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Numerical interconversion between linear viscoelastic material functions with regularization

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Abstract

In this paper, interconversion between linear viscoelastic material functions is studied emphasizing materials with relatively fast rate of relaxation. The aim of this paper is to study the whole material function determination process from a linear viscoelastic experiment to interconversion by taking into account non-ideal loading and noisiness of the data in such an experiment. No assumptions are made concerning the form of the relaxation modulus or the creep compliance. Interconversion is carried out by evaluating numerically the convolution integral. Three different yet similar approaches are studied. In numerical interconversion, the resulting matrix equation is ill-posed. Due to this, Tikhonov regularization is applied to solve the related matrix system. Numerical simulations indicate that reliable results can be obtained with proposed numerical procedures.

Keywords: Linear viscoelasticity; Interconversion; Relaxation test; Non-ideal loading; Regularization

1. Introduction

In order to construct a mathematical model for material, the material parameters have to be determined. In the case of the integral representation of linear viscoelasticity, either relaxation modulus $G(t)$ or creep compliance $J(t)$ has to be modeled. From a typical linear viscoelastic experiment, only either relaxation modulus or creep compliance can be determined directly. After such an experiment, the unknown linear viscoelastic material function can be determined with an interconversion method.

Relaxation modulus and creep compliance are related by a convolution integral. Hopkins and Hamming (1957) divided the convolution integral into a finite number of subintervals. In each subinterval target function is approximated to be constant. The remaining integrals are then evaluated numerically for obtaining a recursive relation for the target function. This approach was further improved by Knoff and Hopkins (1972). They approximated the source and target functions to be piecewise linear for carrying out integration analytically.

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Dooling et al. (1997) proposed a three-step method, which consists of following substeps: (i) the discrete retardation spectrum is fitted to creep data, (ii) the generalized Voigt model is solved numerically to obtain stress relaxation data, (iii) the discrete relaxation spectrum of a generalized Maxwell model is fitted to the relaxation data. Park and Schapery (1999) evaluated the convolution integral analytically by expressing both the source and target function in Prony series to obtain the system of equations for the unknown coefficients of Prony series. Lately, Liu (2001) proposed a direct method where discrete relaxation spectra is obtained directly from creep data. The direct method was based on the assumption that relaxation times are specified by the user, thus are a priori known. Many other related methods are reviewed in Tschoegl (1989).

The direct numerical integration of a discrete set of source data is sensitive to the noisiness of the data. In addition, non-ideal loading can induce large errors. Nevertheless, such methods can be considered to be the most fundamental since no approximations are made on the form of the source or target function. Therefore, it is tempting to attack the interconversion problem with methods where the convolution integral is evaluated numerically.

If either creep or relaxation test is performed to determine viscoelastic material parameter, non-ideal loading is induced due to inertia effects and in practice, step loading cannot be produced. Quite often the relaxation test takes the form of a ramp test, i.e. constant strain rate is followed by constant strain. In that case, no explicit solution exists and the relaxation modulus has to be determined approximately. Zapas and Phillips (1971) derived a simple method to determine the relaxation modulus from non-ideal loading. In their approach time scale is simply shifted when compared to the step loading case. However, Zapas–Phillips method cannot be used for times less than $t_1/2$, where t_1 denotes the ramp time. Lee and Knauss (2000) derived a backward recursive formula for the relaxation modulus in the case of ramp test. Although the derived formula is mathematically exact, there are several drawbacks. Since the method is recursive and contains numerical differentiation of stress it is inherently unstable. Moreover, the initial value has to be approximated. Sorvari and Malinen (2006) improved the Lee–Knauss method by using numerical integration for gaining an explicit formula for the relaxation modulus.

As discussed above, noise in experimental data and non-ideal loading can cause large errors in numerical interconversion and in mathematical sense the problem is said to be ill-posed, (Mead, 1994). For example, if relaxation function is determined with experimental data and interconversion is used to solve the creep compliance, small errors in relaxation function can cause large errors to creep compliance. This is a typical feature of ill-posed problems. Such an ill-posed problem can be solved with regularization methods in which additional criterion is included to problem. In regularization, original ill-posed problem is replaced with nearby well-posed problem which is numerically stable. General discussion concerning ill-posed problems and different regularization methods is given for example by Hansen (1998).

Regularization methods for numerical interconversion have been successfully applied in frequency domain. In the most of the studies, the aim is to compute the relaxation spectrum for polymers. The used regularization methods include Tikhonov regularization by Honerkamp and Weese (1990), in which different methods for choosing the regularization parameter was also studied. A constrained linear regression with regularization was proposed by Mead (1994). Furthermore, quadratic programming regularization (RQP) method was proposed by Ramkumar et al. (1997). As a conclusion, in all of the previous studies regularization was found to be effective to decrease the error when relaxation spectra was estimated from a noisy or incomplete data.

