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Elastica of sandwich panels with a transversely flexible core—A high-order theory approach

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ABSTRACT

The elastica behavior of an extensional sandwich panel with a “soft” core when subjected to in-plane compressive loads is presented and it is compared with the response of its extensional equivalent single layer (ESL) with shear deformations model. The field equations along with the appropriate boundary conditions for the sandwich and the ESL panels have been derived through a variational approach following the High-order SAndwich Panel Theory (HSAPT) approach that takes into account the vertical flexibility of the core. The governing equations include the effects of the extension of the mid-surfaces of the face sheets of the sandwich panel or the mid-plane of the ESL model which the classical elastica approach misses. The results of the elastica response of a clamped-simply-supported sandwich panel and its ESL counterpart are presented and compared. They include the response along the panel, deformed shapes and equilibrium curves of in-plane loads versus structural quantities such as displacements and internal stress resultants and stresses. These results reveal that the predicted buckling load of the ESL panel is larger than that of the sandwich panel and that deep in the non-linear range the upper face sheet wrinkles with increasing overall and edge displacements and a release of the load. Hence, the use of an equivalent single layer panel especially when a sandwich panel with a compliant core is considered may lead to unsafe and unreliable predictions when large displacements and large rotations are considered.

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1. Introduction

Modern sandwich panels, which consist of two face sheets and a compliant low strength compressible core, are used in a large variety of applications within the aerospace, naval and transportation industries. In order to exploit their structural potential and to define their safety their non-linear response that is based on large displacements and large rotations, i.e. elastica, is required. A typical sandwich panel is a layered structure that consists of face sheets made of metal or laminated composite and a core that is made of either metallic or low strength compressible honeycomb or foam. As a result of the core flexibility and compressibility, the shape of the panel is not preserved under deformation along with distortions of its section plane.

In general, the approaches considered for the analysis of sandwich panels can be gathered in two major categories. In the first one the actual layered panel is replaced by an equivalent one with a single layer, denoted by ESL (equivalent single layer) with equivalent properties, see for example Mindlin first-order theory (FOSDT) (1951), and Reddy's high-order theories (1984) and others. In the second category, denoted also as the classical approach, the layered configuration is assume to

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consist of a core that is incompressible and is infinitely stiff in the vertical direction, see textbooks by Allen (1969), Plantema (1966), Zenkert (1995) and Vinson (1999). A different approach where the vertical flexibility of the core is considered along with the localized effects involved has been considered by the author and many others using the High-order SANDwich Panel Theory (HSAPT) approach, see Frostig et al. (1992). It has been applied successfully to a large number of linear and non-linear analyses such as: debonding of face sheets, see Frostig (1992), buckling of sandwich panels, see Frostig and Baruch (1993) and Frostig (1998), non-linear behavior of sandwich panels with rigid and non-rigid interfaces including branching behavior, see Sokolinsky and Frostig (2000), comparisons of the HSAPT model elasticity and FE results see Swanson (1999), indentation resistance analysis using the HSAPT model, see Petras and Sutcliffe (1999, 2000), an experimental and analytical study, see Sokolinsky et al. (2003), non-linear response of sandwich panels, see Frostig and Thomsen (2005), and non-linear response of sandwich shells, see Hohe and Librescu (2003).

Elastica of inextensible beams made of a solid section had attracted the attention of many researchers for many years starting with the pioneering works of Euler and Lagrange, see Dym and Shames (1973), using closed-form solutions with the aid of elliptic integrals. The elastica response of bars with various sections, particular type of loadings and boundary conditions appears in a numerous number of research works and with various type of solution approaches and to mention a few: Wang (1997) dealt with elastica of a clamped-simply-supported beam using perturbation, asymptotic and numerical method; Chucheepsakul and Huang (1997) used FE approach to analyze a beam with a point load between the supports; Ligamov and Ratrou (1999) have analyzed the large displacements of a superconducting cable with fluids inside; temperature effects have been investigated by Coffin and Bloom (1999), Vinogradov and Derrick (2000) investigated the elastica of a bar made of asymmetric laminates layers; Lee (2001, 2002) dealt with a cantilever under distributed and a concentrated load with a linear and non-linear material and Madhusudan et al. (2003) analyzed a cantilever with a variable cross section. In general, the analyses of the elastica of rods, mentioned above, use the equilibrium equations which have been derived through equilibrium of the deformed shape of a differential segment of the rod.

