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The radially nonhomogeneous elastic axisymmentric problem

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ABSTRACT

The plane axisymmetric problem with axisymmetric geometry and loading is analyzed for a radially nonhomogeneous circular cylinder, in linear elasticity. Considering the radial dependence of the stress, the displacements fields and of the stiffness matrix, after a series of admissible functional manipulations, the general differential system solving the problem is developed. The isotopic radially inhomogeneous elastic axisymmetric problem is also analyzed. The exact elasticity solution is developed for a radially nonhomogeneous hollow circular cylinder of exponential Young's modulus and constant Poisson's ratio and of power law Young's modulus and constant Poisson's ratio. For the isotropic elastic axisymmetric problem, a general expression of the stress function is derived. After the satisfaction of the biharmonic equation and making compatible the stress field's expressions, the stress function and the stress and displacements fields of the axisymmetric problem are also deduced. Applications have been made for a radially nonhomogeneous hollow cylinder where the stress and displacements fields are determined.

1. Introduction

The study of the linear axisymmetric problem has already been confronted in Elasticity by many investigators (Love, 1956; Sokolnikoff, 1956; Muskhelishvili, 1963; Timoshenko and Goodier, 1970; Barber, 1974; Ting, 1984; Fabricant, 1990; Barber, 1992; Birman, 1992; Sideridis, 1993; Singh and Kumar, 1994; Gal and Dvorkin, 1995). In investigations in half space problems and in the case of surface tractions (Barber, 1974; Fabricant, 1990), in composite materials determining the stress field (Sideridis, 1993; Gal and Dvorkin, 1995), in shells using high-order theories and analyzing the buckling (Birman, 1992; Kardomateas, 1993), in pressure vessels pipes and in the optimization of the production (Singh and Kumar, 1994; Stampouloglou and Theotokoglou, 2006) the property of axisymmetry arises and the solution of the problems have been simplified considerably. In several studies (Barber, 1974; Fabricant, 1990; Sideridis, 1993; Singh and Kumar, 1994; Gal and Dvorkin, 1995), the solution of the axisymmetric problem is based on the construction of a stress function for determining the stress and displacements fields.

Nonhomogenous materials can be described as two-phase particulate composites, where the volume fraction of its constituents differs continuously in the thickness direction (Bakirtas, 1980; Erdogan and Delale, 1983; Erdogan et al., 1991; Craster and Atkinson, 1994; Aboudi et al., 1995; Zhang and Hasebe, 1999; Horgan and Chan, 1999; Afsar and Sekine, 2002; Paulino et al., 2003; Weng, 2003; Theotokoglou and Stampouloglou, 2004). This implies that the composition profile can be tailored to give appropriate thermo mechanical properties. Radially varying elastic moduli were used by Lutz and Zimmerman (1996) to describe the behaviour of the interphase zone of an infinite body around a chemical inclusion.

In this study, the plane axisymmetric problem with axisymmetric geometry and loading is analyzed in a radially nonhomogeneous hollow circular cylinder. Considering the radial dependence of the stress, the displacements fields and of the

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stiffness matrix **E**, and taking into account the equation of compatibility, after a sequel of admissible functional manipulations, the general differential system is produced for the first time for a radially nonhomogeneous hollow circular cylinder. It is also produced following admissible functional manipulations the differential system in the case of an isotropic radially nonhomogeneous hollow circular cylinder.

The solution of the derived systems may be obtained either numerically or analytically. In general, these systems do not accept an analytic solution. Assuming in the isotropic nonhomogeneous case that the functionally graded material has a constant Poisson's radio and that the Young's modulus is of an exponential or a power law form, an analytical solution results for the stress and displacement fields in terms of hypergeometric functions. From the boundary conditions of the problem, the constants of integration are also obtained.

The solution of the isotropic homogeneous elastic axisymmetric problem may be arised from the differential system of the isotropic radially nonhomogeneous problem. In addition considering that both stresses and displacements are functions of the radial coordinate r, a general form of the stress function occurs. Treating the stress field in terms of the stress function and in terms of the displacements and using the biharmonic equation, a fourth-order homogeneous differential equation results, whose general solution is the stress function of the axisymmetric problem.

The proposed analysis is applied to the elastostatic problem of the hollow nonhomogeneous cylindrical tube under internal loading. Considering either an exponentially varying Young's modulus, or a power law varying Young's modulus the stress and displacements fields are determined.

2. The stress and displacements fields of the radially nonhomogeneous elastic axisymmetric problem

In axisymmetric problems with axisymmetric geometry and loading, the geometry of the body and the loading depend only on the radius r of the polar coordinate system (r, θ), with origin at the centre of axisymmetry. In the case of plane elasticity, the stress and strain fields may be written

$$\sigma_{ij} = \sigma_{ij}(r), \quad \varepsilon_{ij} = \varepsilon_{ij}(r), \qquad i,j = r,\theta, \tag{1a}$$

$$\mathbf{u} = \mathbf{u}(r), \quad \mathbf{u} = u_i\mathbf{e_i}, \qquad i = r,\theta, \tag{1b}$$

where $\mathbf{u}$ is the displacement field and $\boldsymbol{e}_i$ is the polar basis.

It is assumed that the axisymmetric elastic problem is radially inhomogeneous, namely the elastic constants are given in terms of the radial coordinate r, thus

$$\boldsymbol{\sigma} = \mathbf{E}\boldsymbol{\varepsilon}, \quad \mathbf{E} = \mathbf{E}(r), \tag{2}$$

where $\mathbf{E}$ the stiffness matrix.

The equations of equilibrium in the polar coordinate system (Timoshenko and Goodier, 1970; Barber, 1992) are

$$\frac{\partial\sigma_{rr}}{\partial r} + \frac{1}{r}\frac{\partial\sigma_{r\theta}}{\partial\theta} + \frac{\sigma_{rr}-\sigma_{\theta\theta}}{r} = 0, \quad \frac{\partial\sigma_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial\sigma_{\theta\theta}}{\partial\theta} + 2\frac{\sigma_{r\theta}}{r} = 0, \tag{3}$$

and the equation of compatibility of strain is

$$\frac{\partial^2\varepsilon_{\theta\theta}}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2\varepsilon_{rr}}{\partial\theta^2} + \frac{2}{r}\frac{\partial\varepsilon_{\theta\theta}}{\partial r} - \frac{1}{r}\frac{\partial\varepsilon_{rr}}{\partial r} = 2\left(\frac{1}{r}\frac{\partial^2\varepsilon_{r\theta}}{\partial r\partial\theta} + \frac{1}{r^2}\frac{\partial\varepsilon_{r\theta}}{\partial\theta}\right). \tag{4}$$

From relations (1a), we get

$$r\sigma'_{rr} + \sigma_{rr} - \sigma_{\theta\theta} = 0, \quad r\sigma'_{r\theta} + 2\sigma_{r\theta} = 0. \tag{5}$$

Setting, $r = \mathrm{e}^t$, we have

$$\dot{y} = y'r, \quad \ddot{y} - \dot{y} = y''r^2, \quad y' = \frac{\mathrm{d}y}{\mathrm{d}r}, \quad \dot{y} = \frac{\mathrm{d}y}{\mathrm{d}t}. \tag{6}$$

The differential system (5), is written

$$\dot{\sigma}_{rr} + \sigma_{rr} - \sigma_{\theta\theta} = 0, \quad \dot{\sigma}_{r\theta} + 2\sigma_{r\theta} = 0. \tag{7}$$

From the second of (7), it is obtained

$$\ln\sigma_{r\theta} = -2t + \Gamma.$$

Setting, $\Gamma = \ln C_1$, we finally have

$$\sigma_{r\theta} = \frac{C_1}{r^2}, \tag{8}$$

where C_1 constant to be determined.

Thus, the shear stress $\sigma_{r\theta}$ is independent of the inhomogeneity. The first of (7), taking into consideration Polyanin and Zaitsev (p. 81 for $g(t) = 1, f(t) = -1, f_1(t) = \sigma_{\theta\theta}, F(t) = \int f_1(t)/g(t)\mathrm{d}t = -t$), has the solution

$$\sigma_{rr}(t) = Ce^{-t} + e^{-t}\int e^{t}\sigma_{\theta\theta}\,dt = Ce^{-t} + e^{-t}\int \sigma_{\theta\theta}\,d(e^{t}) = \frac{1}{r}\left(C + \int \sigma_{\theta\theta}\,dr\right), \tag{9}$$

where C constant to be determined.

Eq. (4) because of (1) is written

$$\varepsilon''_{\theta\theta} + \frac{2}{r}\varepsilon'_{\theta\theta} - \frac{1}{r}\varepsilon'_{rr} = 0. \tag{10}$$

Taking into consideration the stain components in the axisymmetric case

$$\varepsilon_{rr} = u_{r,r} = u'_r, \quad \varepsilon_{\theta\theta} = \frac{u_r}{r}, \quad \varepsilon_{r\theta} = -\frac{u_\theta}{2r} + \frac{u_{\theta,r}}{2} = \frac{1}{2}\left(u'_\theta - \frac{u_\theta}{r}\right), \tag{11}$$

the differential equation (10) becomes

$$\left(\frac{u''_r}{r} - \frac{2u'_r}{r^2} + \frac{2u_r}{r^3}\right) + \frac{2}{r}\left(\frac{u'_r}{r} - \frac{u_r}{r^2}\right) - \frac{u''_r}{r} \equiv 0.$$

Hence, Eq. (4) is an identity in the case of a radially inhomogeneous axisymmetric problem. Relation (2), because of relations (11) in the case of a body with cylindrical anisotropy in plane conditions subjected to an axial force or a moment with an angle to the principal material directions, thus obtaining monoclinic properties (Ting, 1984), is written as

$$\sigma_{rr} = E_{11}\varepsilon_{rr} + E_{12}\varepsilon_{\theta\theta} + E_{16}\varepsilon_{r\theta} = E_{11}u'_r + E_{12}\frac{u_r}{r} + \frac{E_{16}}{2}\left(u'_\theta - \frac{u_\theta}{r}\right), \tag{12a}$$

$$\sigma_{\theta\theta} = E_{12}\varepsilon_{rr} + E_{22}\varepsilon_{\theta\theta} + E_{26}\varepsilon_{r\theta} = E_{12}u'_r + E_{22}\frac{u_r}{r} + \frac{E_{26}}{2}\left(u'_\theta - \frac{u_\theta}{r}\right), \tag{12b}$$

$$\sigma_{r\theta} = E_{16}\varepsilon_{rr} + E_{26}\varepsilon_{\theta\theta} + E_{66}\varepsilon_{r\theta} = E_{16}u'_r + E_{26}\frac{u_r}{r} + \frac{E_{66}}{2}\left(u'_\theta - \frac{u_\theta}{r}\right). \tag{12c}$$

