} \quad (34)$$

This leads to

$$\dot{\sigma} = -\frac{\sigma_t}{G_f} \sigma \dot{w}_n \quad (35)$$

For uniaxial tension, the loading criterion (18) reduces to $\sigma = (1 - \kappa)^n \sigma_t$, i.e. $\dot{\sigma} = -n(1 - \kappa)^{n-1} \sigma_t \dot{\kappa}$ which together with the expression above results in the following damage evolution

$$\dot{\kappa} = (1 - \kappa) \frac{\sigma_t}{n G_f} \dot{w}_n \quad (36)$$

The general expression for damage evolution is given by (32) where N is unknown, i.e. the quantity A is unknown. In view of the result (36), it is tempting to write A as

$$A = \frac{1}{1 - \kappa} \frac{n G_f}{\sigma_t} S \quad (37)$$

where the quantity S is unknown. With this format, the general damage evolution law (32) reads

$$\dot{\kappa} = (1 - \kappa) \frac{\sigma_t}{n G_f S} (C_1 E_{\text{ceff}} \dot{w}_n + G_{\text{ceff}} \mathbf{C}_2 \cdot \dot{\mathbf{w}}_s) \quad (38)$$

For uniaxial tension $\tau = 0$, i.e. the quantity $\mathbf{C}_2 = 0$, cf. (21), and (38) then reduces to expression (36) valid for uniaxial tension if S is taken as $S = C_1 E_{\text{ceff}}$. However, we may equally well choose S as

$$S = C_1 E_{\text{ceff}} + (\mathbf{C}_2 \cdot \mathbf{C}_2)^{1/2} G_{\text{ceff}} \quad (39)$$

since this format reduces to $S = C_1 E_{\text{ceff}}$ for uniaxial tension. The expression above has the advantage that it preserves a symmetry in the weighting of normal loading and shear loading.

We are now, finally, able to identify the function N explicitly. Insertion of (37) in (31) results in

$$N(\sigma, \tau, \kappa) = \frac{\frac{n G_f}{\sigma_t (1 - \kappa)} S - B}{C_1 \bar{C}_1 E_{\text{ceff}} + \mathbf{C}_2 \cdot \bar{\mathbf{C}}_2 G_{\text{ceff}}} \quad (40)$$

With this result, the formulation of the general constitutive model is finalized.

7. Tangential stiffness matrix

For the calibration of the model, the tangential stiffness matrix will be established and the corresponding rate equations are then solved using standard solvers for ordinary differential equations; a fourth order Runge–Kutta with error control is used.

Before that is derived, we make a summary of some of the equations. The quantity S is defined by (39) and the evolution law for $\dot{\kappa}$ by (38). The evolution laws for $\dot{w}_n^p$ and $\dot{w}_s^p$ are given by (28) which involves the multiplier $\dot{\lambda}$. This quantity is determined by (30) where the term N/A enters. We define the quantity R as $R = N/A$ and obtain from (37) and (40) that

$$R = \frac{N}{A} = \frac{1 - \frac{B(1-\kappa)\sigma_t}{nG_f S}}{C_1 \bar{C}_1 E_{\text{ceff}} + C_2 \cdot \bar{C}_2 G_{\text{ceff}}} \quad (41)$$

With these expressions, the evolution laws (28) for $\dot{\kappa}$, $\dot{w}_n^p$ and $\dot{w}_s^p$ are now completely defined. Insertion of these evolution laws in (22) provides the following tangential stiffness matrix

$$\begin{bmatrix} \dot{\sigma} \\ \dot{\tau} \end{bmatrix} = \begin{bmatrix} k_{11} & \mathbf{k}_{12} \\ \mathbf{k}_{21} & \mathbf{k}_{22} \end{bmatrix} \cdot \begin{bmatrix} \dot{w}_n \\ \dot{w}_s \end{bmatrix} \quad (42)$$

