International Journal of Solids and Structures 51 (2014) 1562–1575

Contents lists available at ScienceDirect

International Journal of Solids and Structures

journal homepage: www.elsevier.com/locate/ijsolstr

An improved theory of laminated Reissner–Mindlin plates

Giovanni Formica [a,*], Marzio Lembo [b], Paolo Podio-Guidugli [c]

[a] Dipartimento di Architettura, Università di Roma Tre, Via della Madonna dei Monti 40, 00184 Roma, Italy
[b] Dipartimento di Ingegneria, Università di Roma Tre, Via Vito Volterra 62, 00146 Roma, Italy
[c] Accademia Nazionale dei Lincei and Department of Mathematics, University of Rome TorVergata, Via della Ricerca Scientifica 1, 00133 Roma, Italy

ARTICLE INFO

Article history:
Received 12 August 2013
Received in revised form 16 December 2013
Available online 15 January 2014

Keywords:
Laminated plates
Reactive stress

ABSTRACT

Theories of laminated plates have been proposed that, although they lead to different plate equations, are based as ours on the assumptions that the three-dimensional deformation of each layer is of Reissner–Mindlin type and that displacement and traction vectors are continuous across layer interfaces. The distinctive feature of our present theory is that reactive stresses are associated with the internal constraints implicit in the assumed kinematics, and exploited to obtain an improved evaluation of the stress field in the three-dimensional layered body for which we propose a two-dimensional model. Application to equilibrium problems for rectangular and circular plates gives results that are in good agreement with the exact three-dimensional solutions of Levinson type we derived in a companion paper.

1. Introduction

On dealing with plate theories, it is convenient to distinguish between a *plate-like body* and a *plate*: the latter, which occupies a flat domain $\mathcal{S}$, is a two-dimensional model of the former, a right cylinder $\mathcal{C}$ of height $2h$ and cross-section $\mathcal{S}$, subject on the top and bottom ends to loads that are to be reduced to $\mathcal{S}$, and on the mantel to loads and boundary conditions that are to be reduced to the boundary $\partial\mathcal{S}$ of $\mathcal{S}$. When plate theories are induced from three-dimensional elasticity, the admissible displacement fields of the plate-like body are often represented *a priori* as products of (unknown) functions of coordinates on $\mathcal{S}$ times (known) functions of the transverse coordinate. Then, plate equations are obtained by means of a procedure which includes integration over $\mathcal{C}$'s thickness of the three-dimensional differential or variational equilibrium equations; as a part of this procedure, the external loads applied to $\mathcal{C}$ are reduced to resultant loads per unit area applied to $\mathcal{S}$ and resultant loads per unit length applied to $\partial\mathcal{S}$. Once the solution of the plate problem is found, and hence the functions that parameterize the representation assumed for the displacement field in the plate-like body are known, the relative three-dimensional strain and stress fields can be constructed. In general, such stress fields do not satisfy the three-dimensional equilibrium equations for $\mathcal{C}$, because one plate problem corresponds to an equivalence class of problems for plate-like bodies, all those problems whose data are reducible to the single set of data of the plate problem in question.

An accurate knowledge of the stresses in $\mathcal{C}$ is always desirable, and especially so when $\mathcal{C}$ is a multilayered body (whose two-dimensional model we refer to as a *laminated plate*), where transverse stress concentrations may occur near material and geometric discontinuities and give rise to damages that are often responsible for service failure. Many theories of laminated plates aiming to evaluate such critical stresses have been proposed. For extensive accounts of the literature on the subject, we refer the reader to a book by Reddy (2004), where many plate theories are presented and discussed; to the papers by Bert (1984) and by Noor and Malik (2000), where different methods to evaluate the three-dimensional stress associated with a laminated-plate solution are compared; and to the review papers by Noor and Burton (1989), Carrera (2000) and Carrera (2002), where different approaches to the modeling of laminated plates and various computational methods are examined.

The importance of plates in structural applications has led to a large amount of studies on the subject, whence the need for a classification of the many proposed theories, a choice of validation criteria, and a hierarchical assessment of theories in the same class. In this connection, it is worth-recalling the accounts by Koiter and Simmonds (1972) and Naghdi (1972), and the historical survey in Altenbach et al. (1998). The two-dimensional equations of a plate theory can be deduced from three-dimensional elasticity, by means of expansions in the thickness coordinate or of *a priori* assumptions on the form of the displacement field; they also can be arrived at by the use of a direct approach, introduced by Cosserat and Cosserat (1909), which consists essentially in postulating that certain two-dimensional principles hold on a surface modeling a plate-like body. An alternative approach to plate theory, that originates with Goodier (1938) and has been applied and extended, e.g., by Gol'denveizer (1962) and Green (1963), is

* Corresponding author. Tel.: +39 0657336253.
E-mail address: formica@uniroma3.it (G. Formica).

0020-7683/$ - see front matter © 2014 Elsevier Ltd. All rights reserved.
http://dx.doi.org/10.1016/j.ijsolstr.2014.01.007

based on asymptotic integration of the three-dimensional equations in an interior region of the plate and in a boundary layer.

For plate theories deduced from three-dimensional elasticity, the usual validation criteria are based on error estimates, with respect to the three-dimensional quantities, for the differential equations and their solutions. Kienzler (2002) has proposed a "uniform-approximation technique", consisting in taking series expansions of both equilibrium equations and solutions, that can be employed to construct 'consistent' plate theories of different order by retaining terms up to different orders in the two expansions. An interesting result in Kienzler's analysis of the lower-order theories obtained by use of this technique, is that the Reissner–Mindlin theory, although based on quite complex *a priori* assumptions, is a consistent second-order theory with respect to a plate parameter depending on the ratio between thickness and a characteristic length of the mid-section.

According to a commonly employed classification (cf. Reddy, 2004), laminated-plate theories based on assumptions concerning the displacement field can be divided in two groups: the so called *equivalent single-layer theories*, that assume for a multilayered plate-like body a suitable single-layer kinematics; and the *layerwise theories*, in which the kinematics is defined layer by layer. Among equivalent single-layer theories, we recall those of Reissner and Stavsky (1961), that employs the same displacement field as the Kirchhoff–Love theory for single-layered plates; of Whitney (1969), that is based on the kinematics of the Reissner–Mindlin theory; and of Reddy (1984), that assumes in-plane displacement components of the third order in the transverse coordinate. Recent contributions to this group of theories have been given by Mittelstedt and Becker (2004), who propose a higher-order single-layer formulation and apply it to evaluate strains and stresses in thermally-loaded laminated plates, and by Lebée and Sab (2011), who generalize Reissner derivation of a single-layer shear-deformable plate theory. Among layerwise theories, besides those by Seide (1980) and DiSciuva (1984) to be discussed later on in this section, we cite those by Lee et al. (1990), that is based on a layerwise cubic variation of the in-plane displacement components, and by Reddy (1987), that is a full layerwise theory, because it employs a layerwise expression also for the transverse displacement component. Recently, layerwise theories have been advanced by Plagianakos and Saravanos (2009), who work with in-plane displacement components having terms up to the third order in the transverse coordinate, and by Cho and Oh (2004), who treat coupling of mechanical, thermal, and electric effects in a laminate plate.

The model of multilayered plate-like body we here derive belongs to the group of layerwise theories, in that it postulates a displacement field of Reissner–Mindlin type in each layer. Besides yielding plate equations that are different from those of other theories moving from this kinematical assumption, our model has two other distinctive characters: the Reissner–Mindlin displacement field is regarded as induced by a set of internal constraints restricting the class of admissible strains; moreover, the reactions accompanying those internal constraints are exploited to improve the evaluation of transverse stresses. Our main aim is to show that, as far as the three-dimensional stress field is concerned, the predictions of a layerwise plate theory based on the Reissner–Mindlin kinematics can be made very close to those obtained by solving the corresponding three-dimensional equilibrium problem, *provided all consequences of assuming a restricted kinematics are taken into account*: this expectation is substantiated in the last part of the paper, where we exemplify how well the stresses resulting from our model match those obtained from exact three-dimensional solutions.

In Podio-Guidugli (1989), with reference to the Kirchhoff–Love theory, the idea was advanced of regarding as internal constraints the kinematical restrictions implicit in the form assumed for the displacement in a plate-like body. The gained benefit is that the presence of internal constraints implies that, in addition to the active (i.e., constitutively determined) stress, a constitutively undetermined reactive stress can be used to restore three-dimensional equilibrium. In the context of classical linearly elastic plate theories, in which shear deformations are not allowed, this approach has been extended in Lembo and Podio-Guidugli (1991) to problems more general than those dealt with in Podio-Guidugli (1989), as well as to multilayered plate-like bodies. The same idea has been exploited for shear-deformable plates, of Reissner–Mindlin type in Lembo and Podio-Guidugli (2007) and transversely extensible in DiCarlo et al. (2001). When trying to extend to those theories the approach used for Kirchhoff–Love's, one realizes that some of the kinematical prescriptions on the displacement field cannot be expressed as restrictions on the first displacement gradient, and hence viewed as first-order internal constraints; the nonconforming prescriptions are formulated in terms of components of the *second* displacement gradient, and are therefore to be viewed as *second-order internal constraints*. But, such higher-order constraints have citizenship in a more complex three-dimensional theory than classic elasticity. It is a theory of this sort that can serve as the right parent theory to induce a plate theory like Reissner–Mindlin's, a parent theory where second-order internal constraints are accompanied by reactive hyperstresses, that add to the ordinary reactive stresses (Lembo and Podio-Guidugli, 2000). In fact, it has been shown in Lembo and Podio-Guidugli (2007) that, if the plate-like body $\mathcal{C}$ is regarded as made of a second-grade material in which the hyperstress is purely reactive, then one can derive a Reissner–Mindlin approximation for the three-dimensional displacement and strain fields in $\mathcal{C}$ and, in addition, one can exploit the reactive stress and hyperstress fields to obtain an improved approximation of the three-dimensional stress field in $\mathcal{C}$. In other words, it is shown in Lembo and Podio-Guidugli (2007) that assuming that the plate is formed of a second-grade material, and that the hyperstress is solely reactive, allows for an improvement in the evaluation of the three-dimensional stress field without altering the plate equations deduced in the context of classical elasticity.

Here we extend this approach to shear-deformable plate-like bodies formed by several layers of transversely isotropic materials. The model of laminated plates we propose is based on the assumptions that each layer can undergo only deformations of Reissner–Mindlin type (Reissner, 1945; Mindlin, 1951), and that displacement and transverse traction vectors are continuous across the interfaces separating adjacent layers: displacement continuity is assumed to guarantee that, when $\mathcal{C}$ deforms, no sliding or detachment of layers occurs; transverse-traction continuity guarantees that part-wise equilibrium holds also for pillbox-shaped parts of $\mathcal{C}$ whose cross-section is a portion of a layer interface.

The equilibrium equations of the proposed model of laminated plate are deduced by integration over the thickness of a three-dimensional principle of virtual work, written for virtual displacements that have a form compatible with the kinematical restrictions imposed to the displacement. The model furnishes an approximation of the stress field in the plate-like body $\mathcal{C}$ that consists of an active part, that is deduced from the plate solution, and a reactive part, that is obtained by solving the three-dimensional equilibrium equations for a body made of second-grade material. We term our plate theory 'improved' because of the availability of such reactive addition, a unique feature among theories of the same type. The accuracy of our model's predictions is evaluated, with quite satisfactory results, in the case of laminated plate-like bodies with rectangular and circular cross-sections, for which we can count on the exact Levinson-type solutions derived in Formica et al. (2013).

Although the main assumptions of our model of a multilayered plate are not new, the resulting equations for the functions entering the parameterization of the displacement field are different from those to be found in papers based on the same assumptions (these are, we recall, that deformations are of Reissner–Mindlin type in each layer, and that displacement and traction vectors are continuous across layer interfaces). To our knowledge, there are two such papers, the one by Seide (1980), the other by DiSciuva (1984). In Seide's paper, the traction-continuity condition is used in a way different from ours, namely, to combine the equilibrium equations of adjacent layers so as to eliminate the interface values of traction components; the equilibrium of a plate-like body consisting of n layers is governed by $(2n+3)$ scalar equations for $(2n+3)$ unknowns, that is, two in-plane displacement components for each interface and for the end faces and one transverse displacement component, the same for the whole body; having determined the displacement components, transverse stresses are calculated from the three-dimensional equilibrium equations. In DiSciuva's paper, the admissible displacements in a multilayered plate-like body are taken the same as in ours. The difference in the equilibrium equations of the two models is due to the fact that they are obtained from principles of virtual work stated for different collections of virtual displacements: we chose to satisfy the displacement continuity condition, but not, as DiSciuva did, the traction continuity condition.

The paper is organized as follows. Firstly, in Section 2, we recapitulate how a plate-like body is modeled as a one-layer Reissner–Mindlin plate; in particular, we discuss what form the active and reactive stresses have as a consequence of the kinematical assumptions of the Reissner–Mindlin theory. Section 3, the bulk of the present work, is where we develop our general theory of laminated Reissner–Mindlin plates. Those simplifications that follow from exploiting certain specialties a laminated plate may have, like mid-plane symmetry, axisymmetry, or a circular cross-section, are worked out in Section 4. As anticipated, the accuracy of our predictions of the three-dimensional stress field in a catalogue of plate-like bodies modeled as laminated plates according to our theory is tested in our final Section 5.

Throughout the paper, we make use of indicial notation and summation convention, with the agreement that Latin and Greek indices have ranges {1, 2, 3} and {1, 2}, respectively; range quantifications are left tacit.

2. Modeling a constrained plate-like body as a Reissner–Mindlin plate

Let $\mathcal{C}$ be a *plate-like body*, that is, a three-dimensional material body that in an undeformed reference configuration has the form of a right cylinder of height $2h$ and mid-section $\mathcal{S}$, with $2h \ll \operatorname{diam}(\mathcal{S})$. Moreover, let $\mathcal{S}^+$ and $\mathcal{S}^-$ denote the end faces of $\mathcal{C}$, $\partial\mathcal{C}$ the boundary of $\mathcal{C}$, and $\partial\mathcal{S}$ the boundary of $\mathcal{S}$. For (x_1, x_2, x_3) the Cartesian coordinates of a typical point of $\mathcal{C}$ in a Cartesian reference $(o; \boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3)$ with origin $o \in \mathcal{S}$ and x_3-axis parallel to the generators of $\mathcal{C}$, let

$$\mathcal{F}(x_1,x_2) := \{(x_1,x_2,x_3)\,|\,(x_1,x_2) \in \mathcal{S},\ x_3 \in (-h,+h)\}$$

denote the material fiber through $(x_1,x_2) \in \mathcal{S}$ that in the reference configuration is straight and parallel to the x_3-direction.

We now summarize an approach to deducing the Reissner–Mindlin plate theory that is exposed in full detail in Lembo and Podio-Guidugli (2007). The pivotal assumption is that *all material fibers* $\mathcal{F}(x_1,x_2)$ *remain straight and suffer no extension.* This is tantamount as accepting the following *a priori* representation for the displacement vector field in $\mathcal{C}$:

$$u_\alpha(x_1,x_2,x_3) = \hat{u}_\alpha(x_1,x_2) + x_3\,\psi_\alpha(x_1,x_2), \quad u_3(x_1,x_2) = w(x_1,x_2). \qquad (1)$$

This representation is nothing but the general integral of the following system of PDEs in terms of components of the displacement vector $\boldsymbol{u}$ and the strain tensor $\boldsymbol{E} = \boldsymbol{E}(\boldsymbol{u})$:

$$E_{33} = u_{3,3} = 0, \quad 2E_{3\alpha,3} = u_{3,\alpha 3} + u_{\alpha,33} = 0. \qquad (2)$$

Of these equations, the first can be seen as the first-order internal constraint of *fiber inextensibility*; the second and third together integrate a second-order internal constraint, *fiber rectilinearity*.[1] Such constraints are maintained by reactive stresses $\boldsymbol{S}^R$ and $\mathbb{S}^R$ of first and second order, respectively, which are all required to do no work in any admissible deformation.