Although regularization methods have been applied in interconversion before, the approach presented in this paper is somewhat different since interconversion is accomplished in time domain. The regularization method used in this study is Tikhonov regularization, formulated by Tikhonov (1963). The additional criterion for the problem is taken as a form of smoothness priori, i.e. material function is decided to be a smooth function of time. This kind of a priori information is valid, since there are no sharp peaks in the material functions, i.e. in relaxation modulus or in creep compliance.

In this work, we present numerical procedures for evaluating creep compliance from relaxation test with non-ideal loading and noise in simulated experimental data. We specially emphasize materials with rather fast relaxation and therefore test times are small. Several ramp tests in which random Gaussian noise is added are simulated. Then the relaxation modulus is determined and after that the creep compliance is computed by interconversion with Tikhonov regularization. In the interconversion we use three different methods. Results from the interconversion methods with and without Tikhonov regularization are compared to analytical solution of the creep compliance. Simulations indicate robustness of presented algorithms even with large noise levels.

2. Theoretical background

The constitutive equation for linear viscoelastic material can be expressed as

$$\sigma(t) = \int_0^t G(t-\tau)\dot{\epsilon}(\tau)\mathrm{d}\tau, \tag{1}$$

where σ is the stress, t is time, ϵ is the strain and $G(t)$ is the relaxation modulus. Alternatively, the constitutive equation can be written as

$$\epsilon(t) = \int_0^t J(t-\tau)\dot{\sigma}(\tau)\mathrm{d}\tau, \tag{2}$$

where $J(t)$ is the creep compliance. Applying the Laplace transform to Eqs. (1) and (2) yields

$$\hat{\sigma}(s) = s\widehat{G}(s)\hat{\epsilon}(s), \tag{3}$$

$$\hat{\epsilon}(s) = s\widehat{J}(s)\hat{\sigma}(s). \tag{4}$$

From Eqs. (3) and (4) we get

$$\widehat{G}(s)\widehat{J}(s) = \frac{1}{s^2}. \tag{5}$$

Applying the inverse Laplace transform to (5) gives

$$t = \int_0^t G(t-\tau)J(\tau)\mathrm{d}\tau, \tag{6}$$

or

$$t = \int_0^t J(t-\tau)G(\tau)\mathrm{d}\tau. \tag{7}$$

Differentiating Eq. (6) with respect to time gives

$$1 = G(0)J(t) + \int_0^t \partial_t G(t-\tau)J(\tau)\mathrm{d}\tau. \tag{8}$$

By setting $t = 0$, we get an initial condition between the relaxation modulus and creep compliance

$$G(0)J(0) = 1. \tag{9}$$

3. Numerical evaluation of the convolution integral

The convolution integral relating the creep compliance and the relaxation modulus, Eq. (6), is the Volterra equation of the first kind. One peculiar feature of such an equation is that convergent numerical integration rules does not necessary lead to a convergent method. If standard quadrature rules are considered, midpoint and Euler methods are numerically stable whereas trapezoidal method is not (Linz, 1985). Also, standard higher order methods such as Gregory and Newton–Cotes methods lead to unstable algorithms (Linz, 1985). To summarize then, the midpoint method, which is second-order accurate, can be regarded as the best standard numerical scheme for the numerical solution of the Volterra equation of the first kind.

Rather than using direct methods, that is, methods where the original equation is directly discretized with a proper numerical integration rule, we consider methods where the target function is assumed to be constant in each subinterval and after that integrals are evaluated numerically. This kind of approach was chosen since such indirect methods are generally used in the interconversion between linear viscoelastic material functions (Tschoegl, 1989). Here we derive three different but basically very similar interconversion methods which are also numerically evaluated in this study. First two methods are also given in Tschoegl (1989).

At time t_n Eq. (6) yields

$$t_n = \int_0^{t_n} G(t_n - \tau)J(\tau)\mathrm{d}\tau = \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} G(t_n - \tau)J(\tau)\mathrm{d}\tau, \tag{10}$$

where $t_0 = 0$.

In the first case (Method 1) the value for the creep compliance is taken to be $J(t_{i-1/2})$, where $t_{i-1/2} = (t_{i-1} + t_i)/2$, in the time interval $t \in [t_{i-1}, t_i]$ and remaining integrals are evaluated via trapezoidal rule. These approximations gives

$$t_n = \sum_{i=1}^{n} J(t_{i-1/2})(G(t_n - t_i) + G(t_n - t_{i-1}))(t_i - t_{i-1})/2, \tag{11}$$

which can be written as

$$J(t_{n-1/2}) = \frac{2t_n - \sum_{i=1}^{n-1} J(t_{i-1/2})(G(t_n - t_i) + G(t_n - t_{i-1}))(t_i - t_{i-1})}{(G(0) + G(t_n - t_{n-1}))(t_n - t_{n-1})}, \tag{12}$$

for $n \geqslant 2$, with

$$J(t_{1/2}) = \frac{2}{G(0) + G(t_1)}. \tag{13}$$

In the second case (Method 2) creep compliance is approximated to be $(J(t_i) + J(t_{i-1}))/2$ rather than $J(t_{i-1/2})$, thus giving