Substituting (8) into (12c), it is obtained

$$\frac{1}{2}\left(u'_\theta - \frac{u_\theta}{r}\right) = \frac{C_1}{E_{66}r^2} - \frac{E_{16}}{E_{66}}u'_r - \frac{E_{26}}{E_{66}}\left(\frac{u_r}{r}\right),$$

and substituting the above equation into relations (12a) and (12b), it is obtained

$$\begin{aligned}\sigma_{rr} &= \left(E_{11} - \frac{E_{16}^2}{E_{66}}\right)u'_r + \left(E_{12} - \frac{E_{16}E_{26}}{E_{66}}\right)\frac{u_r}{r} + \frac{E_{16}}{E_{66}}\left(\frac{C_1}{r^2}\right),\\ \sigma_{\theta\theta} &= \left(E_{12} - \frac{E_{16}E_{26}}{E_{66}}\right)u'_r + \left(E_{22} - \frac{E_{26}^2}{E_{66}}\right)\frac{u_r}{r} + \frac{E_{26}}{E_{66}}\left(\frac{C_1}{r^2}\right).\end{aligned} \tag{13}$$

Taking into consideration relation (13), the first of Eq. (5) becomes

$$\begin{aligned}&\left(E_{11} - \frac{E_{16}^2}{E_{66}}\right)r^2u''_r + \left[r\left(E_{11} - \frac{E_{16}^2}{E_{66}}\right)' + \left(E_{11} - \frac{E_{16}^2}{E_{66}}\right)\right]ru'_r + \left[r\left(E_{12} - \frac{E_{16}E_{26}}{E_{66}}\right)' + \left(E_{22} - \frac{E_{26}^2}{E_{66}}\right)\right]u_r\\ &\quad = \frac{E_{16} + E_{26}}{E_{66}}\left(\frac{C_1}{r}\right) - \left(\frac{E_{16}}{E_{66}}\right)'C_1.\end{aligned} \tag{14}$$

Due to transformation (6), we get

$$\left(E_{11} - \frac{E_{16}^2}{E_{66}}\right)\ddot{u}_r + \left(E_{11} - \frac{E_{16}^2}{E_{66}}\right)^{\cdot}\dot{u}_r + \left[\left(E_{12} - \frac{E_{16}E_{26}}{E_{66}}\right)^{\cdot} - \left(E_{22} - \frac{E_{26}^2}{E_{66}}\right)\right]u_r = C_1e^{-t}\left[\frac{E_{16} + E_{26}}{E_{66}} - \left(\frac{E_{16}}{E_{66}}\right)^{\cdot}\right]. \tag{15}$$

Based on (15), the radial component u_r of the displacements field (1b) is determined. The angular component, u_θ, taking into consideration relations (8) and (12c), is given by

$$ru'_\theta - u_\theta = \frac{2C_1}{rE_{66}} - \frac{2E_{16}}{E_{66}}ru'_r - \frac{2E_{26}}{E_{66}}u_r$$

or due to transformation (6)

$$\dot{u}_\theta - u_\theta = \frac{2C_1e^{-t}}{E_{66}} - \frac{2E_{16}}{E_{66}}\dot{u}_r - \frac{2E_{26}}{E_{66}}u_r. \tag{16}$$

Hence, the displacements field is determined from relations (15) and (16), and the stress field from relations (8) and (13). The constant C_1 as well as the constants arising from the solutions of the differential equations (15) and (16) will be determined from the boundary conditions of the axisymmetric problem (Section 5). Eqs. (15) and (16) constitute the general solution of the radially nonhomogeneous elastic axisymmetric problem, because besides of the solution for the distributed axisymmetric pressure, it may provide the solution for the axisymmetric distributed shear stress where the $\sigma_{r\theta}$ – stress and the u_θ – displacement components are different from zero.

3. The isotropic radially nonhomogeneous elastic axisymmetric problem

In the case of an isotropic radially inhomogeneous axisymmetric material, the stiffness matrix **E** is given (Barber, 1992) by

$$\mathbf{E}(r) = \begin{bmatrix} E_{11}(r) = 2\mu(r) + \lambda^*(r) & E_{12}(r) = \lambda^*(r) & E_{16}(r) = 0 \\ E_{12}(r) = \lambda^*(r) & E_{22}(r) = E_{11}(r) & E_{26}(r) = 0 \\ E_{16}(r) = 0 & E_{26}(r) = 0 & E_{66}(r) = 2\mu(r) \end{bmatrix}, \tag{17}$$

where

$$\lambda^* = \begin{cases} \lambda = \frac{E(r)\nu(r)}{[1+\nu(r)][1-2\nu(r)]} & \text{in plane strain,} \\ \frac{2\lambda(r)\mu(r)}{\lambda(r)+2\mu(r)} = \frac{E(r)\nu(r)}{1-\nu^2(r)} & \text{in generalized plane stress,} \end{cases} \tag{18}$$

with $\mu(r)$ the radially varying shear modulus, $\nu(r)$ the radially varying Poisson's ratio.

The differential equations (15) and (16) for the determination of the displacements field u, because of (17), become

$$E_{11}\ddot{u}_r + \dot{E}_{11}\dot{u}_r + (\dot{E}_{12} - E_{22})u_r = 0, \tag{19a}$$

$$\dot{u}_\theta = u_\theta + \frac{2C_1 \mathrm{e}^{-t}}{E_{66}}. \tag{19b}$$

Let u_{r_0} a partial solution satisfying (19a). Thus, the solution of (19a) is (Polyanin and Zaitsev, 2003, p. 213)

$$u_r(t) = u_{r_0}\left(\Gamma_1 + \Gamma_2 \int \frac{\mathrm{e}^{-F}}{u_{r_0}^2}\,\mathrm{d}t\right), F = \int \frac{\dot{E}_{11}}{E_{11}}\,\mathrm{d}t = \ln E_{11} \tag{20}$$

where Γ_1, Γ_2 constants to be determined.

Hence,

$$u_r(t) = u_{r_0}\left(\Gamma_1 + \Gamma_2 \int \frac{\mathrm{d}t}{E_{11}u_{r_0}^2}\right). \tag{21}$$

Setting, $t = \ln r$, we have

$$u_r(r) = u_{r_0}\left(\Gamma_1 + \Gamma_2 \int \frac{\mathrm{d}r}{rE_{11}u_{r_0}^2}\right). \tag{22}$$

Differential equation (19b) is a Bernoulli equation (Polyanin and Zaitsev, p. 81 for $g(t) = f_1(t) = 1$, and $f_0(t) = (2C_1\mathrm{e}^{-t}/E_{66})(n = 0)$), with solution

$$u_\theta(t) = \Delta_1\mathrm{e}^F + \mathrm{e}^F \int \mathrm{e}^{-F}\frac{f_0(t)}{g(t)}\mathrm{d}t, \quad F(t) = \int \frac{f_1(t)}{g(t)}\mathrm{d}t = \int \mathrm{d}t = t,$$

whence

$$u_\theta(t) = \Delta_1\mathrm{e}^t + 2C_1\mathrm{e}^t \int \frac{\mathrm{e}^{-2t}\,\mathrm{d}t}{E_{66}}.$$

Because, $\Delta_1\mathrm{e}^t = \Delta_1 r$, is a rigid body rotation, it is taken $\Delta_1 = 0$, thus

$$u_\theta(t) = 2C_1\mathrm{e}^t \int \frac{\mathrm{e}^{-2t}\,\mathrm{d}t}{E_{66}}. \tag{23}$$

Setting, $t = \ln r$, Eq. (23), takes the form

$$u_\theta(r) = 2C_1 r \int \frac{\mathrm{d}r}{r^3 E_{66}}. \tag{24}$$

The stress field (8) and (13), because of relations (17) and (22), becomes

$$\begin{aligned} \sigma_{rr} &= \frac{\Gamma_2}{ru_{r_0}} + \left(E_{11}u'_{r_0} + E_{12}\frac{u_{r_0}}{r}\right)\left(\Gamma_1 + \Gamma_2\int\frac{\mathrm{d}r}{rE_{11}u_{r_0}^2}\right), \\ \sigma_{\theta\theta} &= \frac{\Gamma_2 E_{12}}{rE_{11}u_{r_0}} + \left(E_{12}u'_{r_0} + E_{22}\frac{u_{r_0}}{r}\right)\left(\Gamma_1 + \Gamma_2\int\frac{\mathrm{d}r}{rE_{11}u_{r_0}^2}\right), \\ \sigma_{r\theta} &= \frac{C_1}{r^2}. \end{aligned} \tag{25}$$

4. The exponential case $E = E_0 e^{\delta r}$ and v = constant

Let an isotropic radially inhomogeneous elastic body (Fig. 1), where Poisson's ratio v is constant, and modulus E is a function of r such that

$$E = E_0 e^{\delta r}, \tag{26}$$

where E_0 and δ are given constants.

In the plane strain case, where (Barber, 1992)

$$\begin{aligned} E_{11} &= E_{22} = \frac{E(1-v)}{(1+v)(1-2v)} = \frac{E_0(1-v)}{(1+v)(1-2v)} e^{\delta r}, \\ E_{12} &= \frac{Ev}{(1+v)(1-2v)} = \frac{E_0 v}{(1+v)(1-2v)} e^{\delta r}, \end{aligned} \tag{27}$$

the differential equations (14) and (19b), become

$$r^2 u_r'' + r(\delta r + 1)u_r' + \left(\frac{v\delta}{1-v} r - 1\right) u_r = 0, \tag{28a}$$

$$\dot{u}_\theta = u_\theta + \frac{2C_1 e^{-t}}{2\mu} = u_\theta + 2(1+v)C_1 \frac{e^{-t}}{E_0 e^{\delta e^t}}, \quad r = e^t, \tag{28b}$$

whereas in the generalized plane stress, where

$$E_{11} = E_{22} = \frac{E}{1-v^2} = \frac{E_0 e^{\delta r}}{1-v^2}, \quad E_{12} = \frac{Ev}{1-v^2} = \frac{E_0 v e^{\delta r}}{1-v^2}, \tag{29}$$

the differential equations (14) and (19b), become

$$r^2 u_r'' + r(\delta r + 1)u_r' + (v\delta r - 1)u_r = 0, \tag{30a}$$

$$\dot{u}_\theta = u_\theta + \frac{2C_1 e^{-t}}{2\mu} = u_\theta + 2(1+v)C_1 \frac{e^{-t}}{E_0 e^{\delta e^t}}, \quad r = e^t. \tag{30b}$$

Thus, the component u_θ of the displacements field u in the plane strain as much as in the generalized plane stress case, is calculated from the same differential equation (28b) or (30b).