To account for the presence of second-order reactive stress, one assumes that $\mathcal{C}$ is made of a second-grade material (Lembo and Podio-Guidugli, 2000). Moreover, to arrive to plate equations that coincide formally with Reissner–Mindlin's, one assumes that the second-order stress (also called the *hyperstress*) is purely reactive; hence, the active stress consists only of a first-order part $\boldsymbol{S}^A$. All in all (details are found in Lembo and Podio-Guidugli (2001, 2007), where the cases of shear-deformable beams and plates are treated), the stress field is decomposed as follows:

$$\boldsymbol{S} = \boldsymbol{S}^A + \boldsymbol{T}^R, \qquad (3)$$

where the total reactive stress associated with the internal constraints (2) is:

$$\boldsymbol{T}^R = \boldsymbol{S}^R - \operatorname{Div}\mathbb{S}^R. \qquad (4)$$

Here,

$$\boldsymbol{S}^R = \sigma\boldsymbol{e}_3 \otimes \boldsymbol{e}_3, \quad \mathbb{S}^R = \gamma_i \boldsymbol{e}_i \otimes \boldsymbol{e}_3 \otimes \boldsymbol{e}_3 + \eta_\alpha \boldsymbol{e}_3 \otimes (\boldsymbol{e}_\alpha \otimes \boldsymbol{e}_3 + \boldsymbol{e}_3 \otimes \boldsymbol{e}_\alpha), \qquad (5)$$

and the divergence of the third-order tensor $\mathbb{S}^R$ is

$$\operatorname{Div}\mathbb{S}^R = \gamma_{\alpha,3}\boldsymbol{e}_\alpha \otimes \boldsymbol{e}_3 + \eta_{\alpha,3}\boldsymbol{e}_3 \otimes \boldsymbol{e}_\alpha + (\gamma_{3,3} + \eta_{\alpha,\alpha})\boldsymbol{e}_3 \otimes \boldsymbol{e}_3; \qquad (6)$$

hence, according to Eq. (4), the components T^R_{ij} of $\boldsymbol{T}^R$ are

$$[T^R_{ij}] = \begin{bmatrix} 0 & 0 & -\gamma_{1,3} \\ 0 & 0 & -\gamma_{2,3} \\ -\eta_{1,3} & -\eta_{2,3} & \sigma - \gamma_{3,3} - \eta_{\alpha,\alpha} \end{bmatrix}. \qquad (7)$$

The functions $\sigma = \sigma(x_1,x_2,x_3), \gamma_i = \gamma_i(x_1,x_2,x_3)$, and $\eta_\alpha = \eta_\alpha(x_1,x_2,x_3)$, are arbitrary in the sense that they are not constitutively determined; representation (7) shows that the stress $\boldsymbol{T}^R$ is not necessarily symmetric.

Since the hyperstress is purely reactive, the active stress $\boldsymbol{S}^A$ can be constitutively specified as is done in linear elasticity. In that theory, we recall, when a body is internally constrained, the active stress mapping is subjected to the normalization condition of taking its values in the orthogonal complement with respect to the space of all symmetric second-order tensors of the subspace in which the reactive stress lies. Accordingly, in the presence of the constraint $(2)_1$, the constitutive equations of a transversely isotropic material are:

$$\begin{aligned} &S^A_{11} = \mathbb{C}_{1111}E_{11} + \mathbb{C}_{1122}E_{22}, \quad S^A_{22} = \mathbb{C}_{2211}E_{11} + \mathbb{C}_{2222}E_{22}, \\ &S^A_{12} = 2\mathbb{C}_{1212}E_{12}, \quad S^A_{13} = 2\mathbb{C}_{1313}E_{13}, \quad S^A_{23} = 2\mathbb{C}_{2323}E_{23}, \end{aligned} \qquad (8)$$

where

$$\mathbb{C}_{2222} = \mathbb{C}_{1111}, \quad \mathbb{C}_{1111} = \mathbb{C}_{1122} + 2\mathbb{C}_{1212}, \quad \mathbb{C}_{2323} = \mathbb{C}_{1313}. \qquad (9)$$

Remark. In the examples we consider to illustrate our model of laminated plates, we shall find it convenient to replace the transverse-isotropy elasticities $\mathbb{C}_{ijkl}$ with a set of material constants whose definitions are reminiscent of those for the standard technical moduli E, ν, and G, that are used when the material

[1] This terminology is meant to suggest that the image under (1) of fiber $\mathcal{F}(x_1,x_2)$ is a segment of a line whose direction is the same as $(\psi_\alpha(x_1,x_2)\boldsymbol{e}_\alpha + \boldsymbol{e}_3)$.

response is isotropic and unconstrained. We denote by E the Young modulus common to directions x_1 and x_2; by $\overline{E}$ the Young modulus for direction x_3; by ν and $\overline{\nu}$ the Poisson ratios relative to direction pairs x_2, x_1 and x_3, x_1, respectively; finally, $\overline{G}$ denotes the tangential modulus for directions x_1 and x_3. The relationships between the elasticities and these moduli are:

$$\mathbb{C}_{1111} = \frac{E(E-\overline{E}\overline{\nu}^2)}{(1+\nu)(E(1-\nu)-2\overline{E}\overline{\nu}^2)}, \quad \mathbb{C}_{1122} = \frac{E(E\nu+\overline{E}\overline{\nu}^2)}{(1+\nu)(E(1-\nu)-2\overline{E}\overline{\nu}^2)},$$
$$\mathbb{C}_{1212} = \frac{E}{2(1+\nu)}, \quad \mathbb{C}_{1133} = \frac{\overline{E}E\overline{\nu}}{E(1-\nu)-2\overline{E}\overline{\nu}^2},$$
$$\mathbb{C}_{3333} = \frac{\overline{E}E(1-\nu)}{E(1-\nu)-2\overline{E}\overline{\nu}^2}, \quad \mathbb{C}_{1313} = \overline{G}.$$

3. Laminated Reissner–Mindlin plates

3.1. Basic assumptions

A laminated plate is the two-dimensional model of a plate-like body $\mathcal{C}$ composed of n layers of transversely isotropic materials. The k-th layer ($1 \leqslant k \leqslant n$) has thickness $2h^{(k)}$ and occupies the cylindrical subregion $\mathcal{C}^{(k)}$ of $\mathcal{C}$ included between the planes $x_3 = x_3^{(k-1)}$ and $x_3 = x_3^{(k)}$; in particular, $x_3^{(0)} = -h$ and $x_3^{(n)} = h$. Hereafter, as we just did for thickness and subregions, we shall equip all quantities pertaining to k-th layer by a superscript enclosed in parentheses; moreover, we shall leave tacit the quantification: "for all $k \in \{1, \dots, n\}$".

In theories of laminated plates that, like ours, are based on the assumption that (i) each layer deforms as a Reissner–Mindlin plate, it is also assumed that (ii) the displacement vector is continuous across layer interfaces, and that (iii) the traction vector is continuous across layer interfaces. We now discuss these three assumptions, in the order. The first has obvious implications in the writing of the displacement continuity condition. A less immediate implication is that, since it is interpreted as an internal kinematical constraint limiting the class of admissible layer motions, a system of layer reactive stresses must be introduced to maintain that internal constraint, along the lines of reasoning recapitulated in the previous section; such reactive stresses enter the writing of the traction continuity condition.

3.1.1. Layer kinematics

According to assumption (i), in each layer, material fibres that in the reference configuration are parallel to the x_3-direction, remain straight and suffer no extension. Thus, in view of (2), in layer $\mathcal{C}^{(k)}$ we have that

$$E_{33}^{(k)} = 0, \quad E_{3\alpha,3}^{(k)} = 0, \tag{10}$$

and that

$$u_\alpha^{(k)}(x_1,x_2,x_3) = \hat{u}_\alpha^{(k)}(x_1,x_2) + (x_3 - o^{(k)})\,\psi_\alpha^{(k)}(x_1,x_2),$$
$$u_3^{(k)}(x_1,x_2) = w^{(k)}(x_1,x_2). \tag{11}$$

These equations hold for $x_3 \in (x_3^{(k-1)}, x_3^{(k)})$; in particular, $o^{(k)} = (x_3^{(k-1)} + x_3^{(k)})/2$ is the third coordinate of all points in the mid-plane of $\mathcal{C}^{(k)}$, whose displacement components are $(\hat{u}_1^{(k)}, \hat{u}_2^{(k)}, w^{(k)})$. By putting

$$\widetilde{u}_\alpha^{(k)} = \hat{u}_\alpha^{(k)} - o^{(k)}\psi_\alpha^{(k)}, \tag{12}$$

Eqs. $(11)_{1,2}$ become

$$u_\alpha^{(k)}(x_1,x_2,x_3) = \widetilde{u}_\alpha^{(k)}(x_1,x_2) + x_3\psi_\alpha^{(k)}(x_1,x_2). \tag{13}$$

3.1.2. Interlayer continuity of displacements

With the use of Eqs. (13) and $(11)_3$, the displacement field in a n-layer lamitated plate can be expressed in terms of 5n functions of the coordinates x_1 and x_2. Importantly, as we are going to show, the continuity conditions (ii) and (iii) reduce the number of independent parameter functions to 5 in all.

To begin with, displacement continuity across layer interfaces requires that

$$\widetilde{u}_\alpha^{(k)} + x_3^{(k)}\psi_\alpha^{(k)} = \widetilde{u}_\alpha^{(k+1)} + x_3^{(k)}\psi_\alpha^{(k+1)}, \quad w^{(k)} = w^{(k+1)}. \tag{14}$$

Thus, by (13) and $(14)_{1,2}$, the first two components of the displacement in the k-th layer can be written as

$$u_\alpha^{(k)}(x_1,x_2,x_3) = \widetilde{u}_\alpha^{(1)}(x_1,x_2) + \sum_{i=1}^{k-1} x_3^{(i)}(\psi_\alpha^{(i)}(x_1,x_2) - \psi_\alpha^{(i+1)}(x_1,x_2)) + x_3\psi_\alpha^{(k)}(x_1,x_2); \tag{15}$$

moreover, by $(14)_3$, all functions $w^{(k)}(x_1,x_2)$ are equal to one and the same function:

$$u_3^{(k)}(x_1,x_2) = w(x_1,x_2). \tag{16}$$

Relations (15) and (16), which are parameterized by $(3+2n)$ functions of (x_1,x_2), give the form of all displacement fields compatible with the kinematical assumptions (i)–(ii). In view of our use to come of the principle of virtual work, we notice that all variations $\{\delta\widetilde{u}_1^{(k)}, \delta\widetilde{u}_2^{(k)}, \delta\psi_1^{(k)}, \delta\psi_2^{(k)}, \delta w^{(k)}\}$ of functions $\{\widetilde{u}_1^{(k)}, \widetilde{u}_2^{(k)}, \psi_1^{(k)}, \psi_2^{(k)}, w^{(k)}\}$ such that

$$\delta\widetilde{u}_\alpha^{(k)} = \delta\widetilde{u}_\alpha, \quad \delta\psi_\alpha^{(k)} = \delta\psi_\alpha, \quad \delta w^{(k)} = \delta w, \tag{17}$$

are compatible with the restrictions (14) and, thus, with the current expression (15) and (16) for the displacement field. We point out that reactive stresses are workless in the presence of such variations, and hence the resulting balance equations are 'pure', i.e., reaction-free.

3.1.3. Interlayer continuity of traction vector

The condition that the three-dimensional equilibrium equations in integral form hold also for parts of the body $\mathcal{C}$ including one or more interfaces between layers implies (cf, e.g., Truesdell and Toupin, 1960, Sect. 193) that the traction vector $\boldsymbol{s} = \boldsymbol{S}\boldsymbol{e}_3$ must be continuous across the interfaces. Now, combination of (3)–(7) and $(8)_{4,5}$ yields:

$$\boldsymbol{s} = \sum_{\alpha=1}^{2}(\mathbb{C}_{\alpha3\alpha3}E_{\alpha3} - \gamma_{\alpha,3})\,\boldsymbol{e}_\alpha + (\sigma - \gamma_{3,3} - \eta_{\alpha,\alpha})\,\boldsymbol{e}_3.$$

We see that, in the present theory, the third component of $\boldsymbol{s}$, namely,

$$s_3 = \boldsymbol{s}\cdot\boldsymbol{e}_3 = (\sigma - \gamma_{3,3} - \eta_{\alpha,\alpha}),$$

is a reactive quantity; therefore, its continuity can only be imposed after the reaction-free plate equations have been solved, the active stresses determined, and the three-dimensional equilibrium equations used to find the reactive stresses. As to the other two components, we request that their active and reactive parts be both continuous. The continuity of the latter part, $-\sum_{\alpha=1}^{2}\gamma_{\alpha,3}\boldsymbol{e}_\alpha$, can only be imposed in the post-processing phase of calculation of stresses, just as for s_3. The continuity of active parts is expressed by the following equations:

$$S_{3\alpha}^{A(k)}(x_1,x_2,x_3^{(k)}) = S_{3\alpha}^{A(k+1)}(x_1,x_2,x_3^{(k)}), \tag{18}$$

which, in view of $(8)_{4,5}$, $(9)_3$, (15) and (16), become

$$\mathbb{C}_{1313}^{(k)}(w_{,\alpha}(x_1,x_2) + \psi_\alpha^{(k)}(x_1,x_2)) = \mathbb{C}_{1313}^{(k+1)}(w_{,\alpha}(x_1,x_2) + \psi_\alpha^{(k+1)}(x_1,x_2)). \tag{19}$$

It follows that,

$$\psi_\alpha^{(k)} = \chi^{(k)}(w_\alpha + \psi_\alpha^{(1)}) - w_\alpha, \quad \alpha = 1,2, \tag{20}$$

and

$$\chi^{(k)} = \frac{\mathbb{C}_{1313}^{(1)}}{\mathbb{C}_{1313}^{(k)}}; \tag{21}$$

note the role of the *shear-moduli ratios* $\chi^{(k)}$. Eqs. (20) imply that, for a given set of shear-moduli ratios, the $2n$ parameter functions $\psi_\alpha^{(k)}$

are all determined by two of them, $\psi_\alpha^{(1)}$, and by the gradient of the parameter function w.

We choose to represent the admissible displacement fields in $\mathcal{C}$ in terms of the five parameter functions $\widetilde{u}_1, \widetilde{u}_2, \psi_1, \psi_2$, and w, where

$\widetilde{u}_\alpha \equiv \widetilde{u}_\alpha^{(1)}, \quad \psi_\alpha \equiv \psi_\alpha^{(1)}$.