$$J(t_n) = -J(t_{n-1}) + \frac{4t_n - \sum_{i=1}^{n-1}(J(t_i) + J(t_{i-1}))(G(t_n - t_i) + G(t_n - t_{i-1}))(t_i - t_{i-1})}{(G(0) + G(t_n - t_{n-1}))(t_n - t_{n-1})}, \tag{14}$$

for $n \geqslant 2$, with

$$J(t_1) = \frac{3 - G(t_1)/G(0)}{G(0) + G(t_1)}. \tag{15}$$

The third method (Method 3) is derived by using Eq. (8), which yields at time t_n

$$1 = G(0)J(t_n) + \int_0^{t_n} \partial_t G(t_n - \tau)J(\tau)\mathrm{d}\tau \tag{16}$$

$$= G(0)J(t_n) - \int_0^{t_n} \partial_\tau G(t_n - \tau)J(\tau)\mathrm{d}\tau \tag{17}$$

$$= G(0)J(t_n) - \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} \partial_\tau G(t_n - \tau)J(\tau)\mathrm{d}\tau \tag{18}$$

$$\approx G(0)J(t_n) - \sum_{i=1}^{n} \frac{J(t_i) + J(t_{i-1})}{2} \int_{t_{i-1}}^{t_i} \partial_\tau G(t_n - \tau)\mathrm{d}\tau \tag{19}$$

$$= G(0)J(t_n) + \sum_{i=1}^{n} \frac{J(t_i) + J(t_{i-1})}{2} (G(t_n - t_{i-1}) - G(t_n - t_i)) \tag{20}$$

Above formulation leads to a recursive relation

$$J(t_n) = \frac{2 - J(t_{n-1})(G(t_n - t_{n-1}) - G(0))}{G(0) + G(t_n - t_{n-1})} - \frac{\sum_{i=1}^{n-1}(J(t_i) + J(t_{i-1}))(G(t_n - t_{i-1}) - G(t_n - t_i))}{G(0) + G(t_n - t_{n-1})}, \tag{21}$$

for $n \geqslant 2$, with

$$J(t_1) = \frac{3 - G(t_1)/G(0)}{G(0) + G(t_1)}. \tag{22}$$

All three methods can be also written in a matrix form

$$Ax = b, \tag{23}$$

where A is a lower-triangular matrix and vector x contains the unknown values of the creep compliance. For example, the components for Method 1 is given by

$$A_{ij} = \begin{cases} (t_j - t_{j-1})(G(t_i - t_j) + G(t_i - t_{j-1})), & \text{if } j \leqslant i, \\ 0, & \text{if } j > i, \end{cases} \tag{24}$$

$$x_i = J(t_{i-1/2}), \tag{25}$$

$$b_i = 2t_i, \tag{26}$$

for $i,j \in \{1, \ldots, N\}$.

4. Determination of the relaxation modulus

Under step-strain assumption relaxation modulus in a relaxation test is simply given by

$$G(t) = \frac{\sigma(t)}{\epsilon_0}, \tag{27}$$

where ϵ_0 is the constant strain level. If the ramp time, denoted by t_1, is relatively large the step-strain assumption can induce severe errors, especially at the beginning of relaxation. It is commonly assumed that the true and the step-strain response coincides at time $t = 10t_1$, this is known as the factor-of-ten rule (Meissner, 1978). However, the time when true and step-strain response coincides can be significantly larger (Flory and McKenna, 2004). Obviously, if the long-term relaxation behavior is primary focus, then the finite ramp time effect is insignificant. On the contrary, if the initial values of the relaxation modulus should be known accurately, a finite ramp time correction method should be used. If we look at the interconversion methods introduced in the previous section, we see that in all of the three methods the divider in the recursive relation contains term

$$G(0) + G(t_n - t_{n-1}). \tag{28}$$

So, if rather small time steps are used, which is advisable in the sense of accuracy, initial values of the relaxation modulus should be known with good accuracy.

4.1. Finite ramp time correction methods

If the elongation happens under the constant rate of strain, $\dot{\epsilon}_0$, and the constant strain, ϵ_0, is gained at time t_1, then stress at time $t \geqslant t_1$ is given by

$$\sigma(t) = \dot{\epsilon}_0 \int_0^{t_1} G(t - \tau)\mathrm{d}\tau, \tag{29}$$

where $\dot{\epsilon}_0 = \epsilon_0/t_1$.

Differentiating Eq. (29) with respect to time gives

$$\dot{\sigma}(t) = \dot{\epsilon}_0 \int_0^{t_1} \partial_t G(t - \tau)\mathrm{d}\tau = -\dot{\epsilon}_0 \int_0^{t_1} \partial_\tau G(t - \tau)\mathrm{d}\tau = \dot{\epsilon}_0 (G(t) - G(t - t_1)). \tag{30}$$

This leads to a following backward computation formula (Lee and Knauss, 2000):

$$G(t - t_1) = G(t) - \frac{\dot{\sigma}(t)}{\dot{\epsilon}_0}. \tag{31}$$

The time derivative of stress is calculated with some numerical differentiation rule. Since the Lee–Knauss method is recursive, it is highly unstable if the stress data is noisy. Moreover, the time step cannot be set by the user, starting value has to be approximated and the time derivate of stress has to be calculated numerically.

