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Jordan form asymptotic solutions near the tip of a V-shaped notch in Reissner plate

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ABSTRACT

The expressions for the first two order solutions of the asymptotic near-tip fields for V-shaped notch in Reissner plate have been given by the eigenfunction expansion method in the open literature. However, the eigenfunction expansion solutions are incomplete due to the absence of the asymptotic solution corresponding to a crucial eigenvalue. In this paper the asymptotic solution has been derived as a supplement to previous work. Moreover, it is found that the asymptotic solution for the displacement distribution in the plate becomes infinite for some special vertex angles of the notch, this is a paradox. The cases of the paradox are studied, and the corresponding bounded solutions are found to be explained by the Jordan form solution according to the methods of mathematical physics. In another case, Jordan form asymptotic solution also arises where an eigenvalue becomes a double root. By virtue of the methods of mathematical physics, the Jordan form asymptotic solutions for these special cases are derived making use of a rational procedure and specified in explicit form. A numerical example is given in order to prove the validity of the present study and also to discuss the importance of the completeness of the eigenfunction expansion solutions.

1. Introduction

It is well-known that V-shaped notches (or termed as V-notches) frequently occur in elastic plates due to optimization design, manufacturing process and so on. Analytically determining the displacement and stress–strain distributions near the tip of a V-notch is of crucial importance not only for fracture mechanics but also for numerical analysis of any complex problem involving V-notches.

Many previous analytical studies on V-notched plates (or wedges) have been undertaken in the framework of linear elastic fracture mechanics (Qian and Yan, 1985; Steigemann, 2015; Wang, 2013; Yao et al., 1999). The eigenfunction expansion method is one of the most powerful techniques for analyzing the near-tip fields for various types of cracks and notches. It was first proposed by Williams (1951) to analyze the stress singularity problem in an elastic wedge under bending, and then further extended to study stress singularities in angular corners of plates under extension (Williams, 1952). In this method, the solutions of near-tip fields are expanded in asymptotic series form, and the original problem can be simplified into an eigenvalue problem. Liu (1983) employed the eigenfunction expansion method to study the crack problem in Reissner plate, and proposed the expressions for the generalized displacement and internal force fields for the first several orders. Burton and Sinclair (1986) studied stress singularity at the vertex of a V-notch in Reissner plate, and they established the eigenequation in the case of different boundary conditions on the surface of the V-notch. Qian and Long (1992) studied the V-notch problem in Reissner plate using the eigenfunction expansion method while only the first two order terms of the asymptotic solution are given. And then they extended the same method to analyze the three-dimensional notch problem (Qian and Long, 1994).

Through the eigenfunction expansion method, the eigenequation for the V-notch problem in Reissner plate with free-free boundary condition on the surfaces of the notch has been established and a series of eigenvalues with various vertex angles 2α have been determined (Long et al., 2009; Qian and Long, 1992). It is well-known that the smaller the positive real part of the eigenvalue is, the more important the corresponding eigensolution is to determine the near-tip fields for the V-notch problem. Unfortunately, the crucial eigenvalue $\lambda = 1$ was missed in the list of the roots of the eigenequation in the above mentioned literatures, which will be able to result in incompleteness of the eigenfunction expansion solutions and significant error in related

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numerical analysis. As a supplement to their previous work, the eigenfunction expansion solution corresponding to $\lambda = 1$ will be investigated carefully in the present paper.

Furthermore, it is found that there exists $\cos(\lambda + i)\alpha$, ($i = 2k + 1,\ k = 0, 1, 2, \ldots$) in the denominator in the expressions of the eigen expanding terms for the antisymmetric deformation, then the asymptotic solution would become infinite when λ and α satisfy the definite relation, i.e. $\cos(\lambda + i)\alpha = 0$. For example, the eigenvalue $\lambda = 1$ is a root of the eigenequation, and it is found that $\cos 2\alpha = 0$ when $\alpha = 135^\circ$, $\cos 4\alpha = 0$ when $\alpha = 112.5^\circ$ or 157.5°. For these special cases, we will research into the pathological behaviors to obtain the corresponding bounded solutions in the present study.

In fact, there are many abnormal solutions in some special cases, and the phenomenon is usually termed as a "paradox". A well known paradox is that the solution for the stress distribution in an elastic wedge subjected to a concentrated couple at its vertex becomes infinite for every point when the vertex angle 2α equals critical angle $2\tilde{\alpha}$, where $\tan(2\tilde{\alpha}) = 2\tilde{\alpha}$ (Inglis, 1922). Similar paradoxes can be found in wedges subjected to in-plane tractions (Timoshenko and Goodier, 1970), wedges subjected to tractions proportional to r^n (Ding et al., 1998), and flow injected into a wedge region (Moffatt and Duffy, 1980). It is not surprising that these abnormal phenomena attract an extensive discussion and have been investigated and resolved in the open literatures (Dempsey, 1981; Dundurs and Markenscoff, 1989; Markenscoff, 1994; Sternberg and Koiter, 1958; Ting, 1984). On the basis of the symplectic dual approach (Yao et al., 2009), the present author restudied the paradox in elastic wedge subjected to a concentrated couple at its vertex under Hamiltonian system in polar coordinate (Yao and Xu, 2001). It is pointed out that the solution to the paradox just corresponds to Jordan form eigenvector for the eigenvalue $\lambda = -1$, and a rational derivation for solving this kind of problem was proposed.

In this paper, the above mentioned paradox in the eigen expanding term for $\lambda = 1$ when the vertex angle 2α equals some special vertex angles will be solved by a procedure similar to the one employed in Yao and Xu (2001). The expanding terms corresponding to $\lambda = 1$ where paradox arises should be supposed in the Jordan form instead of the original ones. As a result, the explicit expressions of the Jordan form asymptotic solutions for the special cases are specified. In addition, Jordan form asymptotic solution may also arise where an eigenvalue is a double root of the eigenequation. When $\alpha = \tilde{\alpha}$ ($\tan(2\tilde{\alpha}) = 2\tilde{\alpha}$, $\tilde{\alpha} \approx 128.7^\circ$), the eigenvalue $\lambda = 1$ just becomes a double root. In this case, there must exist an extra Jordan form asymptotic solution corresponding to $\lambda = 1$, otherwise, the eigenfunction expansion solutions of the notch-tip fields are incomplete. Again, the first two order terms of the Jordan form asymptotic solution are derived and specified in explicit form.

This paper is organized as follows. After this introduction, the fundamental equations of Reissner plate theory are summarized for the completeness of this paper in Section 2. A restudy of the V-notch problem in Reissner plate is conducted and the eigen expanding terms of the asymptotic solution for general cases are specified in Section 3. Section 4 discusses the asymptotic solutions corresponding to $\lambda = 1$ for some special vertex angles of the notch. The solutions to the paradoxes are explained by the Jordan form asymptotic solution and specified in explicit form. The Jordan form asymptotic solution of $\lambda = 1$ caused by double root is specified in Section 5. Section 6 provides a numerical example to validate the correctness of the formulas. The last section is about summaries.

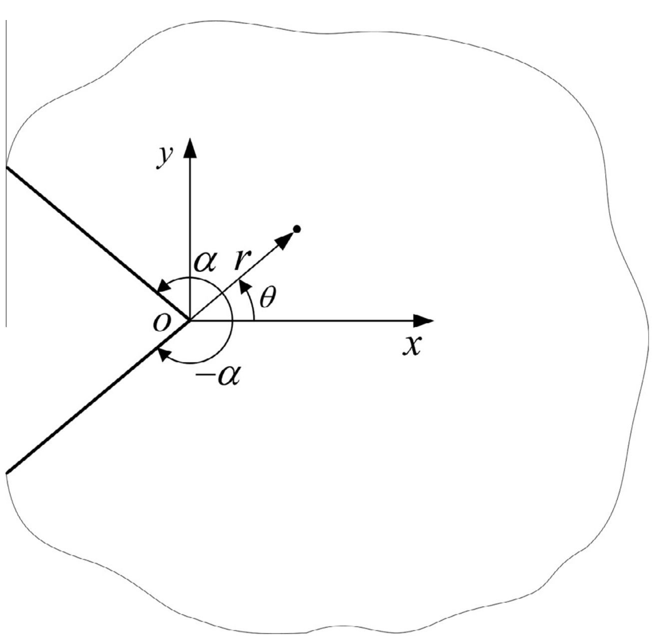

Fig. 1. Coordinate systems defined for a V-shaped notch in Reissner plate.

2. Fundamental equations

Considering a bending plate with a V-notch ($90^\circ < \alpha < 180^\circ$), as shown in Fig. 1, the notch tip is taken as the origin of both the rectangular Cartesian coordinate system and the polar coordinate system.

