

# On configurational forces in multiplicative elastoplasticity

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Received 10 May 2006; received in revised form 23 September 2006  
Available online 26 November 2006

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## Abstract

The main goal of this work consists in the elaboration of the material or rather configurational mechanics in the context of multiplicative elastoplasticity. This nowadays well-established approach, which is inherently related to the concept of a material isomorphism or in other words to a local rearrangement, is adopted as a paradigm for the general modelling of finite inelasticity. The overall motion in space is throughout assumed to be compatible and sufficiently smooth. According to the underlying configurations, namely the material and the spatial configuration as well as what we call the intermediate configuration, different representations of balance of linear momentum are set up for the static case. The underlying flux terms are thereby identified as stress tensors of Piola and Cauchy type and are assumed to derive from a free energy density function, thus taking hyperelastic formats. Moreover, the incorporated source terms, namely the configurational volume forces, are identified by comparison arguments. These quantities include gradients of distortions as well as dislocation density tensors. In particular those dislocation density tensors related to the elastic or plastic distortion do not vanish due to the general incompatibility of the intermediate configuration. As a result, configurational volume forces which are settled in the intermediate configuration embody non-vanishing dislocation density tensors while their material counterparts directly incorporate non-vanishing gradients of distortions. This fundamental property enables us to recover the celebrated Peach–Koehler force for finite inelasticity, acting on a single dislocation, from the intermediate configuration volume forces.

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**Keywords:** Configurational mechanics; Material forces; Peach–Koehler force; Continuum dislocation mechanics; Incompatible configurations; Multiplicative elastoplasticity

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## 1. Introduction

The commonly applied framework of Newtonian mechanics addresses the movement of particles in physical space. Contrary, within Eshelbian or configurational mechanics emphasis is placed on variations of placements of particles in material space. The first approach will consequently be denoted as the spatial motion

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problem in this work while the latter approach is referred to as the material motion problem. The concept of configurational mechanics is particularly suited for the modelling of defects, dislocations, inhomogeneities, heterogeneities, phase boundaries and so forth in, e.g., solid mechanics since these phenomena are driven by configurational forces as present in configurational balance of linear momentum representations. A brief review of fundamental concepts is given by, for instance, [Podio-Guidugli \(2001, 2002\)](#). The consideration of forces acting on defects dates back to the pioneering works by [Eshelby \(1951, 1956\)](#). Nowadays several monographs elaborate the concept of configurational mechanics and mechanics in material space, see, for instance, [Maugin \(1993\)](#), [Šilhavý \(1997\)](#), [Gurtin \(2000\)](#) and references cited therein or the survey article by [Steinmann \(2002b\)](#) and the contributions in [Steinmann and Maugin \(2005\)](#). Related numerical formulations based on purely elastic response date back to the contributions by [Govindjee and Mihalic \(1996\)](#), [Braun \(1997\)](#) and [Steinmann \(2000\)](#). Computational strategies for plastic response have recently been discussed in [Menzel et al. \(2004, 2005a\)](#) and [Nguyen et al. \(2005\)](#). Moreover, the constitutive modelling of finite inelasticity in the present context has been addressed by, e.g., [Svendsen \(2001\)](#), [Cleja-T. frigoiu and Maugin \(2000\)](#), [Menzel and Steinmann \(2003\)](#), [Menzel et al. \(2005b\)](#) and [Gross et al. \(2003\)](#) among others. Eshelby type stress tensors thereby commonly serve as the driving quantity for the plastic distortion rate or for an appropriate symmetric part thereof. For specific applications these stress tensors are often replaced with Mandel type stresses, compare [Mandel \(1974\)](#).

In this contribution, however, particular emphasis is placed on the configurational balance of linear momentum expressions and related configurational volume forces stemming from inelastic deformation processes. Elaborations in this context often incorporate a local rearrangement, or rather material isomorphism, based on an additional linear tangent map or directly apply a multiplicative decomposition of the total deformation gradient; see, for instance, the pioneering works by [Noll \(1958, 1967, 1972\)](#) or [Truesdell and Noll \(2004\)](#), [Wang and Truesdell \(1973\)](#) and [Lee \(1969\)](#) among others. For a review on the underlying basic concepts we refer the reader to the recent monographs by [Haupt \(2000\)](#), [Lubarda \(2002\)](#) and [Bertram \(2005\)](#) as well as to references cited therein. This local rearrangement transformation or the inelastic part of the total deformation gradient, respectively, allows interpretation as plastic distortion. Correlated configurational volume forces have so far been related to the reference or rather material configuration. They usually incorporate, besides contributions stemming from explicit dependencies on material placements, gradients of the plastic distortion. Detailed derivations are highlighted in a series of papers by [Epstein and Maugin \(1990, 1996\)](#), [Maugin and Epstein \(1998\)](#), [Epstein \(2002\)](#) and [Maugin \(2003\)](#), see also [Epstein and Bucataru \(2003\)](#). Skew-symmetric portions of the plastic distortion gradient allow interpretation as dislocation density tensor as related to appropriate Burgers density vectors. The underlying configurational volume forces accordingly possess contributions stemming from geometrically necessary dislocations. This property turns out to be of cardinal importance and establishes a connection on one hand with classical continuum dislocation theories, see, for instance, [Kondo \(1952\)](#), [Bilby et al. \(1955\)](#), [Kosevich \(1979\)](#), [de Wit \(1981\)](#) or the textbooks by [Phillips \(2001\)](#) and [Hull and Bacon \(2001\)](#), and on the other hand with non-local continuum theories, see, for instance, the contributions in [Rogula \(1982\)](#) and [Eringen \(2002\)](#) as well as references cited therein. Early works based on the incorporation of dislocation density tensors date back to the contributions by, e.g., [Seeger \(1955\)](#), [Kröner \(1958, 1960\)](#) and [Kröner and Seeger \(1959\)](#). Several continuum dislocation theories have been advocated since then, whereby special emphasis has in particular been placed on the modelling of geometrically necessary dislocations. Naming solely a brief personal collection of related references, we refer the reader to the contributions by, e.g., [Steinmann \(1996\)](#), [Le and Stumpf \(1996a,b, 1998\)](#), [Menzel and Steinmann \(2000\)](#), [Acharya and Bassani \(2000\)](#), [Cermelli and Gurtin \(2001, 2002\)](#), [Gurtin \(2002, 2004\)](#) and [Gurtin and Needleman \(2005\)](#).

As previously emphasised, we elaborate configurational balance of linear momentum representations with respect to what we call the material, intermediate and spatial configuration and derive correlated volume forces in the following. It turns out that the deduced formats with respect to the intermediate configuration recapture the celebrated Peach–Koehler force, compare [Peach and Koehler \(1950\)](#). Fundamental characteristics of this force, which drives the movement of single dislocations, are reviewed in various publications, monographs and textbooks, see e.g., [Kosevich \(1979\)](#), [Maugin \(1993\)](#), [Nembach \(1997\)](#), [Phillips \(2001\)](#), [Hull and Bacon \(2001\)](#), [Indenbom and Orlov \(1968\)](#), [Ericksen \(1995, 1998\)](#), [Rogula \(1977\)](#) and references cited in these works. In this work, we mainly follow the lines of derivation presented in [Steinmann \(2002a\)](#) and [Menzel and Steinmann \(2005\)](#) and develop a rigorous framework for configurational balance of linear momentum representa-

tions embedded into the finite deformation kinematics of multiplicative elastoplasticity. As a special application, particular volume forces recapture the celebrated Peach–Koehler force which has also been applied to the modelling of plasticity by other authors, see, for instance, the recent publication by Han et al. (2005) among others. For the sake of clarity, the structure of the manuscript is somewhat formal. Conceptually speaking, we set the stage – for essential kinematical concepts, balance of linear momentum, hyperelastic formats and volume forces – by introducing the spatial, material and intermediate motion problem. Sought relations are then obtained by comparison arguments so that derivations of some expression are repeated. Apparently, this highly structured approach turns out to be rather helpful for a systematic and rigorous formulation of configurational balance relations. These derivations are, besides other essential aspects, based on the following key approaches:

- (i) Piola transformations are consequently applied to various two-point tensors, for instance dislocation density tensors and stress tensors – see e.g., Eqs. (4) and (38) – so that appropriate representations of these quantities with respect to solely one single configuration are obtained. The fundamental Piola transformation is highlighted in, for instance, the pioneering monograph by Murnaghan (1951).
- (ii) The commonly applied Piola identity is no longer valid for incompatible configurations. Consequently, divergence operations with respect to the intermediate configuration must be modified according to the underlying incompatibilities so that correlated balance of linear momentum representations take a non-standard format, see e.g., Eqs. (39), (43), (45), (47). A detailed derivation is developed in Appendix C.
- (iii) Based on a free energy density function, stress tensors of hyperelastic format are introduced. It is of cardinal importance to precisely distinguish between those quantities which are fixed and those with respect to which we compute derivatives, see Section 4 and also the related discussion in Ogden (2001). It turns out that both introduced Cauchy stress tensors of the spatial motion problem, as well as both Cauchy type stresses of the intermediate motion problem coincide pairwise. The Cauchy type stress tensors which are related to the material motion problem, however, possess different representations.
- (iv) The introduced hyperelastic stresses are compared within a rather formal scheme in Section 4.4: we give attention to three different combinations of two different Piola type stress tensors which pairwise point to one and the same configuration. Related Cauchy stress tensors can directly be compared with each other since they are also settled in one and the same configuration. Taking finally those Piola type stress tensors into account which point in the opposite direction as those Piola type stresses considered above, renders combinations with their correlated Cauchy stress tensors. The three different types of examined Cauchy stresses are apparently settled in three different configurations, i.e., in what we call the spatial, intermediate and material configuration.
- (v) With these stress tensors in hand, appropriate representations of volume forces in different configurations are derived by the key assumption that the corresponding balances of linear momentum formats are related by pullback or pushforward transformations.

The paper is organised as follows: essential kinematics are reviewed in Section 2. This covers fundamental concepts according to the multiplicative decomposition of the deformation gradient as well as the introduction and comparison of correlated dislocation density tensors. The applied dislocation density tensors, which are introduced as two-point tensors as well as quantities being settled in only one single configuration, capture the description of geometrically necessary dislocations which is usually not explicitly mentioned. Here, and in the subsequent section particular emphasis is placed on the spatial, intermediate and material motion problem and key results are additionally summarised in comprehensive tables for convenience of the reader. Different representations of balance of linear momentum are addressed in Section 3 whereby the intermediate and material motion problem are introduced by analogy with the spatial motion problem. It turns out to be of cardinal importance to take the modified Piola identity into account since the intermediate configuration is in general incompatible. Hyperelastic stress formats for the fluxes in the balance of linear momentum are stated in Section 4 whereby elaborations for the intermediate and material motion problem again follow by arguments of duality. Both, the appropriate definition of an underlying free energy density as well as a precise distinction between those quantities which are fixed and those with respect to which derivatives are computed are essential steps for the definition of these hyperelastic formats. Based on these elaborations, different balance of linear

momentum representations are compared in Section 5. This part constitutes the main body of this work since configurational volume forces, i.e., the source terms in the balance of linear momentum, are explicitly identified. They naturally incorporate gradients of the elastic and plastic distortion, the total deformation gradient or correlated dislocation density tensors. Moreover, the hyperelastic formats introduced in Section 4 are recovered. A classical application is reviewed in Section 6, namely the celebrated Peach–Koehler force acting on a single dislocation. The paper is concluded with a discussion in Section 7 where also the properties and applications of essential balance of linear momentum representations are addressed. Important but technical derivations, as for instance the extension of the classical Piola identity to incompatible configurations, are highlighted in Appendices A–C.

Before we begin, a word on notation: the position vector in Euclidian space is identified with  $\mathbf{X}$ . Furthermore, let  $\mathbf{a}$ ,  $\mathbf{b}$  denote two vectors and  $\mathbf{A}$ ,  $\mathbf{B}$ ,  $\mathbf{C}$  three second-order tensors. The third-order permutation tensor is abbreviated by  $\mathbf{E}$  possessing the properties  $\mathbf{E} : \mathbf{E} = 2\mathbf{I}$ ,  $\mathbf{E} \cdot \mathbf{E} = 2\mathbf{I}^{\text{skw}}$  and  $\mathbf{E} \cdot \mathbf{E} : \mathbf{A} = 2\mathbf{A}^{\text{skw}}$ ; see also Appendix A where further transformations and non-Cartesian representations are additionally reviewed. The tensor  $\mathbf{I}$  symbolises the second order identity and the skew-symmetric fourth-order identity has been introduced as  $\mathbf{I}^{\text{skw}} = \frac{1}{2}[\mathbf{I} \otimes \mathbf{I} - \mathbf{I} \otimes \mathbf{I}]$  with the non-standard dyadic products being defined via  $[\mathbf{A} \otimes \mathbf{B}] \cdot \mathbf{C} = \mathbf{A} \cdot \mathbf{C} \cdot \mathbf{B}^t$  and  $[\mathbf{A} \otimes \mathbf{B}] \cdot \mathbf{C} = \mathbf{A} \cdot \mathbf{C}^t \cdot \mathbf{B}^t$ , respectively, whereby  $\{\bullet\}^t$  characterises the transposed of the second-order tensor  $\{\bullet\}$ . The skew-symmetric part of  $\mathbf{A}$  is consequently determined by  $\mathbf{A}^{\text{skw}} = \mathbf{I}^{\text{skw}} : \mathbf{A} = \frac{1}{2}[\mathbf{A} - \mathbf{A}^t]$ . In the following, differential operators are frequently used, namely the gradient  $\nabla_{\mathbf{X}} \mathbf{A}$ , the divergence  $\nabla_{\mathbf{X}} : \mathbf{A} = \nabla_{\mathbf{X}} \mathbf{A} : \mathbf{I}$  and the curl operation  $\nabla_{\mathbf{X}}^t \times \mathbf{A} := -\nabla_{\mathbf{X}}^{\text{skw}} \mathbf{A} : \mathbf{E} = -\nabla_{\mathbf{X}} \mathbf{A} : \mathbf{E}$ . As an interesting side aspect one observes the relation  $[\nabla_{\mathbf{X}}^t \times \mathbf{A}] \cdot \mathbf{E} = -2\nabla_{\mathbf{X}} \mathbf{A} : \mathbf{I}^{\text{skw}}$ . Furthermore, let  $\partial_{\{\bullet\}} \{\bullet\}$  denote the partial or rather explicit derivative of  $\{\bullet\}$  with respect some (tensorial) quantity  $\{\bullet\}$ . The vector product of two vectors and the outer product of two second order tensors are determined by  $\mathbf{a} \times \mathbf{b} = [\mathbf{a} \otimes \mathbf{b}] \cdot \mathbf{E}$  and  $\mathbf{A} \times \mathbf{B} = [\mathbf{A} \cdot \mathbf{B}^t] : \mathbf{E}$ . Finally, the cofactor is introduced as  $\text{cof}(\mathbf{A}) = \partial_{\mathbf{A}} \det(\mathbf{A}) = \det(\mathbf{A}) \mathbf{A}^{-t}$  with  $\mathbf{A}^{-t} \cdot \mathbf{A}^t = \mathbf{I}$ .

## 2. Essential kinematics

Let  $\mathcal{B}_0 \subset \mathbb{E}^3$  denote some reference or rather material configuration of the body  $\mathcal{B}$  of interest. The corresponding tangent and co-tangent (or dual) spaces at a particular  $\mathbf{X} \in \mathcal{B}_0$  are abbreviated by  $T\mathcal{B}_0$  and  $T^*\mathcal{B}_0$ , respectively. Following standard conventions, let  $\mathcal{B}_t \subset \mathbb{E}^3$  characterise the current or rather spatial configuration of  $\mathcal{B}$  at time  $t \in \mathcal{T} \subset \mathbb{R}$  whereas  $T\mathcal{B}_t$  and  $T^*\mathcal{B}_t$  denominate the correlated tangent spaces at a particular  $\mathbf{x} \in \mathcal{B}_t$ . In line with well-established finite elastoplasticity theories, we assume the existence of a generally incompatible (stress-free) intermediate configuration equipped with the tangent spaces  $T\mathcal{B}_p$  and  $T^*\mathcal{B}_p$ , respectively. Having in mind that the three configurations are finite-dimensional, we can map elements of different tangent spaces in one and the same configuration via appropriate metric tensors.

### 2.1. Spatial motion problem

The commonly considered spatial motion problem takes the interpretation as describing the movement of physical particles through the ambient space while fixing their material position. The corresponding non-linear spatial motion as based on this Lagrangian point of view reads

$$\boldsymbol{\varphi} : \mathcal{B}_0 \times \mathcal{T} \rightarrow \mathcal{B}_t, \quad \mathbf{X} \mapsto \mathbf{x} = \boldsymbol{\varphi}(\mathbf{X}, t) \quad (1)$$

and the correlated deformation gradient is multiplicatively decomposed into the inverse elastic (reversible) distortion and the plastic (irreversible) distortion, respectively, namely

$$\begin{aligned} \nabla_{\mathbf{X}} \boldsymbol{\varphi} &= \mathbf{F} \doteq \mathbf{F}_e \cdot \mathbf{F}_p : T\mathcal{B}_0 \rightarrow T\mathcal{B}_t, & J &= \det(\mathbf{F}) > 0, \\ \mathbf{F}_p &: T\mathcal{B}_0 \rightarrow T\mathcal{B}_p, & J_p &= \det(\mathbf{F}_p) > 0, \\ \mathbf{F}_e &: T\mathcal{B}_p \rightarrow T\mathcal{B}_t, & J_e &= \det(\mathbf{F}_e) > 0 \end{aligned} \quad (2)$$

with  $\nabla_{\mathbf{X}}$  characterising the gradient operator with respect to  $\mathbf{X}$  at fixed time  $t$ .

The spatial configuration is thereby assumed to be compatible, i.e.,

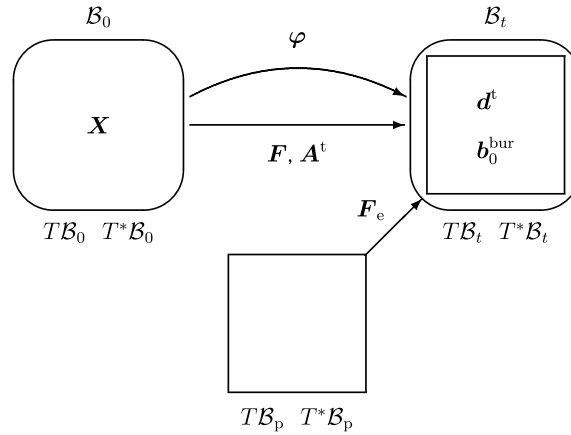


Fig. 1. Spatial motion problem: essential kinematics.

$$\llbracket \varphi \rrbracket = \oint_{c_t} dx = \oint_{c_0} \mathbf{F} \cdot d\mathbf{X} = \int_{\mathcal{A}_0} \nabla_X^t \times \mathbf{F} \cdot \mathbf{N} dA_0 =: \int_{\mathcal{A}_0} \mathbf{A}^t \cdot \mathbf{N} dA_0 =: \int_{\mathcal{A}_0} \mathbf{b}_0^{\text{bur}} dA_0 \doteq \mathbf{0} \quad (3)$$

such that the corresponding local format yields a vanishing dislocation density tensor  $\mathbf{A}^t$  and a vanishing Burgers density vector  $\mathbf{b}_0^{\text{bur}}$ , respectively.<sup>1</sup> Please note, that the two-point tensor  $\mathbf{A}^t$  can be related to the spatial configuration via the appropriate Piola transformation

$$\mathbf{d}^t := \mathbf{A}^t \cdot \text{cof}(\mathbf{F}^{-1}) \quad \text{with} \quad \mathbf{A}^t = \nabla_X^t \times \mathbf{F} = -\nabla_X \mathbf{F} : \mathbf{E}_0 \doteq \mathbf{0} \quad (4)$$

and  $\mathbf{E}_0$  denoting the third-order permutation tensor, compare Appendix A. Further transformations of  $\mathbf{A}^t$  to the material and intermediate configuration or alternative two-point representations are straightforward but omitted for the sake of brevity. Finally, Fig. 1 summarises essential kinematic tensors of the spatial motion problem for convenience of the reader.