Better way to approximate the relaxation modulus is the Zapas–Phillips method. Using the midpoint rule to Eq. (29) gives

$$\sigma(t) = \epsilon_0 G(t - t_1/2) \quad t \geqslant t_1 \tag{32}$$

or (Flory and McKenna, 2004)

$$G(t) = \frac{\sigma(t + t_1/2)}{\epsilon_0} \quad t \geqslant t_1/2, \tag{33}$$

The Zapas–Phillips method is limited since it cannot be used with times shorter than $t_1/2$. The time restriction in the Zapas–Phillips can be avoided in method introduced by Sorvari and Malinen (2006). Using trapezoidal rule to Eq. (29) gives

$$\sigma(t) = \frac{1}{2}\epsilon_0(G(t - t_1) + G(t)). \tag{34}$$

Since Eq. (30) gives

$$G(t) = \frac{\dot{\sigma}(t)}{\dot{\epsilon}_0} + G(t - t_1), \tag{35}$$

we get

$$G(t - t_1) = \frac{\sigma(t)}{\epsilon_0} - \frac{\dot{\sigma}(t)}{2\dot{\epsilon}_0}, \tag{36}$$

for $t \geqslant t_1$ or

$$G(t) = \frac{\sigma(t + t_1)}{\epsilon_0} - \frac{\dot{\sigma}(t + t_1)}{2\dot{\epsilon}_0}, \tag{37}$$

for $t \geqslant 0$. This method is non-recursive and enables the relaxation modulus to be determined in the whole time interval of the relaxation test without using stress history in the time interval $t < t_1$. However, as in the Lee–Knauss method the time derivate of stress has to be evaluated numerically. In the presence of noisy data this can cause undesirable irregularity to the solution.

In this paper, we use the Zapas–Phillips method at time interval of its validity, i.e. $t \geqslant t_1/2$, and the method given in Sorvari and Malinen (2006), with central difference as the numerical differentiation rule of the stress, in the time interval $t < t_1/2$ to determine the relaxation modulus from the relaxation test. Truncation error analysis shows that both of these methods are second-order accurate, i.e. $\varepsilon \sim t_1^2\sigma''(t)$ (Sorvari and Malinen, 2006).

5. Regularization

In this section, Tikhonov regularization and regularization parameter choice methods are briefly discussed. For the detailed analysis of Tikhonov regularization, we refer reader to see references Tikhonov (1963) and Hansen (1998).

As discussed in Section 3, all of the integral approaches for interconversion presented in this paper can be written in the matrix form as

$$Ax = b. \tag{38}$$

In numerical interconversion, the matrix A can be ill-conditioned and the direct inversion can cause large errors to solution x. Due to the ill-conditioning of the problem, Tikhonov regularization is proposed to solve Eq. (38). The Tikhonov regularized solution of Eq. (38) can be written as

$$x_{\text{reg}} = (\gamma R + A^{\mathrm{T}}A)^{-1}A^{\mathrm{T}}b, \tag{39}$$

where $\gamma > 0$ is the regularization parameter and R is the regularization matrix. There are many alternatives to choose regularization matrix. In general, if any kind of a priori information of the solution is known it can be implemented to matrix R. In this study, the interconversion is accomplished in time domain. As a priori information, material functions (creep compliance in simulations) are considered to be smooth functions of time, i.e. there are no sharp peaks in x. In such case, the matrix R is chosen to be the first order difference matrix, i.e.

$$R = \begin{pmatrix} 1 & & & & \\ & -1 & 1 & & \\ & & \ddots & \ddots & \\ & & & -1 & 1 \end{pmatrix}. \tag{40}$$

Another unknown parameter in Tikhonov regularization is the regularization parameter γ. This parameter has a very significant influence to the solution, since if γ is too small, the problem is still ill-conditioned, while too large values of γ smooths solution too much. In general, there are several strategies to choose regularization parameter. These methods include for example discrepancy principle, quasi optimality criterion and L-curve method. In this study we consider L-curve method to choose γ. The main idea in L-curve method is to plot smoothing norm $\|Rx_{\text{reg}}\|$ as a function of corresponding residual norm $\|Ax_{\text{reg}} - b\|$ with varying γ. As these norms are plotted in logarithmic scales, the resulting curve is very often L-shaped. The regularization parameter can be chosen from the solution which is near to L-curve's corner. This graphical method is often used for choosing the regularization parameter in ill-posed problems and the details of the L-curve method is given in Hansen (1998).

When synthetic data is used and the exact solution of the problem is known, it is possible to choose parameter γ according to real estimation error. In this case relative real estimation error, $\|x_{\text{reg}} - x\|/\|x\|$, is plotted as a function of γ and the optimal value of γ is obtained from the minimum value of this function. However, in general case real estimation error is unknown and this method cannot be used.