The fundamental equations, in the absence of body forces and inertia effects, can be expressed in terms of three generalized displacements ψ_r, ψ_θ and w as

$$\begin{cases} D\left[\frac{\partial^2\psi_r}{\partial r^2} + \frac{1}{r}\frac{\partial\psi_r}{\partial r} - \frac{\psi_r}{r^2} + \frac{1-\nu}{2r^2}\frac{\partial^2\psi_r}{\partial\theta^2} + \frac{1+\nu}{2r}\frac{\partial^2\psi_\theta}{\partial r\partial\theta} - \frac{3-\nu}{2r^2}\frac{\partial\psi_\theta}{\partial\theta}\right] + C\left(\frac{\partial w}{\partial r} - \psi_r\right) = 0 \\ D\left[\frac{1+\nu}{2r}\frac{\partial^2\psi_r}{\partial r\partial\theta} + \frac{3-\nu}{2r^2}\frac{\partial\psi_r}{\partial\theta} + \frac{1-\nu}{2}\frac{\partial^2\psi_\theta}{\partial r^2} + \frac{1-\nu}{2r}\frac{\partial\psi_\theta}{\partial r} + \frac{1}{r^2}\frac{\partial^2\psi_\theta}{\partial\theta^2} - \frac{1-\nu}{2r^2}\psi_\theta\right] \\ \quad + C\left(\frac{1}{r}\frac{\partial w}{\partial\theta} - \psi_\theta\right) = 0 \\ C\left[\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial\theta^2} - \left(\frac{\partial\psi_r}{\partial r} + \frac{\psi_r}{r} + \frac{1}{r}\frac{\partial\psi_\theta}{\partial\theta}\right)\right] = 0 \end{cases} \tag{1}$$

where ψ_r and ψ_θ denote rotating angles with respect to straight lines which are perpendicular to the middle plane before deformation in the rOz and θOz plane, respectively. w denotes the deflection of the plate in the z-direction (normal to the middle plane of the plate). D and C are bending and shearing stiffness

$$D = \frac{Eh^3}{12(1-\nu^2)}, \quad C = \frac{5}{6}Gh \tag{2}$$

in which E, G and ν are Young's modulus, shear modulus and Poisson's ratio, and h is plate thickness.

The relations between internal forces and generalized displacements are specified by

$$\begin{cases} M_r = -D\left[\frac{\partial\psi_r}{\partial r} + \nu\left(\frac{1}{r}\frac{\partial\psi_\theta}{\partial\theta} + \frac{\psi_r}{r}\right)\right] \\ M_\theta = -D\left[\frac{1}{r}\frac{\partial\psi_\theta}{\partial\theta} + \frac{\psi_r}{r} + \nu\frac{\partial\psi_r}{\partial r}\right] \\ M_{r\theta} = -\frac{1}{2}(1-\nu)D\left[\frac{1}{r}\frac{\partial\psi_r}{\partial\theta} + \frac{\partial\psi_\theta}{\partial r} - \frac{\psi_\theta}{r}\right] \\ Q_r = C\left(\frac{\partial w}{\partial r} - \psi_r\right) \\ Q_\theta = C\left(\frac{1}{r}\frac{\partial w}{\partial\theta} - \psi_\theta\right) \end{cases} \tag{3}$$

The boundary conditions alone the notch surfaces are assumed to be stress-free and expressed as

$$M_\theta = M_{r\theta} = Q_\theta = 0, \quad \text{at } \theta = \pm\alpha \tag{4}$$

Substituting Eq. (3) into Eq. (4), the above boundary conditions can be rewritten as

$$\begin{cases} \frac{1}{r}\frac{\partial\psi_\theta}{\partial\theta}+\frac{\psi_r}{r}+\nu\frac{\partial\psi_r}{\partial r}=0 \\ \frac{1}{r}\frac{\partial\psi_r}{\partial\theta}+\frac{\partial\psi_\theta}{\partial r}-\frac{\psi_\theta}{r}=0 \\ \frac{1}{r}\frac{\partial w}{\partial\theta}-\psi_\theta=0 \end{cases}, \quad \text{at } \theta=\pm\alpha \tag{5}$$

3. Asymptotic analysis

By adopting the classical eigenfunction expansion method pioneered by Williams (1951), as r approaches zero on R, the above-mentioned three generalized displacements ψ_r, ψ_θ and w are assumed to be expanded in the asymptotic form (Liu, 1983; Qian and Long, 1992) as

$$\begin{cases} \psi_r(r,\theta)=\sum_{n=1}^{\infty}\psi_r^{(\lambda_n)}(r,\theta)=\sum_{n=1}^{\infty}\sum_{i=0}^{\infty}r^{\lambda_n+i}A_i^{(\lambda_n)}(\theta) \\ \psi_\theta(r,\theta)=\sum_{n=1}^{\infty}\psi_\theta^{(\lambda_n)}(r,\theta)=\sum_{n=1}^{\infty}\sum_{i=0}^{\infty}r^{\lambda_n+i}B_i^{(\lambda_n)}(\theta) \\ w(r,\theta)=\sum_{n=1}^{\infty}w^{(\lambda_n)}(r,\theta)=\sum_{n=1}^{\infty}\sum_{i=0}^{\infty}r^{\lambda_n+i}C_i^{(\lambda_n)}(\theta) \end{cases} \tag{6}$$

where λ denotes eigenvalue, λ_n denotes the nth eigenvalue which could be a real or complex number, and the superscript i denotes the ith order term. It is known that the real part of λ_n must be greater than or equal to zero to satisfy the principle that strain energy must be finite as r approaches zero.

After a number of derivation steps, the original problem can be simplified into an eigenvalue problem and the eigenequation for the V-notch problem in Reissner plate is specified by

$$(\sin 2\lambda\alpha+\lambda\sin 2\alpha)(\sin 2\lambda\alpha-\lambda\sin 2\alpha)\sin\lambda\alpha\cos\lambda\alpha=0 \tag{7}$$

More detailed derivation steps are referred to Qian and Long (1992). Obviously, above eigenequation, without loss of any root, can be factorized into four separate equations as follows:

$$D_1(\lambda,\alpha)\equiv\sin 2\lambda\alpha+\lambda\sin 2\alpha=0 \tag{8a}$$

$$D_2(\lambda,\alpha)\equiv\sin 2\lambda\alpha-\lambda\sin 2\alpha=0 \tag{8b}$$

$$D_3(\lambda,\alpha)\equiv\sin\lambda\alpha=0 \tag{8c}$$

$$D_4(\lambda,\alpha)\equiv\cos\lambda\alpha=0 \tag{8d}$$

Regarding the transcendental equations (8a) and (8b), the roots with various vertex angles of the V-notch are given in Long et al. (2009), Yao et al.(2009). Once the root λ is determined, the eigen expanding term of the asymptotic solution corresponding to the nth eigenvalue can be obtained straightforwardly by back substituting the root λ_n into the boundary equations. The derivation procedure is omitted here, and the explicit expressions of the expanding terms are specified directly for a more detailed discussion.

(1) The expanding expressions corresponding to eigenvalues λ_n satisfying eigenvalue equation (8a) are specified by

$$\begin{cases} A_0^{(\lambda_n)}=\beta^{(\lambda_n)}a_0^{(\lambda_n)}=\beta^{(\lambda_n)}[A_{r11}\cos(\lambda_n-1)\theta+A_{r12}\cos(\lambda_n+1)\theta] \\ B_0^{(\lambda_n)}=\beta^{(\lambda_n)}b_0^{(\lambda_n)}=\beta^{(\lambda_n)}[A_{\theta 11}\sin(\lambda_n-1)\theta+A_{\theta 12}\sin(\lambda_n+1)\theta] \\ C_0^{(\lambda_n)}=\beta^{(\lambda_n)}c_0^{(\lambda_n)}=0 \end{cases} \tag{9a}$$

$$\begin{cases} A_1^{(\lambda_n)}=\beta^{(\lambda_n)}a_1^{(\lambda_n)}=0 \\ B_1^{(\lambda_n)}=\beta^{(\lambda_n)}b_1^{(\lambda_n)}=0 \\ C_1^{(\lambda_n)}=\beta^{(\lambda_n)}c_1^{(\lambda_n)}=\beta^{(\lambda_n)}[A_{w11}\cos(\lambda_n-1)\theta+A_{w12}\cos(\lambda_n+1)\theta] \end{cases} \tag{9b}$$

where

$$\begin{cases} A_{r11}=-\frac{\nu\lambda_n+\lambda_n-3+\nu}{\nu\lambda_n+\lambda_n+3-\nu} \\ A_{\theta 11}=1 \\ A_{w11}=\frac{1-\nu}{\nu\lambda_n+\lambda_n+3-\nu} \\ A_{r12}=\frac{1+\nu}{\nu\lambda_n+\lambda_n+3-\nu}(\cos 2\lambda_n\alpha+\lambda_n\cos 2\alpha) \\ A_{\theta 12}=-\frac{1+\nu}{\nu\lambda_n+\lambda_n+3-\nu}(\cos 2\lambda_n\alpha+\lambda_n\cos 2\alpha) \\ A_{w12}=\frac{1}{\nu\lambda_n+\lambda_n+3-\nu}\left[\frac{1+\nu}{\lambda_n+1}(\cos 2\lambda_n\alpha+\lambda_n\cos 2\alpha)-\frac{2\sin(\lambda_n-1)\alpha}{\sin(\lambda_n+1)\alpha}\right] \end{cases} \tag{10}$$