where

$$\begin{aligned} k_{11} &= E_{\text{ceff}} - \frac{\sigma}{\kappa} \frac{m\sigma_t}{nG_f S} C_1 E_{\text{ceff}} - RC_1 \bar{C}_1 E_{\text{ceff}}^2 \\ \mathbf{k}_{22} &= G_{\text{ceff}} \mathbf{I} - \frac{1}{\kappa} \frac{m\sigma_t}{nG_f S} G_{\text{ceff}} \boldsymbol{\tau} \otimes \mathbf{C}_2 - G_{\text{ceff}}^2 R \bar{C}_2 \otimes \mathbf{C}_2 \\ \mathbf{k}_{12} &= -\frac{\sigma}{\kappa} \frac{m\sigma_t}{nG_f S} \mathbf{C}_2 G_{\text{ceff}} - RC_2 \bar{C}_1 E_{\text{ceff}} G_{\text{ceff}} \\ \mathbf{k}_{21} &= -\frac{\boldsymbol{\tau}}{\kappa} \frac{m\sigma_t}{nG_f S} C_1 E_{\text{ceff}} - RC_1 \bar{C}_2 E_{\text{ceff}} G_{\text{ceff}} \end{aligned} \quad (43)$$

For unloading, the tangential response is given by (16).

Observing that κ enters as a denominator in some terms in (43) and that E_{ceff} and G_{ceff} have κ^m in the denominator, a natural question is to investigate what happens when κ is close to zero. In general, for interface modeling the tangential stiffness will therefore be infinitely large when loading is initiated. However, for modeling of cracking a finite tangential stiffness will apply. To identify this tangential stiffness, it is important to observe that the entire crack process starts when the maximum principal stress reaches the uniaxial strength σ_t . For uniaxial tension, the loading condition (18) provides $\sigma = (1 - \kappa)^n \sigma_t$. With (21) and (29) we then obtain

$$C_1 = 2\mu^2 \sigma_t k (1 - \kappa)^n, \quad \mathbf{C}_2 = \bar{\mathbf{C}}_2 = \mathbf{0}, \quad C_3 = 2\mu^2 n \sigma_t^2 k (1 - \kappa)^{2n-1}$$

During uniaxial tension, we have $\dot{w}_n > 0$, $\dot{w}_s = 0$ and from (43) we then get $\dot{\sigma} = k_{11} \dot{w}_n$ and $\dot{\tau} = \mathbf{k}_{21} \dot{w}_n$, but $\mathbf{k}_{21} = \mathbf{0}$ and we are left with $\dot{\sigma} = k_{11} \dot{w}_n$. To evaluate the term k_{11} , we first determine the quantity B defined by (24), i.e.

$$B = 2\mu^2 \sigma_t^2 k (1 - \kappa)^{2n-1} \left(\frac{m}{\kappa} - n \right)$$

With R given by (41), we then arrive at

$$RC_1 \bar{C}_1 E_{\text{ceff}}^2 = E_{\text{ceff}} - \frac{B(1 - \kappa)\sigma_t}{nG_f S} E_{\text{ceff}}$$

Use of these expressions identify the terms k_{11} and we finally end up with

$$\dot{\sigma} = -\frac{\sigma_t}{G_f} \sigma \dot{w}_n$$

We are not surprised that this is exactly the expression for uniaxial tension that we started with when the function N was identified, cf. (35). Moreover, we have demonstrated that as the crack process starts with uniaxial tension, the tangential stiffness is finite.

In the literature, a number of crack/interface models have been proposed where the physical relevance is substantiated by qualitative arguments only. Here, we will compare model predictions with a wide range of experimental data and, as an example, we consider cracking of concrete.

8. Experimental verification of the model

8.1. Combined normal and shear loading

For concrete, Hassanzadeh (1990) performed experiments for combined normal and shear loading. The specimen was first loaded in uniaxial tension so that a crack was initiated. Then, as shown in

Fig. 3 three types of loading directions were investigated; it appears that an angle equal to 90° corresponds to uniaxial tension.

To calibrate the parameters of the model to the experimental data, it is first estimated that the uniaxial strength is $\sigma_t = 2.65 \text{ N/mm}^2$. The fracture energy is taken as $G_f = 0.1 \text{ N/mm}$ ($= 100 \text{ N/m}$) which is a typical value for concrete, cf. for instance Wolinski et al. (1987).