In conclusion, we write the displacement components in the k-th layer as follows:

$$\begin{aligned} u_\alpha^{(k)}(x_1,x_2,x_3) &= \widetilde{u}_\alpha(x_1,x_2) + (H^{(k)} + \chi^{(k)}x_3)\psi_\alpha(x_1,x_2) \\ &\quad + (H^{(k)} + (\chi^{(k)} - 1)x_3)w_{,\alpha}(x_1,x_2), \\ u_3^{(k)}(x_1,x_2) &= w(x_1,x_2), \end{aligned} \tag{22}$$

where

$$H^{(1)} := 0, \quad H^{(k)} := \sum_{i=1}^{k-1} x_3^{(i)}(\chi^{(i)} - \chi^{(i+1)}), \quad 1 < k \leqslant n, \tag{23}$$

3.2. Strain, active stress, and stress resultants

Let us fix our attention on the typical layer $\mathcal{C}^{(k)}$. The nonzero strain components corresponding to expressions (22) for the displacement components are:

$$\begin{aligned} E_{\alpha\beta}^{(k)} &= \frac{1}{2}(\widetilde{u}_{\alpha,\beta} + \widetilde{u}_{\beta,\alpha} + (H^{(k)} + \chi^{(k)}x_3)(\psi_{\alpha,\beta} + \psi_{\beta,\alpha}) \\ &\quad + (H^{(k)} + (\chi^{(k)} - 1)x_3)(w_{,\alpha\beta} + w_{,\beta\alpha})), \\ E_{3\eta}^{(k)} &= \frac{1}{2}\chi^{(k)}(w_{,\eta} + \psi_\eta); \end{aligned} \tag{24}$$

hence, in view of the constitutive equations (8), the nonzero components of the active stress are:

$$\begin{aligned} S_{\alpha\beta}^{A(k)} &= \mathbb{C}_{\alpha\beta\gamma\delta}^{(k)}\frac{1}{2}(\widetilde{u}_{\gamma,\delta} + \widetilde{u}_{\delta,\gamma} + (H^{(k)} + \chi^{(k)}x_3)(\psi_{\gamma,\delta} + \psi_{\delta,\gamma}) \\ &\quad + (H^{(k)} + (\chi^{(k)} - 1)x_3)(w_{,\gamma\delta} + w_{,\delta\gamma})), \\ S_{\eta 3}^{A(k)} &= \mathbb{C}_{1313}^{(k)}\chi^{(k)}(w_{,\eta} + \psi_\eta); \end{aligned} \tag{25}$$

it is understood that in Eq. $(25)_1$ one takes

$$\text{either} \quad \beta = \alpha, \quad \delta = \gamma \quad \text{or} \quad \beta \neq \alpha, \quad \delta \neq \gamma. \tag{26}$$

The stress resultants are obtained by integrating over the layer thickness the stresses (25) and the moments of the stresses $(25)_{1,2}$ with respect to the midplane of the layer. Precisely, the membrane forces $N_{\alpha\beta}^{(k)}$, the shears $Q_\eta^{(k)}$, and the moments $M_{\alpha\beta}^{(k)}$ in the k-th layer are:

$$\begin{aligned} N_{\alpha\beta}^{(k)} &= \int_{x_3^{(k-1)}}^{x_3^{(k)}} S_{\alpha\beta}^{A(k)}\,dx_3 = h^{(k)}\mathbb{C}_{\alpha\beta\gamma\delta}^{(k)}(\widetilde{u}_{\gamma,\delta} + \widetilde{u}_{\delta,\gamma} + (H^{(k)} + \chi^{(k)}o^{(k)})(\psi_{\gamma,\delta} + \psi_{\delta,\gamma}) \\ &\quad + (H^{(k)} + (\chi^{(k)} - 1)o^{(k)})(w_{,\gamma\delta} + w_{,\delta\gamma})), \\ Q_\eta^{(k)} &= \int_{x_3^{(k-1)}}^{x_3^{(k)}} S_{3\eta}^{A(k)}\,dx_3 = 2h^{(k)}\mathbb{C}_{1313}^{(k)}\chi^{(k)}(w_{,\eta} + \psi_\eta) = 2h^{(k)}\mathbb{C}_{1313}^{(1)}(w_{,\eta} + \psi_\eta), \\ M_{\alpha\beta}^{(k)} &= \int_{x_3^{(k-1)}}^{x_3^{(k)}} S_{\alpha\beta}^{A(k)}(x_3 - o^{(k)})\,dx_3 = \frac{2}{3}(h^{(k)})^3\mathbb{C}_{\alpha\beta\gamma\delta}^{(k)}(\chi^{(k)}(\psi_{\gamma,\delta} + \psi_{\delta,\gamma}) \\ &\quad + (\chi^{(k)} - 1)(w_{,\gamma\delta} + w_{,\delta\gamma})). \end{aligned} \tag{27}$$

The stress resultants in the whole plate are:

$$\begin{aligned} N_{\alpha\beta} &= \int_{-h}^{h} S_{\alpha\beta}^A\,dx_3 = \sum_{k=1}^n N_{\alpha\beta}^{(k)} = A_{\alpha\beta\gamma\delta}^{[1]}(\widetilde{u}_{\gamma,\delta} + \widetilde{u}_{\delta,\gamma}) + A_{\alpha\beta\gamma\delta}^{[2]}(\psi_{\gamma,\delta} + \psi_{\delta,\gamma}) \\ &\quad + A_{\alpha\beta\gamma\delta}^{[3]}(w_{,\gamma\delta} + w_{,\delta\gamma}), \\ Q_\eta &= \int_{-h}^{h} S_{3\eta}^A\,dx_3 = \sum_{k=1}^n Q_\eta^{(k)} = B(w_{,\eta} + \psi_\eta), \\ M_{\alpha\beta} &= \int_{-h}^{h} S_{\alpha\beta}^A x_3\,dx_3 = \sum_{k=1}^n (M_{\alpha\beta}^{(k)} + o^{(k)}N_{\alpha\beta}^{(k)}) = D_{\alpha\beta\gamma\delta}^{[1]}(\widetilde{u}_{\gamma,\delta} + \widetilde{u}_{\delta,\gamma}) \\ &\quad + D_{\alpha\beta\gamma\delta}^{[2]}(\psi_{\gamma,\delta} + \psi_{\delta,\gamma}) + D_{\alpha\beta\gamma\delta}^{[3]}(w_{,\gamma\delta} + w_{,\delta\gamma}), \end{aligned} \tag{28}$$

where

$$\begin{aligned} A_{\alpha\beta\gamma\delta}^{[1]} &= \sum_{k=1}^n h^{(k)}\mathbb{C}_{\alpha\beta\gamma\delta}^{(k)}, \\ A_{\alpha\beta\gamma\delta}^{[2]} &= \sum_{k=1}^n h^{(k)}(H^{(k)} + \chi^{(k)}o^{(k)})\mathbb{C}_{\alpha\beta\gamma\delta}^{(k)}, \\ A_{\alpha\beta\gamma\delta}^{[3]} &= \sum_{k=1}^n h^{(k)}(H^{(k)} + (\chi^{(k)} - 1)o^{(k)})\mathbb{C}_{\alpha\beta\gamma\delta}^{(k)}, \end{aligned} \tag{29}$$

$$B = \sum_{k=1}^n 2h^{(k)}\chi^{(k)}\mathbb{C}_{1313}^{(k)} = 2h\mathbb{C}_{1313}^{(1)}, \tag{30}$$

$$\begin{aligned} &D_{\alpha\beta\gamma\delta}^{[1]} = \sum_{k=1}^n o^{(k)}h^{(k)}\mathbb{C}_{\alpha\beta\gamma\delta}^{(k)}, \quad D_{\alpha\beta\gamma\delta}^{[2]} = \sum_{k=1}^n (o^{(k)}h^{(k)}H^{(k)} + ((h^{(k)})^3/3 + (o^{(k)})^2h^{(k)})\chi^{(k)})\mathbb{C}_{\alpha\beta\gamma\delta}^{(k)}, \\ &D_{\alpha\beta\gamma\delta}^{[3]} = \sum_{k=1}^n (o^{(k)}h^{(k)}H^{(k)} + ((h^{(k)})^3/3 + (o^{(k)})^2h^{(k)})(\chi^{(k)} - 1))\mathbb{C}_{\alpha\beta\gamma\delta}^{(k)}. \end{aligned} \tag{31}$$

Eqs. (28) show that, in general, membrane forces and moments depend on all the five functions $\widetilde{u}_1, \widetilde{u}_2, \psi_1, \psi_2$, and w.

3.3. Equilibrium plate equations

The two-dimensional equilibrium plate equations are deduced from the three-dimensional virtual work equation written for virtual displacements $(\delta u_1^{(k)}, \delta u_2^{(k)}, \delta u_3^{(k)})$ having the form of the kinematically admissible displacements (15) and (16), where the functions $\delta\widetilde{u}_1^{(k)}, \delta\widetilde{u}_2^{(k)}, \delta\psi_1^{(k)}, \delta\psi_2^{(k)}, \delta w^{(k)}$ are chosen as in Eqs. (17):

$$\begin{aligned} \delta u_\alpha^{(k)}(x_1,x_2,x_3) &= \delta\widetilde{u}_\alpha(x_1,x_2) + x_3\delta\psi_\alpha(x_1,x_2), \\ \delta u_3^{(k)}(x_1,x_2) &= \delta w(x_1,x_2). \end{aligned} \tag{32}$$

Such three-dimensional virtual work equation reads:

$$\begin{aligned} &\sum_{k=1}^n \int_{\mathcal{S}}\int_{x^{(k-1)}}^{x^{(k)}} (S_{\alpha\beta}^{A(k)}(\delta\widetilde{u}_{\alpha,\beta} + x_3\delta\psi_{\alpha,\beta}) + S_{3\alpha}^{A(k)}(\delta\psi_\alpha + \delta w_{,\alpha}))\,dx_3da \\ &= \int_{\mathcal{S}}\int_{-h}^{h}(b_\alpha(\delta\widetilde{u}_\alpha + x_3\delta\psi_\alpha) + b_3\delta w)\,dx_3da + \int_{\partial\mathcal{S}}\int_{-h}^{h}(t_\alpha(\delta\widetilde{u}_\alpha \\ &\quad + x_3\delta\psi_\alpha) + t_3\delta w)dx_3ds + \int_{\mathcal{S}}(t_\alpha^\pm(\delta\widetilde{u}_\alpha \pm h\delta\psi_\alpha) + t_3^\pm\delta w)\,da, \end{aligned} \tag{33}$$

where b_i, t_i, and $t_i^\pm$ denote the components of, respectively, the force per unit volume $\boldsymbol{b}$ acting on $\mathcal{C}$, the force per unit area $\boldsymbol{t}$ acting on the mantel $\partial\mathcal{S} \times (-h,h)$ of $\mathcal{C}$, and the force per unit area $\boldsymbol{t}^\pm$ acting on the end face $\mathcal{S}^\pm$ of $\mathcal{C}$.[2] After thickness integration, one finds, with a use of the divergence theorem, that

$$\begin{aligned} &\int_{\mathcal{S}}((N_{\alpha\beta,\beta} + q_\alpha)\delta\widetilde{u}_\alpha + (Q_{\alpha,\alpha} + q_3)\delta w + (M_{\alpha\beta,\beta} - Q_\alpha + m_\alpha)\delta\psi_\alpha) \\ &\quad - \int_{\partial\mathcal{S}}((N_{\alpha\beta}n_\beta - f_\alpha)\delta\widetilde{u}_\alpha + (Q_\alpha n_\alpha - f_3)\delta w \\ &\quad + (M_{\alpha\beta}n_\beta - c_\alpha)\delta\psi_\alpha)\,ds = 0, \end{aligned} \tag{34}$$

where q_i and m_α are loads per unit area of $\mathcal{S}$,

$$q_i = \int_{-h}^{h} b_i dx_3 + t_i^+ + t_i^-, \quad m_\alpha = \int_{-h}^{h} b_\alpha x_3 dx_3 + h(t_\alpha^+ - t_\alpha^-), \tag{35}$$

and f_i and c_α are loads per unit length of $\partial\mathcal{S}$,

$$f_i = \int_{-h}^{h} t_i dx_3, \quad c_\alpha = \int_{-h}^{h} t_\alpha x_3 dx_3. \tag{36}$$

[2] The double-sign exponent $\pm$ is meant to suggest a double reading of the sentence where it appears, one for each of the two signs.

It follows from Eq. (34) that the equilibrium plate equations are, in $\mathcal{S}$,

$$N_{\alpha\beta,\beta}+q_\alpha=0,\quad Q_{\alpha,\alpha}+q_3=0,\quad M_{\alpha\beta,\beta}-Q_\alpha+m_\alpha=0, \tag{37}$$

and that they are accompanied by boundary conditions consisting in complementing assignments, along $\partial\mathcal{S}$, of the loads f_α, f_3, and c_α, and of the displacements and rotations $\widetilde{u}_\alpha, w$, and ψ_α.

Since, as observed at the end of SubSection 3.2, membrane forces and moments depend in general on all the functions $\widetilde{u}_1, \widetilde{u}_2, \psi_1, \psi_2$, and w, the equilibrium equations $(37)_{1,2}$ and $(37)_{3,4,5}$ governing, respectively, membranal and flexural deformations of the plate, are coupled. They are not, as is the case for homogeneous single-layer plates, for the mid-plane symmetric laminated plates, whose equilibrium equations we deduce in SubSection 4.1.

3.4. Improved three-dimensional stress field

The five equilibrium equations (37) form a system for the five unknowns $\widetilde{u}_1, \widetilde{u}_2, w, \psi_1$ and ψ_2. Once these functions are found, a corresponding set of displacement, strain and active stress fields in the three-dimensional body $\mathcal{C}$ is obtained, with the use of (22), (24), and (25). In general, such an active stress field does not satisfy the three-dimensional equilibrium equations for $\mathcal{C}$ exactly (as demonstrated, e.g., by the first of boundary conditions $(40)_3$); this may happen because a plate problem does not correspond to a single three-dimensional problem but rather to an equivalence class of equilibrium problems for $\mathcal{C}$. However, as shown in Lembo and Podio-Guidugli (2007) for single-layer plates, one can use the reactive stresses associated with the internal constraints implicit in the Reissner–Mindlin kinematics to strongly improve the active-stress approximation of the three-dimensional stress field.