(2) The expanding expressions corresponding to eigenvalues λ_n satisfying eigenvalue equation (8b) are specified by

$$\begin{cases} A_0^{(\lambda_n)}=\beta^{(\lambda_n)}a_0^{(\lambda_n)}=\beta^{(\lambda_n)}[A_{r21}\sin(\lambda_n-1)\theta+A_{r22}\sin(\lambda_n+1)\theta] \\ B_0^{(\lambda_n)}=\beta^{(\lambda_n)}b_0^{(\lambda_n)}=\beta^{(\lambda_n)}[A_{\theta 21}\cos(\lambda_n-1)\theta+A_{\theta 22}\cos(\lambda_n+1)\theta] \\ C_0^{(\lambda_n)}=\beta^{(\lambda_n)}c_0^{(\lambda_n)}=0 \end{cases} \tag{11a}$$

$$\begin{cases} A_1^{(\lambda_n)}=\beta^{(\lambda_n)}a_1^{(\lambda_n)}=0 \\ B_1^{(\lambda_n)}=\beta^{(\lambda_n)}b_1^{(\lambda_n)}=0 \\ C_1^{(\lambda_n)}=\beta^{(\lambda_n)}c_1^{(\lambda_n)}=\beta^{(\lambda_n)}[A_{w21}\sin(\lambda_n-1)\theta+A_{w22}\sin(\lambda_n+1)\theta] \end{cases} \tag{11b}$$

where

$$\begin{cases} A_{r21}=\frac{\nu\lambda_n+\lambda_n-3+\nu}{\nu\lambda_n+\lambda_n+3-\nu} \\ A_{\theta 21}=1 \\ A_{w21}=-\frac{1-\nu}{\nu\lambda_n+\lambda_n+3-\nu} \\ A_{r22}=\frac{1+\nu}{\nu\lambda_n+\lambda_n+3-\nu}(\cos 2\lambda_n\alpha-\lambda_n\cos 2\alpha) \\ A_{\theta 22}=\frac{1+\nu}{\nu\lambda_n+\lambda_n+3-\nu}(\cos 2\lambda_n\alpha-\lambda_n\cos 2\alpha) \\ A_{w22}=\frac{1}{\nu\lambda_n+\lambda_n+3-\nu}\left[\frac{1+\nu}{\lambda_n+1}(\cos 2\lambda_n\alpha-\lambda_n\cos 2\alpha)+\frac{2\cos(\lambda_n-1)\alpha}{\cos(\lambda_n+1)\alpha}\right] \end{cases} \tag{12}$$

(3) The expanding expressions corresponding to eigenvalues λ_n satisfying eigenvalue equation (8c) are specified by

$$\begin{cases} A_0^{(\lambda_n)}=\beta^{(\lambda_n)}a_0^{(\lambda_n)}=0 \\ B_0^{(\lambda_n)}=\beta^{(\lambda_n)}b_0^{(\lambda_n)}=0 \\ C_0^{(\lambda_n)}=\beta^{(\lambda_n)}c_0^{(\lambda_n)}=\beta^{(\lambda_n)}A_{w31}\cos\lambda_n\theta \end{cases} \tag{13a}$$

$$\begin{cases} A_1^{(\lambda_n)}=\beta^{(\lambda_n)}a_1^{(\lambda_n)}=\beta^{(\lambda_n)}A_{r31}\cos\lambda_n\theta \\ B_1^{(\lambda_n)}=\beta^{(\lambda_n)}b_1^{(\lambda_n)}=\beta^{(\lambda_n)}A_{\theta 31}\sin\lambda_n\theta \\ C_1^{(\lambda_n)}=\beta^{(\lambda_n)}c_1^{(\lambda_n)}=0 \end{cases} \tag{13b}$$

where

$$\begin{cases} A_{r31}=-\frac{\lambda_n}{(\lambda_n+1)(\lambda_n+2)(1-\nu^2)}\frac{2C}{D} \\ A_{\theta 31}=\frac{\nu\lambda_n+\nu+1}{(\lambda_n+1)(\lambda_n+2)(1-\nu^2)}\frac{2C}{D} \\ A_{w31}=1 \end{cases} \tag{14}$$

(4) The expanding expressions corresponding to eigenvalues λ_n satisfying eigenvalue equation (8d) are specified by

$$\begin{cases} A_0^{(\lambda_n)}=\beta^{(\lambda_n)}a_0^{(\lambda_n)}=0 \\ B_0^{(\lambda_n)}=\beta^{(\lambda_n)}b_0^{(\lambda_n)}=0 \\ C_0^{(\lambda_n)}=\beta^{(\lambda_n)}c_0^{(\lambda_n)}=\beta^{(\lambda_n)}A_{w41}\sin\lambda_n\theta \end{cases} \tag{15a}$$

$$\begin{cases} A_1^{(\lambda_n)}=\beta^{(\lambda_n)}a_1^{(\lambda_n)}=\beta^{(\lambda_n)}A_{r41}\sin\lambda_n\theta \\ B_1^{(\lambda_n)}=\beta^{(\lambda_n)}b_1^{(\lambda_n)}=\beta^{(\lambda_n)}A_{\theta 41}\cos\lambda_n\theta \\ C_1^{(\lambda_n)}=\beta^{(\lambda_n)}c_1^{(\lambda_n)}=0 \end{cases} \tag{15b}$$

where

$$\begin{cases} A_{r41} = -\frac{\lambda_n}{(\lambda_n+1)(\lambda_n+2)(1-\nu^2)}\frac{2C}{D} \\ A_{\theta 41} = -\frac{\nu\lambda_n+\nu+1}{(\lambda_n+1)(\lambda_n+2)(1-\nu^2)}\frac{2C}{D} \\ A_{w41} = 1 \end{cases} \tag{16}$$

The symbols $\beta^{(\lambda_n)}$ denote unknown constants and can be determined by the boundary condition at the periphery of structure. The higher order terms of the eigensolutions can be derived using the same procedure. It is clear that Eqs. (9) and (13) represent symmetric deformation with respect to coordinate axis $\theta = 0$, while Eqs. (11) and (15) represent antisymmetric deformation. The complete asymptotic solution of the displacement field can be obtained by substituting all the expanding expressions given above into Eq. (6) and superimposing three rigid-body displacements.

Obviously, it can be found that $\lambda = 1$ is a root of Eq. (8b) for an arbitrary α. Hence, the eigenfunction expansion solution corresponding to $\lambda = 1$ always exists for antisymmetric deformation. Generally, the first two order terms of the asymptotic solution can be specified just by substituting $\lambda_n = 1$ into Eqs. (11) and (12):

$$\begin{cases} A_0^{(1)} = \beta^{(1)}a_0^{(1)} = 0 \\ B_0^{(1)} = \beta^{(1)}b_0^{(1)} = \beta^{(1)} \\ C_0^{(1)} = \beta^{(1)}c_0^{(1)} = 0 \end{cases} \tag{17}$$

$$\begin{cases} A_1^{(1)} = \beta^{(1)}a_1^{(1)} = 0 \\ B_1^{(1)} = \beta^{(1)}b_1^{(1)} = 0 \\ C_1^{(1)} = \beta^{(1)}c_1^{(1)} = \frac{\beta^{(1)}}{2\cos 2\alpha}\sin 2\theta \end{cases} \tag{18}$$

Then the asymptotic expression of the generalized displacements is specified by

$$\begin{cases} \psi_r^{(1)}(r,\theta) = 0 + O(r^3) \\ \psi_\theta^{(1)}(r,\theta) = \beta^{(1)}r + O(r^3) \\ w^{(1)}(r,\theta) = \beta^{(1)}\left(\frac{1}{2\cos 2\alpha}r^2\sin 2\theta\right) + O(r^3) \end{cases} \tag{19}$$

where the symbol $O(r^3)$ represents the truncation error of the asymptotic expansion (similarly hereinafter).

According to a similar derivation procedure, the higher order terms of the eigensolution can be derived. For the sake of the following discussion, the expressions of the third and fourth order terms are also given directly as follows:

$$\begin{cases} A_2^{(1)} = \beta^{(1)}a_2^{(1)}(\theta) = \beta^{(1)}\left[-\frac{1}{6(1-\nu^2)\cos 2\alpha}\frac{C}{D}\sin 2\theta\right] \\ B_2^{(1)} = \beta^{(1)}b_2^{(1)}(\theta) = \beta^{(1)}\left[-\frac{1+3\nu}{12(1-\nu^2)\cos 2\alpha}\frac{C}{D}\cos 2\theta + \frac{1}{4(1-\nu)}\frac{C}{D}\right] \\ C_2^{(1)} = \beta^{(1)}c_2^{(1)}(\theta) = 0 \end{cases} \tag{20}$$

$$\begin{cases} A_3^{(1)} = \beta^{(1)}a_3^{(1)}(\theta) = 0 \\ B_3^{(1)} = \beta^{(1)}b_3^{(1)}(\theta) = 0 \\ C_3^{(1)} = \beta^{(1)}c_3^{(1)}(\theta) = \beta^{(1)}\left[-\frac{1}{24(1+\nu)\cos 2\alpha}\frac{C}{D}\sin 2\theta + \frac{3-\nu}{48(1-\nu^2)\cos 4\alpha}\frac{C}{D}\sin 4\theta\right] \end{cases} \tag{21}$$

It is well-known that the smaller the positive real part of the eigenvalue is, the more important the corresponding eigensolution is to determine the near-tip fields for the V-notch problem. Thus, the asymptotic solution corresponding to $\lambda = 1$ plays an important role in determining the stress and strain fields near the V-notch tip, especially for antisymmetric deformation. However, it should be pointed out that this asymptotic solution was not considered in Long et al. (2009), Qian and Long (1992), which could result in significant error in theoretical and numerical study of such a problem. Unfortunately, all of the numerical examples in Long et al. (2009) are related to the plate under symmetric bending deformation, while the asymptotic solution corresponding to $\lambda = 1$ represents antisymmetric deformation. As a supplement to their previous work, the eigenfunction expansion solution corresponding to $\lambda = 1$ will be investigated carefully in the present paper.