In the experiments, the load paths shown in Fig. 3 are applied after the specimen has been exposed to uniaxial tension and cracking has just been initiated. In the simulations, uniaxial tension was applied until the stress has dropped to $\sigma = 0.95\sigma_t$. Using simplex optimization and for a fourth-order Runge–Kutta solver, the remaining parameters were then identified. In conclusion, the following parameters were adopted:

Helmholz energy :	$E_c = 91.7 \text{ N/mm}^3$, $G_c = E_c/2$
	$m = 0.806$
Loading function :	$\sigma_t = 2.65 \text{ N/mm}^2$, $n = 2.77$,
	$k = 1.32$, $\mu = 1.72$
Plastic potential function :	$\bar{k} = 3.41$, $\bar{\mu} = 6.05$
Damage potential function :	$G_f = 0.1 \text{ N/mm}$ ($= 100 \text{ N/m}$)

(44)

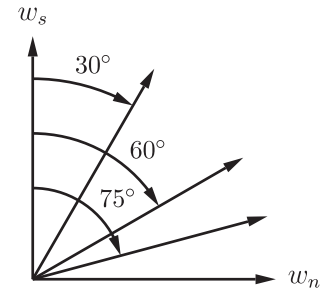


Fig. 3. Loading paths in the experiments of Hassanzadeh (1990); an angle = 90° corresponds to uniaxial tension.

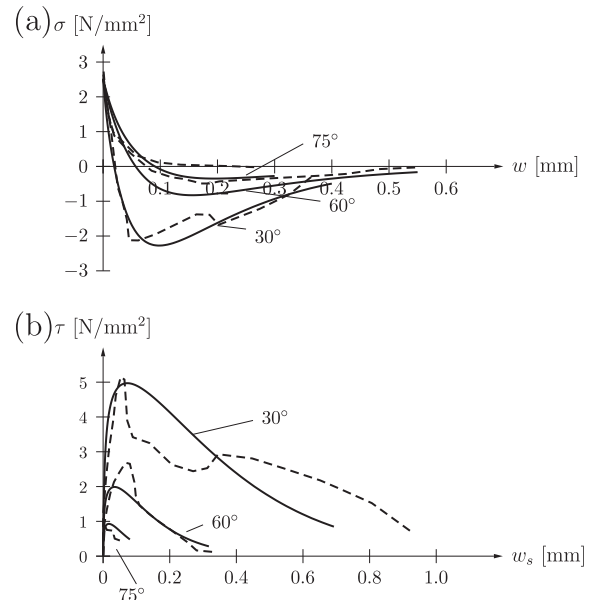


Fig. 4. Experimental data of Hassanzadeh (1990) and predictions; (a) normal stress, (b) shear stress. Solid lines indicate model predictions and dashed lines experimental data.

The experimental data as well as the predictions are shown in Fig. 4(a) and (b) and it appears that a close agreement is achieved.

As mentioned in the Introduction, a large number of crack models have been proposed in the literature. However, only a few of these make comparison with experimental data and two noteworthy examples are the models proposed by Carol et al. (1997) and Nielsen et al. (2010). They also made comparisons with the experimental data of Hassanzadeh (1990) discussed above and good agreement was obtained; however, their verifications were confined to a comparison with these complex load paths.

With the parameters given by (44) we shall now perform further comparisons with experimental data.

8.2. Uniaxial tension

We will now compare the model predictions with the experimental data of Reinhardt et al. (1986) for concrete in uniaxial tension where also the unloading response was identified. In the tests, a slot was made in the specimen so that the position of the crack becomes well-known. The elongations were then measured over a distance of 35 mm and the results are shown in Fig. 5.