Furthermore, using Eq. (24), we have

$$u_\theta(r) = 2C_1 r \int \frac{dr}{r^3 E_{66}} = \frac{2C_1(1+v)}{E_0} r \int \frac{dr}{r^3 e^{\delta r}} = -\frac{C_1(1+v)}{E_0}\left(\frac{e^{-\delta r}}{r}\right). \tag{31}$$

The component u_r of u, is calculated taking into consideration the differential equations (28a) or (30a) for the plane strain or the generalized plane stress cases.

Thus, we may write

$$r^2 u_r'' + r(\delta r + 1)u_r' + (v^* \delta r - 1)u_r = 0, \tag{32}$$

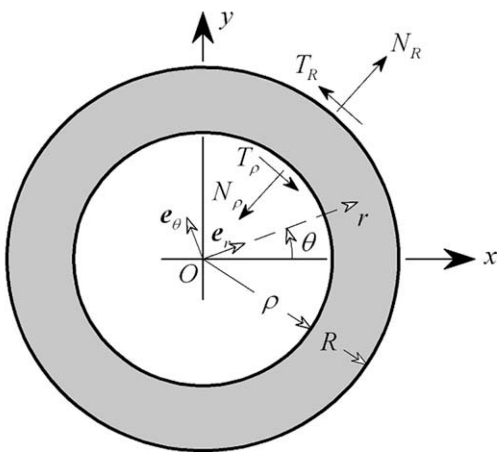

Fig. 1. The nonhogeneous hollow circular cylinder under internal and external loading.

where

$$\begin{array}{lll} E_0^* = E_0/(1+v)(1-2v), & v^* = v/1-v & \text{for plane strain,} \\ E_0^* = E_0/1-v, & v^* = v & \text{for generalized plane stress.} \end{array} \tag{33}$$

The solution of the differential equation (32) is given by (Appendix)

$$u_r(r) = rZ_1\Phi^{(+)} + rZ_2\Psi^{(+)}, \quad \delta > 0, \tag{34}$$

where

$$\Phi^{(+)} = \Phi(v^*+1,3;-\delta r) = \Phi(v^*+1,3;-|\delta|r),$$
$$\Psi^{(+)} = \Psi(v^*+1,3;-\delta r) = \Psi(v^*+1,3;-|\delta|r)$$

or

$$u_r(r) = re^{-\delta r}Z_1\Phi^{(-)} + re^{-\delta r}Z_2\Psi^{(-)} = re^{|\delta|r}Z_1\Phi^{(-)} + re^{|\delta|r}Z_2\Psi^{(-)}, \quad \delta < 0, \tag{35}$$

where

$$\Phi^{(-)} = \Phi(2-v^*,3;\delta r) = \Phi(2-v^*,3;-|\delta|r),$$
$$\Psi^{(-)} = \Psi(2-v^*,3;\delta r) = \Psi(2-v^*,3;-|\delta|r),$$

and, Z_1 and Z_2 constants to be determined, $\Phi(a,b;x)$, $\Psi(a,b;x)$ degenerate hypergeometric functions (Polyanin and Zaitsev, 2003).

The stress field (8), (13), because of relations (17), (33) and (34) or (35), is written

(i) $\delta > 0$

$$\begin{aligned} \sigma_{rr} &= E_0^* e^{\delta r}\left\{Z_1\left[\Phi^{(-)} - \frac{\delta r}{3}\Phi_1^{(+)}\right] + Z_2[\Psi^{(+)} + \delta r\Psi_1^{(+)}]\right\}, \\ \sigma_{\theta\theta} &= E_0^* e^{\delta r}\left\{Z_1\left[\Phi^{(+)} - v^*\frac{\delta r}{3}\Phi_1^{(+)}\right] + Z_2\left[\Psi^{(+)} + v^*\delta r\Psi_1^{(+)}\right]\right\}, \quad \sigma_{r\theta} = \frac{C_1}{r^2}, \end{aligned} \tag{36}$$

where

$$\Phi^{(+)} = \Phi(v^*+1,3;-\delta r), \quad \Phi_1^{(+)} = \Phi(v^*+2,4;-\delta r),$$
$$\Psi^{(+)} = \Psi(v^*+1,3;-\delta r), \quad \Psi_1^{(+)} = \Psi(v^*+2,4;-\delta r).$$

(ii) $\delta < 0$

$$\begin{aligned} \sigma_{rr} &= E_0^*\left\{Z_1\left[\left(1-\frac{\delta r}{1+v^*}\right)\Phi^{(-)} + \left(\frac{\delta r}{3}\right)\frac{2-v^*}{1+v^*}\Phi_1^{(-)}\right] + Z_2\left[\left(1-\frac{\delta r}{1+v^*}\right)\Psi^{(-)} - \delta r\frac{2-v^*}{1+v^*}\Psi_1^{(-)}\right]\right\}, \\ \sigma_{\theta\theta} &= E_0^*\left\{Z_1\left[\left(1-\frac{v^*\delta r}{1+v^*}\right)\Phi^{(-)} + \left(\frac{\delta r}{3}\right)\frac{v^*(2-v^*)}{1+v^*}\Phi_1^{(-)}\right] + Z_2\left[\left(1-\frac{v^*\delta r}{1+v^*}\right)\Psi^{(-)} - \delta r\frac{v^*(2-v^*)}{1+v^*}\Psi_1^{(-)}\right]\right\}, \\ \sigma_{r\theta} &= \frac{C_1}{r^2}, \end{aligned} \tag{37}$$

where

$$\Phi^{(-)} = \Phi(2-v^*,3;\delta r), \quad \Phi_1^{(-)} = \Phi(3-v^*,4;\delta r),$$
$$\Psi^{(-)} = \Psi(2-v^*,3;\delta r), \quad \Psi_1^{(-)} = \Psi(3-v^*,4;\delta r).$$

Relations (36) and (37) can also be reduced to the form

$$\begin{aligned} \sigma_{rr} &= E_0^*[Z_1A_1(r) + Z_2A_2(r)] \\ \sigma_{\theta\theta} &= E_0^*[Z_1B_1(r) + Z_2B_2(r)], \quad \sigma_{r\theta} = \frac{C_1}{r^2}, \end{aligned} \tag{38}$$

where

$$\begin{aligned} A_1(r) &= e^{\delta r}\left(\Phi^{(+)} - \frac{\delta r}{3}\Phi_1^{(+)}\right), \quad A_2(r) = e^{\delta r}(\Psi^{(+)} + \delta r\Psi_1^{(+)}), \qquad \delta > 0, \\ B_1(r) &= e^{\delta r}\left(\Phi^{(+)} - v^*\frac{\delta r}{3}\Phi_1^{(+)}\right), \quad B_2(r) = e^{\delta r}(\Psi^{(+)} + v^*\delta r\Psi_1^{(+)}), \qquad \delta > 0 \end{aligned} \tag{39}$$

with

$$\Phi^{(+)} = \Phi(v^*+1,3;-\delta r), \quad \Phi_1^{(+)} = \Phi(v^*+2,4;-\delta r), \qquad \delta > 0,$$
$$\Psi^{(+)} = \Psi(v^*+1,3;-\delta r), \quad \Psi_1^{(+)} = \Psi(v^*+2,4;-\delta r), \qquad \delta > 0$$

or

$$
\begin{aligned}
A_1(r) &= \left(1 - \frac{\delta r}{1+\nu^*}\right)\Phi^{(-)} + \left(\frac{\delta r}{3}\right)\frac{2-\nu^*}{1+\nu^*}\Phi_1^{(-)}, \quad \delta < 0,\\
A_2(r) &= \left(1 - \frac{\delta r}{1+\nu^*}\right)\Psi^{(-)} - \delta r\frac{2-\nu^*}{1+\nu^*}\Psi_1^{(-)}, \quad \delta < 0,\\
B_1(r) &= \left(1 - \frac{\nu^*\delta r}{1+\nu^*}\right)\Phi^{(-)} + \left(\frac{\delta r}{3}\right)\frac{\nu^*(2-\nu^*)}{1+\nu^*}\Phi_1^{(-)}, \quad \delta < 0,\\
B_2(r) &= \left(1 - \frac{\nu^*\delta r}{1+\nu^*}\right)\Psi^{(-)} - \delta r\frac{\nu^*(2-\nu^*)}{1+\nu^*}\Psi_1^{(-)}, \quad \delta < 0
\end{aligned}
\tag{40}
$$

with

$$
\begin{aligned}
&\Phi^{(-)} = \Phi(2-\nu^*, 3; \delta r), \quad \Phi_1^{(-)} = \Phi(3-\nu^*, 4; \delta r), \qquad \delta < 0,\\
&\Psi^{(-)} = \Psi(2-\nu^*, 3; \delta r), \quad \Psi_1^{(-)} = \Psi(3-\nu^*, 4; \delta r), \qquad \delta < 0.
\end{aligned}
$$

4.1. Boundary conditions

From the boundary conditions of the axisymmetric problem, we determine the constants $C_1,\ Z_1$ and Z_2. Considering the axisymmetric hollow circular cylinder (Fig. 1), loaded at both boundaries, we have for the shear tractions

$$
\int_0^{2\pi} T_\rho \rho^2\,\mathrm{d}\theta = \int_0^{2\pi} T_R R^2\,\mathrm{d}\theta
$$

or

$$
2\pi\rho^2 T_\rho = 2\pi R^2 T_R = M. \tag{41}
$$

Thus, from the boundary conditions

$$
\sigma_{r\theta}(r=\rho) = T_\rho, \quad \sigma_{r\theta}(r=R) = T_R,
$$

and Eq. (8), it follows:

$$
C_1 = M/2\pi, \tag{42}
$$

From the normal tractions (Fig. 1)

$$
\sigma_{rr}(r=\rho) = N_\rho, \quad \sigma_{rr}(r=R) = N_R,
$$

the coefficient, Z_1 and Z_2 are also calculated. Taking into consideration relations (38), it is obtained

$$
Z_1A_1(\rho) + Z_2A_2(\rho) = \frac{N_\rho}{E_0^*}, \quad Z_1A_1(R) + Z_2A_2(R) = \frac{N_R}{E_0^*}. \tag{43}
$$

From the solution of the system (43) occurs

$$
Z_1 = \left(\frac{1}{E_0^*}\right)\frac{N_\rho A_2(R) - N_R A_2(\rho)}{A_1(\rho)A_2(R) - A_1(R)A_2(\rho)}, \quad Z_2 = \left(\frac{1}{E_0^*}\right)\frac{N_R A_1(\rho) - N_\rho A_1(R)}{A_1(\rho)A_2(R) - A_1(R)A_2(\rho)}, \tag{44}
$$

provided that

$$
A_1(\rho)A_2(R) - A_1(R)A_2(\rho) \neq 0.
$$

5. The power law case $E = E_0(r/\rho)^\xi$ and ν = constant

Let an isotropic radially inhomogeneous elastic cylinder (Fig. 1), with constant Poisson's ratio ν and modulus E of the form

$$
E = E_0\left(\frac{r}{\rho}\right)^\xi, \quad \rho \leqslant r \leqslant R, \tag{45}
$$

where E_0 and ξ are given constants.