Consistent with the fact that the reactive stress field we consider maintains also the second-gradient internal constraints $(2)_2$, we try and determine it by the use of the equilibrium equations for *second-grade* material bodies. In our case, the hyperstress $\mathbb{S}$ is purely reactive and the total stress $\boldsymbol{S}$ has the expression (3). Hence, the balance equations at interior points of $\mathcal{C}$ and at its boundary are:

$$\begin{aligned}
&\mathrm{Div}\,(\boldsymbol{S}^A+\boldsymbol{S}^R)-\mathrm{Div}\,(\mathrm{Div}\,\mathbb{S}^R)+\boldsymbol{b}=\boldsymbol{0},\quad \text{in } \mathcal{C},\\
&\pm\,(\boldsymbol{S}^A+\boldsymbol{S}^R)\boldsymbol{e}_3\mp(\mathrm{Div}\,\mathbb{S}^R)\boldsymbol{e}_3\mp{}^s\mathrm{Div}\,(\mathbb{S}^R\boldsymbol{e}_3)=\boldsymbol{t}^\pm,\quad \text{on } \mathcal{S}^\pm,\\
&\boldsymbol{S}^A\boldsymbol{n}-(\mathrm{Div}\,\mathbb{S}^R)\boldsymbol{n}-{}^s\mathrm{Div}\,(\mathbb{S}^R\boldsymbol{n})=\boldsymbol{t},\quad \text{on } \partial\mathcal{S}\times(-h,h),\\
&(\mathbb{S}^R\boldsymbol{n})\boldsymbol{n}=\boldsymbol{p},\quad \text{on } \partial\mathcal{C}\setminus(\partial\mathcal{S}^+\cup\partial\mathcal{S}^-),\\
&(\mathbb{S}^R\boldsymbol{n}_1^\pm)\boldsymbol{v}_1^\pm+(\mathbb{S}^R\boldsymbol{n}_2^\pm)\boldsymbol{v}_2^\pm=\boldsymbol{l}^\pm,\quad \text{on } \partial\mathcal{S}^\pm.
\end{aligned} \tag{38}$$

Here, ${}^s\mathrm{Div}$ denotes the *surface divergence operator*; curve $\partial\mathcal{S}^\pm\equiv\partial\mathcal{S}\times\{\pm h\}$ is the common boundary of the end face $\mathcal{S}^\pm$ and the mantle $\partial\mathcal{S}\times\{-h,+h\}$; at a point of such a curve, the unit vectors $\boldsymbol{n}_1^\pm$ and $\boldsymbol{v}_1^\pm$ denote, respectively, the outer normals to $\mathcal{S}^\pm$ and to $\partial\mathcal{S}^\pm$, while the unit vectors $\boldsymbol{n}_2^\pm$ and $\boldsymbol{v}_2^\pm$ denote, respectively, the outer normals to $\partial\mathcal{S}\times\{-h,+h\}$ and to $\partial\mathcal{S}\times\{\pm h\}$; thus, $\boldsymbol{n}_1^\pm=\pm\boldsymbol{e}_3=\boldsymbol{v}_2^\pm$, $\boldsymbol{v}_1^\pm=\boldsymbol{n}_2^\pm$. Moreover, load $\boldsymbol{p}$ is a couple per unit area, and load $\boldsymbol{l}^\pm$ a force per unit length.[3] We recall that it makes sense to apply such couples and forces to a material body of second grade or higher, but not to a first-grade material body (see e.g. Podio-Guidugli, 2002; Podio-Guidugli and Vianello, 2010); thus, both for simplicity and for consistency with the developments in the Remark that closes this section, we put:

$$\boldsymbol{p}=\boldsymbol{0},\quad \boldsymbol{l}^\pm=\boldsymbol{0}. \tag{39}$$

[3] In the present section, for simplicity, we take $\partial\mathcal{S}$ smooth; had it corners, vertical edges would be found in the mantle, along which forces per unit length could be applied.

Using the expression (7) for the reactive stress, Eqs. (38) become:

$$\begin{aligned}
&\gamma_{\alpha,33}\,\boldsymbol{e}_\alpha+(\gamma_{3,3}+2\eta_{\alpha,\alpha}-\sigma)_{,3}\,\boldsymbol{e}_3=\mathrm{Div}\,\boldsymbol{S}^A+\boldsymbol{b},\quad \text{in } \mathcal{C},\\
&(\gamma_{\alpha,3})^\pm\boldsymbol{e}_\alpha+(\gamma_{3,3}+\eta_{\alpha,\alpha}-\sigma)^\pm\boldsymbol{e}_3=(\boldsymbol{S}^A\boldsymbol{e}_3)^\pm\mp\boldsymbol{t}^\pm,\quad \text{on } \mathcal{S}^\pm,\\
&0=S^A_{\beta\alpha}n_\alpha-t_\beta,\quad 2\eta_{\alpha,3}n_\alpha=S^A_{3\alpha}n_\alpha-t_3,\quad \text{on } \partial\mathcal{S}\times(-h,h),\\
&\gamma_\alpha^\pm=\gamma_3^\pm=0,\quad \text{on } \mathcal{S}^\pm,\\
&\eta_\alpha n_\alpha=0,\quad \text{on } \partial\mathcal{S}^\pm.
\end{aligned} \tag{40}$$

As anticipated, we cannot expect the active stress $\boldsymbol{S}^A$ we have determined by means of our shearable plate theory to satisfy the first of (40). However, for given fields $\boldsymbol{S}^A$ and $\boldsymbol{t}$, the rest of system (40) can be used to evaluate the six parameter functions σ, γ_i, and η_α in the representation (7) of the reactive stress. There is more than one way to do so; here is how we choose to proceed in this paper.

Since

$$\int_{-h}^{h}(h-\zeta)(\mathrm{Div}\,\boldsymbol{S}^A+\boldsymbol{b})_3\,d\zeta=-2h(\boldsymbol{S}^A\boldsymbol{e}_3+\boldsymbol{t})_3^-, \tag{41}$$

by integrating twice the third component of $(40)_1$ on taking $(40)_2$ and $(40)_4$ into account, we obtain:

$$\int_{-h}^{h}(\sigma-2\eta_{\alpha,\alpha})d\zeta=0. \tag{42}$$

We choose:

$$\eta_\alpha(x_1,x_2,x_3)=\gamma_\alpha(x_1,x_2,x_3), \tag{43}$$

an assumption that implies symmetry of the reactive stress $\boldsymbol{T}^R$ (cf. (7)) and that, together with $(40)_4$, makes ${}^s\mathrm{Div}\,(\mathbb{S}^R\boldsymbol{e}_3)$ vanish on the end sections of $\mathcal{C}$, reducing conditions $(40)_{2,3}$ to the form $\boldsymbol{Sn}=\pm\boldsymbol{t}^\pm$ of classical elasticity; in addition, we satisfy Eq. (42) by taking

$$\sigma(x_1,x_2)=\frac{1}{h}\int_{-h}^{h}\eta_{\alpha,\alpha}(x_1,x_2,x_3)\,dx_3. \tag{44}$$

All in all, we are left with a representation for $\boldsymbol{T}^R$ in terms of the three parameter functions γ_i, with the choice of γ_1 and γ_2 restricted by the last of (40), which takes the form:

$$\gamma_\alpha n_\alpha=0\quad \text{on } \partial\mathcal{S}^\pm,$$

due to (43). The components of $\boldsymbol{T}^R$ are obtained by integrating Eq. $(40)_1$ with the conditions $(40)_2$ and $(40)_4$. Making use of (43), we have:

$$\begin{aligned}
&T^{R(1)}_{\alpha3}=-\gamma^{(1)}_{\alpha,3}=-(\boldsymbol{S}^A\boldsymbol{e}_3+\boldsymbol{t})^-_\alpha-\int_{-h}^{x_3}(\mathrm{Div}\boldsymbol{S}^A+\boldsymbol{b})_\alpha d\zeta,\\
&\gamma^{(1)}_\alpha=(x_3+h)(\boldsymbol{S}^A\boldsymbol{e}_3+\boldsymbol{t})^-_\alpha+\int_{-h}^{x_3}(x_3-\zeta)(\mathrm{Div}\boldsymbol{S}^A+\boldsymbol{b})_\alpha d\zeta,\\
&T^{R(1)}_{33}=\sigma-\gamma^{(1)}_{3,3}-\eta^{(1)}_{\alpha,\alpha}=\gamma^{(1)}_{\alpha,\alpha}-(\boldsymbol{S}^A\boldsymbol{e}_3+\boldsymbol{t})^-_3-\int_{-h}^{x_3}(\mathrm{Div}\boldsymbol{S}^A+\boldsymbol{b})_3 d\zeta,
\end{aligned} \tag{45}$$

for the first layer ($k=1$), and

$$\begin{aligned}
&T^{R(k)}_{\alpha3}=-\gamma^{(k)}_{\alpha,3}=-\gamma^{(k-1)}_{\alpha,3}|_{x_3=x_3^{(k-1)}}-\int_{x_3^{(k-1)}}^{x_3}(\mathrm{Div}\boldsymbol{S}^A+\boldsymbol{b})_\alpha d\zeta,\\
&\gamma^{(k)}_\alpha=\gamma^{(k-1)}_\alpha|_{x_3=x_3^{(k-1)}}+(x_3-x_3^{(k-1)})\gamma^{(k-1)}_{\alpha,3}|_{x_3=x_3^{(k-1)}}+\int_{x_3^{(k-1)}}^{x_3}(x_3-\zeta)(\mathrm{Div}\boldsymbol{S}^A+\boldsymbol{b})_\alpha d\zeta,\\
&T^{R(k)}_{33}=T^{R(k-1)}_{33}|_{x_3=x_3^{(k-1)}}-\gamma^{(k-1)}_{\alpha,\alpha}|_{x_3=x_3^{(k-1)}}+\gamma^{(k)}_{\alpha,\alpha}-\int_{x_3^{(k-1)}}^{x_3}(\mathrm{Div}\boldsymbol{S}^A+\boldsymbol{b})_3 d\zeta,
\end{aligned} \tag{46}$$

for all other layers ($k>1$). When the reactive stresses $T^{R(k)}_{i3}$ are component-wise added to the active stresses, an improved approximation of the stress field solving the three-dimensional balance equations in the closure of $\mathcal{C}$ is arrived at. We substantiate this assertion by means of the examples we work out in Section 5.

Remark. One may wonder whether the standard equilibrium equations for *first-grade* material bodies would serve to determine an out-of-plane addition

$$\boldsymbol{R} = R_{\alpha 3}(\boldsymbol{e}_\alpha \otimes \boldsymbol{e}_3 + \boldsymbol{e}_3 \otimes \boldsymbol{e}_\alpha) + R_{33}\boldsymbol{e}_3 \otimes \boldsymbol{e}_3$$

to an in-plane stress field

$$\boldsymbol{S}^P = S_{\alpha\beta}(\boldsymbol{e}_\alpha \otimes \boldsymbol{e}_\beta + \boldsymbol{e}_\beta \otimes \boldsymbol{e}_\alpha),$$

computed by the use of some plate theory, in hope that the stress field $\boldsymbol{A} = \boldsymbol{S}^P + \boldsymbol{R}$ offer a good approximation of the unknown three-dimensional stress field.[4] For $\boldsymbol{S}^P$ chosen equal to our $\boldsymbol{S}^A$, would such $\boldsymbol{R}$ turn out to be the same as our $\boldsymbol{T}^R$ (or better, perhaps)?

This proposition makes sense, because we stipulated (39) and (43) (applied surface couples $\boldsymbol{p}$ and line forces $\boldsymbol{l}$ can be handled only within a theory of grade higher than the first; moreover, when a general second-grade theory is adopted, the reactive stress $\boldsymbol{T}^R$ needs not be symmetric). The additional stress field in question must solve the following system:

$$\begin{aligned} &-\mathrm{Div}\,\boldsymbol{R} = \mathrm{Div}\,\boldsymbol{S}^A + \boldsymbol{b}, \quad \text{in } \mathcal{C}, \\ &-\boldsymbol{R}^\pm \boldsymbol{e}_3 = (\boldsymbol{S}^A\boldsymbol{e}_3)^\pm \mp \boldsymbol{t}^\pm, \quad \text{on } \mathcal{S}^\pm, \\ &-\boldsymbol{R}_{3\alpha} = S^A_{3\alpha}n_\alpha - t_3, \quad \text{on } \partial\mathcal{S}\times(-h,h). \end{aligned} \tag{47}$$

The first two equations in system (47) correspond to the first two in system (40); for the layered plates we study, they lead to a reactive field that has the form detailed in (45) and (46). The boundary conditions on $\partial\mathcal{S}\times(-h,h)$ differ, though. The reason why they do is that their expression is

$$\boldsymbol{Sn} = \boldsymbol{t} \tag{48}$$

within a first-grade theory, and is

$$\boldsymbol{Sn} - {}^s\mathrm{Div}\,(\mathbb{S}\boldsymbol{n}) = \boldsymbol{t}$$

within a second-gradient theory, in which a hyperstress $\mathbb{S}$ integrates the 'ordinary' stress $\boldsymbol{S}$. If in (48) we approximate $\boldsymbol{S}$ by $\boldsymbol{A} = \boldsymbol{S}^A + \boldsymbol{T}^R$, the third component of that equation reads:

$$T^R_{3\alpha}n_\alpha = \eta_{\alpha,3}n_\alpha = S^A_{3\alpha}n_\alpha - t_3,$$

and differs by a factor 2 from the second of $(40)_3$.

In conclusion, the question posed in the beginning of this Remark can be given a semi-positive answer in rather special circumstances and by taking some *ad hoc* measures. But, we believe that it would be better not to use the balance equations of a first-grade theory to approximate the three-dimensional stress field in a plate-like body, layered or not, when the displacement field is given an *a priori* Reissner–Mindlin representation. We offer two reasons: the one, conceptual, is that such a representation can be regarded as an internal constraint that can be completely and exactly satisfied only within the framework of a second-grade theory; the other, practical, is that a second-grade approach sets the stage for an optimal choice of the reactive field, among the many such fields consistent with the given applied loads; we plan to tackle such an optimization problem soon.

4. Laminated Reissner–Mindlin plates with special symmetries

4.1. Mid-plane symmetric plates

A multilayered plate-like body $\mathcal{C}$ is mid-plane symmetric, and its two-dimensional model is called a *mid-plane symmetric plate*, when $\mathcal{C}$ is formed by n pairs of layers having the same thickness, being symmetrically located with respect to the mid-plane of $\mathcal{C}$, and being made of the same transversely isotropic material. We shall denote by one and the same label k the two layers of a given pair. The k layers occupy the regions of $\mathcal{C}$ between planes $x_3 = \pm\overline{x}_3^{(k-1)}$ and $x_3 = \pm\overline{x}_3^{(k)}$ ($\overline{x}_3^{(k-1)}, \overline{x}_3^{(k)}$ are non-negative numbers; in particular, $\overline{x}_3^{(0)} = 0$ and $\overline{x}_3^{(n)} = h$); their thickness is $2h^{(k)}$, and their mid-planes are $x_3 = \pm\overline{o}^{(k)} = \pm(\overline{x}_3^{(k-1)} + \overline{x}_3^{(k)})/2$, with $\overline{o}^{(k)} > 0$. The mid-plane of $\mathcal{C}$ separates the pair of layers labeled with the number 1, whose displacement components are:

$$\begin{aligned} &u_\alpha(x_1,x_2,x_3) = \widetilde{u}_\alpha(x_1,x_2) + x_3\psi_\alpha(x_1,x_2)2, \\ &u_3(x_1,x_2) = w(x_1,x_2). \end{aligned} \tag{49}$$

With this choice, $\widetilde{u}_1, \widetilde{u}_2$, and w are the displacement components of the mid-section of $\mathcal{C}$, while the displacement components of the k layers are:

$$\begin{aligned} u^{(k)}_\alpha(x_1,x_2,x_3) &= \widetilde{u}_\alpha(x_1,x_2) + (\mathrm{sgn}(x_3)\overline{H}^{(k)} + \chi^{(k)}x_3)\psi_\alpha(x_1,x_2) \\ &\quad + (\mathrm{sgn}(x_3)\overline{H}^{(k)} + (\chi^{(k)}-1)x_3)w_{,\alpha}(x_1,x_2), \quad \alpha = 1,2, u^{(k)}_3(x_1,x_2) \\ &= w(x_1,x_2), \end{aligned} \tag{50}$$

where

$$\overline{H}^{(1)} = 0, \quad \overline{H}^{(k)} = \sum_{i=1}^{k-1}\overline{x}_3^{(i)}(\chi^{(i)} - \chi^{(i+1)}), \quad k > 1. \tag{51}$$

The strain components are:

$$\begin{aligned} E^{(k)}_{\alpha\beta} &= \frac{1}{2}(\widetilde{u}_{\alpha,\beta} + \widetilde{u}_{\beta,\alpha} + (\mathrm{sgn}(x_3)\overline{H}^{(k)} + \chi^{(k)}x_3)(\psi_{\alpha,\beta} + \psi_{\beta,\alpha}) \\ &\quad + (\mathrm{sgn}(x_3)\overline{H}^{(k)} + (\chi^{(k)}-1)x_3)(w_{,\alpha\beta} + w_{,\beta\alpha})), \\ E^{(k)}_{3\eta} &= \frac{1}{2}\chi^{(k)}(w_{,\eta} + \psi_\eta). \end{aligned} \tag{52}$$