From the initial slope of this curve, the elastic stiffness k , where $\sigma = k\delta^e$, was determined to $k = 1.18 \cdot 10^3 \text{ N/mm}^3$ and this results in the elastic response over the measuring length shown in Fig. 6(a). From Fig. 5, the uniaxial strength is taken as $\sigma_t = 3.2 \text{ N/mm}^2$, but apart from that we again use the parameters given by (44); the result for the crack response is then given in Fig. 6(a). To determine the unloading behavior, the stress at unloading is given by the loading function as $\sigma = (1 - \kappa)^n \sigma_t$. This determines the damage variable κ and the quantity E_{ceff} that controls unloading, cf. (16), is then known. Adding the elongations shown in Fig. 6(b) and Fig. 6(b) we arrive at Fig. 7 which should be compared with the experimental data in Fig. 5.

It is not surprising that a satisfactory agreement is obtained for loading, since the model predictions are controlled by the expo-

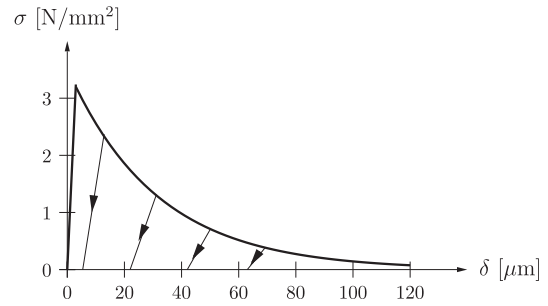


Fig. 7. Predictions that is to be compared with the experimental data of Fig. 5.

nential format given by (33) which we knew, in beforehand, to be of sufficient accuracy. What is of more interest is the excellent agreement also for the unloading stiffnesses and it is emphasized, once more, that we use again the parameters given by (44) – except for the σ_t -value.

8.3. Initial shear stiffness for open cracks

In a number of investigations the concrete specimen is first exposed to uniaxial tension until a certain crack opening w_n is obtained. Then this crack width is held constant and the specimen is now loaded in shear; the paper of Yoshikawa et al. (1989) can be consulted for more information. For the initial shear stiffness, we will now compare experimental data and predictions.

In the model, the crack opening w_n is known and (33) then gives information of the stress σ . Since uniaxial tension occurs initially, the loading function (18) provides $\sigma = (1 - \kappa)^n \sigma_t$, i.e. the damage variable κ is now known. When shearing is applied, we have $\dot{w}_n = 0$ and (42) gives $\dot{\tau} = k_{22} \dot{w}_s$ and as $\tau = 0$ holds initially, $C_2 = \bar{C}_2 = 0$ applies and it follows from (43) that $k_{22} = G_{\text{ceff}} \mathbf{I}$. This implies $\dot{\tau} = G_{\text{ceff}} \dot{w}_s$ and as the loading is plane we get $\dot{\tau} = G_{\text{ceff}} \dot{w}_s$ where G_{ceff} is known.

For the initial shear stiffness, Yoshikawa et al. (1989) collected a number of experimental data from the literature and introduced a correction factor that considers the value of the cylinder strength f'_c . For $f'_c = 35 \text{ N/mm}^2$, these data points are shown in Fig. 8. It should be observed that most of these points refer to large crack openings. As an example, consider the crack opening $w_n = 0.2 \text{ mm}$ in Fig. 8 and compare with the experimental data of Reinhardt et al. (1986) for uniaxial tension as shown in Fig. 5; it is evident that the crack opening is very wide.

In the predictions, the uniaxial strength is chosen as $\sigma_t = 2.5 \text{ N/mm}^2$, but apart from that the parameters given by (44) are again used. The predictions are also shown in Fig. 8 and considering the remarks above, the agreement is fully acceptable.

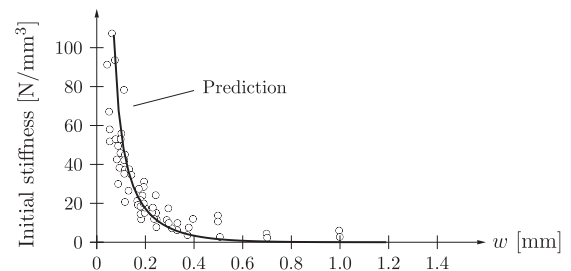


Fig. 8. Initial shear stiffness. Experimental data collected by Yoshikawa et al. (1989) and corrected to cylinder strength $f'_c = 35 \text{ N/mm}^2$. Prediction is also shown indicated by the solid line.