In the plane strain case, where (Barber, 1992),

$$
\begin{aligned}
E_{11} = E_{22} &= \frac{E(1-\nu)}{(1+\nu)(1-2\nu)} = \frac{E_0(1-\nu)}{(1+\nu)(1-2\nu)}\left(\frac{r}{\rho}\right)^\xi,\\
E_{12} &= \frac{E\nu}{(1+\nu)(1-2\nu)} = \frac{E_0\nu}{(1+\nu)(1-2\nu)}\left(\frac{r}{\rho}\right)^\xi,
\end{aligned}
\tag{46}
$$

the differential equations (14) and (19b) finally become

$$r^2u_r'' + r(\xi+1)u_r' + \left(\xi\frac{v}{1-v} - 1\right)u_r = 0, \tag{47a}$$

$$u_\theta = u_\theta + \frac{2C_1 e^{-t}}{2\mu} = u_\theta + 2(1+v)C_1\frac{e^{-t}}{E_0(e^t/\rho)^{\xi}}, \quad r = e^t. \tag{47b}$$

On the other hand, in the generalized plane stress case, where

$$E_{11} = E_{22} = \frac{E}{1-v^2} = \frac{E_0}{1-v^2}\left(\frac{r}{\rho}\right)^{\xi}, \quad E_{12} = \frac{Ev}{1-v^2} = \frac{E_0 v}{1-v^2}\left(\frac{r}{\rho}\right)^{\xi}, \tag{48}$$

the differential equations (14) and (19b) become

$$r^2u_r'' + r(\xi+1)u_r' + (\xi v - 1)u_r = 0, \tag{49a}$$

$$\dot{u}_\theta = u_\theta + \frac{2C_1 e^{-t}}{2\mu} = u_\theta + 2(1+v)C_1\frac{e^{-t}}{E_0(e^t/\rho)^{\xi}}, \quad r = e^t. \tag{49b}$$

In a similar way with the exponential case (Section 4), the component u_θ of the displacements field u, either in the plane strain or the generalized plane stress case, is

$$u_\theta = \frac{2C_1(1+v)}{E_0}\rho^{\xi} r\int\frac{dr}{r^{3+\xi}} = -\frac{2C_1(1+v)}{(2+\xi)E_0\rho}\left(\frac{r}{\rho}\right)^{-1-\xi}. \tag{50}$$

For the calculation of the component u_r of u, we may write

$$r^2u_r'' + r(\xi+1)u_r' + (\xi v^* - 1)u_r = 0, \tag{51}$$

where v^* is given from relation (33).

The differential equation (51) is an Euler type ODE (Polyanin and Zaitsev, 2003) with solution, given by (Appendix)

$$u_r = r^{-\frac{\xi}{2}}\left(Z_1 r^{\frac{1}{2}\sqrt{f(\xi)}} + Z_2 r^{-\frac{1}{2}\sqrt{f(\xi)}}\right), \quad f(\xi) = \xi^2 - 4\xi v^* + 4 > 0, \tag{52}$$

where Z_1 and Z_2 constants to be determined.

The stress field (8) and (13), because of relations (17), (33), (45) and (52), becomes

$$\begin{aligned}\sigma_{rr} &= \frac{E_0^*}{1+v^*}[Z_1P_1(r) + Z_2P_2(r)],\\ \sigma_{\theta\theta} &= \frac{E_0^*}{1+v^*}[Z_1Q_1(r) + Z_2Q_2(r)], \quad \sigma_{r\theta} = \frac{C_1}{r^2}\end{aligned} \tag{53}$$

with

$$\begin{aligned}P_1(r) &= \left(v^* - \frac{\xi}{2} + \frac{\sqrt{f}}{2}\right)\rho^{-\xi}r^{-1+(\xi/2)+(\sqrt{f}/2)}, \quad P_2(r) = \left(v^* - \frac{\xi}{2} - \frac{\sqrt{f}}{2}\right)\rho^{-\xi}r^{-1+(\xi/2)-(\sqrt{f}/2)},\\ Q_1(r) &= \left(1 - v^*\frac{\xi}{2} + v^*\frac{\sqrt{f}}{2}\right)\rho^{-\xi}r^{-1+(\xi/2)+(\sqrt{f}/2)}, \quad Q_2(r) = \left(1 - v^*\frac{\xi}{2} - v^*\frac{\sqrt{f}}{2}\right)\rho^{-\xi}r^{-1+(\xi/2)-(\sqrt{f}/2)},\end{aligned} \tag{54}$$

where

$$f(\xi) = 4 + \xi^2 - 4\xi v^*.$$

5.1. Boundary conditions

The coefficients C_1, Z_1 and Z_2 are determined from the boundary conditions of the axisymmetric problem. A hollow axisymmetric circular cylinder is considered (Fig. 1), loaded at both boundaries with uniformely distributed loads.

The coefficient C_1 is determined from the boundary condition concerning the shear tractions, using relations (41) and (42) (Section 4.1).

From the normal tractions

$$\sigma_{rr}(r=\rho) = N_\rho, \quad \sigma_{rr}(r=R) = N_R,$$

and relations (53) and (54), it is obtained

$$\begin{aligned}Z_1P_1(\rho) + Z_2P_2(\rho) &= N_\rho\frac{1+v^*}{E_0^*},\\ Z_1P_1(R) + Z_2P_2(R) &= N_R\frac{1+v^*}{E_0^*}.\end{aligned} \tag{55}$$

From the solution of system (55), we have

$$Z_1 = \left(\frac{1+\nu^*}{E_0^*}\right)\frac{N_\rho P_2(R) - N_R P_2(\rho)}{P_1(\rho)P_2(R) - P_1(R)P_2(\rho)},$$
$$Z_2 = \left(\frac{1+\nu^*}{E_0^*}\right)\frac{N_R P_1(\rho) - N_\rho P_1(R)}{P_1(\rho)P_2(R) - P_1(R)P_2(\rho)}, \tag{56}$$

provided that

$$\Lambda(\xi,\zeta) = P_1(\rho)P_2(R) - P_1(R)P_2(\rho) \neq 0, \quad \zeta = \frac{R}{\rho}.$$

6. The isotropic homogeneous elastic axisymmetric problem

In the case of an isotropic homogeneous elastic axisymmetric problem, $\delta = 0$, the differential equations (28a) and (31), become

$$r^2 u_r'' + r u_r' - u_r = 0, \tag{57}$$

$$u_\theta(r) = -\frac{C_1}{2\mu r}, \tag{58}$$

where, $\mu = E_0/2(1+\nu)$, the shear modulus.

The general solution of Eq. (57) is

$$u_r(r) = C_{11} r + \frac{C_{21}}{r}, \tag{59}$$

where C_{11} and C_{21} constants to be determined

We may arrive at the same result for the displacements field by considering the Airy-stress function Φ of the problem, which is determined as follows.

The stress field in terms of $\Phi(r,\theta)$ is given by (Timoshenko and Goodier, 1970; Barber, 1992)

$$\sigma_{rr} = \frac{1}{r}\frac{\partial\Phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\Phi}{\partial r^2}, \quad \sigma_{r\theta} = -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\Phi}{\partial\theta}\right), \quad \sigma_{\theta\theta} = \frac{\partial^2\Phi}{\partial r^2}. \tag{60}$$

From relations (60) and (1a), it is obtained

$$\Phi = F(r) + rg(\theta) + h(\theta), \tag{61}$$

where $h(\theta)$ and $g(\theta)$ are unknown functions to be determined. From relations (60) and (61), we have

$$\sigma_{r\theta} = \frac{1}{r^2}h'(\theta), \quad h'(\theta) = \mathrm{d}h(\theta)/\mathrm{d}\theta,$$

and taking into consideration that $\sigma_{r\theta} = \sigma_{r\theta}(r)$, it is furnished that

$$h(\theta) = C_1\theta + C_2, \quad \sigma_{r\theta} = \frac{C_1}{r^2}, \tag{62}$$

where C_1 and C_2 constants to be determined.

From relations (61) and (60), we get

$$\sigma_{rr} = \frac{1}{r}[F'(r) + g(\theta) + g''(\theta)] \tag{63}$$

Because of relations (1a), we have

$$g(\theta) + g''(\theta) = C_3, \quad C_3 \in \mathbb{R}, \tag{64}$$

where C_3 constant to be determined.