## 2.2. Material motion problem

Similar to the spatial motion problem, we next focus on the material motion problem which takes the interpretation as describing the movement of physical particles through the ambient material while fixing their spatial position. The corresponding non-linear material motion as based on this Eulerian type point of view reads

$$\Phi : \mathcal{B}_t \times \mathcal{T} \rightarrow \mathcal{B}_0, \quad \mathbf{x} \mapsto \mathbf{X} = \Phi(\mathbf{x}, t) \quad (5)$$

and the correlated deformation gradient is multiplicatively decomposed into the elastic (reversible) distortion and the inverse plastic (irreversible) distortion, respectively, namely

$$\begin{aligned} \nabla_x \Phi &= \mathbf{f} \doteq \mathbf{f}_p \cdot \mathbf{f}_e : TB_t \rightarrow TB_0, & j &= \det(\mathbf{f}) > 0, \\ \mathbf{f}_e &: TB_t \rightarrow TB_p, & j_e &= \det(\mathbf{f}_e) > 0, \\ \mathbf{f}_p &: TB_p \rightarrow TB_0, & j_p &= \det(\mathbf{f}_p) > 0 \end{aligned} \quad (6)$$

with  $\nabla_x$  characterising the gradient operator with respect to  $\mathbf{x}$  at fixed time  $t$ .

The material configuration is thereby also assumed to be compatible, i.e.,

$$\llbracket \Phi \rrbracket = \oint_{c_t} d\mathbf{X} = \oint_{c_t} \mathbf{f} \cdot d\mathbf{x} = \int_{\mathcal{A}_t} \nabla_x^t \times \mathbf{f} \cdot \mathbf{n} dA_t =: \int_{\mathcal{A}_t} \mathbf{a}^t \cdot \mathbf{n} dA_t =: \int_{\mathcal{A}_t} \mathbf{B}_t^{\text{bur}} dA_t \doteq \mathbf{0} \quad (7)$$

<sup>1</sup> The dislocation density tensors in this work are identified with geometrically necessary dislocations even though we usually do not explicitly place emphasis on this property. Concerning notation, we do not explicitly distinguish between the denomination of the dual and of the transposed of mixed-variant tensors in the following.

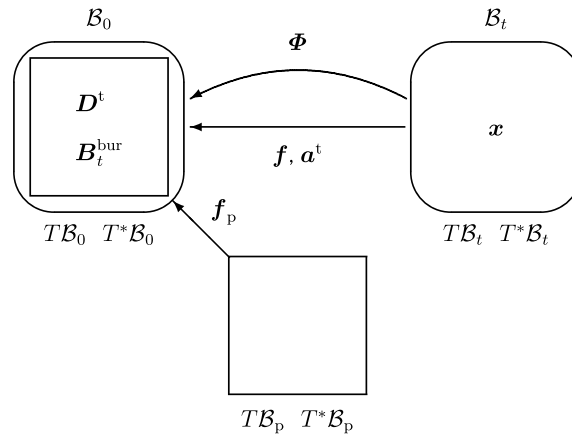


Fig. 2. Material motion problem: essential kinematics.

such that the corresponding local format yields a vanishing dislocation density tensor  $\mathbf{a}^t$  and a vanishing Burgers density vector  $\mathbf{B}_t^{\text{bur}}$ , respectively. Please note, that the two-point tensor  $\mathbf{a}^t$  can be related to the material configuration via the appropriate Piola transformation,

$$\mathbf{D}^t := \mathbf{a}^t \cdot \text{cof}(\mathbf{f}^{-1}) \quad \text{with} \quad \mathbf{a}^t = \nabla_x^t \times \mathbf{f} = -\nabla_x \mathbf{f} : \mathbf{E}_t \doteq \mathbf{0}, \quad (8)$$

whereby the applied permutation tensor  $\mathbf{E}_t$  is defined in Appendix A. Further transformations of  $\mathbf{a}^t$  to the spatial and intermediate configuration or alternative two-point representations are straightforward but omitted for the sake of brevity.

Finally, Fig. 2 summarises essential kinematic tensors of the material motion problem for convenience of the reader.

### 2.3. Intermediate motion problem

Due to the incompatible nature of the considered intermediate configuration, we cannot assume the existence of a differentiable mapping that relates placements of particles in the intermediate configuration to corresponding placements in the material or spatial configuration. Infinitesimal line elements, however, are mapped via

$$d\tilde{\mathbf{x}} := \mathbf{F}_p \cdot d\mathbf{X} = \mathbf{f}_e \cdot d\mathbf{x} =: d\bar{\mathbf{X}} \quad \text{so that} \quad d\tilde{\mathbf{x}} \equiv d\bar{\mathbf{X}}. \quad (9)$$

With these relations in hand, differential operations with respect to the intermediate configuration can formally be defined, namely a ‘gradient’ operator

$$\tilde{\nabla}\{\bullet\} := \nabla_X\{\bullet\} \cdot \mathbf{F}_p^{-1} = \nabla_x\{\bullet\} \cdot \mathbf{f}_e^{-1} =: \bar{\nabla}\{\bullet\} \quad (10)$$

so that the corresponding ‘divergence’ and ‘curl’ operators result in

$$\tilde{\nabla} \cdot \{\bullet\}^t := \tilde{\nabla}\{\bullet\}^t : \mathbf{I}_p^t = \bar{\nabla}\{\bullet\}^t : \mathbf{I}_p^t =: \bar{\nabla} \cdot \{\bullet\}^t \quad (11)$$

as well as

$$\tilde{\nabla}^t \times \{\bullet\} := \tilde{\nabla}\{\bullet\} : \mathbf{E}_p = -\bar{\nabla}\{\bullet\} : \mathbf{E}_p =: \bar{\nabla}^t \times \{\bullet\}, \quad (12)$$

whereby the second order identity  $\mathbf{I}_p$  is explicitly defined in Section 4.3 and the permutation tensor  $\mathbf{E}_p$  is elaborated in Appendix A. Based on these properties, non-vanishing dislocation density tensors are introduced next.

On the one hand, we take the plastic distortion into account and obtain

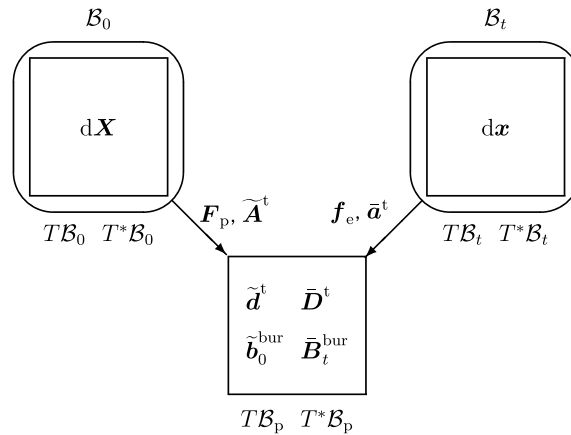


Fig. 3. Intermediate motion problem: essential kinematics.

$$\int_{C_p} d\tilde{\mathbf{x}} = \oint_{C_0} \mathbf{F}_p \cdot d\mathbf{X} = \int_{A_0} \nabla_X^t \times \mathbf{F}_p \cdot \mathbf{N} dA_0 =: \int_{A_0} \tilde{\mathbf{A}}^t \cdot \mathbf{N} dA_0 =: \int_{A_0} \tilde{\mathbf{b}}_0^{\text{bur}} dA_0 \neq \mathbf{0}, \quad (13)$$

where neither the dislocation density tensor  $\tilde{\mathbf{A}}^t$  nor the Burgers density vector  $\tilde{\mathbf{b}}_0^{\text{bur}}$  equal zero. Application of a Piola transformation with respect to the plastic dislocation density tensor in Eq. (13) results in

$$\tilde{\mathbf{d}}^t := \tilde{\mathbf{A}}^t \cdot \text{cof}(\mathbf{F}_p^{-1}) \quad \text{with} \quad \tilde{\mathbf{A}}^t = \nabla_X^t \times \mathbf{F}_p = -\nabla_X \mathbf{F}_p : \mathbf{E}_0 \neq \mathbf{0}. \quad (14)$$

Further transformations of  $\tilde{\mathbf{A}}^t$  to the spatial and material configuration or alternative two-point representations are straightforward but omitted for the sake of brevity.

On the other hand, we take the elastic distortion into account and obtain

$$\int_{C_p} d\bar{\mathbf{x}} = \oint_{C_t} \mathbf{f}_e \cdot d\mathbf{x} = \int_{A_t} \nabla_x^t \times \mathbf{f}_e \cdot \mathbf{n} dA_t := \int_{A_t} \bar{\mathbf{a}}^t \cdot \mathbf{n} dA_t := \int_{A_t} \bar{\mathbf{B}}_t^{\text{bur}} dA_t \neq \mathbf{0}, \quad (15)$$

where neither the dislocation density tensor  $\bar{\mathbf{a}}^t$  nor the Burgers density vector  $\bar{\mathbf{B}}_t^{\text{bur}}$  equal zero. Application of a Piola transformation with respect to the elastic dislocation density tensor in Eq. (15) results in

$$\bar{\mathbf{D}}^t := \bar{\mathbf{a}}^t \cdot \text{cof}(\mathbf{f}_e^{-1}) \quad \text{with} \quad \bar{\mathbf{a}}^t = \nabla_x^t \times \mathbf{f}_e = -\nabla_x \mathbf{f}_e : \mathbf{E}_t \neq \mathbf{0}. \quad (16)$$

Further transformations of  $\bar{\mathbf{a}}^t$  to the material and spatial configuration or alternative two-point representations are straightforward but omitted for the sake of brevity.

In conclusion, we formally defined transformations of gradient and curl operations with respect to different configurations. Concerning the divergence operation, however, it turns out that the fundamental Piola identity is no longer valid as soon as one of the configurations of interest possesses incompatibilities. A detailed review addressing necessary modifications is given in Appendix C.

Finally, Fig. 3 summarises essential kinematic tensors of the intermediate motion problem for convenience of the reader.

#### 2.4. Comparison of the spatial, material and intermediate motion problem

The main objective of the subsequent section consists in deriving relations between the previously introduced deformation gradients, distortions and dislocation density tensors. In this regard, the relation between the spatial and the material motion problem, as reflected by Eqs. (1) and (5), is of cardinal importance in the progressing of the work. Practically speaking, combinations of both motions are assumed to render identity mappings, namely

$$\mathbf{I}_{B_0} \doteq \Phi \circ \varphi \quad \text{and} \quad \mathbf{I}_{B_t} \doteq \varphi \circ \Phi \quad (17)$$



such that the underlying linear tangent maps are related via

$$\mathbf{F}^{-1} \equiv \mathbf{f}, \quad \mathbf{F}_e^{-1} \equiv \mathbf{f}_e, \quad \mathbf{F}_p^{-1} \equiv \mathbf{f}_p \quad \text{with } J^{-1} \equiv j, \quad J_e^{-1} \equiv j_e, \quad J_p^{-1} \equiv j_p. \quad (18)$$

For convenience of the reader, Tables D.1 and D.2 in Appendix D summarise transformations between dislocation density tensors. Nevertheless, detailed derivations are discussed in the following.

#### 2.4.1. Total spatial motion versus total material motion

The relation between the dislocation density tensors  $\mathbf{A}^t$  and  $\mathbf{a}^t$  follows directly from  $\nabla_x(\mathbf{F} \cdot \mathbf{f}) = \mathbf{0}$  or  $\nabla_x(\mathbf{f} \cdot \mathbf{F}) = \mathbf{0}$  which renders

$$-\nabla_x \mathbf{F} = \mathbf{F} \cdot \nabla_x \mathbf{f} : [\mathbf{F} \overline{\otimes} \mathbf{F}], \quad (19)$$

compare Eqs. (4) and (8). As a result we observe that

$$-\nabla_x \mathbf{F} : \mathbf{E}_0 = \mathbf{F} \cdot \nabla_x \mathbf{f} : [\mathbf{F} \overline{\otimes} \mathbf{F}] : \mathbf{E}_0 = J \mathbf{F} \cdot \nabla_x \mathbf{f} : \mathbf{E}_t \cdot \mathbf{f}^t \quad (20)$$

with the sought transformation taking the formats

$$-\mathbf{A}^t = \mathbf{F} \cdot \mathbf{a}^t \cdot \text{cof}(\mathbf{F}) = \mathbf{F} \cdot \mathbf{D}^t \quad \text{and} \quad -\mathbf{a}^t = \mathbf{f} \cdot \mathbf{A}^t \cdot \text{cof}(\mathbf{f}) = \mathbf{f} \cdot \mathbf{d}^t \quad (21)$$

together with  $-\mathbf{D}^t = \mathbf{f} \cdot \mathbf{d}^t \cdot \text{cof}(\mathbf{F})$  as well as  $-\mathbf{d}^t = \mathbf{F} \cdot \mathbf{D}^t \cdot \text{cof}(\mathbf{f})$  now being obvious.

#### 2.4.2. Plastic intermediate motion versus elastic intermediate motion

Following the same lines of derivation for the relations between non-vanishing dislocation density tensors based on the plastic and elastic distortion, the expressions

$$\nabla_x \mathbf{F} = \widetilde{\nabla} \mathbf{F}_e : [\mathbf{F}_p \overline{\otimes} \mathbf{F}_p] + \mathbf{F}_e \cdot \nabla_x \mathbf{F}_p \quad (22)$$

and likewise

$$\nabla_x \mathbf{f} = \widetilde{\nabla} \mathbf{f}_p : [\mathbf{f}_e \overline{\otimes} \mathbf{f}_e] + \mathbf{f}_p \cdot \nabla_x \mathbf{f}_e \quad (23)$$

turn out to be of cardinal importance. Eqs. (22) and (23) motivate in particular the relation between elastic and plastic dislocation density tensors

$$\mathbf{f}_e \cdot \mathbf{A}^t = -\mathbf{f}_e \cdot \nabla_x \mathbf{F} : \mathbf{E}_0 = \widetilde{\mathbf{A}}^t + \mathbf{f}_e \cdot \widetilde{\mathbf{A}}^t \cdot \text{cof}(\mathbf{F}_p) = \mathbf{0} \quad (24)$$

with

$$\widetilde{\mathbf{A}}^t = -\nabla_x \mathbf{F}_p : \mathbf{E}_0 = J_p \mathbf{f}_e \cdot \widetilde{\nabla} \mathbf{F}_e : \mathbf{E}_p \cdot \mathbf{f}_p^t = -\mathbf{f}_e \cdot \widetilde{\mathbf{A}}^t \cdot \text{cof}(\mathbf{F}_p) \quad (25)$$

whereby  $\widetilde{\mathbf{A}}^t = \widetilde{\nabla}^t \times \mathbf{F}_e : T^* \mathcal{B}_p \rightarrow T \mathcal{B}_t$  points opposite to  $\widetilde{\mathbf{a}}^t$ , as well as

$$\mathbf{F}_p \cdot \mathbf{a}^t = -\mathbf{F}_p \cdot \nabla_x \mathbf{f} : \mathbf{E}_t = \widetilde{\mathbf{a}}^t + \mathbf{F}_p \cdot \widetilde{\mathbf{a}}^t \cdot \text{cof}(\mathbf{f}_e) = \mathbf{0} \quad (26)$$

with

$$\widetilde{\mathbf{a}}^t = -\nabla_x \mathbf{f}_e : \mathbf{E}_t = j_e \mathbf{F}_p \cdot \widetilde{\nabla} \mathbf{f}_p : \mathbf{E}_p \cdot \mathbf{F}_e^t = -\mathbf{F}_p \cdot \widetilde{\mathbf{a}}^t \cdot \text{cof}(\mathbf{f}_e) \quad (27)$$

whereby  $\widetilde{\mathbf{a}}^t = \widetilde{\nabla}^t \times \mathbf{f}_p : T^* \mathcal{B}_p \rightarrow T \mathcal{B}_0$  points opposite to  $\widetilde{\mathbf{A}}^t$ . By analogy with Eq. (19) it is straightforward to compute

$$-\nabla_x \mathbf{F}_p = \mathbf{F}_p \cdot \widetilde{\nabla} \mathbf{f}_p : [\mathbf{F}_p \overline{\otimes} \mathbf{F}_p] \quad \text{and} \quad -\nabla_x \mathbf{f}_e = \mathbf{f}_e \cdot \widetilde{\nabla} \mathbf{F}_e : [\mathbf{f}_e \overline{\otimes} \mathbf{f}_e] \quad (28)$$

which constitute the essential kinematic relations for the derivation of the sought expressions

$$\widetilde{\mathbf{A}}^t = \mathbf{F}_p \cdot \widetilde{\nabla} \mathbf{f}_p : [\mathbf{F}_p \overline{\otimes} \mathbf{F}_p] : \mathbf{E}_0 = -\mathbf{F}_p \cdot \widetilde{\mathbf{a}}^t \cdot \text{cof}(\mathbf{F}_p) = \widetilde{\mathbf{a}}^t \cdot \text{cof}(\mathbf{F}) \quad (29)$$

and

$$\widetilde{\mathbf{a}}^t = \mathbf{f}_e \cdot \widetilde{\nabla} \mathbf{F}_e : [\mathbf{f}_e \overline{\otimes} \mathbf{f}_e] : \mathbf{E}_t = -\mathbf{f}_e \cdot \widetilde{\mathbf{A}}^t \cdot \text{cof}(\mathbf{f}_e) = \widetilde{\mathbf{A}}^t \cdot \text{cof}(\mathbf{f}). \quad (30)$$



With these relations in hand, it is possible to compare those distortion-based dislocation density tensors that are entirely settled in the intermediate configuration i.e.,

$$\tilde{\mathbf{d}}^t = \tilde{\mathbf{A}}^t \cdot \text{cof}(\mathbf{f}_p) = -\mathbf{f}_e \cdot \bar{\mathbf{A}}^t = \bar{\mathbf{a}}^t \cdot \text{cof}(\mathbf{F}_e) = \bar{\mathbf{D}}^t \quad (31)$$

and

$$\bar{\mathbf{D}}^t = \bar{\mathbf{a}}^t \cdot \text{cof}(\mathbf{F}_e) = -\mathbf{F}_p \cdot \tilde{\mathbf{a}}^t = \tilde{\mathbf{A}}^t \cdot \text{cof}(\mathbf{f}_p) = \tilde{\mathbf{d}}^t, \quad (32)$$

compare Eqs. (14) and (16). Based on this, we additionally summarise the combinations

$$\bar{\mathbf{A}}^t = -\mathbf{F}_e \cdot \bar{\mathbf{a}}^t \cdot \text{cof}(\mathbf{F}_e) = \mathbf{F} \cdot \tilde{\mathbf{a}}^t = -\mathbf{F}_e \cdot \tilde{\mathbf{A}}^t \cdot \text{cof}(\mathbf{f}_p) = -\mathbf{F}_e \cdot \bar{\mathbf{D}}^t \quad (33)$$

and

$$\tilde{\mathbf{a}}^t = -\mathbf{f}_p \cdot \tilde{\mathbf{A}}^t \cdot \text{cof}(\mathbf{F}_p) = \mathbf{f} \cdot \bar{\mathbf{A}}^t = -\mathbf{f}_p \cdot \bar{\mathbf{a}}^t \cdot \text{cof}(\mathbf{F}_e) = -\mathbf{f}_p \cdot \tilde{\mathbf{d}}^t. \quad (34)$$

**Remark 2.1.** At first glance, the introduction of dislocation density tensors based on the inverse elastic distortion  $\mathbf{F}_e$  for the spatial motion problem and the inverse plastic distortion  $\mathbf{f}_p$  for the material motion problem seems to be straightforward. The incompatibility of the intermediate configuration, however, does not permit the computation of (vanishing) ‘closed’ curve integrals by analogy with Eqs. (3) and (7) as  $\oint_{C_0} d\mathbf{X}$  or  $\oint_{C_t} d\mathbf{x}$ , compare also Eqs. (13) and (15). Applying Stokes’ theorem with respect to incompatible configurations requires modifications of its standard format according to the lines of derivation highlighted for Gauß’ theorem in Appendix C. Nevertheless, (non-vanishing) dislocation density tensors in terms of  $\mathbf{F}_e$  and  $\mathbf{f}_p$  are derived in, e.g., Eqs. (25) and (27).