The stress resultants in the whole plate are:

$$\begin{aligned} &N_{\alpha\beta} = \int_{-h}^{h} S^A_{\alpha\beta}dx_3 = \overline{A}^{[1]}_{\alpha\beta\gamma\delta}(\widetilde{u}_{\gamma,\delta} + \widetilde{u}_{\delta,\gamma}), \quad Q_\eta = \int_{-h}^{h} S^A_{3\eta}dx_3 = B(w_\eta + \psi_\eta), \\ &M_{\alpha\beta} = \int_{-h}^{h} S^A_{\alpha\beta}x_3dx_3 = \overline{D}^{[2]}_{\alpha\beta\gamma\delta}(\psi_{\gamma,\delta} + \psi_{\delta,\gamma}) + \overline{D}^{[3]}_{\alpha\beta\gamma\delta}(w_{\gamma\delta} + w_{\delta\gamma}), \end{aligned} \tag{53}$$

where

$$\begin{aligned} &\overline{A}^{[1]}_{\alpha\beta\gamma\delta} = 2\sum_{k=1}^{n} h^{(k)}\mathbb{C}^{(k)}_{\alpha\beta\gamma\delta}, \quad \overline{B} = 4\sum_{k=1}^{n} h^{(k)}\chi^{(k)}\mathbb{C}^{(k)}_{1313} = 4h\mathbb{C}^{(1)}_{1313}, \\ &\overline{D}^{[2]}_{\alpha\beta\gamma\delta} = 2\sum_{k=1}^{n}(\overline{o}^{(k)}h^{(k)}\overline{H}^{(k)} + ((h^{(k)})^3/3 + (\overline{o}^{(k)})^2h^{(k)})\chi^{(k)})\mathbb{C}^{(k)}_{\alpha\beta\gamma\delta}, \\ &\overline{D}^{[3]}_{\alpha\beta\gamma\delta} = 2\sum_{k=1}^{n}(\overline{o}^{(k)}h^{(k)}\overline{H}^{(k)} + ((h^{(k)})^3/3 + (\overline{o}^{(k)})^2h^{(k)})(\chi^{(k)} - 1))\mathbb{C}^{(k)}_{\alpha\beta\gamma\delta}. \end{aligned} \tag{54}$$

Expressions (53) make clear that, for mid-plane symmetric plates, the membranal equilibrium Eqs. $(37)_{1,2}$ are uncoupled from the flexural equilibrium Eqs. $(37)_{3,4,5}$. We notice that the expressions of the reactive stresses $T^{(k)}_{i3}$ found in Section 3.4 have to be modified in an obvious way to account for the different layer numeration.

4.2. Axisymmetric circular plates

In the next section, we shall apply the proposed model of laminated plates to the study of certain equilibrium problems, which include the case of circular plates subject to axisymmetric deformations. In preparation for these applications, we now specialize for axisymmetric problems the key equations of the previous sections, making use of coordinates (r,θ,z) in a cylindrical system, whose origin and z-axis coincide with the origin and the x_3-axis of the Cartesian system introduced in Section 2, and whose basis vectors are $(\boldsymbol{e}_r, \boldsymbol{e}_\theta, \boldsymbol{e}_z)$.

Let the cylindrical body $\mathcal{C}$ with circular mid-section $\mathcal{S}$ be composed of n layers, the k-th one of those occupying the sub-region

[4] We are grateful to an anonymous reviewer of a former version of this paper for raising this issue.

$\mathcal{C}^{(k)}$ of $\mathcal{C}$ between the planes $z = z^{(k-1)}$ and $z = z^{(k)}$. In the presence of axisymmetric deformations, the displacement components in the k-th layer of $\mathcal{C}$ are:

$$u_r^{(k)}(r,z) = \widetilde{u}^{(k)}(r) + (z - o^{(k)})\psi^{(k)}(r), \quad u_\theta^{(k)}(r,z) = 0, \quad u_z^{(k)}(r) = w^{(k)}(r). \tag{55}$$

In view of the restrictions following from the requested interlayer continuity of displacement and traction vectors, the nonzero components can be written in the form:

$$\begin{aligned} &u_r^{(k)}(r,z) = u(r) + (H^{(k)} + \chi^{(k)}z)\psi(r) + (H^{(k)} + (\chi^{(k)} - 1)z)w_{,r}(r), \\ &w^{(k)}(r) = w(r), \end{aligned} \tag{56}$$

where we denoted by u and ψ, respectively, the functions $\widetilde{u}_r^{(1)}$ and $\psi^{(1)}$, and where

$$\begin{aligned} &\chi^{(k)} = \frac{\mathbb{C}_{rzrz}^{(1)}}{\mathbb{C}_{rzrz}^{(k)}}, \quad k = 1, \dots, n; \quad H^{(1)} = 0, \\ &H^{(k)} = \sum_{i=1}^{k-1} z^{(i)}(\chi^{(i)} - \chi^{(i+1)}), \quad k = 2, \dots, n. \end{aligned} \tag{57}$$

The nonzero strain components in the layer k are:

$$\begin{aligned} &E_{rr}^{(k)} = (u_r^{(k)} + z\psi^{(k)})_{,r} = u_{,r} + (H^{(k)} + \chi^{(k)}z)\psi_{,r} + (H^{(k)} + (\chi^{(k)} - 1)z)w_{,rr}, \\ &E_{\theta\theta}^{(k)} = \frac{1}{r}\,(u_r^{(k)} + z\psi^{(k)}) = \frac{u}{r} + (H^{(k)} + \chi^{(k)}z)\frac{\psi}{r} + (H^{(k)} + (\chi^{(k)} - 1)z)\frac{w_{,r}}{r}, \\ &E_{rz}^{(k)} = \frac{1}{2}(w_{,r}^{(k)} + \psi^{(k)}) = \frac{1}{2}\,\chi^{(k)}(\psi + w_{,r}). \end{aligned} \tag{58}$$

Under the assumption that layers are made of a transversely isotropic material inextensible in the z-direction, the nonzero components of the active stress are:

$$\begin{aligned} &S_{rr}^{A(k)} = \mathbb{C}_{rrrr}^{(k)}E_{rr}^{(k)} + \mathbb{C}_{rr\theta\theta}^{(k)}E_{\theta\theta}^{(k)}, \quad S_{\theta\theta}^{A(k)} = \mathbb{C}_{\theta\theta rr}^{(k)}E_{rr}^{(k)} + \mathbb{C}_{\theta\theta\theta\theta}^{(k)}E_{\theta\theta}^{(k)}, \\ &S_{rz}^{A(k)} = 2\mathbb{C}_{rzrz}^{(k)}E_{rz}^{(k)}, \end{aligned} \tag{59}$$

with $\mathbb{C}_{\theta\theta\theta\theta}^{(k)} = \mathbb{C}_{rrrr}^{(k)}$. The definitions of stress resultants, both in a layer and in the whole plate, are completely analogous to those of Sub-Section 3.2. The nonzero stress resultants in the whole plate can be written in the form:

$$\begin{aligned} &N_{\mathfrak{ab}} = \int_{-h}^{h} S_{\mathfrak{ab}}^{A}\,dz = A_{\mathfrak{ab}rr}^{[1]}u_{,r} + A_{\mathfrak{ab}rr}^{[2]}\psi_{,r} + A_{\mathfrak{ab}rr}^{[3]}w_{,rr} + A_{\mathfrak{ab}\theta\theta}^{[1]}\frac{u}{r} + A_{\mathfrak{ab}\theta\theta}^{[2]}\frac{\psi}{r} + A_{\mathfrak{ab}\theta\theta}^{[3]}\frac{w_{,r}}{r}, \\ &Q_r = \int_{-h}^{h} S_{3r}^{A}\,dz = B(\psi + w_{,r}), \\ &M_{\mathfrak{ab}} = \int_{-h}^{h} S_{\mathfrak{ab}}^{A}z\,dz = D_{\mathfrak{ab}rr}^{[1]}u_r + D_{\mathfrak{ab}rr}^{[2]}\psi_{,r} + D_{\mathfrak{ab}rr}^{[3]}w_{,rr} + D_{\mathfrak{ab}\theta\theta}^{[1]}\frac{u}{r} + D_{\mathfrak{aa}\theta\theta}^{[2]}\frac{\psi}{r} + D_{\mathfrak{ab}\theta\theta}^{[3]}\frac{w_{,r}}{r}, \end{aligned} \tag{60}$$

in which

$$\begin{aligned} &A_{\mathfrak{abcd}}^{[1]} = 2\sum_{k=1}^{n} h^{(k)}\mathbb{C}_{\mathfrak{abcd}}^{(k)}, \quad A_{\mathfrak{abcd}}^{[2]} = 2\sum_{k=1}^{n} h^{(k)}(H^{(k)} + \chi^{(k)}o^{(k)})\mathbb{C}_{\mathfrak{abcd}}^{(k)}, \\ &A_{\mathfrak{abcd}}^{[3]} = 2\sum_{k=1}^{n} h^{(k)}(H^{(k)} + (\chi^{(k)} - 1)o^{(k)})\mathbb{C}_{\mathfrak{abcd}}^{(k)}, \end{aligned} \tag{61}$$

$$B = \sum_{k=1}^{n} 2h^{(k)}\chi^{(k)}\mathbb{C}_{rzrz}^{(k)} = 2h\mathbb{C}_{rzrz}^{(1)}, \tag{62}$$

$$\begin{aligned} D_{\mathfrak{abcd}}^{[1]} &= 2\sum_{k=1}^{n} o^{(k)}h^{(k)}\mathbb{C}_{\mathfrak{abcd}}^{(k)}, \quad D_{\mathfrak{abcd}}^{[2]} = 2\sum_{k=1}^{n}(o^{(k)}h^{(k)}H^{(k)} + (h^{(k)})^3/3 \\ &\quad + (o^{(k)})^2h^{(k)})\chi^{(k)})\mathbb{C}_{\mathfrak{abcd}}^{(k)}, D_{\mathfrak{abcd}}^{[3]} = 2\sum_{k=1}^{n}(o^{(k)}h^{(k)}H^{(k)} \\ &\quad + ((h^{(k)})^3/3 + (o^{(k)})^2h^{(k)})(\chi^{(k)} - 1))\mathbb{C}_{\mathfrak{abcd}}^{(k)}; \end{aligned} \tag{63}$$

indices $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \mathfrak{d}$ can be equal to r and θ, and are such that $\mathfrak{a} = \mathfrak{b}$ and $\mathfrak{c} = \mathfrak{d}$. Due to axial symmetry, the equilibrium plate equations in terms of stress resultants reduce to the following three:

$$\begin{aligned} &N_{rr,r} + \frac{1}{r}(N_{rr} - N_{\theta\theta}) + q_r = 0, \quad Q_{r,r} + \frac{1}{r}Q_r + q_z = 0, \\ &\quad M_{rr,r} + \frac{1}{r}(M_{rr} - M_{\theta\theta}) - Q_r + m = 0, \end{aligned} \tag{64}$$

where the loads per unit area of $\mathcal{S}$ are defined as

$$\begin{aligned} &q_r = \int_{-h}^{h} b_r dz + t_r^+ + t_r^-, \quad q_z = \int_{-h}^{h} b_z dz + t_z^+ + t_z^-, \\ &m = \int_{-h}^{h} q_r z dz + h(t_r^+ - t_r^-). \end{aligned} \tag{65}$$

Moreover, the first-order reactive stress $\boldsymbol{S}^R$ and the reactive hyperstress $\mathbb{S}^R$ reduce to:

$$\begin{aligned} \boldsymbol{S}^R &= \sigma\boldsymbol{e}_z \otimes \boldsymbol{e}_z, \quad \mathbb{S}^R = \gamma_r\boldsymbol{e}_r \otimes \boldsymbol{e}_z \otimes \boldsymbol{e}_z + \gamma_z\boldsymbol{e}_z \otimes \boldsymbol{e}_z \otimes \boldsymbol{e}_z \\ &\quad + \eta_r\boldsymbol{e}_z \otimes (\boldsymbol{e}_r \otimes \boldsymbol{e}_z + \boldsymbol{e}_z \otimes \boldsymbol{e}_r); \end{aligned} \tag{66}$$

the divergence of $\mathbb{S}^R$ is:

$$\mathrm{Div}\,\mathbb{S}^R = \gamma_{r,z}\boldsymbol{e}_r \otimes \boldsymbol{e}_z + \eta_{r,z}\boldsymbol{e}_z \otimes \boldsymbol{e}_r + (\gamma_{z,z} + \eta_{r,r} + \eta_r/r)\boldsymbol{e}_z \otimes \boldsymbol{e}_z \tag{67}$$

and the reactive stress $\boldsymbol{T}^R$ has components:

$$[T_{ij}^R] = \begin{bmatrix} 0 & 0 & -\gamma_{r,z} \\ 0 & 0 & 0 \\ -\eta_{r,z} & 0 & \sigma - \gamma_{z,z} - \eta_{r,r} - \eta_r/r \end{bmatrix}. \tag{68}$$

When $\mathcal{C}$ is regarded as a three-dimensional body made of a second-grade material, in which the hyperstress is purely reactive, subject to null couple per unit area $\boldsymbol{p}$ and to null forces per unit length $\boldsymbol{l}$, the governing balance equations and boundary conditions are:

$$\begin{aligned} &\gamma_{r,zz}\boldsymbol{e}_r + (\gamma_{z,z} + 2(\eta_{r,r} + \eta/r) - \sigma)_{,z}\boldsymbol{e}_z = \mathrm{Div}\boldsymbol{S}^A + \boldsymbol{b}, \ \text{in}\ \mathcal{C}, \\ &(\gamma_{r,z})^{\pm}\boldsymbol{e}_r + (\gamma_{z,z} + \eta_{r,r} + \eta_r/r - \sigma)^{\pm}\boldsymbol{e}_z = (\boldsymbol{S}^A\boldsymbol{e}_z)^{\pm} \mp \boldsymbol{t}^{\pm}, \ \text{on}\ \mathcal{S}^+ \ \text{and}\ \mathcal{S}^-, \\ &2\eta_{r,z}n_r = S_{zr}n_r - t_z, \ \text{on}\ \partial\mathcal{S} \times (-h, h), \\ &\gamma_r^{\pm} = \gamma_z^{\pm} = 0, \ \text{on}\ \mathcal{S}^+ \ \text{and}\ \mathcal{S}^-, \\ &\eta_r n_r = 0, \ \text{on}\ \partial\mathcal{S}^+ \ \text{and}\ \partial\mathcal{S}^-. \end{aligned} \tag{69}$$

We assume that $\eta_{r,z}(r,z) = \gamma_{r,z}(r,z)$, which assures the symmetry of the stress $\boldsymbol{T}^R$, and we take

$$\sigma(r) = \frac{1}{h}\int_{-h}^{h} \eta_{r,r}(r,z)\,dz. \tag{70}$$

By integrating Eqs. $(69)_1$ under conditions $(69)_2$ and $(69)_4$, we obtain

$$\begin{aligned} &T_{rz}^{R(1)} = -\gamma_{r,z}^{(1)} = -(\boldsymbol{S}^A\boldsymbol{e}_z + \boldsymbol{t})_r^- - \int_{-h}^{z}(\mathrm{Div}\boldsymbol{S}^A + \boldsymbol{b})_r d\zeta, \\ &\gamma_r^{(1)} = (z + h)(\boldsymbol{S}^A\boldsymbol{e}_z + \boldsymbol{t})_r^- + \int_{-h}^{z}(z - \zeta)(\mathrm{Div}\boldsymbol{S}^A + \boldsymbol{b})_r d\zeta, \\ &T_{zz}^{R(1)} = \sigma - \gamma_{z,z}^{(1)} - \eta_{r,r}^{(1)} - \eta_r^{(1)}/r = \gamma_{r,r}^{(1)} + \gamma_r^{(1)}/r - (\boldsymbol{S}^A\boldsymbol{e}_z + \boldsymbol{t})_z^- \\ &\qquad - \int_{-h}^{z}(\mathrm{Div}\boldsymbol{S}^A + \boldsymbol{b})_z d\zeta, \end{aligned} \tag{71}$$

for the first layer ($k = 1$), and

$$\begin{aligned} &T_{rz}^{R(k)} = -\gamma_{r,z}^{(k)} = -\gamma_{r,z}^{(k-1)}|_{z=z^{(k-1)}} - \int_{z^{(k-1)}}^{z}(\mathrm{Div}\boldsymbol{S}^A + \boldsymbol{b})_r d\zeta, \\ &\gamma_r^{(k)} = \gamma_r^{(k-1)}|_{z=z^{(k-1)}} + (z - z^{(k-1)})\gamma_{r,z}^{(k-1)}|_{z=z^{(k-1)}} + \int_{z^{(k-1)}}^{z}(z - \zeta)(\mathrm{Div}\boldsymbol{S}^A + \boldsymbol{b})_r d\zeta, \\ &T_{zz}^{R(k)} = \left(T_{zz}^{R(k-1)} - \eta_{r,r}^{(k-1)} - \frac{\eta_r^{(k-1)}}{r}\right)_{z=z^{(k-1)}} + \eta_{r,r}^{(k)} - \int_{z^{(k-1)}}^{z}(\mathrm{Div}\boldsymbol{S}^A + \boldsymbol{b})_z d\zeta, \end{aligned} \tag{72}$$

for all other layers ($k > 1$).