6. Numerical studies

The interconversion methods were numerically tested using synthetic data. The simulations consider the determination of the creep compliance from the relaxation test. The simulation procedure is following. First, the virtual relaxation test with non-ideal loading

$$\epsilon(t) = \begin{cases} \dot{\epsilon}_0 t & t < t_1, \\ \epsilon_0 & t \geqslant t_1, \end{cases} \tag{41}$$

is made from which stress is measured. During this step Gaussian noise with different variance is included to stress. Second, the relaxation modulus is computed from the noisy data as described in Section 4. Third, the creep compliance is determined with methods given in Section 3 with and without Tikhonov regularization when the related matrix equation is solved. This procedure was accomplished in different simulation cases with varying material parameters and noise. In simulations, the relaxation modulus was chosen to be

$$G(t) = G_0 + G_1 \mathrm{e}^{-t/\lambda} \tag{42}$$

with $G_0 = 0.4$ MPa, $G_1 = 0.5$ MPa and $\lambda = 1$, 10 or 100 s. The relaxation modulus with different values of parameter λ is shown in Fig. 1.

In this paper, results from nine case studies are presented. The relaxation time, λ, ramp time, t_1, test termination time, t_f, and length of the time step, h, are given in Table 1. Constant strain rate was chosen to be $\dot{\epsilon}_0 = 0.01\ \mathrm{s}^{-1}$. Simulation cases 1–6 consider longer time intervals, while cases 7–9 consider short time intervals. In each case three different noise levels were studied to simulate the practical measurement system. The noise was simulated as random Gaussian numbers with the variance of ±5%, ±10% or ±20% of the current value of stress and this noise was added to original stress function in each case. The same noise was used for each interconversion method within certain case study in order to get results which are directly comparable. As an example, Gaussian noise for stress was generated for case 1 and then all interconversion methods were simulated using data with the noise levels of 5%, 10% and 20%. For case 2, new noise was generated and all interconversion methods were simulated using this data.

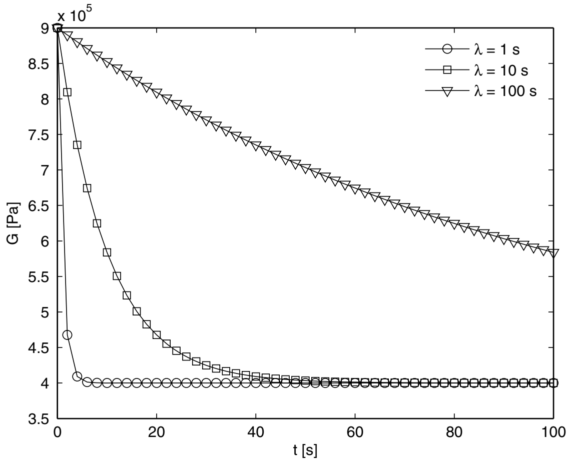

Fig. 1. Relaxation modulus as a function of time with different values of λ.

Table 1
Material and temporal parameters for simulation cases

Case	λ (s)	t_1 (s)	t_f (s)	h (s)
1	1	2	200	0.10
2	1	1	200	0.10
3	10	2	400	0.20
4	10	1	400	0.25
5	100	2	600	0.25
6	100	1	600	0.25
7	1	2	10	0.10
8	10	2	10	0.10
9	100	2	10	0.10

The regularization matrix was chosen to be the 1st order difference matrix, see Eq. (40). This is a natural choice since there are no sharp peaks in the creep function. The regularization parameter was chosen with L-curve method. Since the synthetic data was used, the chosen regularization parameter was verified according to real estimation error. The regularization parameter was chosen using Method 1 and data from case 1 in which Gaussian noise with the variance of $\pm 10\%$ of the current value of stress was included to original stress function. The solution of the problem was computed by varying the value of regularization parameter as $\gamma = [10^0 10^1 10^2, \ldots, 10^{15}]$. The L-curve and real estimation error are shown in Fig. 2. From this figure it can be seen that norm $\|Rx_{\text{reg}}\|$ as a function of norm $\|Ax_{\text{reg}} - b\|$ results in L-curve in log–log scale from which the regularization parameter can be defined (little right from the lower corner of the L-curve). According to Fig. 2, the value for regularization parameter was chosen to be $\gamma = 10^9$ for all simulation cases. From the real estimation error plot in Fig. 2, it can be also seen that the chosen regularization parameter value is near the minimum of the real estimation error, and the choice of the parameter is well justified.