A variational approach that uses Reissner's kinematic relations (see Reissner (1972)), has been considered by Flajs et al. (2003) that used Lagrange multiplier to impose the kinematic relations as constraints and by Pak and Stauffer (1993).

The problem of large displacements and large rotations of unidirectional sandwich panels has been considered by a very small number of researchers assuming that the layered sandwich panel can be replaced by an equivalent single layer with shear deformation, see Huang and Kardomateas (2002) and Bazant and Beghini (2006).

The brief literature survey reveals that the elastica response of sandwich panels made of two face sheets and a compliant core where the vertical flexibility of the core is considered is missing. The approach used here is based on the HSAPT model along with the variational approach. The assumptions adopted follow the "classical" assumptions for sandwich structures with compliant cores: the face sheets possess in-plane and bending rigidities; the face sheets and the core material are assumed be linear elastic; the face sheets include shear deformations following the FOSDT approach and they undergo large displacements and large rotations with moderate strains; the core is considered as a 2D linear elastic continuum that undergoes large rigid body displacements (due to its bond to the adjacent face sheets), but with kinematic relations that correspond to those of small deformations, where the core height may change during deformation and its section plane does not remain plane after deformation; the core possesses only shear and vertical normal stiffness, whereas the in-plane (longitudinal) normal stiffness is neglected; full bond is assumed between the face sheets and the core; and the mechanical loads are applied to the face sheets only; and it is also assumed that the sandwich panel does not loses its integrity as a result of the large deformations and the associated stresses are within the strength capacity of the components of the panel.

The paper consists of a mathematical formulation and a numerical part. In the mathematical formulation, the elastica field and governing equations along with the appropriate boundary, of a sandwich panel that follows the HSAPT model and its ESL counterpart are derived using a variational approach. In the numerical part the elastica response of a real sandwich panel made of three layers and its equivalent single layer (ESL) model with shear deformations are presented. A comparison between the two models is presented and discussed. Finally a summary is included and conclusions are drawn.

2. Mathematical formulation

The mathematical formulation consists of two parts. The first part deals with a real sandwich panel using the HSAPT approach where the shear deformations of the face sheets are considered. The second part is based on the first part and defines the equations for an equivalent single layer sandwich panel with shear deformations. The derivation of the governing field equations of the layered sandwich panel with their appropriate boundary conditions for the face sheets and the core and the stress and the deformations field of the core in a closed-form is presented next.

2.1. Sandwich panel—HSAPT model

The governing equations and the boundary conditions are derived via the variational principle imposed on the total potential energy, as follows:

$$\delta(U + V) = 0 \quad (1)$$

where U and V are the internal and the external potential energy, respectively, and δ denotes the variation operator.

The first variation of the internal potential energy in terms of stresses and strains reads:

$$\delta U = \int_{V_t} (\sigma_{sst} \delta \epsilon_{sst} + \tau_{snt} \delta \gamma'_{snt}) dV + \int_{V_b} (\sigma_{ssb} \delta \epsilon_{ssb} + \tau_{snb} \delta \gamma'_{snb}) dV + \int_{V_c} (\tau_{xz} \delta \gamma'_{sz} + \delta \sigma_{zz} \epsilon_{zz}) dV + \delta U_\lambda \quad (2)$$

where σ_{ssj} and ϵ_{ssj} ($j = t, b$) are the in-plane normal stresses and τ_{snj} and γ'_{snj} ($j = t, b$) are the vertical shear stress and shear angle of the face sheets (in local coordinate directions); τ_{xz} and γ'_{sz} are the vertical shear stresses strains in the core; σ_{zz} and ϵ_{zz} are the normal stresses and strains in the vertical direction of the core (global coordinate directions); V_k ($k = t, b, c$) are the volume of the upper and lower face sheets and the core, respectively; dV denotes the volume of a differential segment and δU_λ is the contribution of imposed conditions, see Eq. (8) ahead.

$$\delta V = - \left(\int_0^L (n_{xt} \delta u_{ot} + q_{zt} \delta w_t + m_t \delta w_{t,s} + n_{xb} \delta u_{ob} + q_{zb} \delta w_b + m_b \delta w_{b,s}) ds \right) - \left(\sum_{i=1}^{NC} \left(\int_0^L (N_{xeti} \delta u_{ot} + P_{zeti} \delta w_t + M_{eti} \delta w_{t,s} + N_{xebi} \delta u_{ob} + P_{zebi} \delta w_b + M_{ebi} \delta w_{b,s}) \delta_d(s - s_i) ds \right) \right) \quad (3)$$