Finally, it results that

$$\Phi = F(r) + rg(\theta) + C_1\theta + C_2 \tag{65}$$

and

$$\sigma_{rr} = \frac{1}{r}[F'(r) + C], \quad \sigma_{\theta\theta} = F''(r), \quad \sigma_{r\theta} = \frac{C_1}{r^2}. \tag{66}$$

Taking into consideration relations (11) and the stress–strain relations (Timoshenko and Goodier, 1970; Barber, 1992)

$$\sigma_{rr} = (2\mu + \lambda^*)\varepsilon_{rr} + \lambda^*\varepsilon_{\theta\theta}, \quad \sigma_{\theta\theta} = \lambda^*\varepsilon_{rr} + (2\mu + \lambda^*)\varepsilon_{\theta\theta}, \quad \sigma_{r\theta} = 2\mu\varepsilon_{r\theta}, \tag{67}$$

where

$$\lambda^* = \begin{cases} \lambda & \text{plane strain,} \\ 2\lambda\mu/(\lambda+2\mu) & \text{plane stress} \end{cases}$$

and λ and μ, are the Lamé constants, it is found that

$$\begin{aligned} \sigma_{rr} &= (2\mu+\lambda^*)u' + \lambda^*\frac{u_r}{r} = \frac{1}{r}[F'(r)+C], \\ \sigma_{\theta\theta} &= \lambda^* u_r' + (2\mu+\lambda^*)\frac{u_r}{r} = F''(r), \\ \sigma_{r\theta} &= \mu\Big(-\frac{u_\theta}{r} + u_\theta'\Big) = \frac{C_1}{r^2}. \end{aligned} \tag{68}$$

From relations (68), relations (57) and (58) may also arise.

In order to determine the stress function Φ, it is considered the equilibrium of the sector $(\rho \leqslant r \leqslant R, -\pi < -\theta_0 \leqslant \theta \leqslant \theta_0 < \pi)$ of an axisymmetric plane problem subjected to the uniform normal internal $p(\rho)(=p_1)$ and external $p(R)(=p_2)$ loadings (Fig. 2). From the equilibrium in the x and y directions, and because of the normal loadings $(C_1 = 0)$, it finally follows for the stress function and the stress and displacements fields, that

$$\begin{aligned} &\Phi = F(r) + rg(\theta) + C_2, \quad g + g'' = C_3, \\ &\sigma_{rr}(r) = \frac{1}{r}(F'(r)+C), \quad \sigma_{\theta\theta}(r) = F''(r), \quad \sigma_{r\theta} = 0, \\ &u_r(r) = C_{11}r + \frac{C_{21}}{r}, \quad u_\theta = 0. \end{aligned} \tag{69}$$

In addition, the stress function Φ must satisfy the biharmonic equation (Timoshenko and Goodier, 1970; Barber, 1992)

$$\nabla^4\Phi = 0,$$

which yields

$$\nabla^4\Phi = F^{(\mathrm{iv})}(r) + \frac{2}{r}F'''(r) - \frac{1}{r^2}F''(r) + \frac{1}{r^3}F'(r) + \frac{C_3}{r^3} = 0 \tag{70}$$

or

$$F_1^{(\mathrm{iv})}(r) + \frac{2}{r}F_1'''(r) - \frac{1}{r^2}F_1''(r) + \frac{1}{r^3}F_1' = 0, \quad F_1(r) = F(r) + C_3 r + C_2. \tag{71}$$

Since the exact form of $g(\theta)$ does not influence the stress and displacements fields (relation (69)), the general solution of the differential equation

$$g(\theta) + g''(\theta) = C_3$$

is not required, and we may proceed with the partial solution

$$g(\theta) = C_3. \tag{72}$$

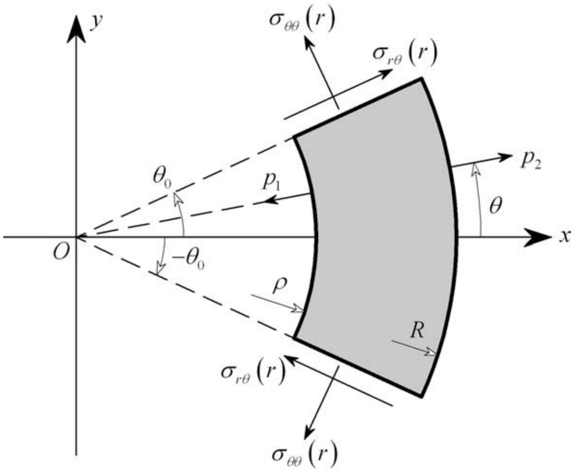

Fig. 2. The equilibrium of a sector in an axisymmetric problem.

Thus, taking into consideration relation (72), the stress function Φ is given by

$$\Phi(r) = F(r) + C_3 r + C_2 = F_1(r). \tag{73}$$

It results that the stress function Φ, given as the general solution of the homogeneous differential equation (71) is

$$\Phi(r) = A\ln r + Br^2 \ln r + \Gamma r^2, \tag{74}$$

where A, B and Γ, constants to be determined.

The stress field due to relation (74) is

$$\sigma_{rr} = \frac{A}{r^2} + B(1 + 2\ln r) + 2\Gamma, \quad \sigma_{\theta\theta} = -\frac{A}{r^2} + B(3 + 2\ln r) + 2\Gamma, \quad \sigma_{r\theta} = 0. \tag{75}$$

From relations (68), (58), (59) and taking into consideration that $C_1 = 0$, we have

$$\sigma_{rr} = -2\mu\frac{C_{21}}{r^2} + 2C_{11}(\mu + \lambda^*), \quad \sigma_{\theta\theta} = 2\mu\frac{C_{21}}{r^2} + 2C_{11}(\mu + \lambda^*), \quad \sigma_{r\theta} = 0. \tag{76}$$

In order that relations (74) and (76) are compatible with relation (60), it is required that

$$A = -2\mu C_{21}, \quad B = 0, \quad \Gamma = C_{11}(\lambda^* + \mu). \tag{77}$$

Hence, the stress function (74) is finally written as

$$\Phi(r) = A\ln r + \Gamma r^2, \tag{78}$$

and the stress and the displacement fields are given by

$$\sigma_{rr} = \frac{A}{r^2} + 2\Gamma, \quad \sigma_{\theta\theta} = -\frac{A}{r^2} + 2\Gamma, \quad \sigma_{r\theta} = 0, \tag{79}$$

$$u_r = \frac{\Gamma r}{\lambda' + \mu} - \frac{A}{2\mu r} = \frac{1}{2\mu}\left[\Gamma(\kappa - 1)r - \frac{A}{r}\right], \quad u_\theta = 0, \tag{80}$$

where $\kappa = (3\mu + \lambda^*)/(\mu + \lambda^*)$ is the Muskhelishvili constant (Muskhelishvili, 1963) and A and Γ, two unknown coefficients determined from the boundary conditions of the problem.

7. Application

Two applications have been made. The first one concern the case where Young's modulus varies exponentially and the second one the case where Young's modulus varies according to a power law.

7.1. The exponential case

Let an isotropic radially nonhomogeneous axisymmetric tube in plane strain conditions ($\rho(= 0.50\ \text{m}) \leqslant r \leqslant R(= 1.00\ \text{m})$), under internal pressure (Fig. 3)

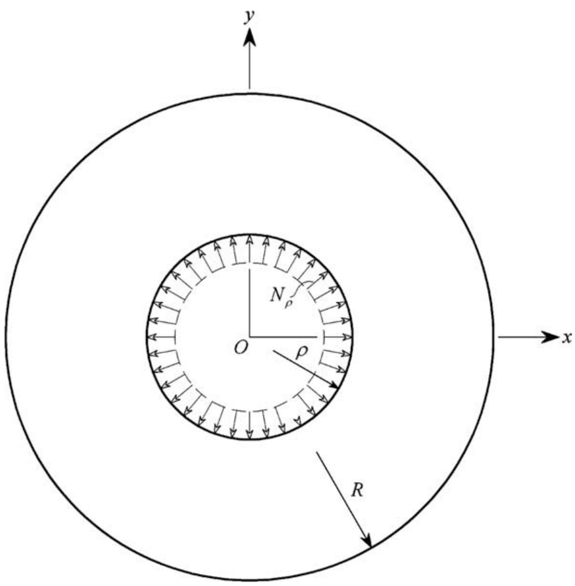

Fig. 3. An isotropic nonhomogeneous axisymmetric cylinder under internal pressure.

$$N_\rho = 1\ \text{GPa}, \quad N_R = T_\rho = T_R = 0,$$

with constant Poisson's ratio, $\nu = 0.34$, and a varying Young's modulus of the form $E = E_0 e^{\delta r}$, $E_0 = 100$ GPa ($\nu^* = 0.515$, $E_0^* = 233.21$ GPa), $-3.0 \leqslant \delta \leqslant 3.0$.

Because the tube is subjected only to normal tractions (Fig. 3), relations (40) ($\delta < 0$), become (Wolfam, 1991)

$$\begin{aligned}
A_1(r) &= \left(1 - \frac{\delta r}{1.515}\right)\Phi(1.485, 3; \delta r) + \frac{\delta r}{3}\left(\frac{1.485}{1.515}\right)\Phi(2.485, 4; \delta r),\\
A_2(r) &= \left(1 - \frac{\delta r}{1.515}\right)\Psi(1.485, 3; \delta r) - \delta r\left(\frac{1.485}{1.515}\right)\Psi(2.485, 4; \delta r),\\
B_1(r) &= \left(1 - \frac{0.515\delta r}{1.515}\right)\Phi(1.485, 3; \delta r) + \frac{\delta r}{3}0.515\left(\frac{1.485}{1.515}\right)\Phi(2.485, 4; \delta r),\\
B_2(r) &= \left(1 - \frac{0.515\delta r}{1.515}\right)\Psi(1.485, 3; \delta r) - \delta r\left(\frac{1.485}{1.515}\right)0.515\Psi(2.485, 4; \delta r),
\end{aligned} \tag{81}$$

where, $\Phi(a, b; z)$, $\Psi(a, b; z)$ the degenerate hypergeometric functions (Polyanin and Zaitsev, 2003, p. 221), with

$$\begin{aligned}
\Psi(1.485, 2+1; \delta r) =& \frac{-1}{2\Gamma(-0.515)}\left\{\Phi(1.485, 3; \delta r)\ln(\delta r) + \sum_{k=0}^{100}\frac{\Pi_{l=0}^{k-1}(1.485+l)}{\Pi_{l=0}^{k-1}(3+l)}[\psi(1.485+k) - \psi(1+k) - \psi(3+k)]\frac{(\delta r)^k}{k!}\right\}\\
&+ \frac{1}{\Gamma(1.485)}\sum_{k=0}^{1}\frac{\Pi_{l=0}^{k-1}(-0.515+l)}{\Pi_{l=0}^{k-1}(-1+l)}\frac{(\delta r)^{k-2}}{k!},\\
\Psi(2.485, 3+1; \delta r) =& \frac{1}{6\Gamma(-0.515)}\left\{\Phi(2.485, 4; \delta r)\ln(\delta r) + \sum_{k=0}^{100}\frac{\Pi_{l=0}^{k-1}(2.485+l)}{\Pi_{l=0}^{k-1}(4+l)}[\psi(2.485+k) - \psi(1+k) - \psi(4+k)]\frac{(\delta r)^k}{k!}\right\}\\
&+ \frac{2}{\Gamma(2.485)}\sum_{k=0}^{2}\frac{\Pi_{l=0}^{k-1}(-0.515+l)}{\Pi_{l=0}^{k-1}(-2+l)}\frac{(\delta r)^{k-3}}{k!}
\end{aligned} \tag{82}$$