4. Solutions to the paradox of $\lambda = 1$

During derivation of the asymptotic solution corresponding to $\lambda = 1$, an interesting phenomenon is found and recognized as a paradox for certain special cases. From Eqs. (18), (20) and (21) it is found that the trigonometric functions $\cos 2\alpha$ and $\cos 4\alpha$ exist in the denominator in the expressions of the eigen expanding terms, and therefore the asymptotic solution may become infinite when the trigonometric functions are equal to zero. For example, $\cos 2\alpha = 0$ when $\alpha = 135^\circ$, $\cos 4\alpha = 0$ when $\alpha = 112.5^\circ$ or 157.5°, $\alpha \in (90^\circ, 180^\circ)$. For these special vertex angles, the original expressions of the asymptotic solution corresponding to $\lambda = 1$ display pathological behavior (termed as paradox) and should be reexamined carefully. According to the author's previous study (Yao and Xu, 2001), it is found that the paradox can be explained by Jordan form solution. By employing the similar procedure, the corresponding Jordan form solutions will be derived in detail in the following sections.

4.1. Jordan form asymptotic solution for $\alpha = 135^\circ$

From Eq. (18) it can be seen that the second order term of the asymptotic solution becomes infinite when $\alpha = 135^\circ$, the original expression of which is obviously invalid at this moment. However, it should be noted that $\lambda = 2$ is also a single root of Eq. (8d) for antisymmetric deformation when $\alpha = 135^\circ$. Considering the asymptotic form of the generalized displacements in Eq. (6), it is found that the asymptotic expansion solutions corresponding to $\lambda = 1$ and $\lambda = 2$ have overlapping terms. For example, the order of r in the second term for $\lambda = 1$ is coincident with that in the first term for $\lambda = 2$, the order of r in the third term for $\lambda = 1$ is coincident with that in the second term for $\lambda = 2$, and the others are in the same manner. Because of the delicate relations, the eigenfunction expansion solution corresponding to $\lambda = 1$ is superseded by the one corresponding to $\lambda = 2$. According to the author's previous study, it is also found that the solution to the paradox of $\lambda = 1$ just corresponds to a special Jordan form solution.

Firstly, according to Eqs. (15) and (16), the first two order terms of the asymptotic eigensolution corresponding to $\lambda = 2$ when $\alpha = 135^\circ$ can be specified by

$$\begin{cases} A_0^{(2)} = \beta^{(2)}a_0^{(2)}(\theta) = 0 \\ B_0^{(2)} = \beta^{(2)}b_0^{(2)}(\theta) = 0 \\ C_0^{(2)} = \beta^{(2)}c_0^{(2)}(\theta) = \beta^{(2)}\sin 2\theta \end{cases} \tag{22}$$

$$\begin{cases} A_1^{(2)} = \beta^{(2)}a_1^{(2)}(\theta) = \beta^{(2)}\left[-\frac{1}{3(1-\nu^2)}\frac{C}{D}\sin 2\theta\right] \\ B_1^{(2)} = \beta^{(2)}b_1^{(2)}(\theta) = \beta^{(2)}\left[-\frac{1+3\nu}{6(1-\nu^2)}\frac{C}{D}\cos 2\theta\right] \\ C_1^{(2)} = \beta^{(2)}c_1^{(2)}(\theta) = 0 \end{cases} \tag{23}$$

Then by virtue of the methods of mathematical physics, the second order term of the asymptotic solution corresponding to $\lambda = 1$ should be supposed in the Jordan form instead of Eq. (18), i.e.

$$\begin{cases} \tilde{A}_{1p}^{(1)} = \beta^{(1)}\left[\tilde{a}_1^{(1)}(\theta) + k_0\ln r a_0^{(2)}(\theta)\right] \\ \tilde{B}_{1p}^{(1)} = \beta^{(1)}\left[\tilde{b}_1^{(1)}(\theta) + k_0\ln r b_0^{(2)}(\theta)\right] \\ \tilde{C}_{1p}^{(1)} = \beta^{(1)}\left[\tilde{c}_1^{(1)}(\theta) + k_0\ln r c_0^{(2)}(\theta)\right] \end{cases} \tag{24}$$

Thus, the corresponding formulas of the generalized displacements are given by

$$\begin{cases} \tilde{\psi}_r^{(1)} = \beta^{(1)}\left\{ra_0^{(1)}(\theta) + r^2\left[\tilde{a}_1^{(1)}(\theta) + k_0 \ln r a_0^{(2)}(\theta)\right]\right\} + O(r^3) \\ \tilde{\psi}_\theta^{(1)} = \beta^{(1)}\left\{rb_0^{(1)}(\theta) + r^2\left[\tilde{b}_1^{(1)}(\theta) + k_0 \ln r b_0^{(2)}(\theta)\right]\right\} + O(r^3) \\ \tilde{w}^{(1)} = \beta^{(1)}\left\{rc_0^{(1)}(\theta) + r^2\left[\tilde{c}_1^{(1)}(\theta) + k_0 \ln r c_0^{(2)}(\theta)\right]\right\} + O(r^3) \end{cases} \tag{25}$$

Substituting above expressions into governing Eqs. (1) and (5), and letting the sum of the coefficients of the first two low orders of r be zero, a set of ordinary linear non-homogeneous differential equations of $\tilde{a}_1^{(1)}(\theta)$, $\tilde{b}_1^{(1)}(\theta)$ and $\tilde{c}_1^{(1)}(\theta)$ can be obtained as follows:

$$\begin{cases} 6\tilde{a}_1^{(1)} + (1-\nu)\ddot{\tilde{a}}_1^{(1)} - (1-3\nu)\dot{\tilde{b}}_1^{(1)} = 0 \\ (5+\nu)\dot{\tilde{a}}_1^{(1)} + 3(1-\nu)\tilde{b}_1^{(1)} + 2\ddot{\tilde{b}}_1^{(1)} = 0 \\ \ddot{\tilde{c}}_1^{(1)} + 4\tilde{c}_1^{(1)} = -4k_0 \sin 2\theta \end{cases} \tag{26}$$

The corresponding boundary conditions are:

$$\begin{cases} (1+2\nu)\tilde{a}_1^{(1)} + \dot{\tilde{b}}_1^{(1)} = 0 \\ \dot{\tilde{a}}_1^{(1)} + \tilde{b}_1^{(1)} = 0 \\ \dot{\tilde{c}}_1^{(1)} = \tilde{b}_0^{(1)} \end{cases}, \quad \text{at } \theta = \pm\alpha \tag{27}$$

in which $(\dot{\cdot}) = \partial/\partial\theta$, $(\ddot{\cdot}) = \partial^2/\partial\theta^2$. Solving Eq. (26) with boundary conditions Eq. (27), the solutions can be obtained as follows:

$$k_0 = \frac{2}{3\pi} \tag{28}$$

and

$$\begin{cases} \tilde{a}_1^{(1)}(\theta) = 0 \\ \tilde{b}_1^{(1)}(\theta) = 0 \\ \tilde{c}_1^{(1)}(\theta) = k \sin 2\theta + \frac{2}{3\pi}\theta\cos 2\theta \end{cases} \tag{29}$$

In the above equations, the term containing constant k is equivalent to superposing the first order term of the asymptotic eigensolution corresponding to $\lambda = 2$, i.e. Eq. (22), hence the constant k can be chosen as $k = 0$ for the sake of simplicity.

It is found from Eq. (20) that the paradox still arises in the third order term. For this case, the corresponding Jordan form solution should be in the form of

$$\begin{cases} \tilde{A}_{2p}^{(1)} = \beta^{(1)}\left[\tilde{a}_2^{(1)}(\theta) + k_0 \ln r a_1^{(2)}(\theta)\right] \\ \tilde{B}_{2p}^{(1)} = \beta^{(1)}\left[\tilde{b}_2^{(1)}(\theta) + k_0 \ln r b_1^{(2)}(\theta)\right] \\ \tilde{C}_{2p}^{(1)} = \beta^{(1)}\left[\tilde{c}_2^{(1)}(\theta) + k_0 \ln r c_1^{(2)}(\theta)\right] \end{cases} \tag{30}$$

After a few derivation steps, the solutions of $\tilde{a}_2^{(1)}(\theta)$, $\tilde{b}_2^{(1)}(\theta)$ and $\tilde{c}_2^{(1)}(\theta)$ are given by

$$\begin{cases} \tilde{a}_2^{(1)}(\theta) = \frac{1}{36(1-\nu^2)}\frac{C}{D}k_0 \sin 2\theta - \frac{1}{3(1-\nu^2)}\frac{C}{D}k_0\theta\cos 2\theta \\ \tilde{b}_2^{(1)}(\theta) = \frac{7+9\nu}{72(1-\nu^2)}\frac{C}{D}k_0\cos 2\theta + \frac{1+3\nu}{6(1-\nu^2)}\frac{C}{D}k_0\theta\sin 2\theta + \frac{1}{4(1-\nu)}\frac{C}{D} \\ \tilde{c}_2^{(1)}(\theta) = 0 \end{cases} \tag{31}$$

Similarly, the higher order terms can be derived in turn. For instance, the solutions of $\tilde{a}_3^{(1)}(\theta)$, $\tilde{b}_3^{(1)}(\theta)$ and $\tilde{c}_3^{(1)}(\theta)$ for the fourth order term are specified by

$$\begin{cases} \tilde{a}_3^{(1)}(\theta) = 0 \\ \tilde{b}_3^{(1)}(\theta) = 0 \\ \tilde{c}_3^{(1)}(\theta) = \frac{1}{144(1+\nu)}\frac{C}{D}k_0\sin 2\theta - \frac{1}{12(1+\nu)}\frac{C}{D}k_0\theta\cos 2\theta - \frac{3-\nu}{48(1-\nu^2)}\frac{C}{D}\sin 4\theta \end{cases} \tag{32}$$