The case studies in Table 1 was simulated with each of the methods described in Section 3. The noise levels of 5%, 10% and 20% were used for each case for each interconversion method. In all simulations, regularization parameter was $\gamma = 10^9$ and regularization matrix was the 1st order difference matrix. Since the same noise was used in certain case for each of the interconversion methods, the results are directly comparable. In addition, synthetic data was used in simulations and the relative error between analytical solution of the creep function and numerical approximation could be computed. The relative error in each case was computed as

$$\varepsilon = \frac{\|\hat{x} - x\|}{\|x\|}, \tag{43}$$

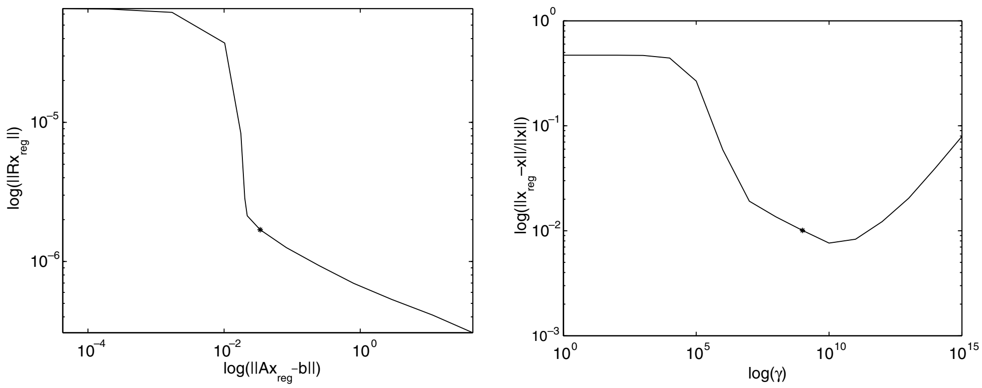

Fig. 2. Left: L-curve for Tikhonov regularization. Right: Real estimation error. The solution according to value $\gamma = 10^9$ is shown with the star (*) in both graphs.

Table 2
Results for Method 1 for different case studies

Case	Error_1 (%)	$\text{Error}_1^{\text{tik}}$ (%)	Error_2 (%)	$\text{Error}_2^{\text{tik}}$ (%)	Error_3 (%)	$\text{Error}_3^{\text{tik}}$ (%)
1	4.24	0.74	47.05	1.11	Inf	2.23
2	0.36	0.35	0.76	0.63	14.31	1.87
3	0.34	0.34	0.59	0.57	3.21	1.35
4	0.27	0.27	0.69	0.68	1.68	1.51
5	0.34	0.34	1.66	0.83	Inf	2.78
6	0.36	0.36	0.81	0.76	44.65	2.41
7	3.29	3.18	11.34	6.93	Inf	8.72
8	5.46	5.17	9.05	6.01	40.50	6.46
9	5.81	3.99	12.22	4.68	Inf	10.66

Errors are computed as a relative error between analytical solution and numerical approximation. For the material and temporal parameters of the cases, see Table 1. Subscripts 1, 2 and 3 refers to noise levels of 5%, 10% and 20%, respectively. Superscript tik refers to Tikhonov regularized solution.

where x is the analytical solution and $\hat{x}$ is the corresponding numerical approximation which is computed with either direct inversion or Tikhonov regularization. The analytical solution for the creep compliance was computed as

$$J(t) = \left[\frac{1}{G_0 + G_1} - \frac{1}{G_0}\right] \mathrm{e}^{-G_0 t/(\lambda(G_0+G_1))} + \frac{1}{G_0} \tag{44}$$

in which G_0, G_1 and λ were computed from the noise free relaxation function.

The results with Method 1 for the case studies are given in Table 2. In all simulated cases the Tikhonov regularization decreases the relative error. In cases 1 and 5 the 20% noise level with the direct inversion results in divergence due the ill-conditioning problem. In addition, even if the step size in time discretization is increased in cases 4–6, the error does not increase significantly, which indicates the robustness of the method. In the very short termination time cases (7–9) the relative error is increased as compared to other cases. However, regularization decreases the approximation errors efficiently, see Fig. 3, especially with higher noise levels.

The results with Method 2 are given in Table 3. As compared to results with Method 1, the relative errors are larger throughout the case studies. With fast tests (cases 7–9) the relative errors are significantly larger in higher noise levels as compared to Method 1.

Simulation results with Method 3 are given in Table 4. From this table it can be seen that the relative error in approximation with Method 3 is larger than with Methods 1 and 2 in all cases. Again, Tikhonov