In the case that $\delta > 0$, from relations (39), we have

$$\begin{aligned}
A_1(r) &= e^{\delta r}\left[\Phi(1.515, 3; -\delta r) - \frac{\delta r}{3}\Phi(2.515, 4; -\delta r)\right],\\
A_2(r) &= e^{\delta r}[\Psi(1.515, 3; -\delta r) + \delta r\Psi(2.515, 4; -\delta r)],\\
B_1(r) &= e^{\delta r}\left[\Phi(1.515, 3; -\delta r) - \nu^*\frac{\delta r}{3}\Phi(2.515, 4; -\delta r)\right],\\
B_2(r) &= e^{\delta r}[\Psi(1.515, 3; -\delta r) + \nu^*\delta r\Psi(2.515, 4; -\delta r)],
\end{aligned} \tag{83}$$

where (Polyanin and Zaitsev, 2003).

$$\begin{aligned}
\Psi(1.515, 2+1; -\delta r) =& \frac{-1}{2\Gamma(-0.485)}\left\{\Phi(1.515, 3; -\delta r)\ln(-\delta r) + \sum_{k=0}^{100}\frac{\Pi_{l=0}^{k-1}(1.515+l)}{\Pi_{l=0}^{k-1}(3+l)}[\psi(1.515+k) - \psi(1+k) - \psi(3+k)]\frac{(-\delta r)^k}{k!}\right\}\\
&+ \frac{1}{\Gamma(1.515)}\sum_{k=0}^{1}\frac{\Pi_{l=0}^{k-1}(-0.485+l)}{\Pi_{l=0}^{k-1}(-1+l)}\frac{(-\delta r)^{k-2}}{k!},\\
\Psi(2.515, 3+1; -\delta r) =& \frac{1}{6\Gamma(-0.485)}\left\{\Phi(2.515, 4; -\delta r)\ln(-\delta r) + \sum_{k=0}^{100}\frac{\Pi_{l=0}^{k-1}(2.515+l)}{\Pi_{l=0}^{k-1}(4+l)}[\psi(2.515+k) - \psi(1+k) - \psi(4+k)]\frac{(-\delta r)^k}{k!}\right\}\\
&+ \frac{2}{\Gamma(2.515)}\sum_{k=0}^{2}\frac{\Pi_{l=0}^{k-1}(-0.485+l)}{\Pi_{l=0}^{k-1}(-2+l)}\frac{(-\delta r)^{k-3}}{k!}
\end{aligned} \tag{84}$$

Having determined $A_i(r)$ and $B_i(r)$ $(i = 1, 2)$ (Wolfam, 1991), the coefficients Z_1, Z_2 are derived from relations (44) for the different values of the exponential coefficient δ.

In the sequel, the displacements and stress fields are determined from relations (34) and (38), respectively. Plots of the above stress and displacements fields are given in Fig. 4 for the values of $\delta(= -3.0, -1.5, -1.0, -0.5, 0.50, 1.0, 1.5, 3.0\ \text{m}^{-1})$ as well as for the homogeneous case ($\delta = 0$) using relations (79) and (80).

7.2. The power law case

Let an isotropic and radially nonhomogeneous axisymmetric cylinder in plane strain conditions ($\rho(= 0.50\ \text{m}) \leqslant r \leqslant R(= 1.00\ \text{m})$), under internal pressure (Fig. 3)

$$N_\rho = 1\ \text{GPa}, \quad N_R = T_\rho = T_R = 0$$

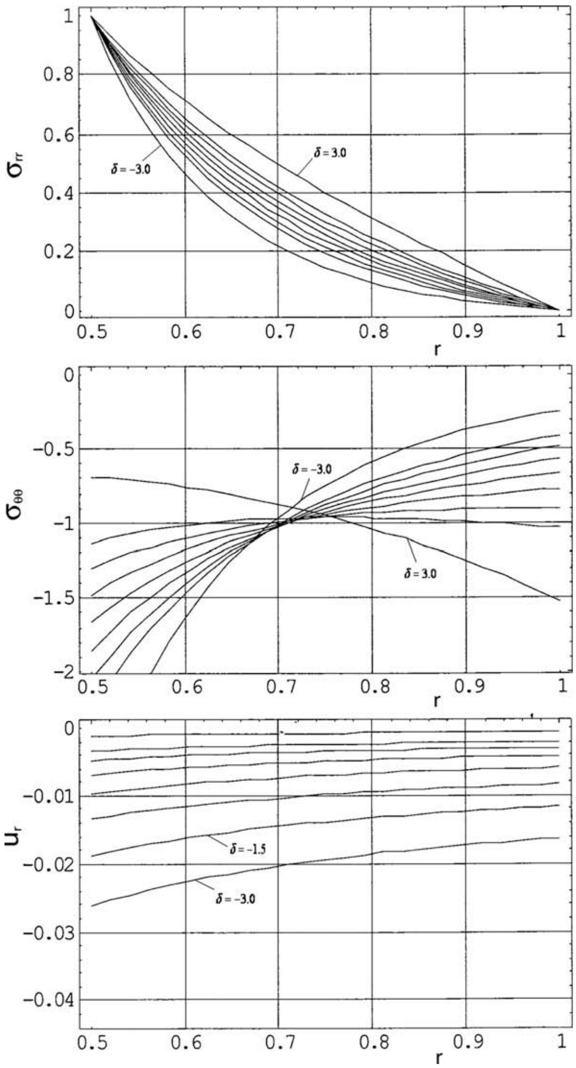

Fig. 4. The $\sigma_{rr},\ \sigma_{r\theta}$ and u_r values ($\rho \leqslant r \leqslant R$) in the exponential case for $v^* = 0.515$, $-3.0 \leqslant \delta \leqslant 3.0$.

with $v = 0.34$, $E = E_0(r/\rho)^{\xi}$, $E_0 = 100$ GPa ($v^* = 0.515,\ E_0^* = 233.21$ GPa), $-3.0 \leqslant \xi \leqslant 3.0$.

From relations (54) we have

$$
\begin{aligned}
&P_1(r) = \left(0.515 - \frac{\xi}{2} + \frac{\sqrt{f}}{2}\right) 0.50^{-\xi} r^{-1+(\xi/2)+(\sqrt{f}/2)},\\
&P_2(r) = \left(0.515 - \frac{\xi}{2} - \frac{\sqrt{f}}{2}\right) 0.50^{-\xi} r^{-1+(\xi/2)-(\sqrt{f}/2)},\\
&Q_1(r) = \left(1 - 0.515\frac{\xi}{2} + 0.515\frac{\sqrt{f}}{2}\right) 0.50^{-\xi} r^{-1+(\xi/2)+(\sqrt{f}/2)},\\
&Q_2(r) = \left(1 - 0.515\frac{\xi}{2} - 0.515\frac{\sqrt{f}}{2}\right) 0.50^{-\xi} r^{-1+(\xi/2)-(\sqrt{f}/2)},\\
&f(\xi) = 4 + \xi^2 - 4\xi * 0.515.
\end{aligned}
\tag{85}
$$

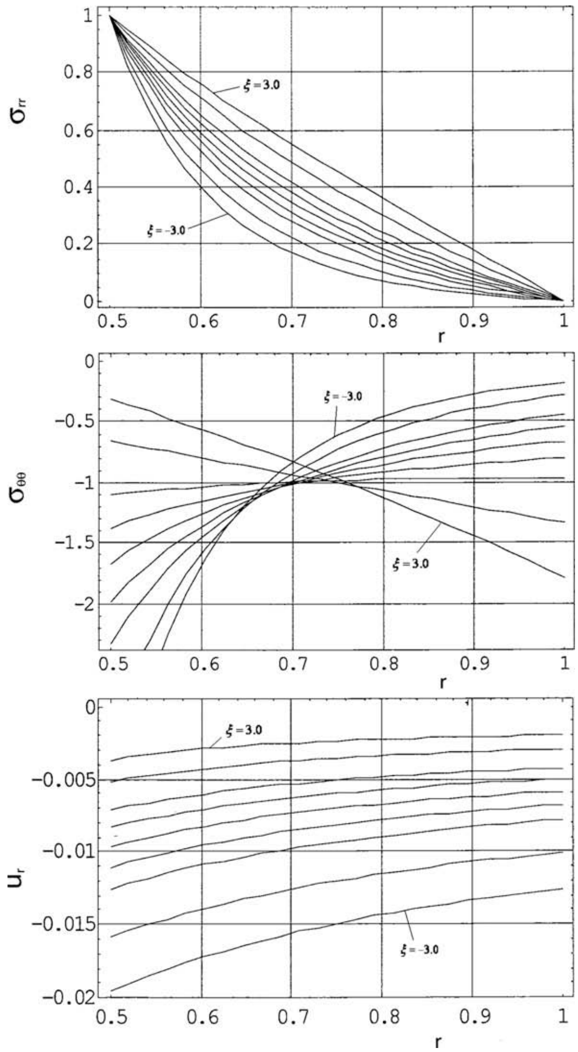

Fig. 5. The σ_{rr}, $\sigma_{r\theta}$ and u_r values ($\rho \leqslant r \leqslant R$) in the power law case for $\nu^* = 0.515$, $-3.0 \leqslant \xi \leqslant 3.0$.

Combining relations (52), (53) and (56) with relations (85), the stress and displacements fields are given by

$$\begin{aligned}
\sigma_{rr} &= \frac{P_1(r)P_2(1.00) - P_1(1.00)P_2(r)}{\Lambda(\xi, \zeta = 2)}, \\
\sigma_{\theta\theta} &= \frac{Q_1(r)P_2(1.00) - P_1(1.00)Q_2(r)}{\Lambda(\xi, \zeta = 2)}, \\
u_r &= \left(\frac{1+\nu^*}{E_0^*}\right) \frac{r^{-(\xi/2)+(\sqrt{f}/2)}P_2(1.00) - P_1(1.00)r^{-(\xi/2)-(\sqrt{f}/2)}}{\Lambda(\xi, \zeta = 2)}.
\end{aligned} \tag{86}$$

Plots of the above fields are given in Fig. 5 for the values of $\xi(= -3.0, -2.0, -1.0, -0.5, 0, 0.5, 1.0, 2.0, 3.0)$.