Finally, for the case $\alpha = 135^\circ$, the asymptotic expressions of the generalized displacements corresponding to $\lambda = 1$ for the first four orders are specified by

$$\begin{cases} \tilde{\psi}_r^{(1)}(r,\theta) = \beta^{(1)}\{r^3[A_{r1}\sin 2\theta + \ln r(A_{r2}\sin 2\theta) + A_{r3}\theta\cos 2\theta]\} + O(r^5) \\ \tilde{\psi}_\theta^{(1)}(r,\theta) = \beta^{(1)}\{rA_{\theta 1} + r^3[A_{\theta 2}\cos 2\theta + \ln r(A_{\theta 3}\cos 2\theta) + A_{\theta 4}\theta\sin 2\theta + A_{\theta 5}]\} \\ \qquad + O(r^5) \\ \tilde{w}^{(1)}(r,\theta) = \beta^{(1)}\{r^2[\ln r(A_{w1}\sin 2\theta) + A_{w2}\theta\cos 2\theta] \\ \qquad + r^4[A_{w3}\sin 2\theta + \ln r(A_{w4}\sin 2\theta) \\ \qquad + A_{w5}\theta\cos 2\theta + A_{w6}\sin 4\theta]\} + O(r^5) \end{cases} \tag{33}$$

where

$$\begin{cases} A_{w1} = \frac{2}{3\pi} \\ A_{w2} = \frac{2}{3\pi} \\ A_{w3} = \frac{1}{216\pi(1+\nu)}\frac{C}{D} \\ A_{w4} = -\frac{1}{18\pi(1+\nu)}\frac{C}{D} \\ A_{w5} = -\frac{1}{18\pi(1+\nu)}\frac{C}{D} \\ A_{w6} = -\frac{3-\nu}{48(1-\nu^2)}\frac{C}{D} \end{cases}, \begin{cases} A_{r1} = \frac{1}{54\pi(1-\nu^2)}\frac{C}{D} \\ A_{r2} = -\frac{2}{9\pi(1-\nu^2)}\frac{C}{D} \\ A_{r3} = -\frac{2}{9\pi(1-\nu^2)}\frac{C}{D} \end{cases}, \begin{cases} A_{\theta 1} = 1 \\ A_{\theta 2} = \frac{7+9\nu}{108\pi(1-\nu^2)}\frac{C}{D} \\ A_{\theta 3} = -\frac{1+3\nu}{9\pi(1-\nu^2)}\frac{C}{D} \\ A_{\theta 4} = \frac{1+3\nu}{9\pi(1-\nu^2)}\frac{C}{D} \\ A_{\theta 5} = \frac{1}{4(1-\nu)}\frac{C}{D} \end{cases} \tag{34}$$

4.2. Jordan form asymptotic solution for $\alpha = 112.5^\circ$ or 157.5°

From Eq. (21) it can be seen that the fourth order term of the asymptotic solution becomes infinite when $\alpha = 112.5^\circ$ or 157.5°, the original expression of which is obviously invalid at this moment. But it is particularly noted that this time $\lambda = 2$ is not a root of Eq. (8d) anymore, while $\lambda = 4$ is a single root at this moment. Considering the asymptotic form of the generalized displacements in Eq. (6), the asymptotic expansion solutions corresponding to $\lambda = 1$ and $\lambda = 4$ have overlapping terms. According to a similar discussion on the paradox for $\alpha = 135^\circ$, the fourth order term of the asymptotic solution should be also supposed in the Jordan form.

According to Eqs. (15) and (16), the first two order terms of the asymptotic eigensolution corresponding to $\lambda = 4$ when $\alpha = 112.5^\circ$ or 157.5° can be specified by

$$\begin{cases} A_0^{(4)} = \beta^{(4)}a_0^{(4)}(\theta) = 0 \\ B_0^{(4)} = \beta^{(4)}b_0^{(4)}(\theta) = 0 \\ C_0^{(4)} = \beta^{(4)}c_0^{(4)}(\theta) = \beta^{(4)}\sin 4\theta \end{cases} \tag{35}$$

$$\begin{cases} A_1^{(4)} = \beta^{(4)}a_1^{(4)} = \beta^{(4)}\left[-\frac{4}{15(1-\nu^2)}\frac{C}{D}\sin 4\theta\right] \\ B_1^{(4)} = \beta^{(4)}b_1^{(4)} = \beta^{(4)}\left[-\frac{1+5\nu}{15(1-\nu^2)}\frac{C}{D}\cos 4\theta\right] \\ C_1^{(4)} = \beta^{(4)}c_1^{(4)} = 0 \end{cases} \tag{36}$$

Then the fourth order term of the asymptotic eigensolution corresponding to $\lambda = 1$ is expanded in the Jordan form instead of Eq. (21), i.e.

$$\begin{cases} \widehat{A}_{3p}^{(1)} = \beta^{(1)}\left[\widehat{a}_3^{(1)}(\theta) + k_1 \ln r a_0^{(4)}(\theta)\right] \\ \widehat{B}_{3p}^{(1)} = \beta^{(1)}\left[\widehat{b}_3^{(1)}(\theta) + k_1 \ln r b_0^{(4)}(\theta)\right] \\ \widehat{C}_{3p}^{(1)} = \beta^{(1)}\left[\widehat{c}_3^{(1)}(\theta) + k_1 \ln r c_0^{(4)}(\theta)\right] \end{cases} \tag{37}$$

And the corresponding formulas of the generalized displacements are given by

$$\begin{cases} \widehat{\psi}_r^{(1)}(r,\theta) = \beta^{(1)}\left\{ra_0^{(1)}(\theta) + r^2a_1^{(1)}(\theta) + r^3a_2^{(1)}(\theta) + r^4\left[\widehat{a}_3^{(1)}(\theta) + k_1\ln r a_0^{(4)}(\theta)\right]\right\} + O(r^5) \\ \widehat{\psi}_\theta^{(1)}(r,\theta) = \beta^{(1)}\left\{rb_0^{(1)}(\theta) + r^2b_1^{(1)}(\theta) + r^3b_2^{(1)}(\theta) + r^4\left[\widehat{b}_3^{(1)}(\theta) + k_1\ln r b_0^{(4)}(\theta)\right]\right\} + O(r^5) \\ \widehat{w}^{(1)}(r,\theta) = \beta^{(1)}\left\{rc_0^{(1)}(\theta) + r^2c_1^{(1)}(\theta) + r^3c_2^{(1)}(\theta) + r^4\left[\widehat{c}_3^{(1)}(\theta) + k_1\ln r c_0^{(4)}(\theta)\right]\right\} + O(r^5) \end{cases} \tag{38}$$

By following a procedure similar to that described for $\alpha = 135^\circ$, the solutions of $\widehat{a}_3^{(1)}(\theta)$, $\widehat{b}_3^{(1)}(\theta)$ and $\widehat{c}_3^{(1)}(\theta)$ can be obtained and specified by

$$k_1 = \frac{3-\nu}{48(1-\nu^2)\alpha}\frac{C}{D} \tag{39}$$

and

$$\begin{cases} \widehat{a}_3^{(1)}(\theta) = 0 \\ \widehat{b}_3^{(1)}(\theta) = 0 \\ \widehat{c}_3^{(1)}(\theta) = k\sin 4\theta - \frac{1}{24(1+\nu)\cos 2\alpha}\frac{C}{D}\sin 2\theta \\ \qquad\qquad + \frac{3-\nu}{48(1-\nu^2)\alpha}\frac{C}{D}\theta\cos 4\theta \end{cases} \tag{40}$$

In the above equation, the term containing constant k is equivalent to superposing the first order term of the asymptotic eigensolution corresponding to $\lambda = 4$, i.e. Eq. (35), hence the constant k can be taken as $k = 0$ for the sake of simplicity.

Finally, for the cases $\alpha = 112.5^\circ$ and 157.5°, the expressions of the generalized displacements corresponding to $\lambda = 1$ for the first four orders are specified by

$$\begin{cases} \widehat{\psi}_r^{(1)}(r,\theta) = \beta^{(1)}\left[r^3(B_{r1}\sin 2\theta)\right] + O(r^5) \\ \widehat{\psi}_\theta^{(1)}(r,\theta) = \beta^{(1)}\left[rB_{\theta 1} + r^3(B_{\theta 2}\cos 2\theta + B_{\theta 3})\right] + O(r^5) \\ \widehat{w}^{(1)}(r,\theta) = \beta^{(1)}\left[r^2 B_{w1}\sin 2\theta + r^4(B_{w2}\sin 2\theta + B_{w3}\ln r\sin 4\theta + B_{w4}\theta\cos 4\theta)\right] + O(r^5) \end{cases} \tag{41}$$

where

$$\begin{matrix} B_{r1} = -\frac{1}{6(1-\nu^2)\cos 2\alpha}\frac{C}{D} & B_{w1} = \frac{1}{2\cos 2\alpha} \\ B_{\theta 1} = 1 & B_{w2} = -\frac{1}{24(1+\nu)\cos 2\alpha}\frac{C}{D} \\ B_{\theta 2} = -\frac{1+3\nu}{12(1-\nu^2)\cos 2\alpha}\frac{C}{D}, & B_{w3} = \frac{3-\nu}{48(1-\nu^2)\alpha}\frac{C}{D} \\ B_{\theta 3} = \frac{1}{4(1-\nu)}\frac{C}{D} & B_{w4} = \frac{3-\nu}{48(1-\nu^2)\alpha}\frac{C}{D} \end{matrix} \tag{42}$$

The higher order terms of the asymptotic eigensolution can be derived according to the similar procedure, and one can solve them where necessary.