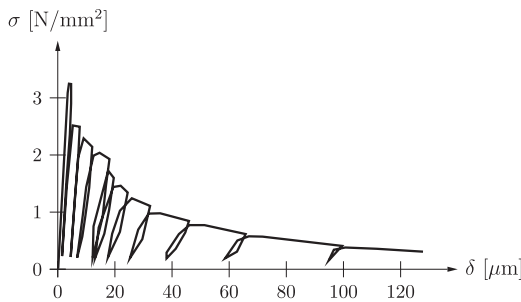


Fig. 5. Experimental data of Reinhardt et al. (1986). The measurements include crack elongation as well as elastic response of the material over the measuring length = 35 mm.

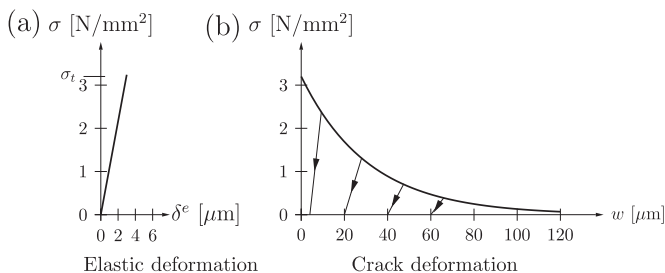


Fig. 6. Predictions. (a) Elastic deformation over the measuring length; (b) crack opening.

9. Conclusions

For small deformations, a thermodynamically based fictitious crack/interface model has been proposed for general 3D normal and shear loading. It will appear that this material model is conceptually very simple as it only involves the following choices: Helmholtz' free energy (9), loading function (18), plastic potential function (26) and damage potential function (27). The latter expression involves the, as yet, unknown function N . This function is identified by choice of an analytic expression $\sigma = \sigma(w_n)$ for the response in uniaxial tension; here we have adopted the simple exponential function given by (33). The model contains 9 parameters and, as a comparison, the model of Carol et al. (1997) involves 11 parameters and the proposal of Nielsen et al. (2010) contains 13 parameters.

An essential feature of the model is its consistency with the concepts behind the fictitious crack/interface model. In particular, no crack deformation occurs before the crack is initiated and when a crack has just been initiated the proposed model provides an unloading stiffness that is infinitely large. Apart from that, it was demonstrated that for the same set of parameters, the model is able to provide predictions that are in close agreement with experimental data for a wide range of loading situations.