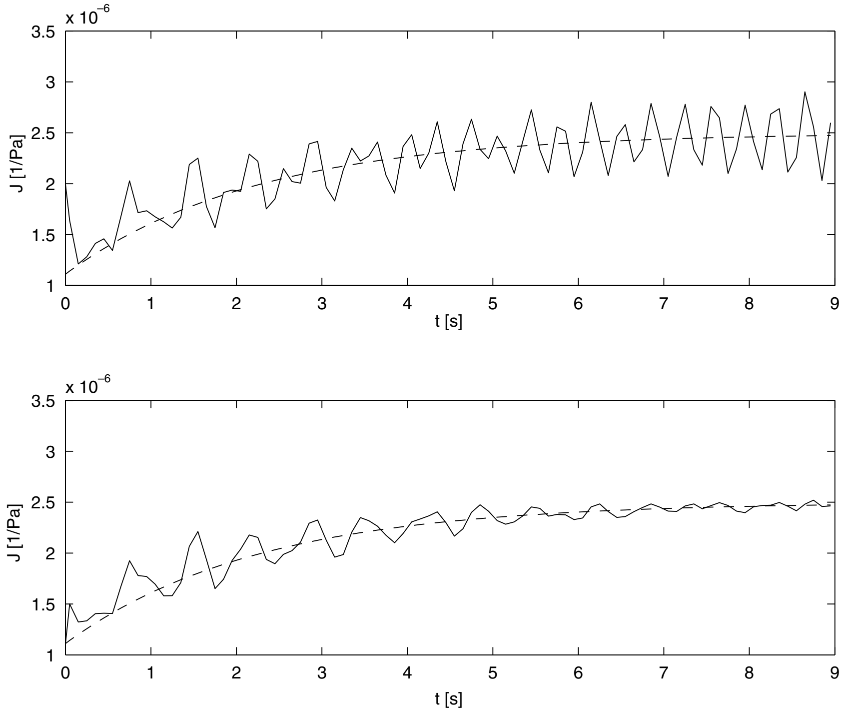

Fig. 3. Analytical (dashed line) and approximated (solid line) creep compliance without (upper figure) and with (lower figure) regularization for Method 1 in simulation case 7 with 10% noise.

Table 3
Results for Method 2 for different case studies

Case	$Error_1$ (%)	$Error_1^{tik}$ (%)	$Error_2$ (%)	$Error_2^{tik}$ (%)	$Error_3$ (%)	$Error_3^{tik}$ (%)
1	6.34	0.85	60.78	1.42	Inf	3.49
2	0.46	0.40	1.10	0.80	16.86	2.57
3	0.39	0.38	0.95	0.75	4.32	2.09
4	0.33	0.32	1.01	0.88	2.19	1.88
5	0.42	0.40	2.27	1.32	Inf	3.65
6	0.52	0.48	1.17	1.00	67.52	3.67
7	3.53	3.45	16.13	7.76	Inf	11.40
8	6.82	6.20	11.41	8.33	63.88	12.41
9	9.97	5.99	23.45	6.12	Inf	14.15

Table 4
Results for Method 3 for different case studies

Case	$Error_1$ (%)	$Error_1^{tik}$ (%)	$Error_2$ (%)	$Error_2^{tik}$ (%)	$Error_3$ (%)	$Error_3^{tik}$ (%)
1	6.34	2.57	60.78	2.98	Inf	8.33
2	0.46	0.46	1.10	1.09	16.78	8.61
3	0.39	0.39	0.95	0.95	4.32	3.94
4	0.33	0.33	1.01	1.01	2.19	2.17
5	0.42	0.42	2.27	2.21	Inf	7.34
6	0.52	0.52	1.17	1.16	67.52	6.80
7	3.53	3.53	16.13	15.72	Inf	66.40
8	6.82	6.82	11.41	11.35	63.83	59.94
9	9.97	9.93	23.45	22.90	Inf	18.95

The variables are as in Table 2.

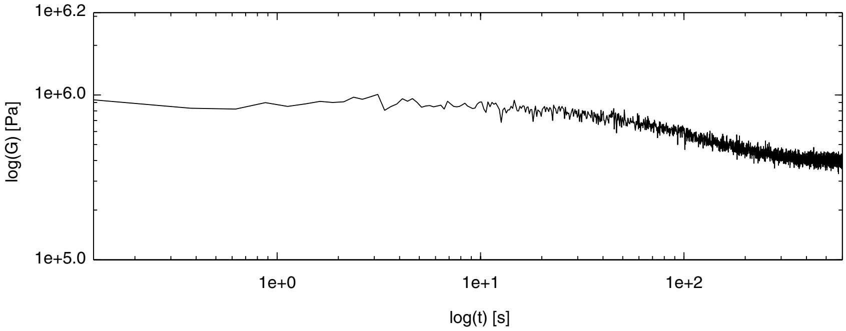

Fig. 4. Relaxation modulus as a function of time for simulation case 6 with 20% noise.