8. Conclusions

The plane axisymmetric problem with axisymmetric geometry and loading of a radially nonhomogeneous hollow circular cylinder was studied in linear elasticity. After a series of admissible functional transformations and considering the radial dependence of the stress, the displacements fields and of the stiffness matrix, the general differential system of the anisotropic and of the isotropic, radially nonhomogeneous elastic axisymmetric problems resulted.

In the case of a nonhomogeneous hollow circular cylinder of constant Poisson's ratio and of Young's modulus of exponential or of power law function of the radial coordinate r, the exact analytic solutions arise for the stress and displacements fields in terms of hypergeometric functions. The advantage of our study relative to the investigation of Zhang and Hasebe

(1999), is the closed form analytic solutions, solving ordinary differential equations without the need of matching a multi-material cylinder with infinite homogenous layers or manipulating numerical solutions.

Our study was based on a more general consideration of the problem where the stiffness matrix of the linear elastic material (isotropic or anisotropic) was a function of r and consequently our solution is valid for a uniformly distributed pressure as well as for a uniformly distributed shear loading. From our analysis the general differential equations (15) and (16) of the displacements fields were formulated. In the sequel three sub-cases in which Eqs. (15) and (16) could be solved analytically, were examined. In the first case Young's modulus depending on r varied exponentially, in the second case Young's modulus varied according to a power law and in the third case the isotropic homogeneous elastic axisymmetric problem was confronted. In the power law case our solution (Eq. (47a)) coincides with the proposed solution by Horgan and Chan (1999, Eq. (2.8)). The solution of Horgan and Chan (1999) is straightforward and it can be more easily managed and programmed than our solution, but our solution is more general and it can be easily extended to the more difficult exponential case (Section 4).

The benefit of our analytical solution (Figs. 4 and 5) is the clarification of the behavior of the $\sigma_{\theta\theta}$ – stress component. It is observed that the variation of the $\sigma_{\theta\theta}$ – stress is increasing or decreasing according to the values of the coefficients δ or ξ. Namely the $\sigma_{\theta\theta}$ – stress takes under internal tractions maximum values in the inner ring when the coefficients δ or ξ decrease negatively or takes maximum values in the outer ring when δ or ξ increase positively.

In addition using the radial dependence of the stress, strain and displacements fields in the isotropic homogeneous case (Section 6), the biharmonic equation of the problem was finally reduced to a fourth order homogeneous differential equation whose general solution is the stress function of the axisymmetric problem. The σ_{zz}-stress in the cylindrical coordinate system (r,θ,z), appearing in plane strain conditions, does not influence the proposed solution because it may result from a linear combination of σ_{rr} and $\sigma_{\theta\theta}$ stresses.

It is observed that the proposed analytical method coincides in the case of the exponentially varying Young's modulus (Fig. 4) with the results of the study of Zhang and Hasebe (1999) in the case of uniformly distributed internal loading. On the other hand, in the case that the Young's modulus varied according to a power law, our results (Fig. 5) coincide with the results proposed by Sladek et al. (2008). In the applications Section 7, the case of distributed pressure was considered. Hence, because of Eq. (42), $C_1 = 0$, and $\sigma_{r\theta} = \varepsilon_{r\theta} = 0$. But with the proposed analysis the solution of equilibrated distributed shear traction ($C_1 \neq 0$) in the inner and the outer ring of the nonhomogeneous axisymmetric tube may also be provided ($\sigma_{r\theta} \neq 0, \varepsilon_{r\theta} \neq 0$).

Appendix A. The solutions of Eqs. (32) and (51)

The differential equation (32)

$$r^2 u_r'' + r(\delta r + 1)u_r' + (v^*\delta r - 1)u_r = 0, \tag{A.1}$$

considering Polyanin and Zaitsev (2003, p. 230), for $a = \delta, n = 1, b = 1, \alpha = 0, \beta = v^*\delta, \gamma = -1$ and the transformation

$$z = r^n = r, \quad \omega = u_r z^{-k} = u_r r^{-k}, \tag{A.2}$$

where k is a root of the equation

$$\eta^2 k^2 + \eta(b-1)k + \gamma = k^2 - 1 = 0$$

or

$$k = \pm 1$$

becomes

$$r\omega'' + (\delta r + 2k + 1)\omega' + (k + v^*)\delta\omega = 0, \quad k = \pm 1, \ \ \omega = u_r r^{-k}. \tag{A.3}$$

The general solution of (A.3) from Polyanin and Zaitsev (2003, p. 225), for $a_2 = 1, b_2 = 0, a_1 = \delta, b_1 = 2k + 1, a_0 = 0, b_0 = (k + v^*)\delta$, provided that

$$a_2 = 1 \neq 0, \quad a_1^2 = \delta^2 \neq 4a_0 a_2 = 0$$

and

$$D = a_1^2 - 4a_0 a_2 = \delta^2, \quad K = \frac{\sqrt{D} - a_1}{2a_2} = \frac{1}{2}(|\delta| - \delta),$$

is investigated in the following two cases.

(i) In the case, $\delta > 0$, the parameters $K, \ \lambda, \ \mu, \ B(K)$, a and b (Polyanin and Zaitsev, 2003, p. 225), become

$$K = \frac{1}{2}(|\delta| - \delta) = 0, \quad \lambda = -\frac{a_2}{2a_2K + a_1} = -\frac{1}{\delta}, \quad \mu = -\frac{b_2}{a_2} = 0,$$
$$B(K) = b_2K^2 + b_1K + b_0 = b_0 = (k + v^*)\delta,$$
$$a = \frac{B(K)}{2\alpha_2K + \alpha_1} = k + v^*, \quad b = (a_2b_1 - a_1b_2)a_2^{-2} = 2k + 1.$$

The general solution of (A.3), is written

$$\omega = e^{Kr}w\left(\frac{r-\mu}{\lambda}\right) = w(-\delta r) = \mathscr{J}(a = k + v^*, b = 2k + 1; z = -\delta r). \tag{A.4}$$

From relation (A.4) occurs

$$\omega = w(-\delta r) = \mathscr{J}(a = v^* + 1, b = 3; z = -\delta r), \quad \delta > 0, \ k = 1, \tag{A.5a}$$
$$\omega = w(-\delta r) = \mathscr{J}(a = v^* - 1, b = -1; z = -\delta r), \quad \delta > 0, \ k = -1. \tag{A.5b}$$

(ii) In the case, $\delta < 0$, the parameters $K, \ \lambda, \ \mu, \ B(K), \ a$ and b (Polyanin and Zaitsev, 2003, p. 225), become

$$K = \frac{1}{2}(|\delta| - \delta) = -\delta, \quad \lambda = -\frac{a_2}{2a_2K + a_1} = \frac{1}{\delta}, \quad \mu = -\frac{b_2}{a} = 0,$$
$$B(K) = b_2K^2 + b_1K + b_0 = \delta(v^* - 1 - k), \quad a = \frac{B(K)}{2a_2K + a_1} = k + 1 - v^*,$$
$$b = (a_2b_1 - a_1b_2)a^{-2} = 2k + 1.$$

The general solution of (A.3) is written

$$\omega = e^{Kr}w\left(\frac{r-\mu}{\lambda}\right) = e^{-\delta r}w(\delta r) = e^{-\delta r}\mathscr{J}(a = k + 1 - v^*, b = 2k + 1; z = \delta r) \tag{A.6}$$

or

$$\omega = e^{-\delta r}w(\delta r) = e^{-\delta r}\mathscr{J}(a = 2 - v^*, b = 3; z = \delta r), \quad \delta < 0, \ k = 1 \tag{A.7a}$$
$$\omega = e^{-\delta r}w(\delta r) = e^{-\delta r}\mathscr{J}(a = -v^*, b = -1; z = \delta r), \quad \delta < 0, \ k = -1. \tag{A.7b}$$

The function $\mathscr{J}(a, b; z)$ in relations (A.4)–(A.7), is derived from the solution of the degenerate hypergeometric equation (Polyanin and Zaitsev, 2003, p. 220)

$$zy'' + (b - z)y' - ay = 0, \quad y' = \frac{dy}{dz}, \quad y'' = \frac{d^2y}{dz^2}. \tag{A.8}$$

The general solution of (A.8) (Polyanin and Zaitsev, 2003, p. 222) is

$$y = \mathscr{J}(a, b; z) = \Delta_1\Phi(a, b; z) + \Delta_2\Psi(a, b; z), \quad b \neq 0, -1, -2, -3, \ldots \tag{A.9}$$

or

$$y = \mathscr{J}(a, b; z) = z^{1-b}[\Delta_1\Phi(a - b + 1, 2 - b; z) + \Delta_2\Psi(a - b + 1, 2 - b; z)], \quad b = 0, -1, -2, -3, \ldots \tag{A.10}$$

Thus, from relations (A.5), (A.7), (A.9) and (A.10), we may write

$$\begin{aligned}
&\omega(r) = \Delta_1\Phi(v^* + 1, 3; -\delta r) + \Delta_2\Psi(v^* + 1, 3; -\delta r), \quad \delta > 0, \ b = 3, \ k = 1,\\
&\omega(r) = \delta^2r^2[\Delta_1\Phi(v^* + 1, 3; -\delta r) + \Delta_2\Psi(v^* + 1, 3; -\delta r)], \quad \delta > 0, \ b = -1, \ k = -1,\\
&\omega(r) = e^{-\delta r}[\Delta_1\Phi(2 - v^*, 3; \delta r) + \Delta_2\Psi(2 - v^*, 3; \delta r)], \quad \delta < 0, \ b = 3, \ k = 1,\\
&\omega(r) = e^{-\delta r}\delta^2r^2[\Delta_1\Phi(2 - v^*, 3; \delta r) + \Delta_2\Psi(2 - v^*, 3; \delta r)], \quad \delta < 0, \ b = -1, \ k = -1.
\end{aligned} \tag{A.11}$$