**Remark 2.2.** The connection between the dislocation density tensors as highlighted in Eqs. (29) and (30) allows alternative derivation via the definition of the corresponding Burgers density vectors as introduced in Eqs. (13) and (15). To be specific, from  $\int_{C_p} d\tilde{\mathbf{x}} \equiv \int_{C_p} d\tilde{\mathbf{X}} \neq \mathbf{0}$  we observe the equivalent relations

$$\int_{\mathcal{A}_0} \tilde{\mathbf{A}}^t \cdot \mathbf{N} dA_0 = \int_{\mathcal{A}_t} \tilde{\mathbf{A}}^t \cdot \text{cof}(\mathbf{f}) \cdot \mathbf{n} dA_t \doteq \int_{\mathcal{A}_t} \bar{\mathbf{a}}^t \cdot \mathbf{n} dA_t \quad (35)$$

and

$$\int_{\mathcal{A}_t} \bar{\mathbf{a}}^t \cdot \mathbf{n} dA_t = \int_{\mathcal{A}_0} \bar{\mathbf{a}}^t \cdot \text{cof}(\mathbf{F}) \cdot \mathbf{N} dA_0 \doteq \int_{\mathcal{A}_0} \tilde{\mathbf{A}}^t \cdot \mathbf{N} dA_0. \quad (36)$$

### 3. Balance of linear momentum

Balance of linear momentum is traditionally based on Newton’s second axiom and set up in the spatial configuration  $(T^*\mathcal{B}_t)$ . The corresponding flux terms (stresses) as based on the spatial motion problem are alternatively two-point tensors or totally settled in the spatial configuration. Thereby, these two possible options are related via appropriate Piola transformations. Nevertheless, we can establish balance equations referring either to the material configuration  $(T^*\mathcal{B}_0)$  or to the intermediate configuration  $(T^*\mathcal{B}_p)$ . These so-called configurational balance equations are consequently based on the material and intermediate motion problem and, together with the corresponding options for the flux terms (stresses), are related via appropriate pullback –, pushforward –, and Piola transformations. For the sake of clarity, we restrict ourselves to the quasi static case and assume conservation of mass.

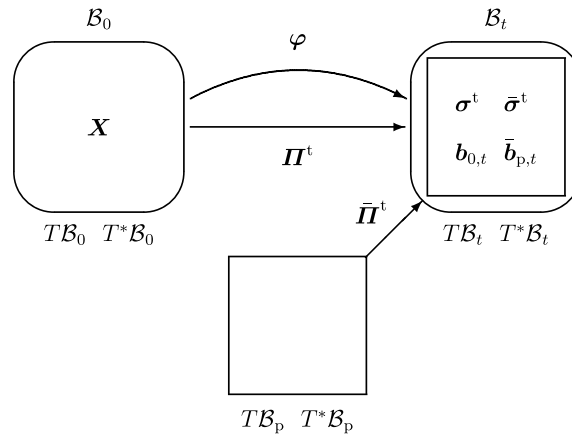


Fig. 4. Spatial motion problem: linear momentum flux and source terms.

### 3.1. Spatial motion problem

Balance of linear momentum for the spatial motion problem is commonly formulated in terms of two different flux terms, namely either the spatial motion Piola stress  $\mathbf{\Pi}^t$  or the spatial motion Cauchy stress  $\boldsymbol{\sigma}^t$ , respectively.<sup>2</sup> The Piola stress thereby constitutes a two-point tensor with respect to the material and spatial configuration. The corresponding balance of linear momentum equations read

$$\nabla_X \cdot \mathbf{\Pi}^t + \mathbf{b}_0 = \mathbf{0} \quad \text{and} \quad \nabla_x \cdot \boldsymbol{\sigma}^t + \mathbf{b}_t = \mathbf{0} \quad \in T^* \mathcal{B}_t \quad (37)$$

with

$$\boldsymbol{\sigma}^t := \mathbf{\Pi}^t \cdot \text{cof}(\mathbf{f}) \quad \text{and} \quad \mathbf{b}_t := j \mathbf{b}_0 \quad (38)$$

whereby  $\mathbf{b}_{0,t} := \mathbf{b}_{0,t}^{\text{int}} + \mathbf{b}_{0,t}^{\text{ext}}$  account for body forces. Alternatively, we can (at first formally) introduce the spatial motion elastic Piola stress  $\bar{\mathbf{\Pi}}^t$  and a corresponding spatial motion elastic Cauchy stress  $\bar{\boldsymbol{\sigma}}^t$ . The Piola stress thereby constitutes a two-point tensor with respect to the (incompatible) intermediate and spatial configuration. The postulated equivalent balance of linear momentum equations read

$$\bar{\nabla} \cdot \bar{\mathbf{\Pi}}^t - \bar{\mathbf{\Pi}}^t \cdot [\mathbf{f}_e \times \bar{\mathbf{A}}] + \bar{\mathbf{b}}_p = \mathbf{0} \quad \text{and} \quad \nabla_x \cdot \bar{\boldsymbol{\sigma}}^t + \bar{\mathbf{b}}_t = \mathbf{0} \in T^* \mathcal{B}_t \quad (39)$$

compare Appendix C, with

$$\bar{\boldsymbol{\sigma}}^t := \bar{\mathbf{\Pi}}^t \cdot \text{cof}(\mathbf{f}_e) \quad \text{and} \quad \bar{\mathbf{b}}_t := j_e \bar{\mathbf{b}}_p \quad (40)$$

whereby  $\bar{\mathbf{b}}_{p,t} := \bar{\mathbf{b}}_{p,t}^{\text{int}} + \bar{\mathbf{b}}_{p,t}^{\text{ext}}$  account for body forces. The modified Piola identity with respect to the intermediate divergence operator  $\bar{\nabla} \cdot \{\bullet\}$  is reviewed in Appendix C.

Fig. 4 displays the stress tensors and volume force vectors introduced above.

### 3.2. Material motion problem

Balance of linear momentum for the material motion problem is, by analogy with Section 3.1, formulated in terms of two different flux terms, namely either the material motion Piola stress  $\boldsymbol{\pi}^t$  or the material motion Cauchy stress  $\boldsymbol{\Sigma}^t$ , respectively.<sup>3</sup> The Piola stress thereby constitutes a two-point tensor with respect to the spatial and material configuration. The corresponding equivalent balance of linear momentum equations read

<sup>2</sup> The transposed is chosen by convention so that traction follow from  $\boldsymbol{\sigma}^t \cdot \mathbf{n}$ , thus the first index of the stress tensor refers to the surface normal.

<sup>3</sup> What we call the material motion Cauchy stress  $\boldsymbol{\Sigma}^t$  is frequently referred to as the Eshelby stress tensor and exhibits the classical energy momentum format, compare Eshelby (1951, 1956).

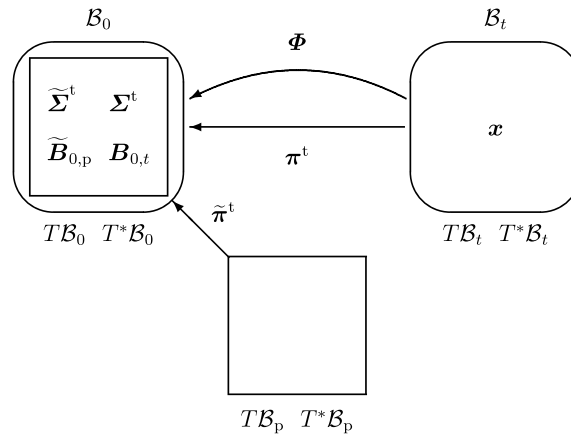


Fig. 5. Material motion problem: linear momentum flux and source terms.

$$\nabla_x \cdot \pi^t + \mathbf{B}_t = \mathbf{0} \quad \text{and} \quad \nabla_X \cdot \Sigma^t + \mathbf{B}_0 = \mathbf{0} \in T^*\mathcal{B}_0 \quad (41)$$

with

$$\Sigma^t := \pi^t \cdot \text{cof}(\mathbf{F}) \quad \text{and} \quad \mathbf{B}_0 := J\mathbf{B}_t, \quad (42)$$

whereby  $\mathbf{B}_{t,0} := \mathbf{B}_{t,0}^{\text{int}} + \mathbf{B}_{t,0}^{\text{ext}}$  account for body forces. Alternatively, we can (again at first formally) introduce the material motion plastic Piola stress  $\tilde{\pi}^t$  and a corresponding material motion plastic Cauchy stress  $\tilde{\Sigma}^t$ . The modified Piola stress thereby constitutes a two-point tensor with respect to the (incompatible) intermediate and material configuration. The postulated equivalent balance of linear momentum equations read

$$\tilde{\nabla} \cdot \tilde{\pi}^t - \tilde{\pi}^t \cdot [\mathbf{F}_p \times \tilde{\mathbf{a}}] + \tilde{\mathbf{B}}_p = \mathbf{0} \quad \text{and} \quad \nabla_X \cdot \tilde{\Sigma}^t + \tilde{\mathbf{B}}_0 = \mathbf{0} \in T^*\mathcal{B}_0, \quad (43)$$

compare Appendix C, with

$$\tilde{\Sigma}^t = \tilde{\pi}^t \cdot \text{cof}(\mathbf{F}_p) \quad \text{and} \quad \tilde{\mathbf{B}}_0 = J_p \tilde{\mathbf{B}}_p \quad (44)$$

whereby  $\tilde{\mathbf{B}}_{p,0} = \tilde{\mathbf{B}}_{p,0}^{\text{int}} + \tilde{\mathbf{B}}_{p,0}^{\text{ext}}$  account for body forces. The Piola identity with respect to the intermediate divergence operator  $\tilde{\nabla} \cdot \{\bullet\}$  is reviewed in Appendix C.

Fig. 5 displays the stress tensors and volume force vectors introduced above.

### 3.3. Intermediate motion problem

Balance of linear momentum for the intermediate motion problem is, by analogy with Sections 3.1 and 3.2, formulated in terms of two different Piola type flux terms, namely either the intermediate motion plastic Piola stress  $\tilde{\Pi}^t$  or the intermediate motion elastic Piola stress  $\tilde{\pi}^t$ , respectively. The setup of the corresponding intermediate motion plastic Cauchy stress  $\tilde{\sigma}^t$  as well as of the intermediate motion elastic Cauchy stress  $\tilde{\Sigma}^t$  is straightforward. The Piola stresses thereby constitute two-point tensors with respect to the material and intermediate configuration for the plastic intermediate motion problem and with respect to the spatial and intermediate configuration for the elastic intermediate motion problem. The postulated equivalent balance of linear momentum equations formally read

$$\nabla_X \cdot \tilde{\Pi}^t + \tilde{\mathbf{b}}_0 = \mathbf{0} \quad \text{and} \quad \tilde{\nabla} \cdot \tilde{\sigma}^t - \tilde{\sigma}^t \cdot [\mathbf{F}_p \times \tilde{\mathbf{a}}] + \tilde{\mathbf{b}}_p = \mathbf{0} \in T^*\mathcal{B}_p, \quad (45)$$

compare Appendix C, with

$$\tilde{\sigma}^t := \tilde{\Pi}^t \cdot \text{cof}(\mathbf{f}_p) \quad \text{and} \quad \tilde{\mathbf{b}}_p := j_p \tilde{\mathbf{b}}_0 \quad (46)$$

and

$$\nabla_x \cdot \tilde{\pi}^t + \tilde{\mathbf{B}}_t = \mathbf{0} \quad \text{and} \quad \tilde{\nabla} \cdot \tilde{\Sigma}^t - \tilde{\Sigma}^t \cdot [\mathbf{f}_c \times \tilde{\mathbf{A}}] + \tilde{\mathbf{B}}_p = \mathbf{0} \in T^*\mathcal{B}_p, \quad (47)$$

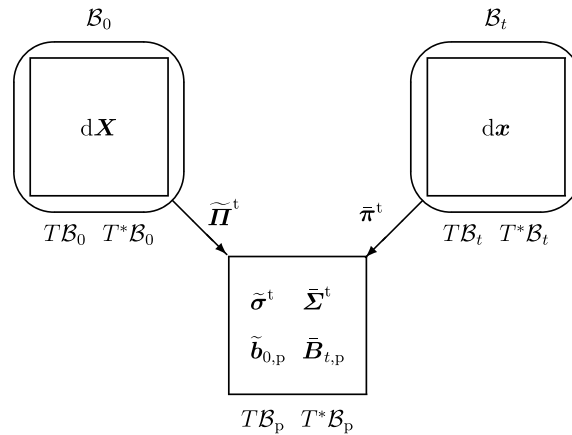


Fig. 6. Intermediate motion problem: linear momentum flux and source terms.

compare Appendix C, with

$$\bar{\Sigma}^t := \bar{\pi}^t \cdot \text{cof}(F_e) \quad \text{and} \quad \bar{B}_p := J_e \bar{B}_t \quad (48)$$

whereby  $\tilde{b}_{0,p} := \tilde{b}_{0,p}^{\text{int}} + \tilde{b}_{0,p}^{\text{ext}}$  and  $\bar{B}_{t,p} := \bar{B}_{t,p}^{\text{int}} + \bar{B}_{t,p}^{\text{ext}}$ , respectively, account for body forces. The modified Piola identity with respect to the intermediate divergence operators  $\tilde{\nabla} \cdot \{\bullet\}$  and  $\bar{\nabla} \cdot \{\bullet\}$  is reviewed in Appendix C.

Fig. 6 displays the stress tensors and volume force vectors introduced above.

#### 4. Hyperelastic stress formats based on a free energy function

In order to compare the various representations of balance of linear momentum proposed in Section 3, we first introduce hyperelastic formats for the corresponding stresses in the following Section 4 and second address derivations of correlated volume forces in Section 5. Based on the standard argumentation of rational thermomechanics, the adopted hyperelastic formats are based on the idea, that constitutive equations are defined in terms of a free energy density. Without loss of generality we restrict ourselves to the isothermal case and do not elaborate particular evolution or balance equations for, e.g., the plastic distortion for sake of conceptual clarity. As the key assumption the postulated free energy density potential depends only on the elastic distortion and possibly on the material placement of particles. In addition, invariance requirements further constrain how the free energy density depends on its arguments. We will not place special emphasis on invariance conditions under superposed orientation preserving spatial isometries. Translational invariance, however, is of cardinal importance for the problem at hand. As a result, the free energy density must not depend on the spatial placement of particles,  $\mathbf{x} = \boldsymbol{\varphi}(\mathbf{X}, t)$ , but solely on material placements,  $\mathbf{X} = \boldsymbol{\Phi}(\mathbf{x}, t)$ . Moreover, we consequently obtain vanishing internal volume or rather self forces for the spatial motion problem,  $\mathbf{b}_{0,t}^{\text{int}} = \mathbf{0}$ .

Following these preliminary statements, let the free energy density be defined as

$$W_0 = W_0(\mathbf{F}, \mathbf{F}_p; \mathbf{X}) = W_0(\mathbf{F} \cdot \mathbf{f}_p; \mathbf{X}) = W_0(\mathbf{F}_e; \mathbf{X}). \quad (49)$$

In order to relate this free energy density per material unit volume to intermediate and spatial unit volumes, we additionally mention the transformations

$$W_0 = J_p W_p = J W_t \quad \text{s.t.} \quad W_p = j_p W_0 = J_e W_t \quad \text{or} \quad W_t = j_e W_p = j W_0. \quad (50)$$

##### 4.1. Spatial motion problem

The commonly accepted constitutive equation for the spatial motion Piola stress reads

$$\boldsymbol{\Pi}^t = \partial_{\mathbf{F}} W_0|_{\mathbf{F}_p}. \quad (51)$$

From Eqs. (38) and (50) we additionally observe that the spatial motion Cauchy stress allows representation in energy momentum format

$$\boldsymbol{\sigma}^t = \partial_F W_0|_{F_p} \cdot \text{cof}(\mathbf{f}) = W_t \mathbf{I}_t^t - \mathbf{f}^t \cdot \partial_f W_t|_{f_p} \quad (52)$$

whereby  $\mathbf{I}_t : T\mathcal{B}_t \rightarrow T\mathcal{B}_t$  denotes the second order identity in the spatial configuration and use of the relation  $\partial_F W_t|_{F_p} = -\mathbf{f}^t \cdot \partial_f W_t|_{f_p} \cdot \mathbf{f}^t$  has been made.

For the second family of stresses, we define the spatial motion elastic Piola stress to take the format

$$\bar{\boldsymbol{\Pi}}^t := \partial_{F_e} W_p. \quad (53)$$

From Eqs. (40) and (50) we additionally observe that the spatial motion elastic Cauchy stress then allows representation in energy momentum format

$$\bar{\boldsymbol{\sigma}}^t = \partial_{F_e} W_p \cdot \text{cof}(\mathbf{f}_e) = W_t \mathbf{I}_t^t - \mathbf{f}_e^t \cdot \partial_{f_e} W_t, \quad (54)$$

whereby use of the relation  $\partial_{F_e} W_t = -\mathbf{f}_e^t \cdot \partial_{f_e} W_t \cdot \mathbf{f}_e^t$  has been made.