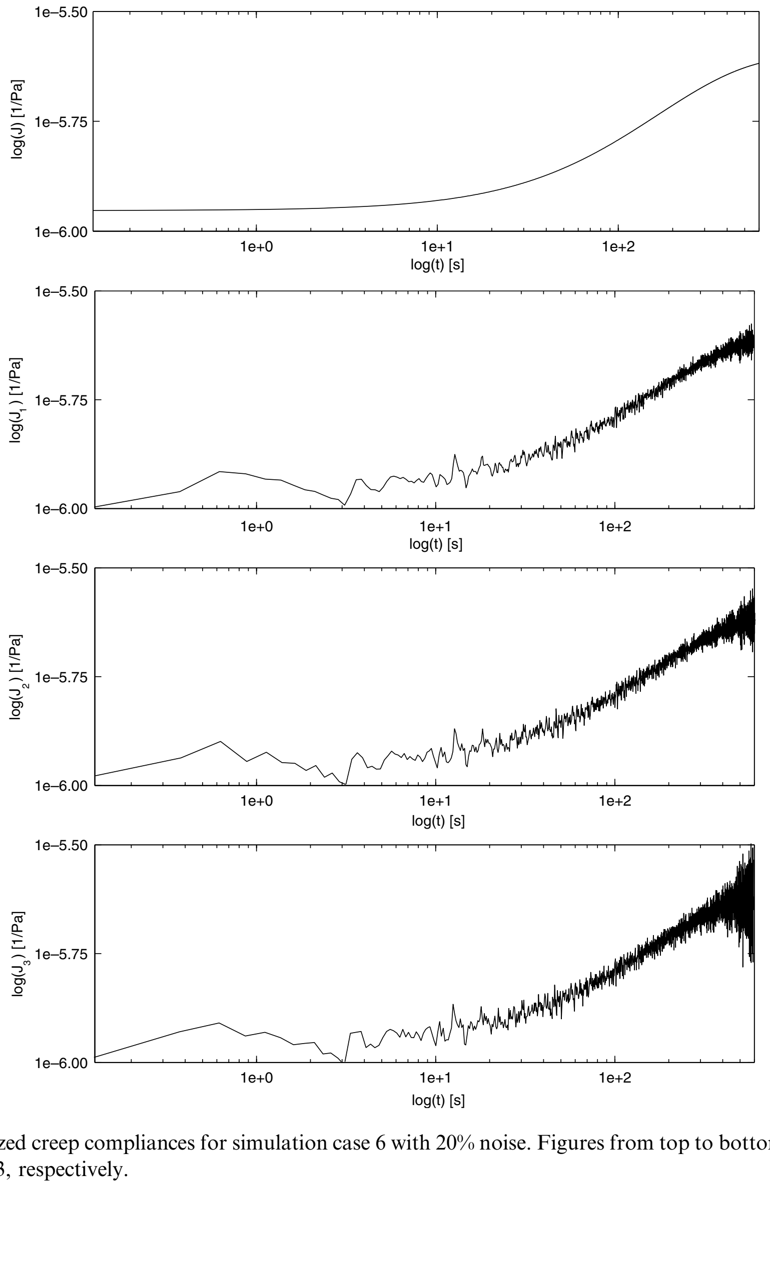

Fig. 5. Tikhonov regularized creep compliances for simulation case 6 with 20% noise. Figures from top to bottom correspond to analytical solution and Methods 1–3, respectively.

regularization decreases efficiently the relative error in cases 1–6. In cases 7–9 the regularization does not affect significantly to the approximation error.

As an example, the relaxation modulus from case 6 with 20% noise level is shown in Fig. 4. The analytical solution and Tikhonov regularized solutions for creep compliance with all interconversion methods for simulation case 6 with 20% noise level is shown in Fig. 5. This figure clearly shows how noise affects to simulated interconversion methods. As it can be seen, high noise level causes rapid oscillation to computed creep compliance with Method 3, while Methods 1 and 2 results in much smoother creep compliance approximation.

As a conclusion, Method 1 gives the best numerical estimates when creep compliance is solved by interconversion from relaxation modulus data. Results indicate that the relative error in the approximation of creep compliance was decreased with all methods in cases 1–6 with Tikhonov regularization. However, when very fast tests are considered (cases 7–9), Method 1 is the most accurate while regularization does not affect much to the results computed with Method 3.

7. Conclusions

In this paper, three different interconversion methods were studied in order to determine creep compliance from non-ideal relaxation test. In practice, there is always measurement noise in viscoelastic experiment. Due to this and the fact that the convolution integral relating creep compliance and relaxation modulus is numerically ill-posed problem, regularization is necessity to obtain reliable results. The Tikhonov regularization was proposed to solve the related matrix equation numerically to avoid the ill-conditioning problem.

The proposed numerical procedures were tested using synthetic data with different noise levels. The regularization matrix was determined to be 1st order difference matrix and the regularization parameter was defined with L-curve method. Although there are several alternatives to choose regularization parameter, the L-curve was found to be suitable in this case, and the value of the parameter was justified by computing the real estimation error. The simulation results from different case studies indicate that regularization decreases the approximation error with all interconversion methods. Especially, when noise level increases, the ill-conditioning problem comes more evident and regularization is the only alternative to get the reliable result. As interconversion methods are compared, Method 1 was found to be the most accurate. In addition, the computation time is almost the same for all methods so Method 1 should be used in practice. Especially, during the very fast tests, Method 1 gives the most accurate approximation while results with Method 3 remains inaccurate in spite of regularization.

It was shown that interconversion methods based on numerical evaluation of the convolution integral are potential candidates to solve creep compliance from relaxation test even with presence of uncertainties such as noise, non-ideal loading and short test times. If the relaxation test is very long, the dimensions of the related matrix equation becomes very large. In this case logarithmic time scale can be used or interconversion can be accomplished in the frequency domain. However, the purpose of this paper was to present robust algorithms which can be used in time domain with materials exhibiting relatively fast rate of relaxation.

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