where n_{xj} , q_{zj} and m_j ($j = t, b$) are the in-plane and vertical distributed loads (in the global coordinates) and the bending moment distributed loads, respectively; N_{xeji} , P_{zeji} and M_{eji} ($j = t, b$) are external concentrated loads in the in-plane, vertical direction (global coordinates) and the concentrated moment respectively exerted at $s = s_i$; NC denotes the number of concentrated loads; $\delta_d(s - s_i)$ are the delta of Dirac function; u_{oj} , w_j and $w_{j,s}$ ($j = t, b$) are the in-plane and the vertical displacement (in global coordinates directions) and the rotation of each face sheets, respectively and s is the longitudinal coordinate. Geometry and sign convention of stresses, displacements, and loads appear in Fig. 1.

The displacements pattern of the face sheets takes into account the shear deformations following the FOSDT approach, see Mindlin (1951), and they read ($j = t, b$):

$$u_j(s, z_j) = u_{oj}(s) - z_j \Psi_j(s) \quad (4)$$

where z_j are the vertical coordinates of each face sheet independently and they are measured downwards from the centroid of each face sheet (see Fig. 1a); u_{oj} are the in-plane displacements at the centroid of the face sheets and $\Psi_j(s)$ is the rotation angle of the section plane of the face sheets and is related to the shear angle, $\gamma_j(s)$ by the following expression:

$$\gamma_j(s) = -\Psi_j(s) + \frac{d}{ds} w_j(s) \quad (5)$$

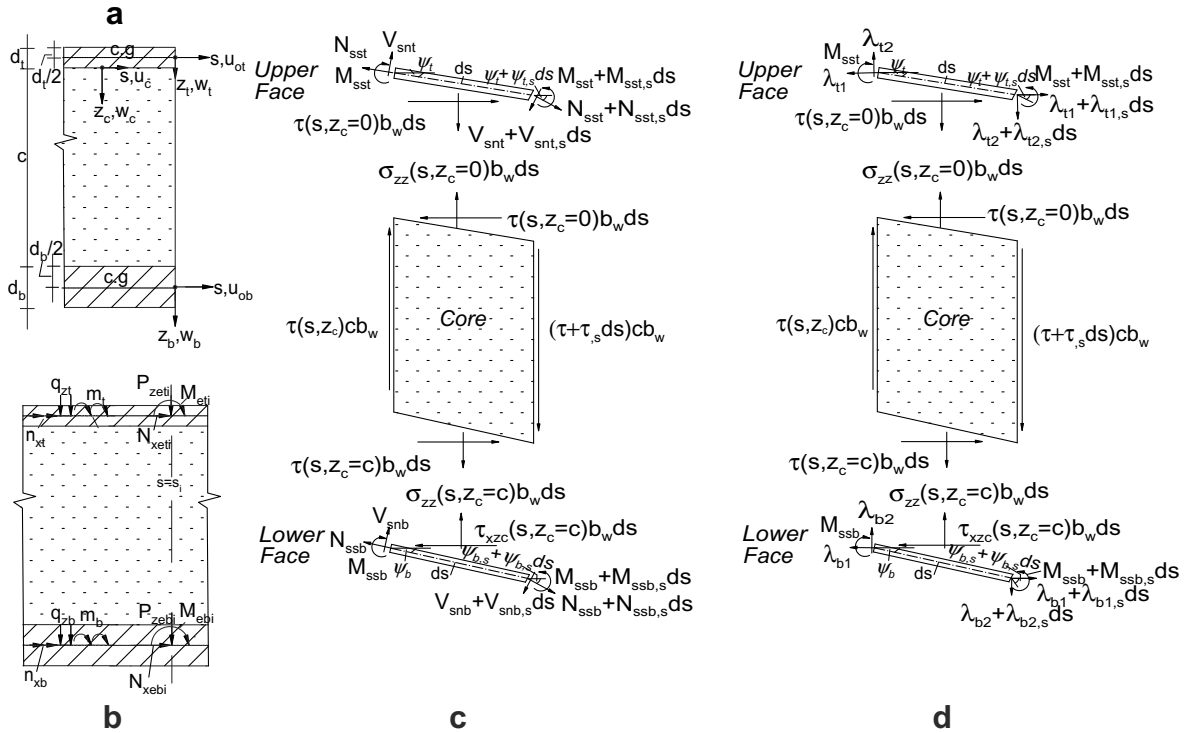


Fig. 1. Dimensions and signs conventions of a typical sandwich panel. (a) Geometry; (b) loads at face sheets. Internal stress resultants in (c) local coordinate directions and (d) in global coordinate directions.