#### 4.2. Material motion problem

The, by now, well-accepted constitutive equation for the material motion Piola stress reads

$$\boldsymbol{\pi}^t = \partial_f W_t|_{f_p}. \quad (55)$$

From Eqs. (42) and (50) we additionally observe that the material motion Cauchy stress allows representation in energy momentum format

$$\boldsymbol{\Sigma}^t = \partial_f W_t|_{f_p} \cdot \text{cof}(\mathbf{F}) = W_0 \mathbf{I}_0^t - \mathbf{F}^t \cdot \partial_F W_0|_{F_p}, \quad (56)$$

whereby  $\mathbf{I}_0 : T\mathcal{B}_0 \rightarrow T\mathcal{B}_0$  denotes the second-order identity in the material configuration and use of the relation  $\partial_f W_0|_{f_p} = -\mathbf{F}^t \cdot \partial_F W_0|_{F_p} \cdot \mathbf{F}^t$  has been made.

For the second family of stresses, we define the material motion plastic Piola stress to take the format

$$\tilde{\boldsymbol{\pi}}^t := \partial_{f_p} W_p|_f. \quad (57)$$

From Eqs. (44) and (50) we additionally observe that the material motion plastic Cauchy stress then allows representation in energy momentum format

$$\tilde{\boldsymbol{\Sigma}}^t = \partial_{f_p} W_p|_f \cdot \text{cof}(\mathbf{F}_p) = W_0 \mathbf{I}_0^t - \mathbf{F}_p^t \cdot \partial_{F_p} W_0|_F, \quad (58)$$

whereby use of the relation  $\partial_{f_p} W_0|_F = -\mathbf{F}_p^t \cdot \partial_{F_p} W_0|_F \cdot \mathbf{F}_p^t$  has been made.

#### 4.3. Intermediate motion problem

The constitutive equation for the intermediate motion plastic Piola stress is defined as

$$\tilde{\boldsymbol{\Pi}}^t := \partial_{F_p} W_0|_F. \quad (59)$$

From Eqs. (46) and (50) we additionally observe that the intermediate motion plastic Cauchy stress then allows representation in energy momentum format

$$\tilde{\boldsymbol{\sigma}}^t = \partial_{F_p} W_0|_F \cdot \text{cof}(\mathbf{f}_p) = W_p \mathbf{I}_p^t - \mathbf{f}_p^t \cdot \partial_{f_p} W_p|_f \quad (60)$$

whereby  $\mathbf{I}_p : T\mathcal{B}_p \rightarrow T\mathcal{B}_p$  denotes the second order identity in the intermediate configuration and use of the relation  $\partial_{F_p} W_p|_F = -\mathbf{f}_p^t \cdot \partial_{f_p} W_p|_f \cdot \mathbf{f}_p^t$  has been made.

For the second family of stresses, we define the intermediate motion elastic Piola stress to take the format

$$\bar{\boldsymbol{\pi}}^t := \partial_{f_e} W_t. \quad (61)$$

From Eqs. (48) and (50) we additionally observe that the material motion elastic Cauchy stress then allows representation in energy momentum format

$$\bar{\Sigma}^t = \partial_{f_e} W_t \cdot \text{cof}(\mathbf{F}_e) = W_p \mathbf{I}_p^t - \mathbf{F}_e^t \cdot \partial_{F_e} W_p, \quad (62)$$

whereby use of the relation  $\partial_{f_e} W_p = -\mathbf{F}_e^t \cdot \partial_{F_e} W_p \cdot \mathbf{F}_e^t$  has been made.

#### 4.4. Comparison of the spatial, material and intermediate motion problem

The main objective of the subsequent section consists in the derivation of relations between the previously introduced stresses. As point of departure, Piola and Cauchy stress tensors contribution to a balance of linear momentum representation in one and the same configuration, i.e., either the spatial, intermediate or material configuration, are compared. The corresponding energy momentum formats as well as correlated Piola stress tensors, are also addressed. Adhering to this procedure, some derivations are, apparently, repeated. For completeness, however, we prefer to apply this highly structured approach.

For convenience of the reader, Tables D.3–D.5 in Appendix D summarise transformations between representative stress tensors. Nevertheless, detailed derivations are discussed in the following.

##### 4.4.1. Total and elastic spatial motion versus total and elastic material motion

First, we compare the spatial motion Piola stress tensors  $\Pi^t$  and  $\bar{\Pi}^t$  with the spatial motion Cauchy stresses  $\sigma^t$  and  $\bar{\sigma}^t$  as introduced in Sections 4.1–4.3. The material motion counterparts  $\pi^t$ ,  $\bar{\pi}^t$  are also addressed. When placing emphasis on Piola type stresses associated with the spatial motion problem but related to different free energy densities, we expect their relation to be determined via Piola transformations and consequently observe from Eqs. (51), (53), (40)

$$\Pi^t = J_p \partial_F W_p|_{F_p} = J_p \partial_{F_e} W_p \cdot \mathbf{f}_p^t = \bar{\Pi}^t \cdot \text{cof}(\mathbf{F}_p) = \bar{\sigma}^t \cdot \text{cof}(\mathbf{F}). \quad (63)$$

Based on the definitions in Section 3.1, it thus turns out that the spatial motion Cauchy stress and the spatial motion elastic Cauchy stress coincide, i.e.,  $\sigma^t = \bar{\sigma}^t$ . The relations of the spatial motion Cauchy stresses to the material and intermediate motion Piola stresses, however, take the common configurational format. For the material motion problem, Eqs. (52) and (55) render

$$\sigma^t = W_t \mathbf{I}_t^t - \mathbf{f}^t \cdot \pi^t, \quad (64)$$

whereas for the intermediate motion problem, Eqs. (54) and (61) lead to

$$\bar{\sigma}^t = W_t \mathbf{I}_t^t - \mathbf{f}_e^t \cdot \bar{\pi}^t. \quad (65)$$

Moreover, the comparison of Eqs. (64) and (65) results in  $\pi^t = \mathbf{F}_p^t \cdot \bar{\pi}^t$ . Note that  $\pi^t$  and  $\bar{\pi}^t$  are defined with respect to the same free energy density and consequently transform via a tangent map and not via a Piola transformation. Furthermore, the above considerations are in line with derivations in finite elasticity since the intermediate configuration allows interpretation as a stress-free reference configuration.

##### 4.4.2. Plastic and elastic intermediate motion versus plastic material and elastic spatial motion

Second, we compare the intermediate motion plastic and elastic Piola stresses  $\tilde{\Pi}^t$  and  $\bar{\pi}^t$  with the intermediate motion plastic and elastic Cauchy stresses  $\tilde{\sigma}^t$  and  $\bar{\Sigma}^t$  as introduced in Sections 4.1–4.3. The spatial and material motion counterparts  $\bar{\Pi}^t$ ,  $\tilde{\pi}^t$  are also addressed. When placing emphasis on Piola type stresses associated with the intermediate motion problem but related to different free energy densities, we expect their relation to be determined via Piola transformations and consequently observe from Eqs. (59), (61), (48)

$$\tilde{\Pi}^t = J \partial_{F_p} W_t|_F = J \partial_{f_e} W_t \cdot \mathbf{f}^t = \bar{\pi}^t \cdot \text{cof}(\mathbf{F}) = \bar{\Sigma}^t \cdot \text{cof}(\mathbf{F}_p). \quad (66)$$

Based on the definitions in Section 3.3, it thus turns out that the intermediate motion plastic Cauchy stress and the intermediate motion elastic Cauchy stress coincide, i.e.,  $\tilde{\sigma}^t = \bar{\Sigma}^t$ . The relations of the intermediate motion Cauchy stresses to the material motion plastic and spatial motion elastic Piola stresses, however, take the common configurational format. For the material motion problem, Eqs. (60) and (57) render

$$\tilde{\sigma}^t = W_p \mathbf{I}_p^t - \mathbf{f}_p^t \cdot \tilde{\pi}^t, \quad (67)$$

whereas for the spatial motion problem, Eqs. (62) and (53) lead to

$$\tilde{\Sigma}^t = W_p \mathbf{I}_p^t - \mathbf{F}_c^t \cdot \tilde{\Pi}^t. \quad (68)$$

Moreover, the comparison of Eqs. (67) and (68) results in  $\tilde{\pi}^t = \mathbf{F}^t \cdot \tilde{\Pi}^t$ . Note that  $\tilde{\pi}^t$  and  $\tilde{\Pi}^t$  are defined with respect to the same free energy density and consequently transform via a tangent map and not via a Piola transformation.

#### 4.4.3. Plastic and total material motion versus plastic and total spatial motion

Third, we compare the material motion plastic and total Piola stress tensors  $\tilde{\pi}^t$  and  $\pi^t$  with the material motion plastic and total Cauchy stresses  $\tilde{\Sigma}^t$  and  $\Sigma^t$  as introduced in Sections 4.1–4.3. The intermediate and spatial motion counterparts  $\tilde{\Pi}^t$ ,  $\Pi^t$  are also addressed. When placing emphasis on Piola type stresses associated with the material motion problem but related to different free energy densities, we expect their relation to be determined via Piola transformations. For the problem at hand, however, it is not the elastic distortion which serves as a dual quantity for one of the considered Piola type stresses. Consequently, we begin with relating  $\tilde{\pi}^t$  to Cauchy type stresses located in the intermediate configuration and then proceed by incorporating Piola type stresses which are referred to the elastic distortion and the total material motion gradient, respectively, to be specific

$$\tilde{\pi}^t = W_p \mathbf{F}_p^t - \mathbf{F}_p^t \cdot \tilde{\sigma}^t = W_p \mathbf{F}_p^t - \mathbf{F}_p^t \cdot \tilde{\Sigma}^t = W_p \mathbf{F}_p^t - \mathbf{F}_p^t \cdot \tilde{\pi}^t \cdot \text{cof}(\mathbf{F}_c) = W_p \mathbf{F}_p^t - \pi^t \cdot \text{cof}(\mathbf{F}_c) \quad (69)$$

compare Eq. (67) as well as Eqs. (61) and (62). Based on the definitions in Section 3.2 and Eq. (69), it thus turns out that the material motion plastic and total Cauchy stress do not coincide but are related by

$$\tilde{\Sigma}^t = \tilde{\pi}^t \cdot \text{cof}(\mathbf{F}_p) = W_0 \mathbf{I}_0^t - \pi^t \cdot \text{cof}(\mathbf{F}) = W_0 \mathbf{I}_0^t - \Sigma^t. \quad (70)$$

The relations between the material motion stresses and the corresponding spatial motion stresses in Eq. (69), however, take the common configurational format besides a Piola type transformation. For the intermediate motion problem, Eqs. (58) and (59) render

$$\tilde{\Sigma}^t = W_0 \mathbf{I}_0^t - \mathbf{F}_p^t \cdot \tilde{\Pi}^t, \quad (71)$$

whereas for the spatial motion problem, Eqs. (56), (51) and Eq. (69) lead to

$$\Sigma^t = W_0 \mathbf{I}_0^t - \mathbf{F}^t \cdot \Pi^t. \quad (72)$$

Moreover, the comparison of Eqs. (70)–(72) results in

$$\tilde{\Pi}^t = W_0 \mathbf{f}_p^t - \mathbf{F}_c^t \cdot \Pi^t. \quad (73)$$

Note that  $\tilde{\Pi}^t$  and  $\Pi^t$  are defined with respect to one and the same free energy density but transform by analogy to Eq. (69); the alternative derivation verifying Eq. (73) consequently reads

$$\tilde{\Pi}^t = \tilde{\sigma}^t \cdot \text{cof}(\mathbf{F}_p) = \tilde{\Sigma}^t \cdot \text{cof}(\mathbf{F}_p) = W_0 \mathbf{f}_p^t - \mathbf{F}_c^t \cdot \tilde{\Pi}^t \cdot \text{cof}(\mathbf{F}_p) = W_0 \mathbf{f}_p^t - \mathbf{F}_c^t \cdot \Pi^t \quad (74)$$

compare Eq. (46) as well as Eqs. (53), (62), (63).

According to the fact that  $\Sigma^t \neq \tilde{\Sigma}^t$ , which is in contrast to the previous elaborations in Sections 4.4.1 and 4.4.2 where  $\sigma^t \equiv \tilde{\sigma}^t$  and  $\tilde{\sigma}^t \equiv \tilde{\Sigma}^t$ , we finally mention the interesting relations

$$\tilde{\Sigma}^t = \mathbf{F}^t \cdot \Pi^t, \quad \Sigma^t = \mathbf{F}_p^t \cdot \tilde{\Pi}^t, \quad (75)$$

which directly follow from Eq. (70).

**Remark 4.1.** Apparently, the derived set of stress tensors is by far not complete in the sense that further stress tensor are frequently applied in continuum mechanics and theory of materials – for instance Kirchhoff and Mandel type stress tensors. With the relations summarised in Tables D.3–D.5 in hand, however, the derivation of these additional stresses becomes obvious. In this regard, we consider the elaborated stress tensors as essential quantities which serve as a platform to examine other stresses as well. For the sake of clarity, we do not place emphasis on the illustration of further stresses since these tensors would not yield any additional (physical) inside for the problem at hand.



**Remark 4.2.** A similar modelling concept as the multiplicative decomposition, which is adopted in this work, was pioneered by Noll and is often denoted as a material isomorphism or a local rearrangement, compare Section 1: let  $\mathbf{F}_k : T\mathcal{B}_k \rightarrow T\mathcal{B}_0$  with  $j_k = \det^{-1}(\mathbf{F}_k) > 0$  characterise this mapping such that the free energy density is defined in terms of  $W_0 = j_k W_k(\mathbf{F} \cdot \mathbf{F}_k; \mathbf{X})$ . When identifying  $\mathbf{F}_k$  with  $\mathbf{f}_p$  it is straightforward to introduce dislocation density tensors in terms of  $\nabla_X^t \times \mathbf{F}_k^{-1}$ . The incorporation of  $\nabla_x^t \times (\mathbf{F}_k \cdot \mathbf{f})$ , however, is commonly not addressed.

## 5. Volume forces

As previously mentioned at the beginning of Section 4, we next identify configurational volume forces which have formally been introduced in Section 3. Apparently, these elaborations are intrinsically related to the hyperelastic formats developed in Section 4 and constitute the key contribution of this work. The cardinal importance of these forces is underpinned by the fact that they involve driving forces acting on defects, inhomogeneities, heterogeneities and so forth. As point of departure, we apply as the fundamental requirement that appropriate pullback and pushforward transformations of different representations of balance of linear momentum coincide, which enables us to relate the various volume forces. It turns out that gradients of the elastic and plastic distortion and especially dislocation density tensors, as introduced in Section 2, are naturally incorporated. To set the stage, we preliminarily review pullback and pushforward transformations well known from finite elasticity and thereby discuss the basic derivation procedure in detail, see Appendix B. The same modus operandi will then be applied in general to express the various volume forces, e.g., particularly in the intermediate configuration.

### 5.1. Starting from spatial motion balance of linear momentum representations

The main objective of this section consists in the derivation of representations for the volume forces  $\mathbf{B}_{0,t}$  and  $\mathbf{B}_{t,p}$  introduced in Eqs. (41) and (47).

#### 5.1.1. Pullback to the material configuration

First, recall that the balance of linear momentum representations in Eqs. (37) and (41) are related via

$$\begin{aligned} \mathbf{0} &= -\mathbf{F}^t \cdot [\nabla_X \cdot \mathbf{\Pi}^t + \mathbf{b}_0^{\text{ext}}] \\ &=: \nabla_X \cdot \mathbf{\Sigma}^t + \mathbf{B}_0 \end{aligned} \quad (76)$$

compare Appendix C, with  $\mathbf{b}_0^{\text{int}} = \mathbf{0}$ . Now, we reformulate the transformed divergence of  $\mathbf{\Pi}^t$  by

$$-\mathbf{F}^t \cdot [\nabla_X \cdot \mathbf{\Pi}^t] = -\nabla_X \cdot (\mathbf{F}^t \cdot \mathbf{\Pi}^t) + \nabla_X \mathbf{F}^t : \mathbf{\Pi}^t \quad (77)$$

in order to relate both material divergence operations in Eq. (76). Note that  $\nabla_X \mathbf{F}$  contributes to the dislocation density tensor introduced in Eq. (4) which, however, vanishes for the compatible spatial configuration. The second term on the right hand side of Eq. (77) allows representation as

$$\nabla_X \mathbf{F}^t : \mathbf{\Pi}^t = \mathbf{\Pi}^t : \nabla_X \mathbf{F} - \mathbf{\Pi} \times \mathbf{A}, \quad (78)$$

see Appendix B for a detailed derivation. As a result, the dislocation density tensor  $\mathbf{A}^t$  is automatically incorporated. The first term on the right hand side of Eq. (78) is next rewritten as

$$\mathbf{\Pi}^t : \nabla_X \mathbf{F} = \nabla_X \cdot (W_0 \mathbf{I}_0^t) - \tilde{\mathbf{\Pi}}^t : \nabla_X \mathbf{F}_p - \partial_X W_0 \quad (79)$$

whereby the hyperelastic formats in Eqs. (51) and (59) and Eq. (49) are included. Assembling terms then renders

$$\begin{aligned} \mathbf{0} &= \nabla_X \cdot (W_0 \mathbf{I}_0^t - \mathbf{F}^t \cdot \mathbf{\Pi}^t) - \mathbf{\Pi} \times \mathbf{A} - \tilde{\mathbf{\Pi}}^t : \nabla_X \mathbf{F}_p - \partial_X W_0 - \mathbf{F}^t \cdot \mathbf{b}_0^{\text{ext}} \\ &=: \nabla_X \cdot \mathbf{\Sigma}^t + \mathbf{B}_0 \end{aligned} \quad (80)$$

from which we identify the sought expressions for the material motion Cauchy stress

$$\Sigma^t = W_0 I_0^t - F^t \cdot \Pi^t \quad (81)$$

and the corresponding volume force

$$B_0^{\text{int}} = -\Pi \times A - \tilde{\Pi}^t : \nabla_X F_p - \partial_X W_0 \quad (82)$$

with  $B_0^{\text{ext}} = -F^t \cdot b_0^{\text{ext}}$ . Note that the definition of  $\Sigma^t$  recaptures Eq. (72).