Theoretically, the paradox may be found when the difference between two eigenvalues is an integer according to the asymptotic form of the generalized displacements in Eq. (6), while this is only a necessary condition. According to the expressions of the expanding terms of the eigensolution corresponding to $\lambda = 1$, it is found that the trigonometric function $\cos(1+i)\alpha$, $(i = 2k+1,\ k = 0, 1, 2, \ldots)$ exists in the denominator, then the eigensolution would become infinite when $\cos(1+i)\alpha = 0$. In addition to $\cos 2\alpha = 0$ and $\cos 4\alpha = 0$ that are studied in the present paper, many other cases can also result in the existence of the paradox, such as $\cos 6\alpha = 0$, $\cos 8\alpha = 0$, and so on. In other words, it is found that paradox arises when the wedge angle satisfies $\cos(1+i)\alpha = 0$ and $\alpha \in (90^\circ, 180^\circ)$. For example, it is found that $\cos 6\alpha = 0$ when $\alpha = 105^\circ$, 135° or 165°, at this moment the expanding terms of the eigensolution become infinite from the beginning of the sixth order term, and the solutions to the paradoxes should be supposed in the Jordan form based on the expanding terms of the eigensolution corresponding to $\lambda = 6$. For the sake of simplicity, more discussion will not be given here, and one can solve the related Jordan form solution where necessary.

5. Jordan form asymptotic solution for the double eigenvalue $\lambda = 1$

When $\alpha = \tilde{\alpha}$ $(\tan(2\tilde{\alpha}) = 2\tilde{\alpha},\ \tilde{\alpha} \approx 128.7^\circ)$, it is found that the eigenvalue $\lambda = 1$ becomes a double root of Eq. (8b) by reason that $\lambda = 1$ is also a root of $\partial D_2(\lambda, \alpha)/\partial\lambda = 0$. In accordance with the derivation above there is only one unknown constant $\beta^{(1)}$ to be determined for $\lambda = 1$, in other word, there is only one basic asymptotic solution for $\lambda = 1$. Hence, for the case $\tilde{\alpha} \approx 128.7^\circ$, there must exist an extra Jordan form asymptotic solution in addition to the basic asymptotic solution Eq. (19), otherwise, the eigenfunction expansion solutions of the notch-tip fields are incomplete.

By virtue of the methods of mathematical physics, the first two order terms of the Jordan form asymptotic eigensolution should be supposed, respectively, in the form of

$$\begin{cases} A_{0J}^{(1)} = \beta_J^{(1)}\left[a_{0J}^{(1)} + \ln r a_0^{(1)}\right] \\ B_{0J}^{(1)} = \beta_J^{(1)}\left[b_{0J}^{(1)} + \ln r b_0^{(1)}\right] \\ C_{0J}^{(1)} = \beta_J^{(1)}\left[c_{0J}^{(1)} + \ln r c_0^{(1)}\right] \end{cases} \tag{43}$$

$$\begin{cases} A_{1J}^{(1)} = \beta_J^{(1)}\left[a_{1J}^{(1)} + \ln r a_1^{(1)}\right] \\ B_{1J}^{(1)} = \beta_J^{(1)}\left[b_{1J}^{(1)} + \ln r b_1^{(1)}\right] \\ C_{1J}^{(1)} = \beta_J^{(1)}\left[c_{1J}^{(1)} + \ln r c_1^{(1)}\right] \end{cases} \tag{44}$$

And the corresponding formulas of the generalized displacements are given by

$$\begin{cases} \psi_{rJ}^{(1)}(r,\theta) = \beta_J^{(1)}\left\{r\left[a_{0J}^{(1)}(\theta) + \ln r a_0^{(1)}(\theta)\right] + r^2\left[a_{1J}^{(1)}(\theta) + \ln r a_1^{(1)}(\theta)\right]\right\} + O(r^3) \\ \psi_{\theta J}^{(1)}(r,\theta) = \beta_J^{(1)}\left\{r\left[b_{0J}^{(1)}(\theta) + \ln r b_0^{(1)}(\theta)\right] + r^2\left[b_{1J}^{(1)}(\theta) + \ln r b_1^{(1)}(\theta)\right]\right\} + O(r^3) \\ w_J^{(1)}(r,\theta) = \beta_J^{(1)}\left\{r\left[c_{0J}^{(1)}(\theta) + \ln r c_0^{(1)}(\theta)\right] + r^2\left[c_{1J}^{(1)}(\theta) + \ln r c_1^{(1)}(\theta)\right]\right\} + O(r^3) \end{cases} \tag{45}$$

Substituting above expressions into governing Eq. (1) and Eq. (5), and letting the sum of the coefficients of r with the same power order be zero, the equation sets about $a_{iJ}^{(1)}(\theta)$, $b_{iJ}^{(1)}(\theta)$ and $c_{iJ}^{(1)}(\theta)$ $(i = 0, 1)$ for each order can be obtained, and the corresponding boundary conditions can also be obtained. For instance, a set of ordinary linear differential equations of $a_{0J}^{(1)}(\theta)$, $b_{0J}^{(1)}(\theta)$ and $c_{0J}^{(1)}(\theta)$ is formed by letting the sum of the coefficients of r^{-1} term be zero, i.e.

$$\begin{cases} \frac{1-\nu}{2}\ddot{a}_{0J}^{(1)} + (-1+\nu)\dot{b}_{0J}^{(1)} = 0 \\ 2\dot{a}_{0J}^{(1)} + \ddot{b}_{0J}^{(1)} + (1-\nu) = 0 \\ \ddot{c}_{0J} + c_{0J} = 0 \end{cases} \tag{46}$$

The corresponding boundary conditions are

$$\begin{cases} (1+\nu)a_{0J}^{(1)} + \dot{b}_{0J}^{(1)} = 0 \\ \dot{a}_{0J}^{(1)} + 1 = 0 \\ \dot{c}_{0J}^{(1)} = 0 \end{cases}, \text{ at } \theta = \pm\tilde{\alpha} \tag{47}$$

Solving Eq. (46) with boundary conditions Eq. (47), the solutions of $a_{0J}^{(1)}(\theta)$, $b_{0J}^{(1)}(\theta)$ and $c_{0J}^{(1)}(\theta)$ can be obtained:

$$\begin{cases} a_{0J}^{(1)} = -\frac{1+\nu}{4\cos 2\tilde{\alpha}}\sin 2\theta - \frac{1-\nu}{2}\theta \\ b_{0J}^{(1)} = -\frac{1+\nu}{4\cos 2\tilde{\alpha}}\cos 2\theta + k \\ c_{0J}^{(1)} = 0 \end{cases} \tag{48}$$

In the above equation, the term containing constant k is equivalent to superposing the first order term of the basic eigensolution corresponding to $\lambda = 1$, i.e. Eq. (17), hence the constant k can be taken as $k = 0$ for the sake of simplicity.

In the same way the solutions of $a_{1J}^{(1)}(\theta)$, $b_{1J}^{(1)}(\theta)$ and $c_{1J}^{(1)}(\theta)$ for the second order term of the Jordan form eigensolution can be specified by

$$\begin{cases} a_{1J}^{(1)} = 0 \\ b_{1J}^{(1)} = 0 \\ c_{1J}^{(1)} = -\frac{1+\nu-4\tilde{\alpha}^2}{4\cos 2\tilde{\alpha}}\sin 2\theta + \frac{1}{2\cos 2\tilde{\alpha}}\theta\cos 2\theta - \frac{1-\nu}{4}\theta \end{cases} \tag{49}$$

Finally, for the case $\alpha = \tilde{\alpha}$ ($\tan(2\tilde{\alpha}) = 2\tilde{\alpha}$, $\tilde{\alpha} \approx 128.7^\circ$), the Jordan form asymptotic expressions of the generalized displacements corresponding to $\lambda = 1$ for the first two orders are specified by

$$\begin{cases} \psi_{rJ}^{(1)}(r,\theta) = \beta_J^{(1)} r\left(-\frac{1+\nu}{4\cos 2\tilde{\alpha}}\sin 2\theta - \frac{1-\nu}{2}\theta\right) + O(r^3) \\ \psi_{\theta J}^{(1)}(r,\theta) = \beta_J^{(1)} r\left(-\frac{1+\nu}{4\cos 2\tilde{\alpha}}\cos 2\theta + \ln r\right) + O(r^3) \\ w_J^{(1)}(r,\theta) = \beta_J^{(1)} r^2\Big(-\frac{1+\nu-4\tilde{\alpha}^2}{4\cos 2\tilde{\alpha}}\sin 2\theta + \frac{1}{2\cos 2\tilde{\alpha}}\theta\cos 2\theta \\ -\frac{1-\nu}{4}\theta + \ln r\frac{1}{2\cos 2\tilde{\alpha}}\sin 2\theta\Big) + O(r^3) \end{cases} \tag{50}$$

The higher order terms of the Jordan form eigensolution can be derived according to the similar procedure, and one can solve them where necessary.