Notice that the shear strain is independent of the vertical coordinate of the face sheet following the FOSDT computational model. In addition when the shear deformation is neglected the slope of the section plane equals $\Psi_j(s) = \frac{d}{ds} w_j(s)$.

The kinematic relations of large displacements and large rotations of the face sheets with shear deformations read:

$$\varepsilon_{ssj}(s, Z) = \varepsilon_{oj}(s) - z_j \left(\frac{d}{ds} \Psi_j(s) \right) \quad (6)$$

where $\varepsilon_{oj}(s)$ is the mid-plane strain and $\gamma_j(s)$ is the shear angle of the face sheets. They can be defined using the displacements pattern that appears in Fig. 2 for an isolated differential segment of a typical face sheet and they read ($j = t, b$):

$$\begin{aligned} \frac{d}{ds} u_{oj}(s) &= (1 + \varepsilon_{oj}(s)) (\cos(\Psi_j(s)) \cos(\gamma(s)) - \sin(\Psi_j(s)) \sin(\gamma(s))) \\ &\approx (1 + \varepsilon_{oj}(s)) \cos(\Psi_j(s)) - \sin(\Psi_j(s)) \gamma(s) \\ \frac{d}{ds} w_j(s) &= (1 + \varepsilon_{oj}(s)) \sin(\Psi_j(s)) \cos(\gamma(s)) + \cos(\Psi_j(s)) \sin(\gamma(s)) \\ &\approx (1 + \varepsilon_{oj}(s)) \sin(\Psi_j(s)) + \gamma_j(s) \cos(\Psi_j(s)) \end{aligned} \quad (7)$$

where $u_{oj}(s)$ and $w_j(s)$ are the in-plane and vertical displacements, respectively, in the global coordinates, of the face sheets, respectively. Notice that the approximated relations, see right equations of Eq. (7), equal those denoted as Reissner strains, see Reissner (1972) and they correspond to the kinematic relations of elastica with moderate strains. They are implemented in the analysis through Lagrange multipliers, λ_{j1} and λ_{j2} , ($j = t, b$) that are added to the internal potential energy of the panel, see Eq. (2), as follows:

$$\begin{aligned} \delta U_\lambda &= \left[\int_0^L \lambda_{j1}(s) \left(\frac{d}{ds} u_{oj}(s) - (1 + \varepsilon_{oj}(s)) \cos(\Psi_j(s)) + \gamma_j(s) \sin(\Psi_j(s)) \right) ds \right. \\ &\quad \left. + \int_0^L \lambda_{j2}(s) \left(\frac{d}{ds} w_{oj}(s) - (1 + \varepsilon_{oj}(s)) \sin(\Psi_j(s)) - \gamma_j(s) \cos(\Psi_j(s)) \right) ds \right] \Big|_{j=t,b} \quad (8) \end{aligned}$$

The kinematic relations of the core are those of small displacements and they read:

$$\varepsilon_{zzc}(s, Z_c) = \frac{\partial}{\partial Z_c} w_c(s, Z_c), \quad \gamma_{szc}(s, Z_c) = \frac{\partial}{\partial Z_c} u_c(s, Z_c) + \frac{\partial}{\partial s} w_c(s, Z_c) \quad (9)$$

where $u_c(s, Z_c)$ and $w_c(s, Z_c)$ are the longitudinal and the vertical displacements of the core, respectively, in the global coordinates directions; z_c is the vertical coordinate of the core measured downwards from upper core-face interface (see Fig. 1a) and $\varepsilon_{zzc}(s, Z_c)$ and $\gamma_{szc}(s, Z_c)$ are the vertical normal strain and the shear angle in the global vertical direction of the core, respectively.

The requirements of full bond at the upper and the lower face-core interfaces are expressed through the following compatibility conditions, see Eq. (4), at the upper and the lower face-core interface, respectively:

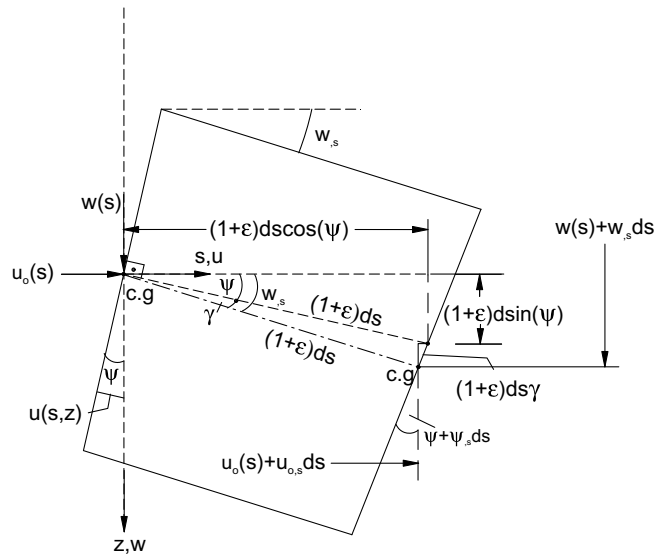


Fig. 2. Displacements pattern of a typical differential segment of a face sheet with large displacements and large rotations.

$$u_c(s, z_c = 0) = u_{ot}(s) - \frac{1}{2} d_t \Psi_t(s), \quad w_c(s, z_c = 0) = w_t(s) \quad (10)$$

$$u_c(s, z_c = 0) = u_{ob}(s) + \frac{1}{2} d_b \Psi_b(s), \quad w_c(s, z_c = c) = w_b(s) \quad (11)$$

where c is the height of the core and $d_j (j = t, b)$ are the thickness of the upper and the lower face sheets, respectively, see Fig. 1a.

The field equations are derived using Eqs. (1) and (2) with Eqs. (8) and (3) along with the kinematic relations of the face sheets and the core, Eqs. (6) and (9), and the compatibility conditions at the face-core interfaces, Eqs. (10) and (11). After some integration by parts and some algebraic manipulations they read:

For the face sheets ($j = t, b$):

$$-\left(\frac{d}{ds} \lambda_{j1}(s)\right) - \alpha \tau_{cj}(s) b_w - n_{xj}(s) = 0 \quad (12)$$

$$-\left(\frac{d}{ds} \lambda_{j2}(s)\right) - \alpha \sigma_{zzcj}(s) b_w - q_{zj}(s) = 0 \quad (13)$$

$$(-\lambda_{j2}(s) \epsilon_j(s) + \lambda_{j1}(s) \gamma_j(s) - \lambda_{j2}(s)) \cos(\Psi_j(s)) + (\lambda_{j2}(s) \gamma_j(s) + \lambda_{j1}(s) + \lambda_{j1}(s) \epsilon_j(s)) \sin(\Psi_j(s)) - m_j(s) + \frac{d}{ds} M_{ssj}(s) + \frac{1}{2} \tau_{cj}(s) b_w d_j = 0 \quad (14)$$

$$\frac{d}{ds} u_{oj}(s) - \cos(\Psi_j(s)) - \cos(\Psi_j(s)) \epsilon_j(s) + \gamma_j(s) \sin(\Psi_j(s)) = 0 \quad (15)$$

$$\frac{d}{ds} w_j(s) - \sin(\Psi_j(s)) - \sin(\Psi_j(s)) \epsilon_j(s) - \gamma_j(s) \cos(\Psi_j(s)) = 0 \quad (16)$$

$$-\lambda_{j2}(s) \sin(\Psi_j(s)) - \lambda_{j1}(s) \cos(\Psi_j(s)) + N_{ssj}(s) = 0 \quad (17)$$

$$-\lambda_{j2}(s) \cos(\Psi_j(s)) + \lambda_{j1}(s) \sin(\Psi_j(s)) + V_{snj}(s) = 0 \quad (18)$$

where $\tau_{cj}(s)$ and $\sigma_{zzcj}(s)$ ($j = t, b$) are the interfacial shear and vertical normal stresses at the upper ($z_c = 0$) and at the lower ($z_c = c$) face-core interfaces, respectively; $\alpha = 1$ when $j = t$ and -1 when $j = b$; N_{ssj} , V_{snj} and M_{ssj} ($j = t, b$) are the in-plane, shear and bending moments stress resultants of the upper and the lower face sheets, respectively, see Fig. 1c. Notice that the field equations of the face sheets, Eqs. (12)–(14), are the equilibrium equations of the face sheets in the global longitudinal and vertical directions and the moment, see Fig. 1d. In addition, notice that the Lagrange multipliers, $\lambda_{j1}(s)$ and $\lambda_{j2}(s)$, can be defined in terms of the in-plane and shear stress resultants of the face sheets through the solution of Eqs. (17) and (18) and they read:

$$\begin{aligned} \lambda_{j1}(s) &= -\sin(\Psi_j(s)) V_{snj}(s) + \cos(\Psi_j(s)) N_{ssj}(s) \\ \lambda_{j2}(s) &= \cos(\Psi_j(s)) V_{snj}(s) + N_{ssj}(s) \sin(\Psi_j(s)) \end{aligned} \quad (19)$$