### 5.1.2. Pullback to the intermediate configuration

Second, we relate the balance of linear momentum representations in Eqs. (39) and (47) via

$$\begin{aligned} \mathbf{0} &= -F_e^t \cdot [\bar{\nabla} \cdot \bar{\Pi}^t - \bar{\Pi}^t \cdot [f_e \times \bar{A}] + \bar{b}_p^{\text{ext}}] \\ &=: \bar{\nabla} \cdot \bar{\Sigma}^t - \bar{\Sigma}^t \cdot [f_e \times \bar{A}] + \bar{B}_p \end{aligned} \quad (83)$$

compare Appendix C, with  $\bar{b}_p^{\text{int}} = \mathbf{0}$  and  $\bar{A}^t \neq \mathbf{0}$ . Now, we reformulate the transformed divergence of  $\bar{\Pi}^t$  by

$$-F_e^t \cdot [\bar{\nabla} \cdot \bar{\Pi}^t] = -\bar{\nabla} \cdot (F_e^t \cdot \bar{\Pi}^t) + \bar{\nabla} F_e^t : \bar{\Pi}^t \quad (84)$$

in order to relate both intermediate divergence operations in Eq. (83). Note that  $\bar{\nabla} F_e$  contributes to the dislocation density tensor introduced in Eq. (25) which does not vanishes for the incompatible intermediate configuration. The second term on the right hand side of Eq. (84) allows representation as

$$\bar{\nabla} F_e^t : \bar{\Pi}^t = \bar{\Pi}^t : \bar{\nabla} F_e - \bar{\Pi} \times \bar{A}, \quad (85)$$

see Appendix B for a detailed derivation. As a result, the dislocation density tensor  $\bar{A}^t$  is automatically incorporated. The first term on the right hand side of Eq. (85) is next rewritten as

$$\bar{\Pi}^t : \bar{\nabla} F_e = \bar{\nabla} \cdot (W_p I_p^t) - f_p^t \cdot \partial_X W_p \quad (86)$$

whereby the hyperelastic formats in Eq. (53) and Eq. (49) are included. Assembling terms then renders

$$\begin{aligned} \mathbf{0} &= \bar{\nabla} \cdot (W_p I_p^t - F_e^t \cdot \bar{\Pi}^t) + F_e^t \cdot \bar{\Pi}^t \cdot [f_e \times \bar{A}] - \bar{\Pi} \times \bar{A} - f_p^t \cdot \partial_X W_p - F_e^t \cdot \bar{b}_p^{\text{ext}} \\ &=: \bar{\nabla} \cdot \bar{\Sigma}^t - \bar{\Sigma}^t \cdot [f_e \times \bar{A}] + \bar{B}_p \end{aligned} \quad (87)$$

from which we identify the sought expressions for the material motion elastic Cauchy stress

$$\bar{\Sigma}^t = W_p I_p^t - F_e^t \cdot \bar{\Pi}^t \quad (88)$$

and the corresponding volume force

$$\bar{B}_p^{\text{int}} = -\bar{\Sigma} \times \bar{D} - f_p^t \cdot \partial_X W_p \quad (89)$$

with  $\bar{B}_p^{\text{ext}} = -F_e^t \cdot \bar{b}_p^{\text{ext}}$  and wherein use of Eqs. (33) and (68) and Appendix B has been made. Note that the definition of  $\bar{\Sigma}^t$  recaptures Eq. (68). A fundamental difference between the derived volume forces in Eq. (82) and Eq. (89) is given by the fact that  $\bar{B}_{p,i}^{\text{int}}$  incorporate non-vanishing gradients of distortions solely via dislocation density tensors.

## 5.2. Starting from material motion balance of linear momentum representations

The main objective of this section consists in the derivation of representations for the volume forces  $b_{t,0}$  and  $\tilde{b}_{p,0}$  introduced in Eqs. (41) and (43).

### 5.2.1. Pushforward to the spatial configuration

First, recall that the balance of linear momentum representations in Eqs. (41) and (37) are related via

$$\begin{aligned} \mathbf{0} &= -f^t \cdot [\nabla_x \cdot \pi^t + B_t] \\ &=: \nabla_x \cdot \sigma^t + b_t^{\text{ext}} \end{aligned} \quad (90)$$

compare Appendix C, with  $b_t^{\text{int}} = \mathbf{0}$ . Now, we reformulate the transformed divergence of  $\pi^t$  by

$$-\mathbf{f}^t \cdot [\nabla_x \cdot \boldsymbol{\pi}^t] = -\nabla_x \cdot (\mathbf{f}^t \cdot \boldsymbol{\pi}^t) + \nabla_x \mathbf{f}^t : \boldsymbol{\pi}^t \quad (91)$$

in order to relate both spatial divergence operations in Eq. (90). Note that  $\nabla_x \mathbf{f}$  contributes to the dislocation density tensor introduced in Eq. (8) which, however, vanishes for the compatible material configuration. The second term on the right hand side of Eq. (91) allows representation as

$$\nabla_x \mathbf{f}^t : \boldsymbol{\pi}^t = \boldsymbol{\pi}^t : \nabla_x \mathbf{f} - \boldsymbol{\pi} \times \mathbf{a}, \quad (92)$$

see Appendix B for a detailed derivation. As a result, the dislocation density tensor  $\mathbf{a}^t$  is automatically incorporated. The first term on the right hand side of Eq. (92) is next rewritten as

$$\boldsymbol{\pi}^t : \nabla_x \mathbf{f} = \nabla_x \cdot (W_t \mathbf{I}_t^t) - [j_e \tilde{\boldsymbol{\pi}}^t - W_t \mathbf{F}_p^t] : \nabla_x \mathbf{f}_p - \mathbf{f}^t \cdot \partial_X W_t, \quad (93)$$

whereby the hyperelastic formats in Eqs. (55) and (57) and Eq. (49) are included. Assembling terms then renders

$$\begin{aligned} \mathbf{0} &= \nabla_X \cdot (W_t \mathbf{I}_t^t - \mathbf{f}^t \cdot \boldsymbol{\pi}^t) - \boldsymbol{\pi} \times \mathbf{a} + [W_t \mathbf{F}_p^t - j_e \tilde{\boldsymbol{\pi}}^t] : \nabla_x \mathbf{f}_p - \mathbf{f}^t \cdot \partial_X W_t - \mathbf{f}^t \cdot \mathbf{B}_t \\ &=: \nabla_x \cdot \boldsymbol{\sigma}^t + \mathbf{b}_t^{\text{ext}} \end{aligned} \quad (94)$$

from which we identify the sought expressions for the spatial motion Cauchy stress

$$\boldsymbol{\sigma}^t = W_t \mathbf{I}_t^t - \mathbf{f}^t \cdot \boldsymbol{\pi}^t \quad (95)$$

and the corresponding volume force

$$\mathbf{b}_t^{\text{ext}} = -\boldsymbol{\pi} \times \mathbf{a} - j \tilde{\boldsymbol{\Pi}}^t : \nabla_x \mathbf{F}_p - \mathbf{f}^t \cdot \partial_X W_t - \mathbf{f}^t \cdot \mathbf{B}_t \quad (96)$$

with  $\mathbf{b}_t^{\text{ext}} = -\mathbf{f}^t \cdot \mathbf{B}_t^{\text{ext}}$  and wherein use of Eqs. (44), (71), (4), (8) and basic kinematic considerations has been made. Note that the definition of  $\boldsymbol{\sigma}^t$  recaptures Eq. (64).

### 5.2.2. Pushforward to the intermediate configuration

Second, we relate the balance of linear momentum representations in Eqs. (43) and (45) via

$$\begin{aligned} \mathbf{0} &= -\mathbf{f}_p^t \cdot \tilde{\nabla} \cdot \tilde{\boldsymbol{\pi}}^t - \tilde{\boldsymbol{\pi}}^t \cdot [\mathbf{F}_p \times \tilde{\mathbf{a}}] + \tilde{\mathbf{B}}_p \\ &=: \tilde{\nabla} \cdot \tilde{\boldsymbol{\sigma}}^t - \tilde{\boldsymbol{\sigma}}^t \cdot [\mathbf{F}_p \times \tilde{\mathbf{a}}] + \tilde{\mathbf{b}}_p \end{aligned} \quad (97)$$

compare Appendix C, with  $\tilde{\mathbf{a}} \neq \mathbf{0}$ . Now, we reformulate the transformed divergence of  $\tilde{\boldsymbol{\pi}}^t$  by

$$-\mathbf{f}_p^t \cdot \tilde{\nabla} \cdot \tilde{\boldsymbol{\pi}}^t = -\tilde{\nabla} \cdot (\mathbf{f}_p^t \cdot \tilde{\boldsymbol{\pi}}^t) + \tilde{\nabla} \mathbf{f}_p^t : \tilde{\boldsymbol{\pi}}^t. \quad (98)$$

in order to relate both intermediate divergence operations in Eq. (97). Note that  $\tilde{\nabla} \mathbf{f}_p$  contributes to the dislocation density tensor introduced in Eq. (27) which does not vanishes for the incompatible intermediate configuration. The second term on the right hand side of Eq. (98) allows representation as

$$\tilde{\nabla} \mathbf{f}_p^t : \tilde{\boldsymbol{\pi}}^t = \tilde{\boldsymbol{\pi}}^t : \tilde{\nabla} \mathbf{f}_p - \tilde{\boldsymbol{\pi}} \times \tilde{\mathbf{a}}, \quad (99)$$

see Appendix B for a detailed derivation. As a result, the dislocation density tensor  $\tilde{\mathbf{a}}^t$  is automatically incorporated. The first term on the right hand side of Eq. (99) is next rewritten as

$$\tilde{\boldsymbol{\pi}}^t : \tilde{\nabla} \mathbf{f}_p = \tilde{\nabla} \cdot (W_p \mathbf{I}_p^t) + [W_p \mathbf{F}^t - J_e \boldsymbol{\pi}^t] : \tilde{\nabla} \mathbf{f} - \mathbf{f}_p^t \cdot \partial_X W_p \quad (100)$$

whereby the hyperelastic formats in Eqs. (55) and (57) and Eq. (49) are included. Assembling terms then renders

$$\begin{aligned} \mathbf{0} &= \tilde{\nabla} \cdot (W_p \mathbf{I}_p^t - \mathbf{f}_p^t \cdot \tilde{\boldsymbol{\pi}}^t) + \mathbf{f}_p^t \cdot \tilde{\boldsymbol{\pi}}^t \cdot [\mathbf{F}_p \times \tilde{\mathbf{a}}] - \tilde{\boldsymbol{\pi}} \times \tilde{\mathbf{a}} + [W_p \mathbf{F}^t - J_e \boldsymbol{\pi}^t] : \tilde{\nabla} \mathbf{f} - \mathbf{f}_p^t \cdot \partial_X W_p - \mathbf{f}_p^t \cdot \tilde{\mathbf{B}}_p \\ &=: \tilde{\nabla} \cdot \tilde{\boldsymbol{\sigma}}^t - \tilde{\boldsymbol{\sigma}}^t \cdot [\mathbf{F}_p \times \tilde{\mathbf{a}}] + \tilde{\mathbf{b}}_p \end{aligned} \quad (101)$$

from which we identify the sought expressions for the intermediate motion plastic Cauchy stress

$$\tilde{\boldsymbol{\sigma}}^t = W_p \mathbf{I}_p^t - \mathbf{f}_p^t \cdot \tilde{\boldsymbol{\pi}}^t \quad (102)$$

and the corresponding volume force

$$\tilde{\mathbf{b}}_p = [\tilde{\boldsymbol{\sigma}} \cdot \mathbf{F}_p] \times \tilde{\mathbf{a}} + J_c[\mathbf{F}^t \cdot \boldsymbol{\sigma}^t] : \tilde{\nabla} \mathbf{f} - \mathbf{f}_p^t \cdot \partial_X W_p - \mathbf{f}_p^t \cdot \tilde{\mathbf{B}}_p \quad (103)$$

wherein use of Eq. (64) has been made. Note that the definition of  $\tilde{\boldsymbol{\sigma}}^t$  recaptures Eq. (67). By analogy with the representation of, for instance,  $\tilde{\mathbf{B}}_p^{\text{int}}$  in Eq. (89), we can also rewrite Eq. (103) in a more compact format which is elaborated below, see Section 5.4.

### 5.3. Starting from intermediate motion balance of linear momentum representations

The main objective of this section consists in the derivation of representations for the volume forces  $\tilde{\mathbf{B}}_{0,p}$  and  $\tilde{\mathbf{b}}_{t,p}$  introduced in Eqs. (43) and (39).

#### 5.3.1. Pullback to the material configuration

First, we relate the balance of linear momentum representations in Eqs. (45) and (43) via

$$\begin{aligned} \mathbf{0} &= -\mathbf{F}_p^t \cdot [\nabla_X \cdot \tilde{\boldsymbol{\Pi}}^t + \tilde{\mathbf{b}}_0] \\ &=: \nabla_X \cdot \tilde{\boldsymbol{\Sigma}}^t + \tilde{\mathbf{B}}_0 \end{aligned} \quad (104)$$

compare Appendix C, with  $\tilde{\mathbf{a}}^t \neq \mathbf{0}$ . Now, we reformulate the transformed divergence of  $\tilde{\boldsymbol{\Pi}}^t$  by

$$-\mathbf{F}_p^t \cdot [\nabla_X \cdot \tilde{\boldsymbol{\Pi}}^t] = -\nabla_X \cdot (\mathbf{F}_p^t \cdot \tilde{\boldsymbol{\Pi}}^t) + \nabla_X \mathbf{F}_p^t : \tilde{\boldsymbol{\Pi}}^t \quad (105)$$

in order to relate both material divergence operations in Eq. (104). Note that  $\nabla_X \mathbf{F}_p$  contributes to the dislocation density tensor introduced in Eq. (14) which does not vanishes for the incompatible intermediate configuration. The second term on the right hand side of Eq. (105) allows representation as

$$\nabla_X \mathbf{F}_p^t : \tilde{\boldsymbol{\Pi}}^t = \tilde{\boldsymbol{\Pi}}^t : \nabla_X \mathbf{F}_p - \tilde{\boldsymbol{\Pi}} \times \tilde{\mathbf{A}}, \quad (106)$$

see Appendix B for a detailed derivation. As a result, the dislocation density tensor  $\tilde{\mathbf{A}}^t$  is automatically incorporated. The first term on the right hand side of Eq. (106) is next rewritten as

$$\tilde{\boldsymbol{\Pi}}^t : \nabla_X \mathbf{F}_p = \nabla_X \cdot (W_0 \mathbf{I}_0^t) - \boldsymbol{\Pi}^t : \nabla_X \mathbf{F} - \partial_X W_0 \quad (107)$$

whereby the hyperelastic formats in Eqs. (59) and (51) and Eq. (49) are included. Assembling terms then renders

$$\begin{aligned} \mathbf{0} &= \nabla_X \cdot (W_0 \mathbf{I}_0^t - \mathbf{F}_p^t \cdot \tilde{\boldsymbol{\Pi}}^t) - \tilde{\boldsymbol{\Pi}} \times \tilde{\mathbf{A}} - \boldsymbol{\Pi}^t : \nabla_X \mathbf{F} - \partial_X W_0 - \mathbf{F}_p^t \cdot \tilde{\mathbf{b}}_0 \\ &=: \nabla_X \cdot \tilde{\boldsymbol{\Sigma}}^t + \tilde{\mathbf{B}}_0 \end{aligned} \quad (108)$$

from which we identify the sought expressions for the material motion plastic Cauchy stress

$$\tilde{\boldsymbol{\Sigma}}^t = W_0 \mathbf{I}_0^t - \mathbf{F}_p^t \cdot \tilde{\boldsymbol{\Pi}}^t \quad (109)$$

and the corresponding volume force

$$\tilde{\mathbf{B}}_0 = -\tilde{\boldsymbol{\Pi}} \times \tilde{\mathbf{A}} - \boldsymbol{\Pi}^t : \nabla_X \mathbf{F} - \partial_X W_0 - \mathbf{F}_p^t \cdot \tilde{\mathbf{b}}_0. \quad (110)$$

Note that the definition of  $\tilde{\boldsymbol{\Sigma}}^t$  recaptures Eq. (71).

#### 5.3.2. Pushforward to the spatial configuration

Second, we relate the balance of linear momentum representations in Eqs. (47) and (39) via

$$\begin{aligned} \mathbf{0} &= -\mathbf{f}_c^t \cdot [\nabla_x \cdot \tilde{\boldsymbol{\pi}}^t + \tilde{\mathbf{B}}_t] \\ &=: \nabla_x \cdot \tilde{\boldsymbol{\sigma}}^t + \tilde{\mathbf{b}}_t^{\text{ext}} \end{aligned} \quad (111)$$

compare Appendix C, with  $\tilde{\mathbf{b}}_t^{\text{int}} = \mathbf{0}$  and  $\tilde{\mathbf{A}}^t \neq \mathbf{0}$ . Now, we reformulate the transformed divergence of  $\tilde{\boldsymbol{\pi}}^t$  by

$$-\mathbf{f}_e^t \cdot [\nabla_x \cdot \bar{\boldsymbol{\pi}}^t] = -\nabla_x \cdot (\mathbf{f}_e^t \cdot \bar{\boldsymbol{\pi}}^t) + \nabla_x \mathbf{f}_e^t : \bar{\boldsymbol{\pi}}^t \quad (112)$$

in order to relate both spatial divergence operations in Eq. (111). Note that  $\nabla_x \mathbf{f}_e$  contributes to the dislocation density tensor introduced in Eq. (16) which does not vanishes for the incompatible intermediate configuration. The second term on the right hand side of Eq. (112) allows representation as

$$\nabla_x \mathbf{f}_e^t : \bar{\boldsymbol{\pi}}^t = \bar{\boldsymbol{\pi}}^t : \nabla_x \mathbf{f}_e - \bar{\boldsymbol{\pi}} \times \bar{\mathbf{a}}, \quad (113)$$

see Appendix B for a detailed derivation. As a result, the dislocation density tensor  $\bar{\mathbf{A}}^t$  is automatically incorporated. The first term on the right hand side of Eq. (113) is next rewritten as

$$\bar{\boldsymbol{\pi}}^t : \nabla_x \mathbf{f}_e = \nabla_x \cdot (W_t \mathbf{I}_t^t) - \mathbf{f}^t \cdot \partial_X W_t \quad (114)$$

where the hyperelastic formats in Eqs. (61) and (49) are included. Assembling terms then renders

$$\begin{aligned} \mathbf{0} &= \nabla_X \cdot (W_t \mathbf{I}_t^t - \mathbf{f}_e^t \cdot \bar{\boldsymbol{\pi}}^t) - \bar{\boldsymbol{\pi}} \times \bar{\mathbf{a}} - \mathbf{f}^t \cdot \partial_X W_t - \mathbf{f}_e^t \cdot \bar{\mathbf{B}}_t \\ &=: \nabla_x \cdot \bar{\boldsymbol{\sigma}}^t + \bar{\mathbf{b}}_t^{\text{ext}} \end{aligned} \quad (115)$$

from which we identify the sought expressions for the spatial motion elastic Cauchy stress

$$\bar{\boldsymbol{\sigma}}^t = W_t \mathbf{I}_t^t - \mathbf{f}_e^t \cdot \bar{\boldsymbol{\pi}}^t \quad (116)$$

and the corresponding volume force

$$\bar{\mathbf{b}}_t^{\text{ext}} = -\bar{\boldsymbol{\pi}} \times \bar{\mathbf{a}} - \mathbf{f}^t \cdot \partial_X W_t - \mathbf{f}_e^t \cdot \bar{\mathbf{B}}_t \quad (117)$$

with  $\bar{\mathbf{b}}_t^{\text{ext}} = -\mathbf{f}_e^t \cdot \bar{\mathbf{B}}_t^{\text{ext}}$ . Note that the definition of  $\bar{\boldsymbol{\sigma}}^t$  recaptures Eq. (65).