As it has been done in Section 4.1 in the case of a generic cross-section, the layers of axisymmetric circular plates that are also mid-plane symmetric are labeled in pairs, with the label $k=1$ assigned to the pair adjacent to the mid-plane. The additional symmetry implies that the expressions (60)$_1$ and (60)$_3$ for membrane forces and moments become simpler:

$$
\begin{aligned}
N_{\mathfrak{ab}} &= \int_{-h}^{h} S^{A}_{\mathfrak{ab}} dz = 2\overline{A}^{[1]}_{\mathfrak{ab}rr} u_{,r} + 2\overline{A}^{[1]}_{\mathfrak{ab}\theta\theta} \frac{u}{r},\\
M_{\mathfrak{ab}} &= \int_{-h}^{h} S^{A}_{\mathfrak{ab}} z dz = \overline{D}^{[2]}_{\mathfrak{aa}rr} \psi_{,r} + \overline{D}^{[3]}_{\mathfrak{aa}rr} w_{,rr} + \overline{D}^{[2]}_{\mathfrak{aa}\theta\theta} \frac{\psi}{r} + \overline{D}^{[3]}_{\mathfrak{aa}\theta\theta} \frac{w_r}{r};
\end{aligned}
\tag{73}
$$

with

$$
\begin{aligned}
&\overline{A}^{[1]}_{\mathfrak{abcd}} = 4\sum_{k=1}^{n} h^{(k)} \mathbb{C}^{(k)}_{\mathfrak{abcd}}, \quad \overline{D}^{[2]}_{\mathfrak{abcd}} = 4\sum_{k=1}^{n} (\overline{o}^{(k)} h^{(k)} \overline{H}^{(k)} + ((h^{(k)})^3/3 + (\overline{o}^{(k)})^2 h^{(k)})\chi^{(k)}) \mathbb{C}^{(k)}_{\mathfrak{abcd}},\\
&\overline{D}^{[3]}_{\mathfrak{abcd}} = 4\sum_{k=1}^{n} (\overline{o}^{(k)} h^{(k)} \overline{H}^{(k)} + ((h^{(k)})^3/3 + (\overline{o}^{(k)})^2 h^{(k)})(\chi^{(k)} - 1)) \mathbb{C}^{(k)}_{\mathfrak{abcd}};
\end{aligned}
\tag{74}
$$

indices $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \mathfrak{d}$ can be equal to r and θ, and are such that $\mathfrak{a} = \mathfrak{b}$ and $\mathfrak{c} = \mathfrak{d}$. For this class of plate problems, as a glance to the resultants in (73) makes clear, the membranal and flexural equilibrium Eqs. (64)$_{2,3}$ and (64)$_1$ decouple. Once again, it is important to keep in mind that formulae (71) and (72) for the reactive stresses have to be modified to take account of the different layer numeration.

5. Examples

In this section, we apply our plate model to study some equilibrium problems for a multilayered plate-like body $\mathcal{C}$ having rectangular or circular cross-section. For each of these problems, the plate solution is used to construct the three-dimensional active stress field in $\mathcal{C}$, which is improved by addition of the reactive stress field. Then, the total stress is compared with the stress of the exact three-dimensional Levinson-type solution (the deduction of the Levinson solutions we exploit has been presented in Formica et al. (2013)). In all examples, the only external loads acting on $\mathcal{C}$ are surface loads on the end faces. In the case of a circular cross-section, we also apply those radial tractions on the mantel that are needed to guarantee the existence of a Levinson-type solution and, for simplicity, we restrict attention to axisymmetric loadings and deformations.

5.1. Plate-like bodies with rectangular cross-section

Let the cross-section of $\mathcal{C}$ be a rectangle with sides parallel to the axes x_1 and x_2, of length l_1 and l_2, and let $\mathcal{C}$ consist of either two, three, or five layers of equal thickness. In the last two cases, let the layers symmetrically located with respect to the mid-plane of $\mathcal{C}$ be formed by the same transversely isotropic material; thus, according to the definitions of Section 4.1, $\mathcal{C}$ is a mid-plane symmetric multilayered plate-like body made of $2n$ layers, with n equal to 2 and 3, respectively. Furthermore, let the boundary conditions on the mantel of $\mathcal{C}$ be of the type assumed in the Levinson problem (Levinson, 1985), namely, such that both transverse and tangential displacements are prevented and normal tractions are null: that is to say that, on the sides whose normal is parallel to $\boldsymbol{e}_1$, the geometric boundary conditions are $w=0$ and, in addition, $\widetilde{u}_2 = 0$ and $\psi_2 = 0$, while the natural conditions are that both the membrane force N_{11} and the bending moment M_{11} vanish; and that, on the sides whose normal is parallel to $\boldsymbol{e}_2$, analogous conditions prevail, modulo the interchange of the index values. All in all, no matter the layer number, the boundary conditions are:

$$
\begin{aligned}
&\text{for } x_1 = 0 \text{ and } x_1 = l_1: \quad \widetilde{u}_2 = 0,\ w = 0,\ \psi_2 = 0,\ N_{11} = 0,\ M_{11} = 0;\\
&\text{for } x_2 = 0 \text{ and } x_2 = l_2: \quad \widetilde{u}_1 = 0,\ w = 0,\ \psi_1 = 0,\ N_{22} = 0,\ M_{22} = 0.
\end{aligned}
\tag{75}
$$

The external loads consist in normal tractions applied to $\mathcal{S}^{\pm}$, having the following expressions:

$$
\begin{aligned}
t^{+}_{3pq}(x_1, x_2) &= \sigma^{+}_{pq} \sin\frac{p\pi x_1}{l_1} \sin\frac{q\pi x_2}{l_2},\\
t^{-}_{3pq}(x_1, x_2) &= \sigma^{-}_{pq} \sin\frac{p\pi x_1}{l_1} \sin\frac{q\pi x_2}{l_2},
\end{aligned}
\tag{76}
$$

in which σ^{+}_{pq} and σ^{-}_{pq} are constants. These tractions can be thought of as terms of series expansions of loads acting on the end faces of $\mathcal{C}$; the corresponding plate loads are:

$$
\begin{aligned}
q_{3pq}(x_1, x_2) &= q_{pq} \sin\frac{p\pi x_1}{l_1} \sin\frac{q\pi x_2}{l_2}\\
&= (\sigma^{+}_{pq} + \sigma^{-}_{pq}) \sin\frac{p\pi x_1}{l_1} \sin\frac{q\pi x_2}{l_2}.
\end{aligned}
\tag{77}
$$

The unknown functions $\widetilde{u}_1, \widetilde{u}_2, w, \psi_1$, and ψ_2 are assumed to have the representations:

$$
\begin{aligned}
&\widetilde{u}_1(x_1, x_2) = \widetilde{u}_{1pq} \cos\frac{p\pi x_1}{l_1} \sin\frac{q\pi x_2}{l_2}, \quad \widetilde{u}_2(x_1, x_2) = \widetilde{u}_{2pq} \sin\frac{p\pi x_1}{l_1} \cos\frac{q\pi x_2}{l_2},\\
&w(x_1, x_2) = w_{pq} \sin\frac{p\pi x_1}{l_1} \sin\frac{q\pi x_2}{l_2},\\
&\psi_1(x_1, x_2) = \psi_{1pq} \cos\frac{p\pi x_1}{l_1} \sin\frac{q\pi x_2}{l_2}, \quad \psi_2(x_1, x_2) = \psi_{2pq} \sin\frac{p\pi x_1}{l_1} \cos\frac{q\pi x_2}{l_2},
\end{aligned}
\tag{78}
$$

where $\widetilde{u}_{1pq}, \widetilde{u}_{2pq}, w_{pq}, \psi_{1pq}$, and ψ_{2pq} are coefficients to be determined. Introduction of (78) into (37) produces a linear algebraic system for the unknown coefficients in (78), that can be given the form:

$$
\begin{bmatrix}
A_{11} & A_{12} & A_{13} & A_{14} & A_{15}\\
A_{21} & A_{22} & A_{23} & A_{24} & A_{25}\\
A_{31} & A_{32} & A_{33} & A_{34} & A_{35}\\
A_{41} & A_{42} & A_{43} & A_{44} & A_{45}\\
A_{51} & A_{52} & A_{53} & A_{54} & A_{55}
\end{bmatrix}
\begin{bmatrix}
\widetilde{u}_{1pq}\\ \widetilde{u}_{2pq}\\ w_{pq}\\ \psi_{1pq}\\ \psi_{2pq}
\end{bmatrix}
=
\begin{bmatrix}
0\\ 0\\ -q_{pq}\\ 0\\ 0
\end{bmatrix}.
\tag{79}
$$

Then, the total stresses are evaluated and compared with those furnished by the exact three-dimensional Levinson-type solution of the same problem, as given in Formica et al. (2013).

In the three examples we consider, the thickness of $\mathcal{C}$ is $2h = 100$ mm, the lengths of the sides of $\mathcal{S}$ are $l_1 = 1500$ mm and $l_2 = 2000$ mm, and the values of the elastic moduli of a layer are always taken such that

$$
\overline{E} = 0.75 \times E, \quad \nu = \overline{\nu} = 0.25, \quad \overline{G} = 0.25 \times E.
\tag{80}
$$

With this choice, specification of the Young modulus E for in-plane directions suffices to define all the moduli of a layer. In Figs. 1–5, exact solutions are plotted with continuous lines, our model with dotted lines; dashed lines represent the constant active shear stresses corresponding to the plate solutions.

5.1.1. Two-layer rectangular plate

Layers are assumed to have the same thickness, and are numbered from the bottom to top. The Young moduli of the layers for the in-plane directions are $E^{(1)} = 1.7 \times 10^5$ N/mm^2 and $E^{(2)} = E^{(1)}/25$. The loads applied to $\mathcal{C}$ are those of Eqs. (76) with $p = q = 1$ and $\sigma^{+}_{pq} = \sigma^{-}_{pq} = 10$ N/mm^2. Plots in Fig. 1 give the stresses along the fiber parallel to x_3 intersecting $\mathcal{S}$ at $(x_1, x_2) = (l_1/3, l_2/3)$; they show a fairly good agreement between the predictions of our model and the exact solution.

5.1.2. Three-layer, mid-plane symmetric rectangular plate

The mid-plane symmetric body $\mathcal{C}$ is formed by two layers labeled by the number 1 and located between the planes $x_3 = 0$ and $x_3 = \pm h/3$, and two layers labeled by the number 2 and located between the planes $x_3 = \pm h/3$ and $x_3 = \pm h$ (thus, $\mathcal{C}$ can also be regarded as formed by three layers of equal thickness $2h/3$). The

Young moduli of the layers for the in-plane directions are $E^{(1)} = 1.7 \times 10^5$ N/mm^2 and $E^{(2)} = E^{(1)}/25$. The external load are those of Eqs. (76), with $p = q = 1$ and $\sigma^+_{pq} = \sigma^-_{pq} = 10$ N/mm^2. Results are presented in Fig. 2, where stresses on the fiber parallel to x_3 intersecting $\mathcal{S}$ at $(x_1, x_2) = (l_1/3, l_2/3)$ are plotted; again, our model's predictions turn out to be fairly accurate.

5.1.3. Five-layer, mid-plane symmetric rectangular plate

This example considers a mid-plane symmetric body $\mathcal{C}$ that is formed by two layers labeled by the number 1 and located between the planes $x_3 = 0$ and $x_3 = \pm h/5$, two layers labeled by the number 2 and located between the planes $x_3 = \pm h/5$ and $x_3 = \pm 3h/5$, and two layers labeled by the number 3 located between the planes $x_3 = \pm 3h/5$ and $x_3 = \pm h$ (thus, $\mathcal{C}$ can also be regarded as formed by five layers of equal thickness $2h/5$). The Young moduli of the layers for the in-plane directions are $E^{(1)} = 1.7 \times 10^5$ N/mm^2, $E^{(2)} = E^{(1)}/25$, and $E^{(3)} = E^{(1)}$. The loads applied to $\mathcal{C}$ have the expressions of Eqs. (76), with $p = q = 1$ and $\sigma^+_{pq} = \sigma^-_{pq} = 10$ N/mm^2. In Fig. 3, the stresses on the fiber parallel to x_3 and intersecting $\mathcal{S}$ at $(x_1, x_2) = (l_1/3, l_2/3)$ are plotted; they show the good accuracy achieved by the model. Some small differences appear on the external layers for the in-plane stresses, which are all active in nature and arise from to the plate solution, because there are no corresponding reactive stresses.