From transformation (A.2), it finally occurs

$$\begin{aligned}
&u_r(r) = r\Delta_1\Phi(v^* + 1, 3; -\delta r) + r\Delta_2\Psi(v^* + 1, 3; -\delta r), \quad \delta > 0, \ b = 3, \ k = 1,\\
&u_r(r) = r\delta^2\Delta_1\Phi(v^* + 1, 3; -\delta r) + r\delta^2\Delta_2\Psi(v^* + 1, 3; -\delta r), \quad \delta > 0, \ b = -1, \ k = -1,\\
&u_r(r) = re^{-\delta r}\Delta_1\Phi(2 - v^*, 3; \delta r) + re^{-\delta r}\Delta_2\Psi(2 - v^*, 3; \delta r), \quad \delta < 0, \ b = 3, \ k = 1,\\
&u_r(r) = re^{-\delta r}\delta^2\Delta_1\Phi(2 - v^*, 3; \delta r) + re^{-\delta r}\delta^2\Delta_2\Psi(2 - v^*, 3; \delta r), \quad \delta < 0, \ b = -1, \ k = -1.
\end{aligned} \tag{A.12}$$

The hypergeometric functions $\Phi(a, b; z)$ and $\Psi(a, b; z)$ are specified (Polyanin and Zaitsev, 2003, pp. 220, 221, 753) as follows:

$$\Phi(a, b; z) = 1 + \sum_{l=1}^{\infty}\frac{(a)_l}{(b)_l}\left(\frac{z^l}{l!}\right), \quad b \neq 0, -1, -2, -3, \ldots \tag{A.13}$$

with

$$(\alpha)_l = \alpha(\alpha+1)(\alpha+2)\dots(\alpha+l-1), \quad (\alpha)_0 = 1.$$

In the case that $b > a > 0$

$$\Phi(a,b;z) = \frac{\Gamma(b)}{\Gamma(a)\Gamma(b-a)} \int_0^1 e^{zt} t^{\alpha-1}(1-t)^{b-\alpha-1} dt, \quad b > a > 0, \tag{A.14}$$

where the gamma function is defined as

$$\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} dt. \tag{A.15}$$

We also have (Polyanin and Zaitsev, 2003, p. 221)

$$\Psi(a,b;z) = \frac{\Gamma(1-b)}{\Gamma(a-b+1)} \Phi(a,b;z) + \frac{\Gamma(b-1)}{\Gamma(a)} z^{1-b} \Phi(a-b+1, 2-b; z). \tag{A.16}$$

In the limit where $b \to n(= 1,2,3,4,5,\dots)$, from relation (A.16) occurs

$$\begin{aligned}\Psi(a,n+1;z) = {} & \frac{(-1)^{n-1}}{n!\Gamma(a-n)} \left\{ \Phi(a,n+1;z)\ln z + \sum_{l=0}^{\infty} \frac{(a)_l}{(n+1)_l} [\psi(a+l) - \psi(1+l) - \psi(1+n+l)] \frac{z^l}{l!} \right\} \\ & + \frac{(n-1)!}{\Gamma(a)} \sum_{l=0}^{n-1} \frac{(a-n)_l}{(1-n)_l} \left(\frac{z^{l-n}}{l!} \right),\end{aligned} \tag{A.17}$$

where, $\psi(t) = (\ln \Gamma(z))'_z$, the logarithmic derivative of the gamma function, and

$$\psi(1) = -\gamma, \ \psi(n) = -\gamma \sum_{l=1}^{n-1} l^{-1}, \quad \gamma = 0.5772 \ \text{(the Euler gamma)}. \tag{A.18}$$

The following properties are also valid (Polyanin and Zaitsev, 2003):

$$\Phi(a,b;z) = e^z \Phi(b-a,b;z), \quad \Psi(a,b;z) = z^{1-b} \Psi(1+a-b, 2-b; z), \tag{A.19}$$

$$\frac{d}{dz} \Phi(a,b;z) = \Phi'(a,b;z) = \frac{a}{b} \Phi(a+1, b+1; z), \tag{A.20a}$$

$$\frac{d^n}{dz^n} \Phi(a,b;z) = \frac{(a)_n}{(b)_n} \Phi(a+n, b+n; z), \tag{A.20b}$$

$$\frac{d}{dz} \Psi(a,b;z) = \Psi'(a,b;z) = -a\Psi(a+1,b+1;z), \tag{A.21a}$$

$$\frac{d^n}{dz^n} \Psi(\alpha,b;z) = (-1)^n (\alpha)_n \Psi(\alpha+n, b+n; z). \tag{A.21b}$$

The differential equation (51)

$$r^2 u''_r + r(\xi+1)u'_r + (\xi v^* - 1)u_r = 0 \tag{A.22}$$

is an Euler type ODE (Polyanin and Zaitsev, 2003, p. 226), with parameters

$$a = \xi + 1, \quad b = \xi v^* - 1, \quad (1-a)^2 - 4b = \xi^2 - 4\xi v^* + 4 = f(\xi). \tag{A.23}$$

It is observed that for the range of the values of v^* $(0 \leqslant v^* \leqslant 1)$, $f(\xi) > 0$, the general solution of (A.22) is given by

$$u_r = |r|^{(1-\alpha)/2}(Z_1|r|^\mu + Z_2|r|^{-\mu}), \quad \mu = \frac{1}{2}\sqrt{(1-a)^2 - 4b}$$

or

$$u_r = r^{-\frac{\xi}{2}}\left(Z_1 r^{\frac{1}{2}\sqrt{f(\xi)}} + Z_2 r^{-\frac{1}{2}\sqrt{f(\xi)}}\right), \quad f(\xi) = \xi^2 - 4\xi v^* + 4 > 0. \tag{A.24}$$

References

Aboudi, J., Pindera, M.-Z., Arnold, S.M., 1995. Thermo-inelastic response of functionally graded composites, I. Journal of Solids and Structures 32, 1675–1710.
Afsar, A.M., Sekine, H., 2002. Inverse problems of material distributions for prescribed apparent fracture Toughness in FGM coatings around a circular hole in infinite elastic media. Composites Science and Technology 62, 1063–1077.
Bakirtas, I., 1980. The Problem of a Rigid Punch on a Nonhomogeneous Elastic Half Space. International Journal of Engineering Science 18, 597–610.

Barber, J.R., 1974. Some mixed boundary value problems for the semi-infinite elastic solid subjected to tangential surface tractions. International Journal of Mechanical Science 16, 727–734.

Barber, J.R., 1992. Elasticity. Kluwer Academic Publishers, Dordrecht.

Birman, V., 1992. Exact solution of axisymmetric problems of laminated cylindrical shells with arbitrary boundary conditions – higher order theory. Mechanics Research Communications. 19, 219–225.

Craster, R.V., Atkinson, C., 1994. Mixed boundary value problems in nonhomogeneous elastic materials. Quarterly Journal of Mechanics and Applied Mathematics 47, 183–206.

Erdogan, F., Delale, F., 1983. The crack problem for a Nonhomogeneous plane. ASME, Journal of Applied Mechanics 56, 609–614.

Erdogan, F., Kaya, A.C., Joseph, P.F., 1991. The crack problem in bonded nonhomogeneous materials. ASME, Journal of Applied Mechanics 58, 410–418.

Fabricant, V.I., 1990. Axisymmetric bonded punch problem: a complete solution. Ingenieur-Archives 60, 213–224.

Gal, D., Dvorkin, J., 1995. Stresses in anisotropic cylinders. Mechanics Research Communications 22, 109–113.

Horgan, C.O., Chan, A.M., 1999. The pressurized hollow cylinder or disk problem for functionally graded isotropic linearly elastic materials. Journal of Elasticity 55, 43–59.

Kardomateas, G.A., 1993. Buckling of thick orthotropic cylindrical-shells under external pressure. ASME Journal of Applied Mechanis 60, 195–202.

Love, A.E.H., 1956. A Treatise on the Mathematical Theory of Elasticity, fourth ed. Cambridge University Press, Cambridge.

Lutz, M.P., Zimmerman, R.W., 1996. Effect of the interphase zone on the bulk modulus of a particular composite. ASME, Journal of Applied Mechanics 63, 855–861.

Muskhelishvili, N.I., 1963. Some Basic Problems of the Mathematical Theory of Elasticity, Fourth ed. P. Noordhoff, Groningen, The Netherlands.

Paulino, G.H., Jin, Z.-H., Dodds Jr., R.H., 2003. Failure of functionally graded materials. In: Karihaloo, B. et al. (Eds.), Encyclopedia of Comprehensive Structural Integrity, vol. 2. Elsevier, Amsterdam, pp. 607–642.

Polyanin, A.D., Zaitsev, V.F., 2003. Handbook of Exact Solutions for Ordinary Differential Equations, Second ed. CRC Press, Boca Raton, FL.

Sideridis, E., 1993. The transverse elastic modulus of fiber-reinforced composite, as defined by the concept of interphase. Journal of Applied Polymer Science 48, 243–255.

Singh, S., Kumar, A., 1994. Stresses due to flow past a circular cylinder. Internationl Journal of Pressure Vessels and Piping 60, 37–48.

Sladek, V., Sladek, J., Zhang, C., 2008. Computation of stresses in non-homogeneous elastic solids by local integral equation method: a comparative study. Computational Mechanics 41, 827–845.

Sokolnikoff, I.S., 1956. Mathematical Theory of Elasticity. McGraw-Hill, New York.

Stampouloglou, I.H., Theotokoglou, E.E., 2006. Investigation of the problem of a plane axisymmetric cylindrical tube under internal and external pressures. International Journal of Mechanics and Materials in Design. 3, 59–71.

Theotokoglou, E.E., Stampouloglou, I.H., 2004. The plane wedge problem loaded at its apex – the self similarity property and the characteristic vector. Journal of Elasticity 76, 21–43.

Timoshenko, S.P., Goodier, J.N., 1970. Theory of Elasticity. McGraw-Hill, Tokyo.

Ting, T.C.T., 1984. Anisotropic Elasticity: Theory and Applications. Oxford Science, New York.

Weng, G.J., 2003. Effective bulk moduli of two functionally graded composites. Acta Mechanica 166, 57–67.

Wolfam, S., 1991. Mathematica: A System for Doing Mathematics by Computer, second ed. Addison-Wesley, Redwood City, CA.

Zhang, X., Hasebe, N., 1999. Elasticity solution for a radially nonhomogeneous hollow cylinder. Journal of Applied Mechanics 66, 598–606.