A fundamental difference between the derived volume forces in Eq. (110) and Eq. (117) is given by the fact that  $\bar{\mathbf{b}}_{t,p}^{\text{ext}}$  incorporate gradients of distortions solely via dislocation density tensors.

#### 5.4. Comparison of the spatial, material and intermediate motion problem

The main objective of the subsequent section consists in the examination of further relations between the various volume forces. We perform the subsequent elaborations in order to (double) check the previously obtained expressions which have been identified via (i) setting up different balance of linear momentum representations; (ii) introducing hyperelastic stress formats; (iii) relating different balance of linear momentum representations by means of pullback or pushforward transformations; (iv) arguments of comparison. As such, the following Sections 5.4.1–5.4.3 are not mandatory and those readers mainly interested in the obtained results might skip these final verifications and continue with Section 6. As point of departure, relations between Cauchy type stress tensors together with the corresponding balance of linear momentum representations serve for the identification of the coherence between different volume forces.

For convenience of the reader, Table D.6 in Appendix D summarises transformations between representative configurational volume forces whereby we restrict ourselves to (as compact as possible) illustrations in terms of  $j\bar{\mathbf{b}}_0^{\text{ext}} = \mathbf{b}_t^{\text{ext}} = \bar{\mathbf{b}}_t^{\text{ext}} = j_e \bar{\mathbf{b}}_p^{\text{ext}}$ . As a fundamental property, we observe that  $\bar{\mathbf{b}}_{0,p}, \bar{\mathbf{B}}_{p,t} \in T^* \mathcal{B}_p$  incorporate gradients of distortions solely via dislocation density tensors whereas these quantities directly occur within the representations of  $\bar{\mathbf{B}}_{p,0}, \mathbf{B}_{0,t} \in T^* \mathcal{B}_0$ . As an interesting side aspect,  $\bar{\mathbf{B}}_{p,0}$  do not include any inhomogeneities stemming from explicit dependencies of the free energy density on material placements. Nevertheless, further derivations are discussed in the following.

##### 5.4.1. Spatial versus intermediate and material volume forces

First, recall that  $\mathbf{b}_t = \bar{\mathbf{b}}_t$  explicitly follows from  $\boldsymbol{\sigma}^t = \bar{\boldsymbol{\sigma}}^t$ , compare Eqs. (37) and (39), whereby  $\bar{\mathbf{b}}_{0,t}^{\text{int}}$  and  $\bar{\mathbf{b}}_{p,t}^{\text{int}}$  vanish due to translational invariance in physical space. Based on the representations highlighted in Eqs. (96) and (117), we observe

$$\begin{aligned} \bar{\mathbf{b}}_t^{\text{ext}} &= -\boldsymbol{\sigma} \times \mathbf{d} - j\tilde{\boldsymbol{\Pi}}^t : \nabla_x \mathbf{F}_p - \mathbf{f}^t \cdot \partial_X W_t - \mathbf{f}^t \cdot \mathbf{B}_t \\ &= -\bar{\boldsymbol{\pi}} \times \bar{\mathbf{a}} - \mathbf{f}^t \cdot \partial_X W_t - \mathbf{f}_e^t \cdot \bar{\mathbf{B}}_t \end{aligned} \quad (118)$$

with the computation of  $\mathbf{b}_0^{\text{ext}}$  and  $\bar{\mathbf{b}}_p^{\text{ext}}$  being obvious. Besides recapturing the relations  $\boldsymbol{\sigma} \times \mathbf{d} = \boldsymbol{\pi} \times \mathbf{a} = \mathbf{0}$  as well as  $[\bar{\boldsymbol{\pi}} \cdot \mathbf{f}_e] \times \mathbf{d} = -\bar{\boldsymbol{\pi}} \times \bar{\mathbf{a}}$ , which directly follows from  $\bar{\mathbf{d}}^t = \bar{\mathbf{A}}^t \cdot \text{cof}(\mathbf{f}_e) = -\mathbf{F}_e \cdot \bar{\mathbf{a}}^t$ , we are now able to associate for instance  $\mathbf{B}_t$  with  $\bar{\mathbf{B}}_t$ ; to give an example

$$\mathbf{B}_t = -\nabla_x \mathbf{F}_p^t : \bar{\boldsymbol{\pi}}^t + \mathbf{F}_p^t \cdot \bar{\mathbf{B}}_t, \quad (119)$$

compare Eqs. (25), (30), (66) and Appendix B.

#### 5.4.2. Intermediate versus spatial and material volume forces

Second, recall that  $\bar{\mathbf{b}}_p = \bar{\mathbf{B}}_p$  follows directly from  $\bar{\boldsymbol{\sigma}}^t = \bar{\boldsymbol{\Sigma}}^t$ , compare Eqs. (45) and (47). Based on the representations highlighted in Eqs. (89) and (103) we observe

$$\begin{aligned} \bar{\mathbf{B}}_p &= -\bar{\boldsymbol{\Sigma}} \times \bar{\mathbf{D}} - \mathbf{f}_p^t \cdot \partial_X W_p - \mathbf{F}_e^t \cdot \bar{\mathbf{b}}_p^{\text{ext}} \\ &= \tilde{\mathbf{b}}_p = -\tilde{\boldsymbol{\sigma}} \times \tilde{\mathbf{d}} - j_p \boldsymbol{\Pi}^t : \tilde{\nabla} \mathbf{F} - \mathbf{f}_p^t \cdot \partial_X W_p - \mathbf{f}_p^t \cdot \tilde{\mathbf{B}}_p \end{aligned} \quad (120)$$

with the computation of  $\bar{\mathbf{B}}_t$  and  $\tilde{\mathbf{b}}_0$  being obvious. Besides recapturing the relations  $[\tilde{\boldsymbol{\sigma}} \cdot \mathbf{F}_p] \times \tilde{\mathbf{a}} = -\tilde{\boldsymbol{\sigma}} \times \tilde{\mathbf{d}}$  as well as  $J_e[\mathbf{F}^t \cdot \boldsymbol{\sigma}^t] : \tilde{\nabla} \mathbf{f} = -j_p \boldsymbol{\Pi}^t : \tilde{\nabla} \mathbf{F}$ , which directly follow from Eqs. (34) and (38) and basic kinematic considerations, we are now able to associate for instance  $\bar{\mathbf{b}}_p^{\text{ext}}$  with  $\tilde{\mathbf{B}}_p$ ; to give an example

$$\bar{\mathbf{b}}_p^{\text{ext}} = j_p \boldsymbol{\Pi}^t : \nabla_x \mathbf{F} + \mathbf{f}_p^t \cdot \tilde{\mathbf{B}}_p \quad (121)$$

compare Eq. (31).

#### 5.4.3. Material versus spatial and intermediate volume forces

Third, recall that  $\mathbf{B}_0 \neq \tilde{\mathbf{B}}_0$  follows directly from  $\boldsymbol{\Sigma}^t \neq \tilde{\boldsymbol{\Sigma}}^t$ , compare Eqs. (41), (43), (70). Based on the representations highlighted in Eq. (82) and Eq. (110) we observe

$$\mathbf{B}_0 = -\boldsymbol{\Sigma} \times \mathbf{D} - \tilde{\boldsymbol{\Pi}}^t : \nabla_x \mathbf{F}_p - \partial_X W_0 - \mathbf{F}^t \cdot \mathbf{b}_0^{\text{ext}} \quad (122)$$

together with

$$\tilde{\mathbf{B}}_0 = -\tilde{\boldsymbol{\Pi}} \times \tilde{\mathbf{A}} - \boldsymbol{\Pi}^t : \nabla_x \mathbf{F} - \partial_X W_0 - \mathbf{F}_p^t \cdot \tilde{\mathbf{b}}_0 \quad (123)$$

with the computation of  $\mathbf{B}_t$  and  $\tilde{\mathbf{B}}_p$  being obvious. Apparently, we recaptured the relations  $\boldsymbol{\Pi} \times \mathbf{A} = \boldsymbol{\Sigma} \times \mathbf{D} = \mathbf{0}$  as well as  $\boldsymbol{\Sigma} \times \mathbf{D} = -\tilde{\boldsymbol{\Pi}} \times \tilde{\mathbf{A}}$  and  $J[\mathbf{F}^t \cdot \boldsymbol{\sigma}^t] : \nabla_x \mathbf{f} = -\boldsymbol{\Pi}^t : \nabla_x \mathbf{F}$ , which directly follow from  $\tilde{\mathbf{D}}^t = \tilde{\mathbf{a}}^t \cdot \text{cof}(\mathbf{F}_p) = -\mathbf{f}_p \cdot \tilde{\mathbf{A}}^t$  and Eqs. (75) and (38) together with basic kinematic relations. In contrast to Sections 5.4.1 and 5.4.2, however, the correlation between  $\mathbf{B}_0$  and  $\tilde{\mathbf{B}}_0$  as introduced in Eqs. (41) and (43) must additionally be identified via Eq. (70), namely

$$\nabla_x \cdot \boldsymbol{\Sigma}^t + \mathbf{B}_0 = \nabla_x \cdot (W_0 \mathbf{I}_0^t - \tilde{\boldsymbol{\Sigma}}^t) + \mathbf{B}_0 = \nabla_x \cdot (W_0 \mathbf{I}_0^t - \boldsymbol{\Sigma}^t) + \tilde{\mathbf{B}}_0 = \nabla_x \cdot \tilde{\boldsymbol{\Sigma}}^t + \tilde{\mathbf{B}}_0 \quad (124)$$

such that

$$\mathbf{B}_0 = \nabla_x \cdot (W_0 \mathbf{I}_0^t - 2 \boldsymbol{\Sigma}^t) + \tilde{\mathbf{B}}_0 = \nabla_x \cdot (2 \tilde{\boldsymbol{\Sigma}}^t - W_0 \mathbf{I}_0^t) + \tilde{\mathbf{B}}_0. \quad (125)$$

This relation finally enables us to conclude

$$\tilde{\mathbf{b}}_0 = -\mathbf{f}_p^t \cdot [\tilde{\boldsymbol{\Pi}} \times \tilde{\mathbf{A}}] - \mathbf{f}_p^t \cdot \partial_X W_0 - \mathbf{F}_e^t \cdot \mathbf{b}_0^{\text{ext}}, \quad (126)$$

whereby use of  $\mathbf{A}^t = \mathbf{0}$  and Eqs. (107) and (43) has been made. An alternative derivation of Eq. (126) can be based on the relation  $\mathbf{b}_0^{\text{ext}} = J_p \bar{\mathbf{b}}_p^{\text{ext}} (\bar{\mathbf{B}}_p = \tilde{\mathbf{b}}_p)$  and Eqs. (38), (40), (89).

**Remark 5.1.** The elaborations above enable us to review all relations between the derived volume forces – similar to the dislocation density tensors and stresses in Tables D.1, D.2, D.5. For physical reasons, however, we are in particular interested in representations of configurational volume forces with respect to  $\mathbf{b}_{t,0}^{\text{ext}}$  and  $\bar{\mathbf{b}}_{t,p}^{\text{ext}}$ , respectively, as summarised in Table D.6. The underlying derivations are straightforward, as e.g.,

$$\begin{aligned}
\tilde{\mathbf{b}}_p &= -\tilde{\boldsymbol{\sigma}} \times \tilde{\mathbf{d}} - j_p \boldsymbol{\Pi}^t : \tilde{\nabla} \mathbf{F} - \mathbf{f}_p^t \cdot \partial_X W_p - \mathbf{f}_p^t \cdot \tilde{\mathbf{B}}_p \\
&= -\tilde{\boldsymbol{\sigma}} \times \tilde{\mathbf{d}} - j_p \boldsymbol{\Pi}^t : \tilde{\nabla} \mathbf{F} - \mathbf{f}_p^t \cdot \partial_X W_p + j_p \boldsymbol{\Pi}^t : \bar{\nabla} \mathbf{F} - \mathbf{F}^t \cdot \bar{\mathbf{b}}_p^{\text{ext}} \\
&= -\tilde{\boldsymbol{\sigma}} \times \tilde{\mathbf{d}} - \mathbf{f}_p^t \cdot \partial_X W_p - \mathbf{F}^t \cdot \bar{\mathbf{b}}_p^{\text{ext}}
\end{aligned} \tag{127}$$

recall Eq. (121), with  $\tilde{\boldsymbol{\sigma}} \times \tilde{\mathbf{d}} = \bar{\boldsymbol{\Sigma}} \times \bar{\mathbf{D}}$  being obvious from Eq. (31).

## 6. Peach–Koehler force

A classical example of a material or rather configurational force is highlighted in the subsequent section, namely a representation of the celebrated Peach–Koehler force that we propose to express in the intermediate configuration. Since this configurational volume force should be located in the incompatible intermediate configuration, we consequently take volume forces in  $T^* \mathcal{B}_p$  into account; see the summary in Table D.6. Let  $\tilde{\mathbf{F}} = \bar{\mathbf{F}} \in T^* \mathcal{B}_p$  denote such a configurational force, so that the following alternative expressions apply

$$\begin{aligned}
\tilde{\mathbf{F}} &= \int_{V_0} \tilde{\mathbf{b}}_0 dV_0 = - \int_{V_0} \mathbf{f}_p^t \cdot [\tilde{\boldsymbol{\Pi}} \times \tilde{\mathbf{A}}] dV_0 \\
&= \int_{V_p} \tilde{\mathbf{b}}_p dV_p = - \int_{V_p} \tilde{\boldsymbol{\sigma}} \times \tilde{\mathbf{d}} dV_p \\
&= \int_{V_p} \bar{\mathbf{B}}_p dV_p = - \int_{V_p} \bar{\boldsymbol{\Sigma}} \times \bar{\mathbf{D}} dV_p \\
&= \int_{V_t} \bar{\mathbf{B}}_t dV_t = - \int_{V_t} \mathbf{F}_c^t \cdot [\bar{\boldsymbol{\pi}} \times \bar{\mathbf{a}}] dV_t = \bar{\mathbf{F}}
\end{aligned} \tag{128}$$

wherein any dependence of the free energy density on material placements,  $\partial_X W_0$ , as well as external volume forces,  $\mathbf{b}_t^{\text{ext}}$ , have been neglected for conceptual simplicity. Please note that Eq. (128) incorporates solely dislocation density tensors in contrast to volume forces in  $T^* \mathcal{B}_0$ , compare Table D.6, which therefore are less suitable concerning the derivation of a Peach–Koehler force. This quantity, as a paradigm of a configurational volume force, takes the interpretation as driving a single dislocation. Accordingly, we represent such a single dislocation in the domain of interest via  $\tilde{\mathbf{d}}^t \doteq \delta_p \mathbf{b}_p^{\text{bur}} \otimes \bar{\mathbf{T}}_p \equiv \delta_p \bar{\mathbf{B}}_p^{\text{bur}} \otimes \bar{\mathbf{T}}_p \doteq \bar{\mathbf{D}}^t$  wherein  $\delta_p$  denotes a Dirac-delta distribution, so that

$$\begin{aligned}
\tilde{\mathbf{F}}^{\text{PK}} &= - \int_{V_p} \tilde{\boldsymbol{\sigma}} \times [\delta_p \tilde{\mathbf{b}}_p^{\text{bur}} \otimes \bar{\mathbf{T}}_p]^t dV_p \\
&= - \int_{\mathcal{L}_p} [\tilde{\boldsymbol{\sigma}} \cdot \tilde{\mathbf{b}}_p^{\text{bur}}] \times \bar{\mathbf{T}}_p dL_p \\
&= - \int_{V_p} \bar{\boldsymbol{\Sigma}} \times [\delta_p \bar{\mathbf{B}}_p^{\text{bur}} \otimes \bar{\mathbf{T}}_p]^t dV_p \\
&= - \int_{\mathcal{L}_p} [\bar{\boldsymbol{\Sigma}} \cdot \bar{\mathbf{B}}_p^{\text{bur}}] \times \bar{\mathbf{T}}_p dL_p = \bar{\mathbf{F}}^{\text{PK}}
\end{aligned} \tag{129}$$

with  $j_p \tilde{\mathbf{b}}_0^{\text{bur}} = \tilde{\mathbf{b}}_p^{\text{bur}} \equiv \bar{\mathbf{B}}_p^{\text{bur}} = J_e \bar{\mathbf{B}}_t^{\text{bur}} \in T \mathcal{B}_p$  being defined by means of Eqs. (13) and (15), see also Remark 2.2, and  $\bar{\mathbf{T}}_p \equiv \bar{\mathbf{T}}_p \in T \mathcal{B}_p$  characterising the unit tangent vector according to the dislocation line  $\mathcal{L}_p$ . Note the familiar format of the Peach–Koehler force per unit length,  $[\tilde{\boldsymbol{\sigma}} \cdot \tilde{\mathbf{b}}_p^{\text{bur}}] \times \bar{\mathbf{T}}_p \equiv [\bar{\boldsymbol{\Sigma}} \cdot \bar{\mathbf{B}}_p^{\text{bur}}] \times \bar{\mathbf{T}}_p$ , nevertheless in terms of intermediate motion plastic or elastic Cauchy stresses that allow representation in energy momentum format; compare the discussion in Ericksen (1998).

## 7. Discussion

The main goal of this contribution was the elaboration of configurational balance of linear momentum representations embedded into an inelastic and finite deformation framework. A multiplicative decomposition of



the deformation gradient has been adopted and served as a general setting for the modelling of finite inelasticity. Application of the developed approach to viscoelastic behaviour or growth phenomena, for instance, is straightforward even though special emphasis has been placed on elastoplastic response in this work. Compatibility of the spatial and material configuration has been assumed throughout while the underlying intermediate configuration is generally incompatible. Due to these basic geometrical properties, one obtains non-vanishing dislocation density tensors in terms of the elastic or plastic distortion. Several versions of these quantities have been highlighted whereby two-point formats as well as representations entirely settled in one configuration have been developed. From the physical or rather material modelling point of view, these tensors are commonly related to geometrically necessary dislocations. Moreover, dislocation density tensors also directly contributed to particular configurational balance of linear momentum representations since the normally applied Piola identity is not valid in its standard format as soon as incompatible configurations are considered.

Motivated by the assumption of a free energy density which depends on the elastic distortion, hyperelastic stress formats have been introduced. Piola and Cauchy stress tensors with respect to different configurations were discussed in detail and also their relations have been highlighted. As an interesting aspect, it turned out that different versions of material motion Cauchy stresses do not coincide while their spatial and intermediate motion counterparts are pairwise identical. The particular representations of the constitutive stress functions were recaptured when deriving the corresponding configurational volume forces by comparing balance of linear momentum representations of different motion problems. External spatial volume forces thereby allowed interpretation as applied loads and internal spatial volume forces must vanish due to spatial translational invariance. The derived formats of intermediate and material volume forces, however, additionally incorporate appropriate combinations of stresses and gradients of distortions besides possible terms stemming from dependencies of the free energy density on material placements of particles. It thereby turned out that gradients of distortions directly contribute to material volume forces while these quantities are incorporated via dislocation density tensors into intermediate volume forces. This property is of cardinal importance and reflects the idea that the celebrated Peach–Koehler force is settled in the incompatible intermediate configuration for the kinematical framework at hand. Consequently, the derived representation of the Peach–Koehler would not have been accessible when restricting the present configurational approach to material balance of linear momentum expressions.