References

- Allix, O., Corigliano, A., 1999. Geometrical and interfacial non-linearities in the analysis of delamination in composites. *International Journal of Solids and Structures* 36, 2189–2216.
- Carol, I., Prat, P.C., 1990. A statically constrained microplane model for the smeared analysis of concrete cracking. In: Bićanić, N., Mang, H. (Eds.), *Computer Aided Analysis and Design of Concrete Structures*. Pineridge Press, Swansea, UK, pp. 919–930.
- Carol, I., Prat, P.C., López, C.M., 1997. Normal/shear cracking model: application to discrete crack analysis. *Journal of Engineering Mechanics* 123, 765–773.
- Červenka, J., 1994. Discrete crack modeling in concrete structure. Ph.D. thesis, University of Colorado at Boulder.
- Červenka, J., Kishen, J.M.C., Saouma, V.E., 1998. Mixed mode fracture of cementitious bimaterial interfaces; part II: Numerical simulation. *Engineering Fracture Mechanics* 60, 95–107.
- de Borst, R., 2003. Numerical aspects of cohesive-zone models. *Engineering Fracture Mechanics* 70, 1743–1757.
- Fagerström, M., Larsson, R., 2006. Theory and numerics for finite deformation fracture modelling using strong discontinuities. *International Journal for Numerical Methods in Engineering* 66, 911–948.
- Gálvez, J.C., Červenka, J., Cendón, D., Saouma, V., 2002a. A discrete crack approach to normal/shear cracking of concrete. *Cement and Concrete Research* 32, 1567–1585.
- Gálvez, J.C., Cendón, D.A., Planas, J., 2002b. Influence of shear parameters on mixed-mode fracture of concrete. *International Journal of Fracture* 118, 163–189.
- Gopalaratnam, V.S., Shah, S.P., 1985. Softening response of plain concrete in direct tension. *ACI Journal* 82, 310–323.
- Hassanzadeh, M., 1990. Determination of fracture zone properties in mixed mode I and II. *Engineering Fracture Mechanics* 35, 845–853.
- Hillerborg, A., Modér, M., Petersson, P.-E., 1976. Analysis of crack formation and crack growth in concrete by means of fracture mechanics and finite elements. *Cement and Concrete Research* 6, 773–782.
- Jirásek, M. and Belytschko, T., 2002. Computational resolution of strong discontinuities. In: M.H.A., R.F.G., (Eds.), *Proceeding World Congress on Computational Mechanics, WCCM V*.
- Jirásek, M., Zimmermann, T., 2001. Embedded crack model: I. Basic formulation. *International Journal for Numerical Methods in Engineering* 50, 1269–1290.
- Larsson, R., Jansson, N., 2002. Geometrically non-linear damage interface based on regularized strong discontinuity. *International Journal for Numerical Methods in Engineering* 54, 473–497.
- Mergheim, J., Kuhl, E., Steinmann, P., 2007. Towards the algorithmic treatment of 3D strong discontinuities. *Communications in Numerical Methods in Engineering* 23, 97–108.
- Mosler, J., Scheider, I., 2011. A thermodynamically and variationally consistent class of damage-type cohesive models. *Journal of the Mechanics and Physics of Solids* 59, 1647–1668.
- Needleman, A., 1987. A continuum model for void nucleation by inclusion debonding. *Journal of Applied Mechanics* 54, 525–531.
- Nielsen, L.O., Mougard, J.F., Jacobsen, J.S., Poulsen, P.N., 2010. A mixed mode model for fracture in concrete. In: *Fracture Mechanics of Concrete and Concrete Structures, Proceedings of the 7th International Conference on Fracture Mechanics of Concrete and Concrete Structures*, Korea Concrete Institute, Seoul, pp. 231–237.
- Ortiz, M., Pandolfi, A., 1999. Finite-deformation irreversible cohesive elements for three-dimensional crack-propagation analysis. *International Journal for Numerical Methods in Engineering* 44, 1267–1282.
- Ottosen, N.S., 1986. Thermodynamic consequences of strain softening in tension. *Journal of Engineering Mechanics* 112, 1152–1164.
- Ottosen, N.S., Ristinmaa, M., 2005. *The Mechanics of Constitutive Modeling*. Elsevier.
- Park, K., Paulino, G.H., Roesler, J.R., 2009. A unified potential-based cohesive model of mixed-mode fracture. *Journal of the Mechanics and Physics of Solids* 57, 891–908.
- Reinhardt, H.W., Cornelissen, H.A.W., Hordijk, D.A., 1986. Tensile test and failure analysis of concrete. *Journal of Structural Engineering* 112, 2462–2477.
- van der Bosch, M.J., Schreurs, P.J.G., Geers, M.G.D., 2006. An improved description of the exponential Xu and Needleman cohesive zone law for mixed-mode decohesion. *Engineering Fracture Mechanics* 73, 1220–1234.
- Wolinski, S., Hordijk, D.A., Reinhardt, H.W., Cornelissen, H.A.W., 1987. Influence of aggregate size on fracture mechanics parameters of concrete. *International Journal of Cement Composites and Lightweight Concrete* 9, 95–103.
- Xu, X.-P., Needleman, A., 1993. Void nucleation by inclusion depending in a crystal matrix. *Modelling and Simulation in Materials Science and Engineering* 1, 111–132.
- Yoshikawa, H., Wu, Z., Tanabe, T.-A., 1989. Analytical model for shear slip of cracked concrete. *Journal of Structural Engineering* 115, 771–788.