After the Jordan form asymptotic solutions are obtained, the complete expressions of the generalized displacements for $\alpha = \tilde{\alpha}$ can be given by superimposing the basic and the Jordan form asymptotic solutions corresponding to $\lambda = 1$ on all other asymptotic solutions corresponding to λ_n (unequal to 1) and three rigid-body displacements, i.e.

$$\begin{cases} \psi_r(r,\theta) = \sum_{\substack{n=1 \\ (\lambda_n \neq 1)}}^{\infty} \psi_r^{(\lambda_n)}(r,\theta) + \psi_r^{(1)}(r,\theta) + \psi_{rJ}^{(1)}(r,\theta) + a_0\cos\theta + b_0\sin\theta \\ \psi_\theta(r,\theta) = \sum_{\substack{n=1 \\ (\lambda_n \neq 1)}}^{\infty} \psi_\theta^{(\lambda_n)}(r,\theta) + \psi_\theta^{(1)}(r,\theta) + \psi_{\theta J}^{(1)}(r,\theta) - a_0\sin\theta + b_0\cos\theta \\ w(r,\theta) = \sum_{\substack{n=1 \\ (\lambda_n \neq 1)}}^{\infty} w^{(\lambda_n)}(r,\theta) + w^{(1)}(r,\theta) + w_J^{(1)}(r,\theta) + a_0 r\cos\theta + b_0 r\sin\theta + w_0 \end{cases} \tag{51}$$

in which $\psi_r^{(1)}$, $\psi_\theta^{(1)}$ and $w^{(1)}$ given in Eq. (19) represent the basic form asymptotic eigensolutions corresponding to $\lambda = 1$; $\psi_{rJ}^{(1)}$, $\psi_{\theta J}^{(1)}$ and $w_J^{(1)}$ given in Eq. (50) represent the Jordan form asymptotic eigensolutions corresponding to $\lambda = 1$.

6. Numerical example

The asymptotic expansion solutions are very important to many numerical methods, while incomplete eigenfunction expansion will result in significant error. In order to prove this, the following numerical example is given, and numerical results obtained with and without the asymptotic eigensolution corresponding to $\lambda = 1$ are compared with benchmark results, respectively. In the numerical example, a kind of singular element method for solving crack or V-shaped notch problems in Reissner plate proposed by the authors is employed (Yao et al., 2014). The singular element method employs the asymptotic expansion solutions to describe the displacement fields near the tip of a V-notch, so the eigenfunction expansion plays a crucial role in the performance of this numerical method. Finite element method (FEM) results with dense meshes are given as reference solutions.

As shown in Fig. 2, a square plate containing a center rhombic hole is considered. The plate is subjected to a twist moment m_0 uniformly distributed at its all sides, which leads to an antisymmetric deformation state. The geometric parameters are: l is half of the length of left–right diagonal line of the rhombic hole, α is half of the left–right vertex angles, h is plate thickness. In this example, let the length of the plate side $2L = 20l$, the plate thickness $h = 2.0l$ and Poisson's ratio $\nu = 0.3$. Because of structural symmetry, only the right half of the plate is used for computation. And a singular element with radius R is installed on the notch-tip (see

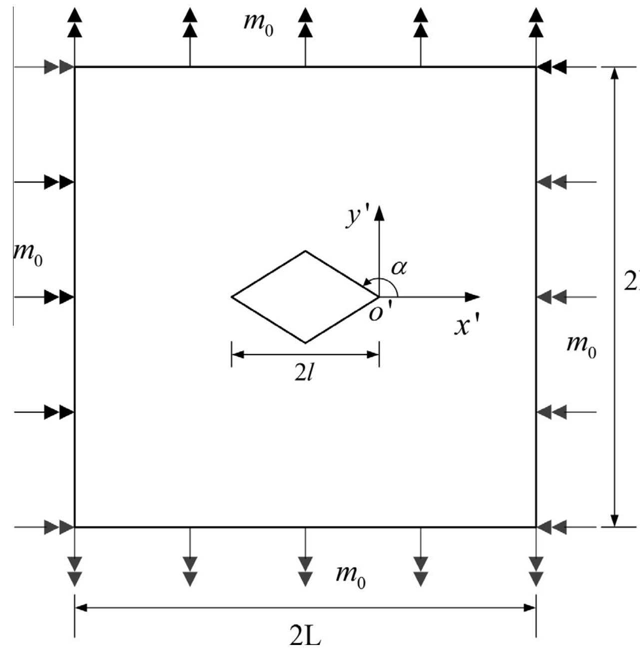

Fig. 2. A square plate with a center rhombic hole subjected to uniform twisting moment.

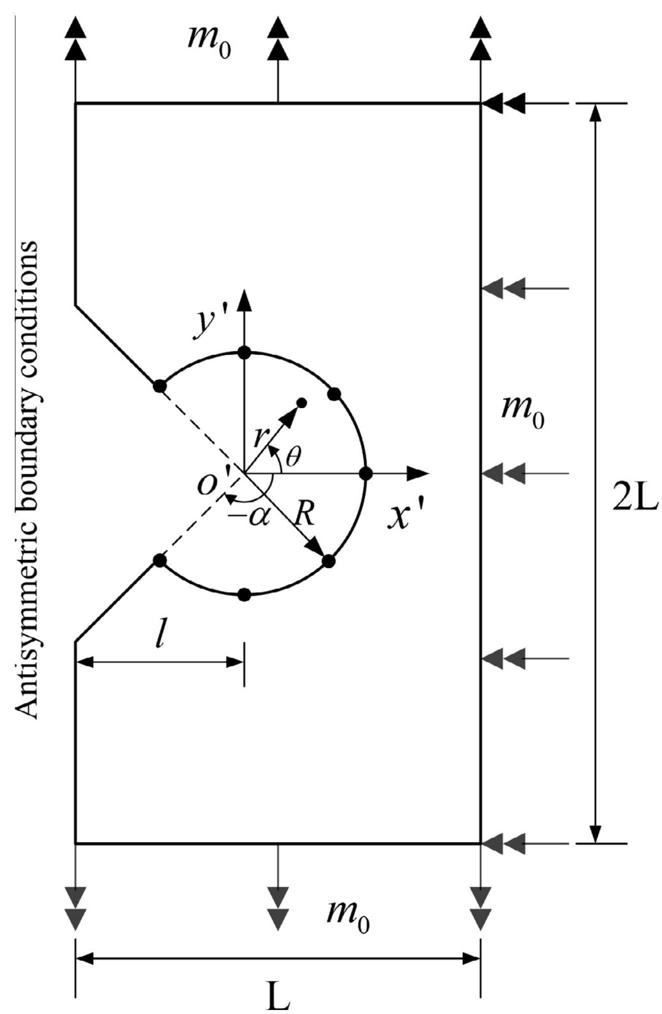

Fig. 3. Half of the plate with a singular element installed on the notch-tip.

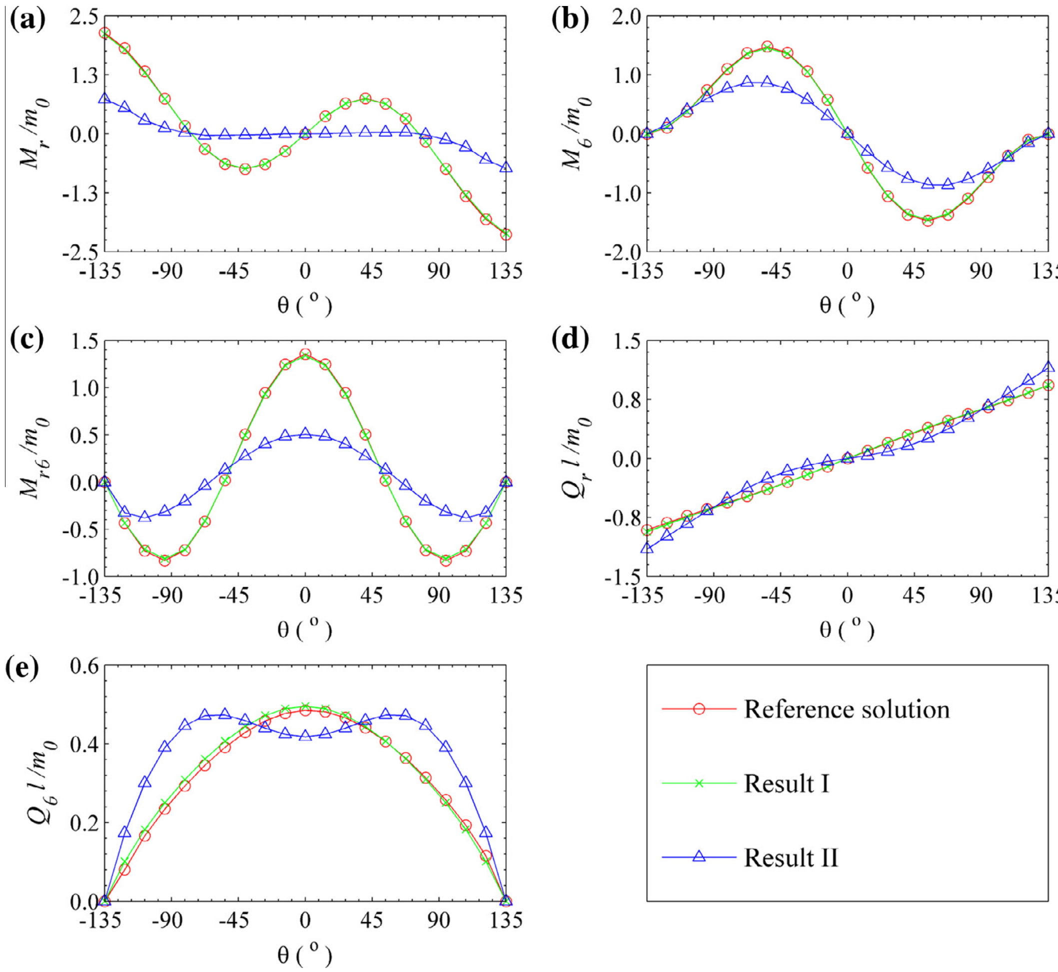

Fig. 4. Angular variations of the internal forces near the tip of a V-shaped notch in Reissner plate for $\alpha = 135^\circ$.