Conceptually speaking, a non-vanishing Burgers density vector is introduced with respect to the incompatible intermediate configuration. The formulation is either based on the incorporation of the plastic or elastic distortion. By analogy with these two definitions, two lines of deriving intermediate volume forces come into the picture which apparently refer to the plastic and elastic intermediate motion balance of linear momentum. The related Cauchy stresses and volume forces consequently coincide and the Peach–Koehler force takes a classical format similar to the small strain setting. In this regard, the definition of this celebrated configurational force within an inelastic finite deformation framework seems to be clarified. Furthermore, not only the small strain case is naturally included within the presented framework but also purely elastic finite deformations. To be specific, by setting either the plastic or elastic distortion equal to the identity – so that either the spatial motion gradient coincides with the inverse elastic distortion or the material motion gradient equals the inverse plastic distortion – renders non-vanishing Burgers density vectors in the material or spatial configuration, respectively. The total motion, however, is then no longer compatible but apart from that all equations derived in this contribution can, e.g., be carried over to the case of elastic incompatibilities.

In conclusion, we discussed and derived four fundamentally different representations of balance of linear momentum:

- (i) The commonly applied quasi static equilibrium condition in  $T^*\mathcal{B}_t$  is illustrated by Eqs. (37) and (39), namely

$$\mathbf{0} = \nabla_X \cdot (\mathbf{\Pi}^t \cdot \text{cof}(\mathbf{f})) + \mathbf{b}_t = \nabla_x \cdot (\bar{\mathbf{\Pi}}^t \cdot \text{cof}(\mathbf{f}_c)) + \bar{\mathbf{b}}_t$$

$$\text{with } \mathbf{\Pi}^t = \partial_F W_0|_{F_p}, \quad \bar{\mathbf{\Pi}}^t = \partial_{F_c} W_p$$

$$\text{and } \mathbf{b}_t = \mathbf{b}_t^{\text{ext}} = \bar{\mathbf{b}}_t^{\text{ext}} = \bar{\mathbf{b}}_t.$$

Note that the divergence operation acts on identical Cauchy stresses. Moreover, internal volume force contributions vanish due to spatial translational invariance and the remaining terms take the interpretation as external loads.

(ii) The balance of linear momentum representations in  $T^*\mathcal{B}_p$  are illustrated by Eqs. (45) and (47), namely

$$\begin{aligned} \mathbf{0} &= \widetilde{\nabla} \cdot \left( \widetilde{\Pi}^t \cdot \text{cof}(\mathbf{f}_p) \right) - \left[ \widetilde{\Pi}^t \cdot \text{cof}(\mathbf{f}_p) \right] \cdot [\mathbf{F}_p \times \widetilde{\mathbf{a}}] + \widetilde{\mathbf{b}}_p \\ &= \overline{\nabla} \cdot (\widetilde{\pi}^t \cdot \text{cof}(\mathbf{F}_e)) - [\widetilde{\pi}^t \cdot \text{cof}(\mathbf{F}_e)] \cdot [\mathbf{f}_e \times \bar{\mathbf{A}}] + \bar{\mathbf{B}}_p \\ \text{with } \widetilde{\Pi}^t &= \partial_{F_p} W_0|_F, \quad \widetilde{\pi}^t = \partial_{f_e} W_t \\ \text{and } \widetilde{\mathbf{b}}_p &= -\widetilde{\sigma}^t \times \widetilde{\mathbf{d}} - \mathbf{f}_p^t \cdot \partial_X W_p - J_e \mathbf{F}_e^t \cdot \mathbf{b}_t^{\text{ext}} \\ &= -\widetilde{\Sigma}^t \times \bar{\mathbf{D}} - \mathbf{f}_p^t \cdot \partial_X W_p - J_e \mathbf{F}_e^t \cdot \mathbf{b}_t^{\text{ext}} = \bar{\mathbf{B}}_p. \end{aligned}$$

Note that the divergence operations act on identical Cauchy type stresses whereby additional contributions stemming from the incompatibility of the intermediate configuration are acknowledged. Moreover,  $\mathbf{F}_p \times \widetilde{\mathbf{a}} = \mathbf{f}_e \times \bar{\mathbf{A}}$  must hold throughout since  $\widetilde{\mathbf{b}}_p = \bar{\mathbf{B}}_p$  and  $\widetilde{\Pi}^t \cdot \text{cof}(\mathbf{f}_p) = \widetilde{\pi}^t \cdot \text{cof}(\mathbf{F}_e)$ . This condition results in  $\mathbf{F}_p \cdot \widetilde{\mathbf{a}}^t = \mathbf{f}_e \cdot \bar{\mathbf{A}}^t$  which is verified by Eqs. (33) and (34). Apparently, dislocation density tensors are also incorporated into the corresponding volume forces which enabled us to derive Peach–Koehler force formats embedded into a framework for finite inelasticity.

(iii) On the one hand, an established material balance of linear momentum representations in  $T^*\mathcal{B}_0$  is illustrated by Eq. (41), namely

$$\begin{aligned} \mathbf{0} &= \nabla_X \cdot (\mathbf{F}_p^t \cdot \widetilde{\Pi}^t) + \mathbf{B}_0 \\ \text{with } \widetilde{\Pi}^t &= \partial_{F_p} W_0|_F \\ \text{and } \mathbf{B}_0 &= -\widetilde{\Pi}^t : \nabla_X \mathbf{F}_p - \partial_X W_0 - J \mathbf{F}^t \cdot \mathbf{b}_t^{\text{ext}}. \end{aligned}$$

Note that the commonly derived material balance of linear momentum version, as based on the framework of local rearrangements, follows by analogy with (iii); compare Remark 4.2. The underlying stress tensor, which we call material motion Cauchy stress  $\widetilde{\Sigma}^t = \mathbf{F}_p^t \cdot \widetilde{\Pi}^t$ , is commonly referred to as the Eshelby stress or energy momentum tensor. Moreover, the obtained volume force in (iii) directly incorporates the gradient of the underlying material isomorphisms or, in the present context, plastic distortion. It is also interesting to recall the relation  $\widetilde{\Pi}^t : \nabla_X \mathbf{F}_p = \widetilde{\Sigma}^t : [\mathbf{f}_p \cdot \nabla_X \mathbf{F}_p]$  with  $\mathbf{f}_p \cdot \nabla_X \mathbf{F}_p$  taking the interpretation as the corresponding connection.

(iv) On the other hand, an alternative material balance of linear momentum representations in  $T^*\mathcal{B}_0$  is illustrated by Eq. (43), namely

$$\begin{aligned} \mathbf{0} &= \nabla_X \cdot (\mathbf{F}^t \cdot \Pi^t) + \widetilde{\mathbf{B}}_0 \\ \text{with } \Pi^t &= \partial_F W_0|_{F_p} \\ \text{and } \widetilde{\mathbf{B}}_0 &= -\Pi^t : \nabla_X \mathbf{F} - J \mathbf{F}^t \cdot \mathbf{b}_t^{\text{ext}}, \end{aligned}$$

which seems to be unrecognised in the literature so far. Note that the underlying stress tensor, which we call material motion plastic Cauchy stress  $\widetilde{\Sigma}^t = \mathbf{F}^t \cdot \Pi^t$ , is commonly referred to as the (material) Mandel stress. Moreover, the obtained volume force in (iii) directly incorporates the gradient of the deformation gradient while no dependencies on material placements of particles occur. It is also interesting to recall the relation  $\Pi^t : \nabla_X \mathbf{F} = \widetilde{\Sigma}^t : [\mathbf{f} \cdot \nabla_X \mathbf{F}]$  with  $\mathbf{f} \cdot \nabla_X \mathbf{F}$  taking the interpretation as the corresponding torsion-free connection due to overall compatibility. This balance of linear momentum representation is in particular suitable for numerical applications since (i) solely standard stresses as the spatial motion Piola stress tensor need to be computed; (ii) neither gradients of the plastic and elastic distortion nor any directly related dislocation density tensor are incorporated. Practically speaking, besides the computation of  $\nabla_X \mathbf{F}$  conventional techniques can be adopted within, for instance, a standard finite element setting.

In conclusion, we derived material and, what we call, intermediate representations of balance of linear momentum in addition to the spatial format which classically defines equilibrium in physical space. Two

key steps within this derivation are (i) the extension of the Piola identity to incompatible configurations, Appendix C; (ii) the relation between Piola stresses based on the plastic distortion and the total motion stresses, Eqs. (69) and (74). The obtained intermediate and material volume forces recaptured the celebrated Peach–Koehler force and gave rise to the incorporation of the gradient of the plastic distortion and total deformation gradient. Apparently, these configurational volume forces serve as a platform for the definition of appropriate evolution equations for, e.g., slip systems, continuum dislocations, defects, inhomogeneities and so forth.

## Appendix A. Transformations of third-order permutation tensors

Within a finite deformation context, different representations of (isotropic) third-order permutation tensors can be introduced, namely purely contra- or co-variant and various mixed-variant formats. In this work, particular use of the material representations

$$E_0 : T^*B_0 \times T^*B_0 \times T^*B_0 \rightarrow \mathbb{R} \quad \text{and} \quad e_0 : TB_0 \times TB_0 \times TB_0 \rightarrow \mathbb{R} \quad (\text{A.1})$$

is made, with

$$E_0^{ijk} = e_{0ijk}^{-1} = \begin{cases} \det^{\frac{1}{2}}(\mathbf{G}^{-1}) & \text{if } \{i, j, k\} \text{ is an even permutation of } \{1, 2, 3\}, \\ -\det^{\frac{1}{2}}(\mathbf{G}^{-1}) & \text{if } \{i, j, k\} \text{ is an odd permutation of } \{1, 2, 3\}, \\ E_0^{ijk} = e_{0ijk} = 0 & \text{otherwise,} \end{cases} \quad (\text{A.2})$$

wherein  $\mathbf{G}$  denotes the co-variant metric in  $B_0$  so that

$$E_0 = \det(\mathbf{G}^{-1}) [\mathbf{G}^{-1} \otimes \mathbf{G}^{-1}] : e_0 \cdot \mathbf{G}^{-1}, \\ e_0 = \det(\mathbf{G}) [\mathbf{G} \otimes \mathbf{G}] : E_0 \cdot \mathbf{G}. \quad (\text{A.3})$$

It is now straightforward to prove the relations,

$$E_0 : E_0 = \frac{2}{\det(\mathbf{G}^{-1})} \mathbf{G}^{-1}, \quad e_0 : e_0 = \frac{2}{\det(\mathbf{G})} \mathbf{G}, \quad E_0 : e_0 = 2I_0, \\ E_0 \cdot E_0 = \frac{2}{\det(\mathbf{G}^{-1})} \mathbf{G}^{-1\text{skw}}, \quad e_0 \cdot e_0 = \frac{2}{\det(\mathbf{G})} \mathbf{G}^{\text{skw}}, \quad E_0 \cdot e_0 = 2I^{\text{skw}} \quad (\text{A.4})$$

with  $\mathbf{G}^{-1\text{skw}} = \frac{1}{2} [\mathbf{G}^{-1} \otimes \mathbf{G}^{-1} - \mathbf{G}^{-1} \underline{\otimes} \mathbf{G}^{-1}]$ , etc.

Based on these definitions, we next place emphasis on the demanded transformations of the permutation tensors in Eq. (A.1) to the intermediate and spatial configuration. The coefficients of the permutation tensors are thereby weighted with the determinant of the corresponding linear tangent map, compare for instance Ogden (1997), so that the relations

$$E_p = j_p [F_p \otimes F_p] : E_0 \cdot F_p^t, \quad e_p = J_p [f_p^t \otimes f_p^t] : e_0 \cdot f_p, \\ E_t = j [F \otimes F] : E_0 \cdot F^t, \quad e_t = J [f^t \otimes f^t] : e_0 \cdot f \quad (\text{A.5})$$

are obtained similarly to Eq. (A.3). For completeness, we finally conclude

$$E_t = j_e [F_e \otimes F_e] : E_p \cdot F_e^t, \quad e_t = J_e [f_e^t \otimes f_e^t] : e_p \cdot f_e \quad (\text{A.6})$$

with the reverse representations of Eqs. (A.5) and (A.6) being obvious. Apparently, one could also relate  $E_p$ ,  $E_t$  and  $e_p$ ,  $e_t$  by analogy with Eq. (A.3) which, however, are omitted for the sake of brevity.

## Appendix B. A note on incorporating dislocation density tensors into configurational volume forces

It is self-evident that gradients of tangent maps are directly related to – or rather define dislocation density tensors. The double contraction of these quantities with appropriate stress tensors is frequently applied in this work and therefore discussed in detail for the simplest case in the sequel, namely in terms of the spatial motion gradient  $\mathbf{F}$ . Similar derivations with respect to the material motion gradient or the elastic and plastic distortions follow by analogy and are omitted for the sake of brevity.

In this context, a full length derivation of Eq. (78) reads

$$\begin{aligned}
 \nabla_X \mathbf{F}^t : \mathbf{\Pi}^t &= \mathbf{\Pi}^t : \nabla_X \mathbf{F} + 2\mathbf{\Pi}^t : [\nabla_X \mathbf{F} : \mathbf{I}^{\text{skw}}] \\
 &= \mathbf{\Pi}^t : \nabla_X \mathbf{F} - \mathbf{\Pi}^t : [\nabla_X^t \times \mathbf{F} \cdot \mathbf{e}_0] \\
 &= \mathbf{\Pi}^t : \nabla_X \mathbf{F} + \mathbf{\Pi}^t : [\mathbf{A}^t \cdot \mathbf{e}_0] \\
 &= \mathbf{\Pi}^t : \nabla_X \mathbf{F} - [\mathbf{\Pi} \cdot \mathbf{A}^t] : \mathbf{e}_0 \\
 &= \mathbf{\Pi}^t : \nabla_X \mathbf{F} - \mathbf{\Pi} \times \mathbf{A}
 \end{aligned} \tag{B.1}$$

whereby use of the notation introduced in Section 1, Eq. (4) and Appendix A has been made. Note that the left hand side of Eq. (B.1) also allows representation in terms of a spatial gradient operator, namely  $\nabla_X \mathbf{F}^t : \mathbf{\Pi}^t = [\nabla_X \mathbf{F}^t \cdot \mathbf{F}] : \mathbf{\Pi}^t$ . These elaborations together with Eqs. (4) and (38) finally enable us to reformulate the second term on the right hand side of Eq. (B.1) as

$$\begin{aligned}
 \mathbf{\Pi} \times \mathbf{A} &= [\mathbf{\Pi} \cdot \mathbf{A}^t] : \mathbf{e}_0 = [\text{cof}(\mathbf{F}^t) \cdot \boldsymbol{\sigma} \cdot \mathbf{d}^t \cdot \text{cof}(\mathbf{F})] : \mathbf{e}_0 \\
 &= J^2 [\boldsymbol{\sigma} \cdot \mathbf{d}^t] : [\mathbf{f}^t \otimes \mathbf{f}^t] : \mathbf{e}_0 \\
 &= J [\boldsymbol{\sigma} \cdot \mathbf{d}^t] : \mathbf{e}_t \cdot \mathbf{F} \\
 &= J \mathbf{F}^t \cdot [\boldsymbol{\sigma} \times \mathbf{d}],
 \end{aligned} \tag{B.2}$$

compare Appendix A. Note that the stress tensor  $\boldsymbol{\sigma}^t$  and the dislocation density tensor  $\mathbf{d}^t$  on the right hand side of Eq. (B.2) are entirely settled in one configuration, here the spatial setting, while  $\mathbf{\Pi}^t$  and  $\mathbf{A}^t$  in Eq. (B.1) are two-point tensors. Along the same lines of derivation, similar relations as for instance

$$\mathbf{f} \times \mathbf{A}^t = [\mathbf{f} \cdot \mathbf{d}^t \cdot \text{cof}(\mathbf{F})] : \mathbf{e}_0 = \mathbf{d}^t : \mathbf{e}_t \cdot \mathbf{F} = \mathbf{F}^t \cdot [\mathbf{I}_t \times \mathbf{d}] \tag{B.3}$$

follow straightforwardly.