5.2. Plate-like bodies with circular cross-section

We consider two examples of axisymmetric deformations of mid-plane symmetric plate-like bodies $\mathcal{C}$ with circular cross-section. As observed at the end of Section 3.3, in such cases the equations describing membranal and flexural deformations decouple. We assume that the external loads applied to the plate-like body at $\mathcal{S}^\pm$ are:

$$t^+_z(r) = \sigma^+_m J_0(\kappa_m r/R), \quad t^-_z(r) = \sigma^-_m J_0(\kappa_m r/R), \tag{81}$$

where σ^+_m and σ^-_m are constant, J_0 is the Bessel function of the first kind and order 0, and κ_m are the positive zeros of J_0. The loads (81) represent terms of series expansions of loads acting on the end faces

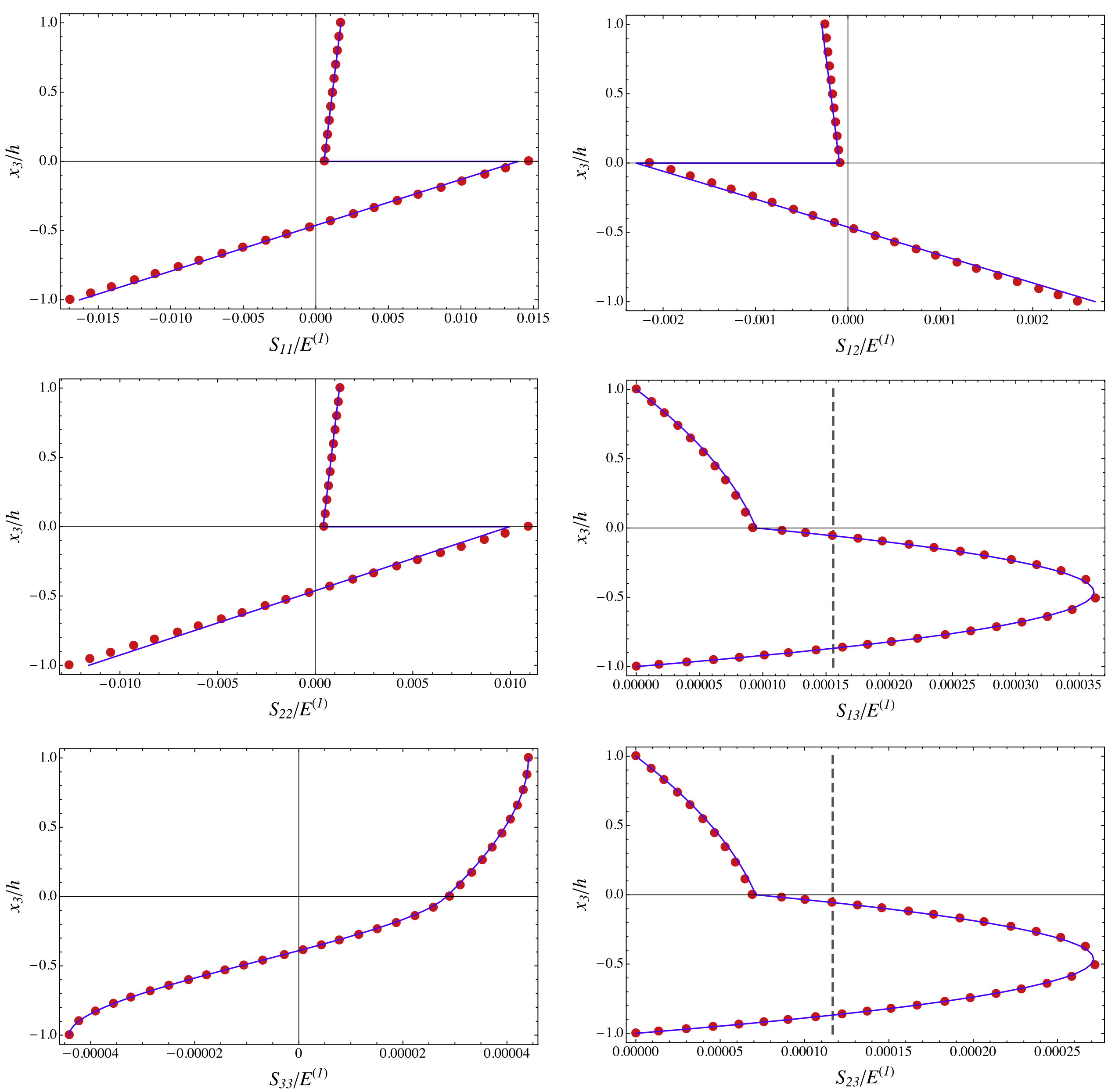

Fig. 1. (Two-layer rectangular plate) Non-dimensional stresses along the transverse fiber at $x_1/l_1 = 1/3$, $x_2/l_2 = 1/3$.

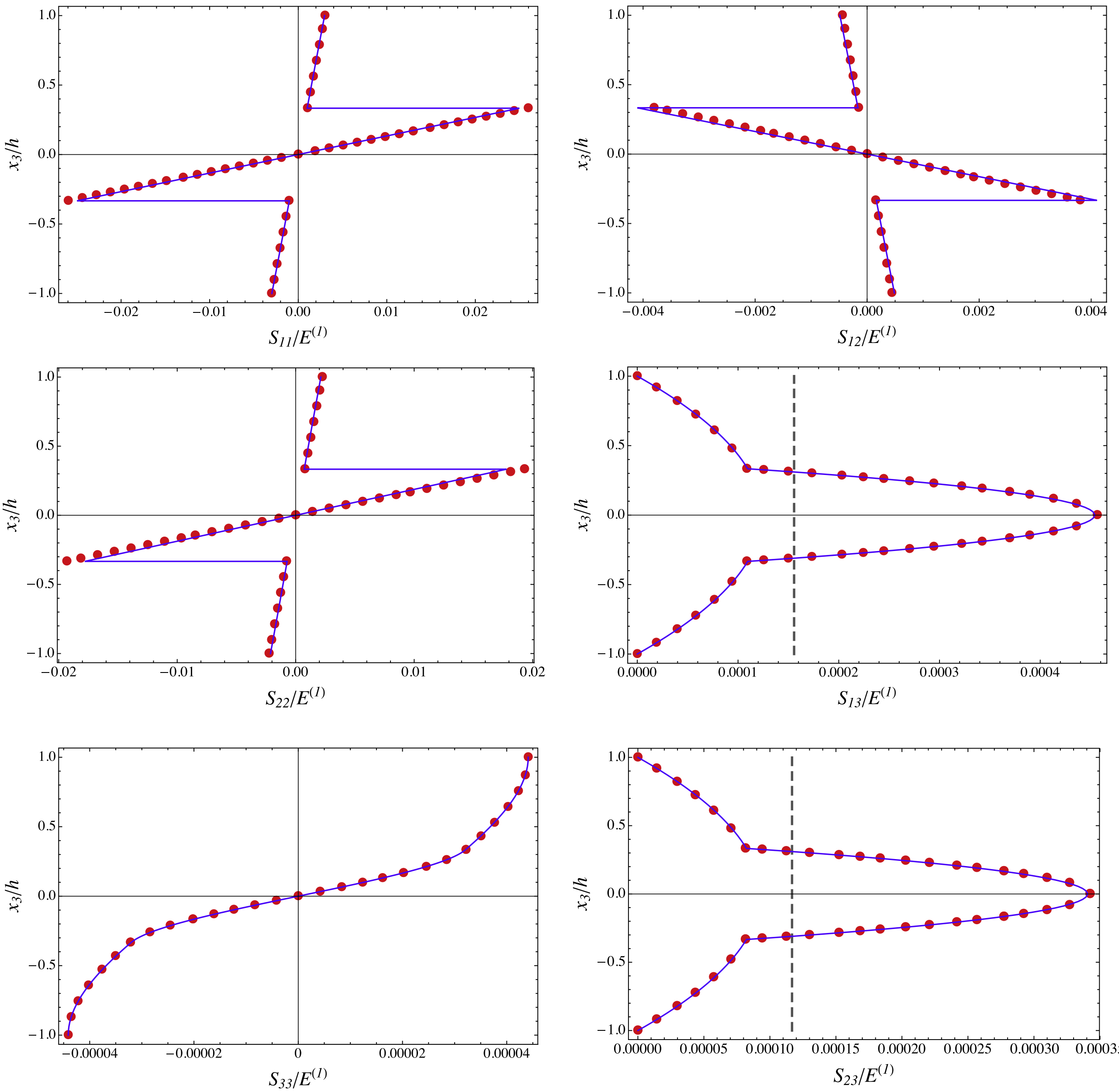

Fig. 2. (Three-layers rectangular plate) Non-dimensional stresses along the transverse fiber at $(x_1/l_1 = 1/3, x_2/l_2 = 1/3)$.

of $\mathcal{C}$; they are accompanied, on the mantel of $\mathcal{C}$, by those normal tractions that are necessary for the equilibrium problem of plate-like bodies with circular cross-section to have a solution of Levinson type (cf. Nicotra et al., 1999; Formica et al., 2013); we recall that such normal tractions have null resultant and non-null resultant moment $\mathcal{M}_m$ along each segment parallel to the axis of $\mathcal{C}$. The loads (81) produce the plate load q_z per unit area,

$$q_z = q_m J_0(\kappa_m r/R) = (\sigma_m^+ + \sigma_m^-) J_0(\kappa_m r/R). \tag{82}$$

The solution of the plate problem can be determined as follows (cf. Formica et al., 2011). Equilibrium is governed by equations $(64)_{2,3}$, that, making use of Eqs. (60), can be written in terms of the functions w and ψ:

$$Br^{-1}(r(w_r+\psi))_r + q_z = 0, \quad \overline{D}^{[2]}_{rrrr}(r^{-1}(r(w_r+\psi))_r)_r \\ + (\overline{D}^{[3]}_{rrrr} - \overline{D}^{[2]}_{rrrr})(r^{-1}(rw_r)_r)_r - B(w_r+\psi) = 0. \tag{83}$$

It follows from Eq. $(83)_1$ that

$$w_{,r} + \psi = \frac{1}{Br}\int_0^r q_z \rho\, d\rho; \tag{84}$$

substitution of this in Eq. $(83)_1$ yields

$$(r^{-1}(rw_r)_r)_r = \frac{1}{(\overline{D}^{[3]}_{rrrr} - \overline{D}^{[2]}_{rrrr})r}\int_0^r q_z\rho\, d\rho + \frac{\overline{D}^{[2]}_{rrrr}}{b(\overline{D}^{[3]}_{rrrr} - \overline{D}^{[2]}_{rrrr})} q_{z,r}. \tag{85}$$

The general solution of the homogeneous equation associated with (85) is

$$w(r) = c_1 + c_2 r^2 + c_3 \ln r, \tag{86}$$

where c_3 must be zero to have a finite displacement at $r = 0$. When the load per unit area q_z has the expression (82), Eq. (85) has the particular solution

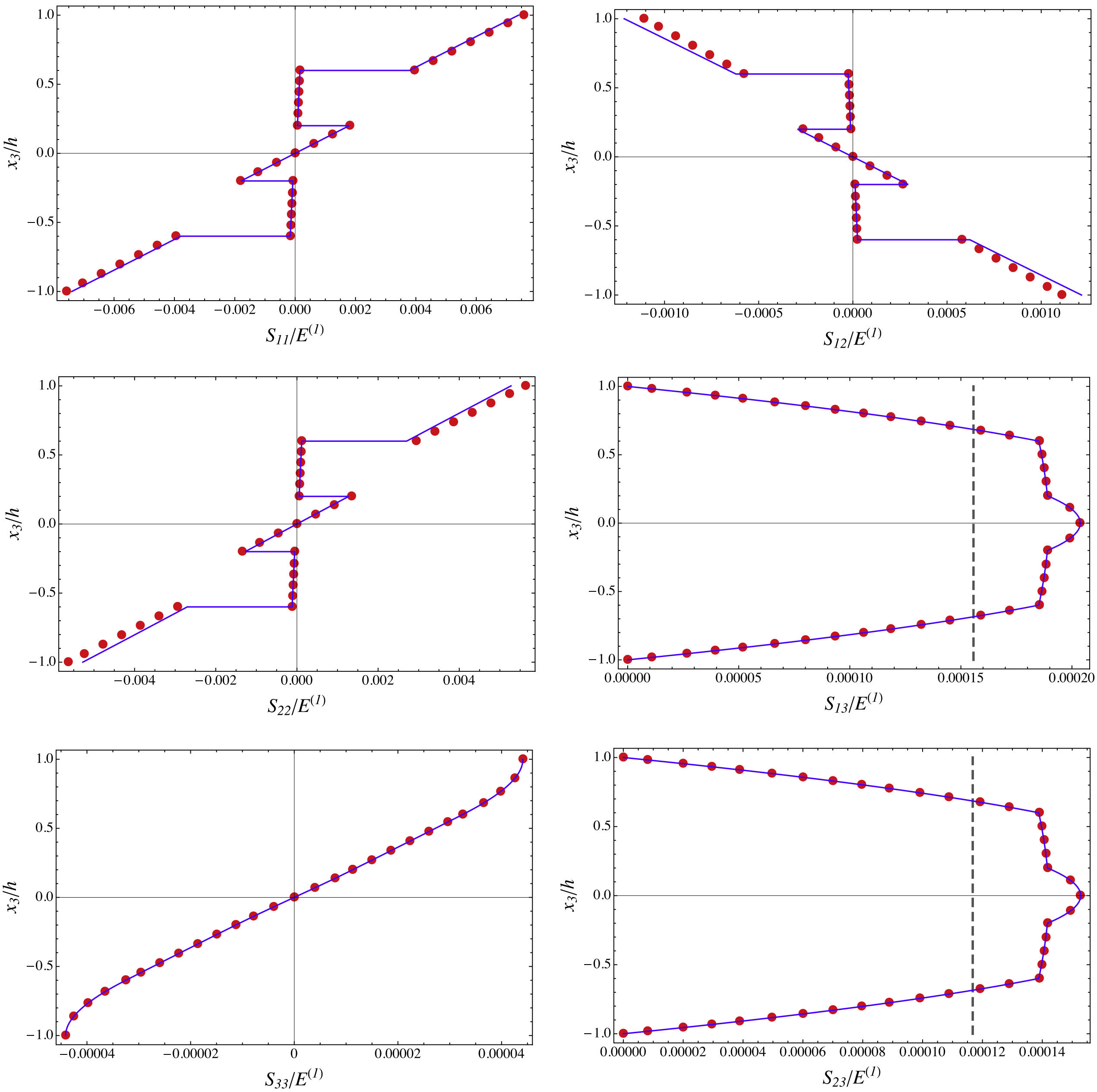

Fig. 3. (Five-layer rectangular plate) Non-dimensional stresses along the transverse fiber at $(x_1/l_1 = 1/3, x_2/l_2 = 1/3)$.

$$w(r) = \frac{q_m}{(\overline{D}^{[3]}_{rrrr} - \overline{D}^{[2]}_{rrrr})}\left(\frac{\overline{D}^{[2]}_{rrrr}\kappa_m}{BR} + \frac{R}{\kappa_m}\right)\left(\frac{R}{\kappa_m}\right)^3 J_0(\kappa_m r/R). \tag{87}$$

Thus, the general solution of (85) is known, and it follows from $(83)_1$ that

$$\psi(r) = -2c_2 r - q_m\left(\frac{R^2}{(\overline{D}^{[3]}_{rrrr} - \overline{D}^{[2]}_{rrrr})\kappa_m^2}\left(\frac{\overline{D}^{[2]}_{rrrr}\kappa_m}{BR} + \frac{R}{\kappa_m}\right) - \frac{R}{B\kappa_m}\right)J_1(\kappa_m r/R), \tag{88}$$

with J_1 the Bessel function of the first kind and order 1, and where the integration constants c_1 and c_2 are determined from the conditions $w(R) = 0$ e $M_{rr}(R) = \mathcal{M}_m$.

In both examples we present, the cross-section of $\mathcal{C}$ has radius $R = 1000$ mm and the thickness is $2h = 100$ mm.

5.2.1. Three-layer, mid-plane symmetric circular plate

Here $\mathcal{C}$ consists of two layers labeled by the number 1 and located between the planes $z = 0$ and $z = \pm h/3$, and two layers labeled by the number 2 and located between the planes $z = \pm h/3$ and $z = \pm h$ (so that $\mathcal{C}$ can be regarded as formed by three layers of equal thickness $2h/3$). The Young moduli of the layers for the in-plane directions are $E^{(1)} = 1.7 \times 10^5$ N/mm^2 and $E^{(2)} = E^{(1)}/25$. The external loads on $\mathcal{C}$ are those of Eqs. (81) with $m = 1$ and $\sigma_m^+ = \sigma_m^- = 10$ N/mm^2; the external loads on the plate are: the surface load q_z, given by Eq. (82) and acting on $\mathcal{S}$, and the bending moment $\mathcal{M}_m$, acting on $\partial\mathcal{S}$ and due to the tractions necessary on the mantel of $\mathcal{C}$ to have a Levinson-type exact solution. In Fig. 4 a comparison of the stresses on a fiber parallel to z and intersecting $\mathcal{S}$ at $r = R/2$ is given; it shows that the results of our model are in very good agreement with those of the exact solution.