Fig. 3). During all calculations, we keep the radius of the singular element $R = 0.3l$, and the number of export nodes $n = 17$.

Based on the asymptotic expansion solutions where the first four order expansion terms for each eigenvalue are employed, a singular element is constructed. In this example, two sets of results obtained by using the singular element method are presented, in which one is denoted by "result I" calculated by employing the asymptotic solution of $\lambda = 1$, whereas the other one is denoted by "result II" calculated by neglecting the eigenfunction expansion of $\lambda = 1$.

In order to facilitate comparison, the dimensionless quantities $\tilde{M}_r = M_r/m_0$, $\tilde{M}_\theta = M_\theta/m_0$, $\tilde{M}_{r\theta} = M_{r\theta}/m_0$, $\tilde{Q}_r = Q_r l/m_0$ and $\tilde{Q}_\theta = Q_\theta l/m_0$ are introduced. When $\alpha = 135^\circ$, numerical results of the dimensionless bending moments $\tilde{M}_r$, $\tilde{M}_\theta$ and $\tilde{M}_{r\theta}$, and the dimensionless shear forces $\tilde{Q}_r$ and $\tilde{Q}_\theta$ distributed along the circumference $r/R = 0.5$ are plotted in Fig. 4(a)–(e), respectively. Actually, the results indicate the distribution of the bending moments and the shear forces near the notch-tip in the angular direction. Subsequently, results along a radius line at $\theta = 0^\circ$ are chosen when $\alpha = 157.5^\circ$. Because of symmetry, only bending moment $M_{r\theta}$ and shear force Q_θ are nonzero along this direction. The numerical results of the dimensionless bending moment $\tilde{M}_{r\theta}$ and the dimensionless shear force $\tilde{Q}_\theta$ are plotted in Fig. 5(a) and (b), respectively. Results calculated by conventional FEM using dense finite meshes with 6480 elements are provided as reference solutions. It can be seen that "result I" is in excellent agreement with the reference solutions, while "result II" has a significant difference with them, especially for the bending moments. The great deviation is mainly caused by neglecting the eigenfunction expansion solution corresponding to $\lambda = 1$ in the asymptotic expressions for the displacement field near the tip of a notch. On the other hand, it is shown that the Jordan form asymptotic solutions presented in the previous sections are correct and the present study is proven to be valid.

For general vertex angles of the notch, numerical results of the dimensionless bending moment $\tilde{M}_{r\theta}$ and the dimensionless shear force $\tilde{Q}_\theta$ at $r/R = 0.5$, $\theta = 0^\circ$ for different half opening angles of the notch are plotted in Fig. 6(a) and (b), respectively. It can be seen that "result I" is in excellent agreement with reference results while "result II" has great difference. Again, it illustrates that the asymptotic eigensolution corresponding to $\lambda = 1$ can make a significant impact on the numerical results not only for the special cases, but also for the general case. In other words, the eigensolution corresponding to $\lambda = 1$ plays an important role in determining the near-tip fields for the V-notch, especially for antisymmetric deformation.

7. Discussions and conclusions

In the present paper the asymptotic expansion solutions for the notch-tip field in Reissner plate have been reexamined and derived systematically using the eigenfunction expansion method. For an

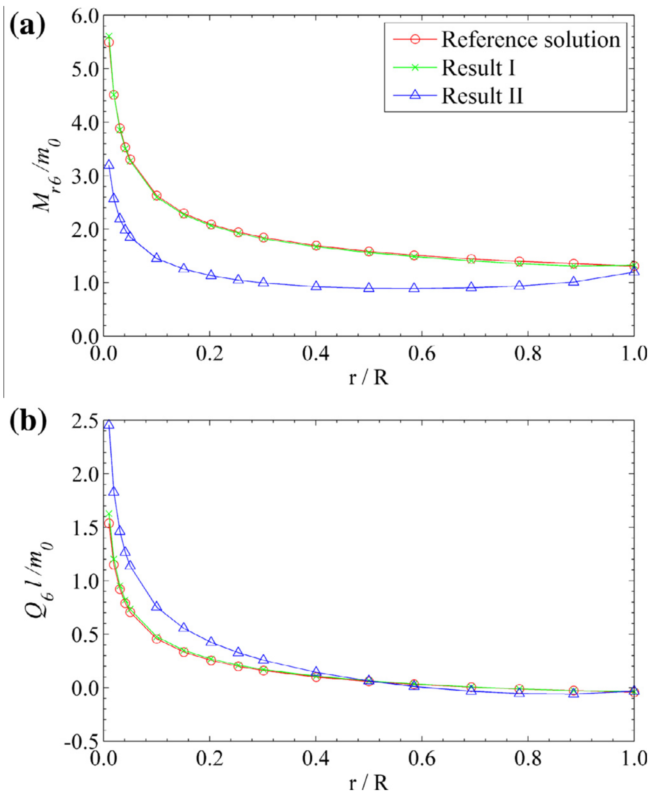

Fig. 5. Variations of the internal forces along $\theta = 0^\circ$ in front of the V-notch tip for $\alpha = 157.5^\circ$.

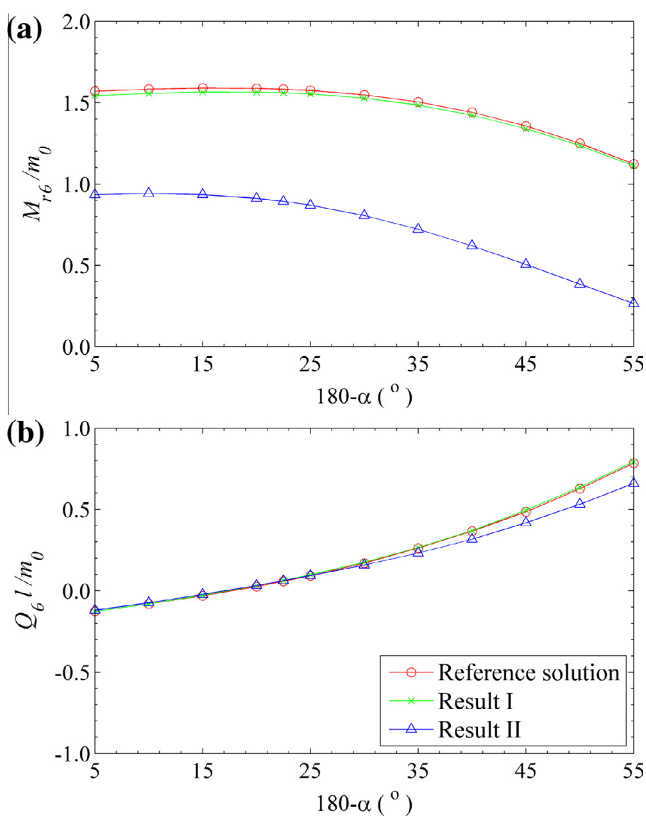

Fig. 6. Variations of the internal forces $M_{r\theta}$ and Q_θ at $r/R = 0.5$, $\theta = 0^\circ$ for different half opening angles of the notch.

arbitrary vertex angle of the notch, the eigenvalue $\lambda = 1$ is always a root of the eigenequation for the V-notch problem in Reissner plate, and the corresponding eigenfunction expansion solution plays an important role in determining the stress and strain fields near the notch-tip, especially for antisymmetric deformation. As a supplement to previous work, the crucial asymptotic solution is derived in the present study. Moreover, a paradox is found to arise in the expansion terms of the asymptotic solution for some special vertex angles (e.g. $\alpha = 135^\circ$, 112.5° or 157.5°). The cases of the paradox are studied, and the corresponding bounded solutions are found to be explained by the Jordan form solution. Theoretically, the paradox may be found when the difference between two eigenvalues is an integer, while this is only a necessary condition. After a rational derivation procedure, the Jordan form asymptotic solutions are obtained and specified in explicit form, and the higher order terms are suggested to be solved through a similar procedure. Furthermore, Jordan form solution is also found to arise in another special case where $\tan(2\alpha) = 2\alpha$, $\alpha \approx 128.7^\circ$, because $\lambda = 1$ becomes a double root of the eigenequation in this case. Again, the Jordan form asymptotic solution is specified in explicit form. A numerical example is given to illustrate the validity of the present study. Two kinds of numerical results calculated based on the asymptotic expansion solutions, respectively, with and without the eigensolution corresponding to $\lambda = 1$, together with the FEM results with dense meshes are given. It is observed that the asymptotic expansion solution corresponding to $\lambda = 1$ is important for the distribution of the displacement and stress fields near the notch tip, and a significant error could be resulted in when the solution is neglected in related numerical studies. On the other hand, it is shown that the Jordan form asymptotic solutions presented in the present paper are correct.

The present study intends to contribute to the completeness of the eigenfunction expansion solutions of the asymptotic near-tip fields for V-shaped notch in Reissner plate, especially for the special cases, and propose a rational method to derive the Jordan form solutions of similar problems.

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