### Appendix C. Divergence operation with respect to incompatible configurations

In the following, we discuss the Piola identity in the context of an incompatible configuration. Special emphasis is placed on the application of the divergence operation to a second-order tensor as based on the differential operations introduced in Eqs. (10) and (11). To set the stage, recall that Gauß' theorem results in view of a sufficiently smooth overall motion in

$$\int_{\mathcal{A}_0} \boldsymbol{\Xi} \cdot \mathbf{N} dA_0 = \int_{\mathcal{V}_0} \nabla_X \cdot \boldsymbol{\Xi} dV_0 = \int_{\mathcal{V}_t} \nabla_x \cdot \boldsymbol{\xi} dV_t = \int_{\mathcal{A}_t} \boldsymbol{\xi} \cdot \mathbf{n} dA_t \tag{C.1}$$

wherein  $\boldsymbol{\Xi}$  denotes either a two-point or entirely referential second order tensors. The tensor  $\boldsymbol{\xi}$  is related to  $\boldsymbol{\Xi}$  via the Piola transformation  $\boldsymbol{\Xi} = \boldsymbol{\xi} \cdot \text{cof}(\mathbf{F})$ . Consequently,  $\mathbf{N}$  and  $\mathbf{n}$  characterise outward unit vectors in  $\mathcal{B}_0$  and  $\mathcal{B}_t$ , respectively, so that  $\mathbf{N} dA_0 = \text{cof}(\mathbf{f}) \cdot \mathbf{n} dA_t$ . The underlying relation for Eq. (C.1) is given by the Piola identity

$$\nabla_X \cdot \boldsymbol{\Xi} = \nabla_X \cdot (\boldsymbol{\xi} \cdot \text{cof}(\mathbf{F})) = [\nabla_X \cdot \boldsymbol{\xi}] \cdot \text{cof}(\mathbf{F}^t) = J \nabla_x \cdot \boldsymbol{\xi}, \tag{C.2}$$

i.e., the divergence of the corresponding cofactor vanishes identically

$$\begin{aligned}
 \nabla_X \cdot \text{cof}(\mathbf{F}) &= \mathbf{f}^t \cdot \nabla_X J + J \nabla_X \cdot \mathbf{f}^t \\
 &= \partial_f J : \nabla_X \mathbf{f} \cdot \mathbf{f} + J \nabla_X \mathbf{f}^t : \mathbf{F}^t \\
 &= -J \mathbf{F}^t : \nabla_X \mathbf{f} + J \mathbf{F}^t : \nabla_X \mathbf{f} - J \mathbf{F} \times \mathbf{a} \\
 &= \mathbf{0} \quad \text{for } \mathbf{a}^t = \mathbf{0};
 \end{aligned} \tag{C.3}$$

see, e.g., the contribution by Ericksen (1960), or the monographs by Marsden and Hughes (1994), Ciarlet (1988) and Šilhavý (1997). The incorporation of the dislocation density tensor in Eq. (C.3) is based on similar

considerations as highlighted in Eq. (B.1) and  $\nabla_X \cdot \text{cof}(\mathbf{f}) = \mathbf{0}$  follows by analogy with the above elaborations. By deriving the Piola identity, however, compatibility of the overall motion has been assumed, namely  $\mathbf{a}^t = \mathbf{0}$ ; compare Eq. (7). Apparently, dislocation density tensors with respect to the plastic or elastic distortion do not vanish in general so that the relation between the intermediate divergence and the material or spatial divergence of  $\mathbf{a}$ , for instance, second order tensor possesses a different format compared to Eq. (C.1). To be specific, we obtain on the one hand

$$\int_{A_0} \mathbf{r} \cdot \mathbf{N} dA_0 = \int_{V_0} \nabla_X \cdot \mathbf{r} dV_0 = \int_{V_p} \tilde{\nabla} \cdot \tilde{\mathbf{r}} - \tilde{\mathbf{r}} \cdot [\mathbf{F}_p \times \tilde{\mathbf{a}}] dV_p \quad (\text{C.4})$$

wherein  $\mathbf{r} = \tilde{\mathbf{r}} \cdot \text{cof}(\mathbf{F}_p)$  denotes either a two-point or entirely referential second-order tensor and  $\tilde{\mathbf{a}}^t$  has been introduced in Eq. (27). On the other hand one similarly observes

$$\int_{A_t} \mathbf{v} \cdot \mathbf{n} dA_t = \int_{V_t} \nabla_x \cdot \mathbf{v} dV_t = \int_{V_p} \bar{\nabla} \cdot \bar{\mathbf{v}} - \bar{\mathbf{v}} \cdot [\mathbf{f}_e \times \bar{\mathbf{A}}] dV_p \quad (\text{C.5})$$

wherein  $\mathbf{v} = \bar{\mathbf{v}} \cdot \text{cof}(\mathbf{f}_e)$  characterises either a two-point or entirely spatial second-order tensor and  $\bar{\mathbf{A}}^t$  has been introduced in Eq. (25). In contrast to Eq. (C.2), the corresponding local representations consequently result in

$$\nabla_X \cdot \mathbf{r} = J_p \tilde{\nabla} \cdot \tilde{\mathbf{r}} - J_p \tilde{\mathbf{r}} \cdot [\mathbf{F}_p \times \tilde{\mathbf{a}}], \quad \nabla_x \cdot \mathbf{v} = j_e \bar{\nabla} \cdot \bar{\mathbf{v}} - j_e \bar{\mathbf{v}} \cdot [\mathbf{f}_e \times \bar{\mathbf{A}}], \quad (\text{C.6})$$

or equivalently in

$$\nabla_X \cdot \mathbf{r} + \mathbf{r} \cdot \mathbf{F}_p^t \cdot [\mathbf{F}_p \times \tilde{\mathbf{a}}] = J_p \tilde{\nabla} \cdot \tilde{\mathbf{r}}, \quad \nabla_x \cdot \mathbf{v} + \mathbf{v} \cdot \mathbf{f}_e^t \cdot [\mathbf{f}_e \times \bar{\mathbf{A}}] = j_e \bar{\nabla} \cdot \bar{\mathbf{v}}. \quad (\text{C.7})$$

#### Appendix D. Transformation relations of dislocation density tensors, stress measures, and volume forces

See Tables D.1–D.6.

Table D.1

Transformations related to the dislocation density tensors  $\mathbf{A}^t = \nabla_x^t \times \mathbf{F}$  and  $\mathbf{a}^t = \nabla_x^t \times \mathbf{f}$

	$\mathbf{A}^t$	$\mathbf{d}^t$	$\mathbf{a}^t$	$\mathbf{D}^t$
$\mathbf{A}^t$	•	$\mathbf{d}^t \cdot \text{cof}(\mathbf{F})$	$-\mathbf{F} \cdot \mathbf{a}^t \cdot \text{cof}(\mathbf{F})$	$-\mathbf{F} \cdot \mathbf{D}^t$
$\mathbf{d}^t$	$\mathbf{A}^t \cdot \text{cof}(\mathbf{f})$	•	$-\mathbf{F} \cdot \mathbf{a}^t$	$-\mathbf{F} \cdot \mathbf{D}^t \cdot \text{cof}(\mathbf{f})$
$\mathbf{a}^t$	$-\mathbf{f} \cdot \mathbf{A}^t \cdot \text{cof}(\mathbf{f})$	$-\mathbf{f} \cdot \mathbf{d}^t$	•	$\mathbf{D}^t \cdot \text{cof}(\mathbf{f})$
$\mathbf{D}^t$	$-\mathbf{f} \cdot \mathbf{A}^t$	$-\mathbf{f} \cdot \mathbf{d}^t \cdot \text{cof}(\mathbf{F})$	$\mathbf{a}^t \cdot \text{cof}(\mathbf{F})$	•

Table D.2

Transformations related to the dislocation density tensors  $\tilde{\mathbf{A}}^t = \nabla_x^t \times \mathbf{F}_p$  and  $\tilde{\mathbf{a}}^t = \nabla_x^t \times \mathbf{f}_e$

	$\tilde{\mathbf{A}}^t$	$\tilde{\mathbf{d}}^t \equiv \bar{\mathbf{D}}^t$	$\tilde{\mathbf{a}}^t$	$\bar{\mathbf{A}}^t$	$\tilde{\mathbf{a}}^t$
$\tilde{\mathbf{A}}^t$	•	$\tilde{\mathbf{d}}^t \cdot \text{cof}(\mathbf{F}_p)$	$\tilde{\mathbf{a}}^t \cdot \text{cof}(\mathbf{F})$	$-\mathbf{f}_e \cdot \bar{\mathbf{A}}^t \cdot \text{cof}(\mathbf{F}_p)$	$-\mathbf{F}_p \cdot \tilde{\mathbf{a}}^t \cdot \text{cof}(\mathbf{F}_p)$
$\tilde{\mathbf{d}}^t \equiv \bar{\mathbf{D}}^t$	$\tilde{\mathbf{A}}^t \cdot \text{cof}(\mathbf{f}_p)$	•	$\tilde{\mathbf{a}}^t \cdot \text{cof}(\mathbf{F}_e)$	$-\mathbf{f}_e \cdot \bar{\mathbf{A}}^t$	$-\mathbf{F}_p \cdot \tilde{\mathbf{a}}^t$
$\tilde{\mathbf{a}}^t$	$\tilde{\mathbf{A}}^t \cdot \text{cof}(\mathbf{f})$	$\tilde{\mathbf{d}}^t \cdot \text{cof}(\mathbf{f}_e)$	•	$-\mathbf{f}_e \cdot \bar{\mathbf{A}}^t \cdot \text{cof}(\mathbf{f}_e)$	$-\mathbf{F}_p \cdot \tilde{\mathbf{a}}^t \cdot \text{cof}(\mathbf{f}_e)$
$\bar{\mathbf{A}}^t$	$-\mathbf{F}_e \cdot \tilde{\mathbf{A}}^t \cdot \text{cof}(\mathbf{f}_p)$	$-\mathbf{F}_e \cdot \tilde{\mathbf{d}}^t$	$-\mathbf{F}_e \cdot \tilde{\mathbf{a}}^t \cdot \text{cof}(\mathbf{F}_e)$	•	$\mathbf{F} \cdot \tilde{\mathbf{a}}^t$
$\tilde{\mathbf{a}}^t$	$-\mathbf{f}_p \cdot \tilde{\mathbf{A}}^t \cdot \text{cof}(\mathbf{f}_p)$	$-\mathbf{f}_p \cdot \tilde{\mathbf{d}}^t$	$-\mathbf{f}_p \cdot \tilde{\mathbf{a}}^t \cdot \text{cof}(\mathbf{F}_e)$	$\mathbf{f} \cdot \bar{\mathbf{A}}^t$	•

Table D.3

Transformations related to the Piola type stresses  $\Pi^t = \partial_F W_0|_{F_p}$ ,  $\bar{\Pi}^t = \partial_{F_e} W_p$  and  $\tilde{\Pi}^t = \partial_{F_p} W_0|_F$ 

	$\Pi^t$	$\bar{\Pi}^t$	$\tilde{\Pi}^t$
$\Pi^t$	•	$\bar{\Pi}^t \cdot \text{cof}(F_p)$	$W_0 f^t - f_e^t \cdot \tilde{\Pi}^t$
$\bar{\Pi}^t$	$\Pi^t \cdot \text{cof}(f_p)$	•	$W_p f_e^t - f_e^t \cdot \tilde{\Pi}^t \cdot \text{cof}(f_p)$
$\tilde{\Pi}^t$	$W_0 f_p^t - F_e^t \cdot \Pi^t$	$W_0 f_p^t - F_e^t \cdot \bar{\Pi}^t \cdot \text{cof}(F_p)$	•
$\pi^t$	$W_i F^t - F^t \cdot \Pi^t \cdot \text{cof}(f)$	$W_i F^t - F^t \cdot \bar{\Pi}^t \cdot \text{cof}(f_e)$	$F_p^t \cdot \tilde{\Pi}^t \cdot \text{cof}(f)$
$\tilde{\pi}^t$	$F^t \cdot \Pi^t \cdot \text{cof}(f_p)$	$F^t \cdot \bar{\Pi}^t$	$W_p F_p^t - F_p^t \cdot \tilde{\Pi}^t \cdot \text{cof}(f_p)$
$\bar{\pi}^t$	$W_i F_e^t - F_e^t \cdot \Pi^t \cdot \text{cof}(f)$	$W_i F_e^t - F_e^t \cdot \bar{\Pi}^t \cdot \text{cof}(f_e)$	$\tilde{\Pi}^t \cdot \text{cof}(f)$
$\sigma^t$	$\Pi^t \cdot \text{cof}(f)$	$\bar{\Pi}^t \cdot \text{cof}(f_e)$	$W_i I_i^t - f_e^t \cdot \tilde{\Pi}^t \cdot \text{cof}(f)$
$\bar{\sigma}^t$	$W_p I_p^t - F_e^t \cdot \Pi^t \cdot \text{cof}(f_p)$	$W_p I_p^t - F_e^t \cdot \bar{\Pi}^t$	$\tilde{\Pi}^t \cdot \text{cof}(f_p)$
$\Sigma^t$	$W_0 I_0^t - F^t \cdot \Pi^t$	$W_0 I_0^t - F^t \cdot \bar{\Pi}^t \cdot \text{cof}(F_p)$	$F_p^t \cdot \tilde{\Pi}^t$
$\tilde{\Sigma}^t$	$F^t \cdot \Pi^t$	$F^t \cdot \bar{\Pi}^t \cdot \text{cof}(F_p)$	$W_0 I_0^t - F_p^t \cdot \tilde{\Pi}^t$

Table D.4

Transformations related to the Piola type stresses  $\pi^t = \partial_f W_i|_{f_p}$ ,  $\tilde{\pi}^t = \partial_{f_p} W_p|_f$  and  $\bar{\pi}^t = \partial_{f_e} W_i$ 

	$\pi^t$	$\tilde{\pi}^t$	$\bar{\pi}^t$
$\Pi^t$	$W_0 f^t - f^t \cdot \pi^t \cdot \text{cof}(F)$	$f^t \cdot \tilde{\pi}^t \cdot \text{cof}(F_p)$	$W_0 f^t - f_e^t \cdot \bar{\pi}^t \cdot \text{cof}(F)$
$\bar{\Pi}^t$	$W_p f_e^t - f^t \cdot \pi^t \cdot \text{cof}(F_e)$	$f^t \cdot \tilde{\pi}^t$	$W_p f_e^t - f_e^t \cdot \bar{\pi}^t \cdot \text{cof}(F_e)$
$\tilde{\Pi}^t$	$f_p^t \cdot \pi^t \cdot \text{cof}(F)$	$W_0 f_p^t - f_p^t \cdot \tilde{\pi}^t \cdot \text{cof}(F_p)$	$\pi^t \cdot \text{cof}(F)$
$\pi^t$	•	$W_i F^t - \tilde{\pi}^t \cdot \text{cof}(f_e)$	$F_p^t \cdot \bar{\pi}^t$
$\tilde{\pi}^t$	$W_p F_p^t - \pi^t \cdot \text{cof}(F_e)$	•	$W_p F_p^t - F_p^t \cdot \bar{\pi}^t \cdot \text{cof}(F_e)$
$\bar{\pi}^t$	$f_p^t \cdot \pi^t$	$W_i F_e^t - f_p^t \cdot \tilde{\pi}^t \cdot \text{cof}(f_e)$	•
$\sigma^t$	$W_i I_i^t - f^t \cdot \pi^t$	$f^t \cdot \tilde{\pi}^t \cdot \text{cof}(f_e)$	$W_i I_i^t - f_e^t \cdot \bar{\pi}^t$
$\bar{\sigma}^t$	$f_p^t \cdot \pi^t \cdot \text{cof}(F_e)$	$W_p I_p^t - f_p^t \cdot \tilde{\pi}^t$	$\bar{\pi}^t \cdot \text{cof}(F_e)$
$\Sigma^t$	$\pi^t \cdot \text{cof}(F)$	$W_0 I_0^t - \tilde{\pi}^t \cdot \text{cof}(F_p)$	$F_p^t \cdot \bar{\pi}^t \cdot \text{cof}(F)$
$\tilde{\Sigma}^t$	$W_0 I_0^t - \pi^t \cdot \text{cof}(F)$	$\tilde{\pi}^t \cdot \text{cof}(F_p)$	$W_0 I_0^t - F_p^t \cdot \bar{\pi}^t \cdot \text{cof}(F)$

Table D.5

Transformations related to the Cauchy type stresses  $\sigma^t = \Pi^t \cdot \text{cof}(f) = \bar{\Pi}^t \cdot \text{cof}(f_e) = \bar{\sigma}^t$ ,  $\tilde{\sigma}^t = \tilde{\Pi}^t \cdot \text{cof}(f_p) = \tilde{\pi}^t \cdot \text{cof}(F_e) = \tilde{\Sigma}^t$ ,  $\Sigma^t = \pi^t \cdot \text{cof}(F)$  and  $\tilde{\Sigma}^t = \tilde{\pi}^t \cdot \text{cof}(F_p)$ 

	$\sigma^t \equiv \bar{\sigma}^t$	$\tilde{\sigma}^t \equiv \tilde{\Sigma}^t$	$\Sigma^t$	$\tilde{\Sigma}^t$
$\Pi^t$	$\sigma^t \cdot \text{cof}(F)$	$W_0 f^t - f_e^t \cdot \tilde{\sigma}^t \cdot \text{cof}(F_p)$	$W_0 f^t - f^t \cdot \Sigma^t$	$f^t \cdot \tilde{\Sigma}^t$
$\bar{\Pi}^t$	$\sigma^t \cdot \text{cof}(F_e)$	$W_p f_e^t - f_e^t \cdot \tilde{\sigma}^t$	$W_p f_e^t - f^t \cdot \Sigma^t \cdot \text{cof}(f_p)$	$f^t \cdot \tilde{\Sigma}^t \cdot \text{cof}(f_p)$
$\tilde{\Pi}^t$	$W_0 f_p^t - F_e^t \cdot \sigma^t \cdot \text{cof}(F)$	$\tilde{\sigma}^t \cdot \text{cof}(F_p)$	$f_p^t \cdot \Sigma^t$	$W_0 f_p^t - f_p^t \cdot \tilde{\Sigma}^t$
$\pi^t$	$W_i F^t - F^t \cdot \sigma^t$	$F_p^t \cdot \tilde{\sigma}^t \cdot \text{cof}(f_e)$	$\Sigma^t \cdot \text{cof}(f)$	$W_i F^t - \Sigma^t \cdot \text{cof}(f)$
$\tilde{\pi}^t$	$F^t \cdot \sigma^t \cdot \text{cof}(F_e)$	$W_p F_p^t - F_p^t \cdot \tilde{\sigma}^t$	$W_p F_p^t - \Sigma^t \cdot \text{cof}(f_p)$	$\tilde{\Sigma}^t \cdot \text{cof}(f_p)$
$\bar{\pi}^t$	$W_i F_e^t - F_e^t \cdot \sigma^t$	$\tilde{\sigma}^t \cdot \text{cof}(F_e)$	$f_p^t \cdot \Sigma^t \cdot \text{cof}(f)$	$W_i F_e^t - f_p^t \cdot \tilde{\Sigma}^t \cdot \text{cof}(f)$
$\sigma^t$	•	$W_i I_i^t - f_e^t \cdot \tilde{\sigma}^t \cdot \text{cof}(f_e)$	$W_i I_i^t - f^t \cdot \Sigma^t \cdot \text{cof}(f)$	$f^t \cdot \tilde{\Sigma}^t \cdot \text{cof}(f)$
$\bar{\sigma}^t$	$W_p I_p^t - F_e^t \cdot \sigma^t \cdot \text{cof}(F_e)$	•	$f_p^t \cdot \Sigma^t \cdot \text{cof}(f_p)$	$W_p I_p^t - f_p^t \cdot \tilde{\Sigma}^t \cdot \text{cof}(f_p)$
$\Sigma^t$	$W_0 I_0^t - F^t \cdot \sigma^t \cdot \text{cof}(F)$	$F_p^t \cdot \tilde{\sigma}^t \cdot \text{cof}(F_p)$	•	$W_0 I_0^t - \tilde{\Sigma}^t$
$\tilde{\Sigma}^t$	$F^t \cdot \sigma^t \cdot \text{cof}(F)$	$W_0 I_0^t - F_p^t \cdot \tilde{\sigma}^t \cdot \text{cof}(F_p)$	$W_0 I_0^t - \Sigma^t$	•

Table D.6

Transformations related to the spatial motion volume forces  $b_i^{\text{ext}} = \bar{b}_i^{\text{ext}}$ 

	$b_i^{\text{ext}} = \bar{b}_i^{\text{ext}} \in T^* \mathcal{B}_i$
$\bar{b}_0 \in T^* \mathcal{B}_p$	$-f_p^t \cdot [\tilde{\Pi} \times \tilde{A}] - f_p^t \cdot \partial_X W_0 - F_e^t \cdot b_0^{\text{ext}}$
$\bar{b}_p \in T^* \mathcal{B}_p = \bar{\mathcal{B}}_p$	$-\tilde{\sigma} \times \tilde{d} - f_p^t \cdot \partial_X W_p - F_e^t \cdot b_p^{\text{ext}} = -\tilde{\Sigma} \times \tilde{D} - f_p^t \cdot \partial_X W_p - F_e^t \cdot b_p^{\text{ext}}$
$\bar{b}_i \in T^* \mathcal{B}_p$	$-F_e^t \cdot [\tilde{\pi} \times \tilde{a}] - f_p^t \cdot \partial_X W_i - F_e^t \cdot b_i^{\text{ext}}$
$\bar{B}_0 \in T^* \mathcal{B}_0$	$-\Pi^t : \nabla_X F + F^t \cdot b_0^{\text{ext}}$
$B_0 \in T^* \mathcal{B}_0$	$-\tilde{\Pi}^t : \nabla_X F_p - \partial_X W_0 - F^t \cdot b_0^{\text{ext}}$

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