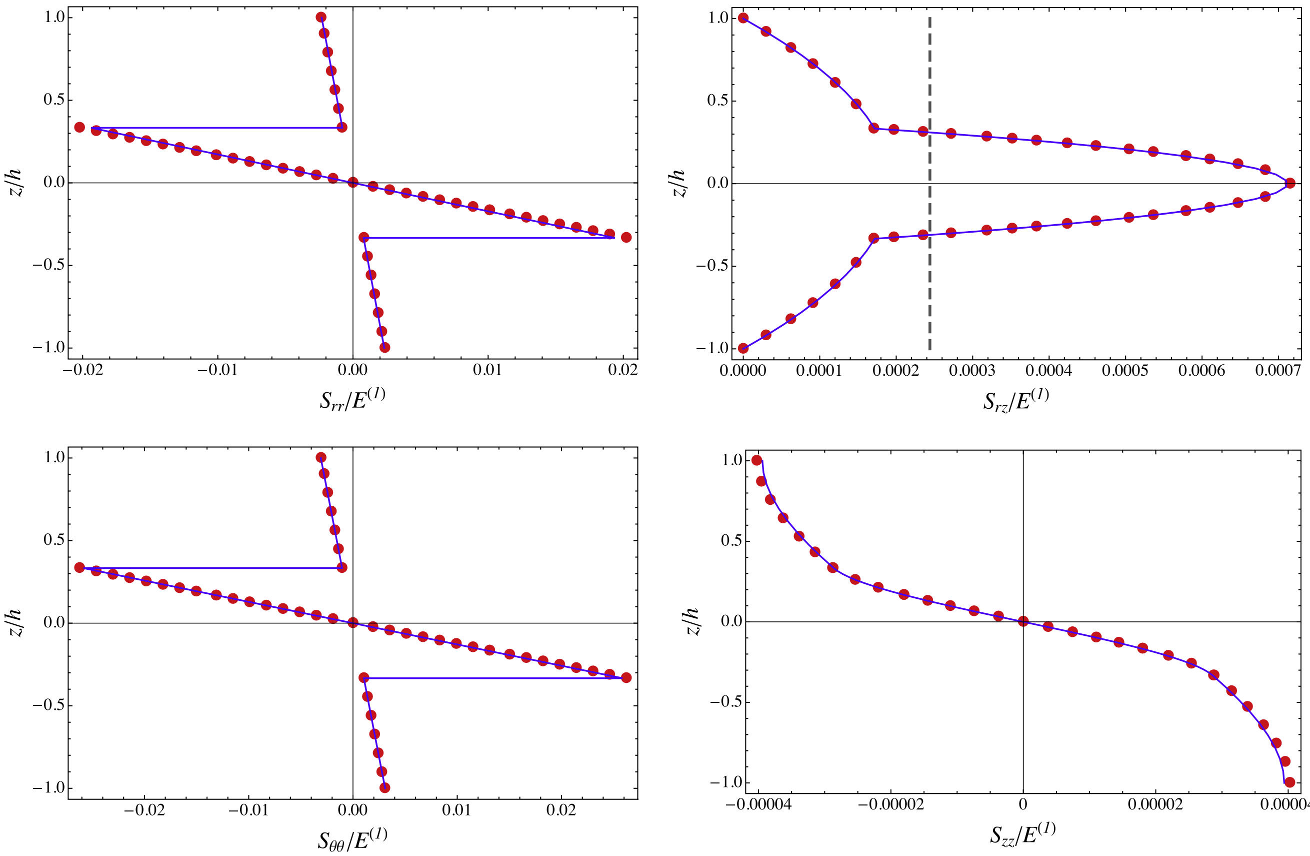

Fig. 4. (Three-layer circular plate) Non-dimensional stresses along the transverse fiber at $r/R = 1/2$.

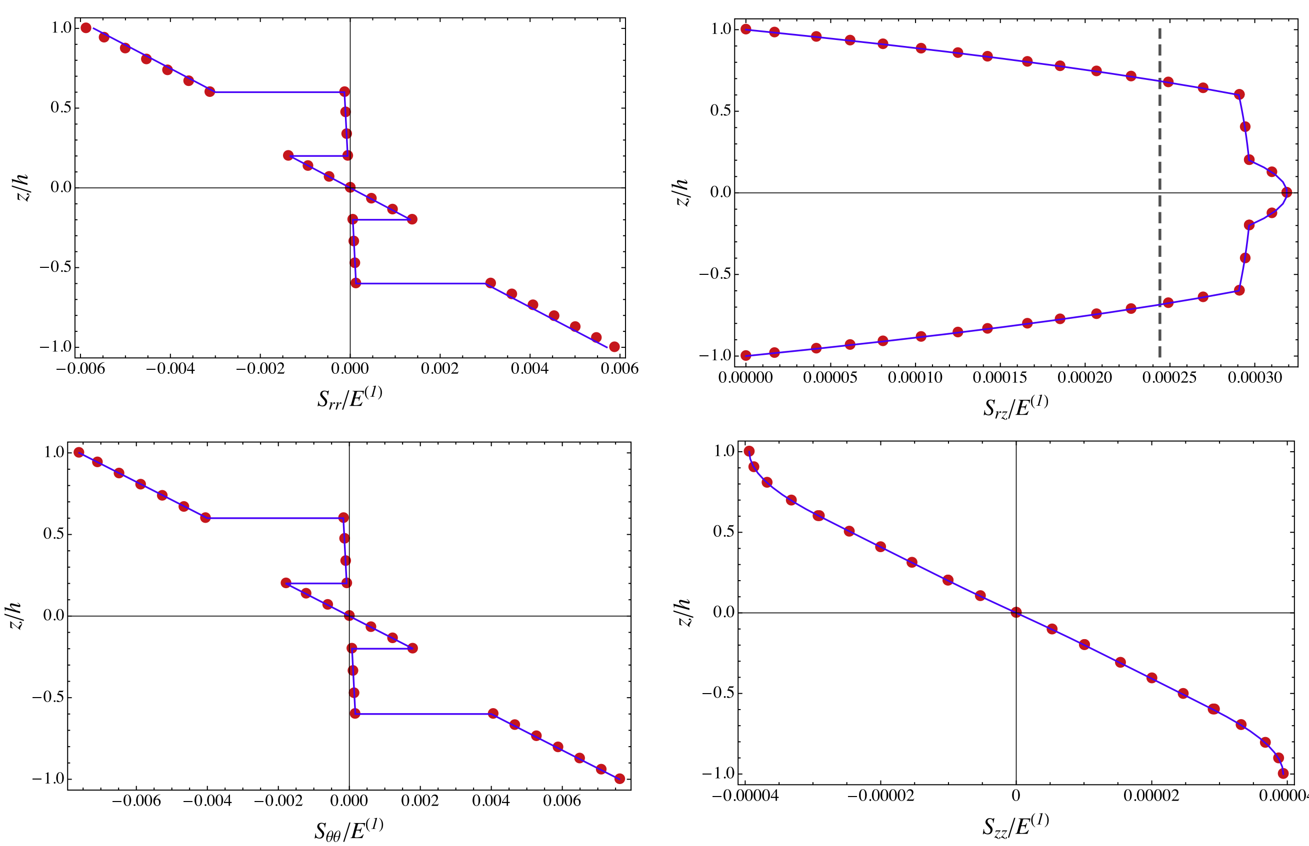

Fig. 5. (Five-layer circular plate) Non-dimensional stresses along the transverse fiber at $r/R = 1/2$.

5.2.2. Five-layer, mid-plane symmetric circular plate

In this example, $\mathcal{C}$ is composed of two layers labeled by the number 1 and located between the planes $z = 0$ and $z = \pm h/5$; two layers, labeled by the number 2 and located between the planes $z = \pm h/5$ and $z = \pm 3h/5$, and two layers labeled by the number 3 and located between the planes $z = \pm 3h/5$ and $z = \pm h$ (so that $\mathcal{C}$ can be regarded as formed by five layers of equal thickness $2h/5$). The Young moduli of the layers for the in-plane directions are $E^{(1)} = 1.7 \times 10^5\ \mathrm{N/mm}^2$, $E^{(2)} = E^{(1)}/25$, and $E^{(3)} = E^{(1)}$. The external loads on $\mathcal{C}$ are those of Eq. (81), with $m = 1$ and $\sigma_m^+ = \sigma_m^- = 10\ \mathrm{N/mm}^2$; the external loads on the plate are: the surface load q_z, given by Eq. (82) and acting on $\mathcal{S}$, and the bending moment $\mathcal{M}_m$, acting on $\partial\mathcal{S}$ and due to the tractions on the mantel of $\mathcal{C}$ necessary in a Levinson-type exact solution. Fig. 5 gives a comparison of the stresses on a fiber parallel to z intersecting $\mathcal{S}$ at $r = R/2$, and shows that the accuracy of the results of our model is again good.

References

Altenbach, H., Altenbach, J., Naumenko, K., 1998. Ebene Flächentragwerke. Springer-Verlag, Berlin.
Bert, C.W., 1984. A critical evaluation of new plate theories applied to laminated composite. Compos. Struct. 2, 329–347.
Carrera, E., 2000. An assessment of mixed and classical theories on global and local response of multilayered orthotropic plates. Compos. Struct. 50, 183–198.
Carrera, E., 2002. Theories and finite elements for multilayered, anisotropic, composite plates and shells. Arch. Comput. Methods Eng. 9, 87–140.
Cho, M., Oh, J., 2004. Higher order zig-zag theory for fully coupled thermo-electric-mechanical smart composite plates. Int. J. Solids Struct. 41, 1331–1356.
Cosserat, E., Cosserat, F., 1909. Théorie des corps déformables. A. Herrmann et Fils, Paris.
DiCarlo, A., Podio-Guidugli, P., Williams, W.O., 2001. Shell with thickness distension. Int. J. Solids Struct. 38, 1201–1225.
DiSciuva, M., 1984. A refinement of the transverse shear deformation theory for multilayered orthotropic plates. L'Aerotecnica, Missili e Spazio 63, 84–92.
Formica, G., Lembo, M., Podio-Guidugli, P., 2011. In-plane strain and stress fields in theories of shearable laminated plates subject to transverse loads. In: Altenbach, H., Emereyev, V.A. (Eds.), Shell-like Structures – Non Classical Theories and Applications. Springer, Berlin, pp. 699–714.
Formica, G., Lembo, M., Podio-Guidugli, P., 2013. Explicit and exact Levinson-type solutions for multilayered plates. Arch. Mech. 65 (5), 445–466.
Gol'denveizer, A.L., 1962. Derivation of an approximate theory of bending of a plate by the method of asymptotic integrations of the equations of the theory of elasticity. J. Appl. Math. Mech. 26, 1000–1025.
Goodier, J.N., 1938. On the problems of the beam and the plate in the theory of elasticity. Trans. R. Soc. Can. 32, 65–88, Se.III.
Green, A.E., 1963. Boundary layer equations in the theory of thin elastic shells. Proc. R. Soc. London Ser. A 269, 481–491.
Kienzler, R., 2002. On consistent plate theories. Arch. Appl. Mech. 72, 229–247.
Koiter, W.T., Simmonds, J.G., 1972. Foundations of shell theory. Department of Mechanical Engineering, Delft University of Technology, Report Nr.473, Delft.
Lebée, A., Sab, K., 2011. A bending-gradient model for thick plates, part I: theory. Int. J. Solids Struct. 65, 2878–2888.
Lee, K.H., Senthilnathan, N.R., Lim, S.P., Chow, S.T., 1990. An improved zig-zag model for the bending of laminated composite plates. Compos. Struct. 15, 137–148.
Lembo, M., Podio-Guidugli, P., 1991. Plate theory as an exact consequence of three-dimensional linear elasticity. Eur. J. Mech. A/Solids 10, 485–516.
Lembo, M., Podio-Guidugli, P., 2000. Second-gradient constraints in second-grade materials. In: K. Wilmanski's 60th Anniversary Volume, W.I.A.S. Berlin, pp. 200–206.
Lembo, M., Podio-Guidugli, P., 2001. Internal constraints, reactive stresses, and the Timoshenko beam theory. J. Elast. 65, 249–263.
Lembo, M., Podio-Guidugli, P., 2007. How to use reactive stresses to improve plate-theory approximations of the stress field in a linearly elastic plate-like body. Int. J. Solids Struct. 44, 1337–1369.
Levinson, M., 1985. The simply supported rectangular plate: an exact, three dimensional, linear elasticity solution. J. Elast. 15, 283–291.
Mindlin, R.D., 1951. Influence of rotatory inertia and shear on flexural motions of isotropic elastic plates. J. Appl. Mech. 18, 31–38.
Mittelstedt, C., Becker, W., 2004. A single-layer theory approach to stress concentration phenomena in layered plates. Compos. Sci. Technol. 64, 1737–1748.
Naghdi, P.M., 1972. The theory of shells and plates. In: Truesdell, C. (Ed.), . Handbuch der Physik, vol.VIa. Springer-Verlag, Berlin.
Nicotra, V., Podio-Guidugli, P., Tiero, A., 1999. Exact equilibrium solutions for linearly elastic plate-like bodies. J. Elast. 56, 231–245.
Noor, A.K., Burton, W.S., 1989. Assessment of shear deformation theories for multilayered composite plates. Appl. Mech. Rev. 42, 1–13.
Noor, A.K., Malik, M., 2000. An assessment of five modeling approaches for thermo-mechanical stress analysis of laminated composite panels. Comput. Mech. 25, 45–58.
Plagianakos, T.S., Saravanos, D.A., 2009. Higher-order layerwise laminate theory for the prediction of interlaminar shear stresses in thick composite and sandwich composite plates. Compos. Struct. 87, 23–35.
Podio-Guidugli, P., 1989. An exact derivation of thin plate equation. J. Elast. 22, 121–133.
Podio-Guidugli, P., 2002. Contact interactions, stress, and material symmetry, for nonsimple elastic materials. Theor. Appl. Mech. 28–29, 261–276.
Podio-Guidugli, P., Vianello, M., 2010. Hypertractions and hyperstresses convey the same mechanical information. Continuum Mech. Thermodyn. 22 (3), 163–176.
Reddy, J.N., 1984. A simple higher-order theory for laminated composite plates. J. Appl. Mech. 51, 745–752.
Reddy, J.N., 1987. A generalization of two-dimensional theories of laminated composite plates. Commun. Appl. Numer. Methods 3, 173–180.
Reddy, J.N., 2004. Mechanics of laminated composite plates and shells: theory and analysis, second ed. CRC Press, Boca Raton, Florida.
Reissner, E., 1945. The effect of transverse shear deformation on the bending of elastic plates. J. Appl. Mech. 12, 69–77.
Reissner, E., Stavsky, Y., 1961. Bending and stretching of certain types of heterogeneous aelotropic elastic plates. J. Appl. Mech. 28, 402–408.
Seide, P., 1980. An improved approximate theory for bending of laminated plates. Mech. Today 5, 451–466.
Truesdell, C., Toupin, R., 1960. The classical field theories. In: Flügge, S. (Ed.), Handbuch der Physik. Springer-Verlag, Berlin.
Whitney, J.M., 1969. The effect of transverse shear deformation in the bending of laminated plates. J. Compos. Mater. 3, 534–547.