Thus, these Lagrange multipliers are actually components of the axial and shear stress resultants in the longitudinal and the vertical direction of the global coordinate system, see Fig. 1d. Eqs. (15) and (16) are the imposed strain relations due to the elastica deformation pattern, see approximated equations in Eq. (7). Please notice that the mid in-plane strain and the shear angle of the face sheets are defined through the solution of Eqs. (15) and (16) and they read:

$$\begin{aligned} \epsilon_j(s) &= \left(\frac{d}{ds} w_j(s)\right) \sin(\Psi_j(s)) - 1 + \cos(\Psi_j(s)) \left(\frac{d}{ds} u_{oj}(s)\right) \\ \gamma_j(s) &= -\sin(\Psi_j(s)) \left(\frac{d}{ds} u_{oj}(s)\right) + \cos(\Psi_j(s)) \left(\frac{d}{ds} w_j(s)\right) \end{aligned} \quad (20)$$

In addition, notice that when substituting Eq. (20) into the moment equilibrium equation, see Eq. (14), it changes into the following simplified form:

$$-\lambda_{j2}(s) \left(\frac{d}{ds} u_{oj}(s)\right) + \lambda_{j1}(s) \left(\frac{d}{ds} w_j(s)\right) - m_j + \frac{d}{ds} M_{ssj}(s) + \frac{1}{2} \tau_{cj}(s) b_w d_j = 0 \quad (21)$$

For the core:

$$-\left(\frac{\partial}{\partial z_c} \tau(s, z_c)\right) = 0, \quad -\left(\frac{\partial}{\partial s} \tau(s, z_c)\right) - \left(\frac{\partial}{\partial z_c} \sigma_{zzc}(s, z_c)\right) = 0 \quad (22)$$

where $\tau(s, z_c)$ and $\sigma_{zzc}(s, z_c)$ are the shear and the vertical normal stresses within the core and they coincide with those of the linear high-order sandwich model theory, see Frostig et al. (1992).

The solution of the fields differential equation of the core, see Eq. (22) yields that the shear stresses through the depth of the core are uniform and the vertical normal stresses are linear as follows:

$$\tau(s, z_c) = \tau(s), \quad \sigma_{zzc}(s, z_c) = -\left(\frac{d}{ds} \tau(s)\right) z_c + C_{wl}(s) \quad (23)$$

where $C_{w1}(s)$ is a constant of integration to be determined by the compatibility condition at the upper face-core interface. The uniform distribution of the shear stress within the core yields that $\tau_{ct}(s) = \tau_{cb}(s) = \tau(s)$, see Eqs. (12) and (13).

The boundary conditions at the edges of the panel ($s_e = 0, L$), which are a by product of the variational approach and are defined by the variational terms at each face sheet and the core at the edges of the panel, read:

For the face sheets ($j = t, b$):

$$\begin{aligned} -N_{xej}(s_e) + \alpha\lambda_{j1}(s_e) &= 0 \quad \text{or} \quad u_{odj}(s_e) = u_{eoj} \\ -M_{ej} - \alpha M_{ssj}(s_e) &= 0 \quad \text{or} \quad \Psi_j(s_e) = \Psi_{ej} \\ \alpha\lambda_{j2}(s_e) - P_{zej} &= 0 \quad \text{or} \quad w_j(s_e) = w_{ej} \end{aligned} \quad (24)$$

where u_{eoj} , Ψ_{ej} and w_{ej} are the prescribed in-plane displacement, rotation and vertical displacement at the edges of the upper and the lower face sheets (in the global coordinate directions), respectively; N_{xej} , M_{ej} and P_{zej} are the external concentrated loads, in the longitudinal direction, bending moment and external vertical forces at the edges of the face sheets respectively (in the global coordinate directions); $\alpha = 1$ when $s_e = L$ and $\alpha = -1$ when $s_e = 0$ and $u_{odj}(s_e) = u_{oj}(s_e) - s_e$ is the in-plane displacement. See Fig. 3a for sign conventions, loading and stress resultants directions.

For the core:

$$\tau(s_e) = 0 \quad \text{or} \quad w_c(s_e, z_c) = w_{ec}(z_c) \quad (25)$$

where $w_{ec}(z_c)$ is the prescribed vertical displacement distribution, in the global coordinate direction, through the depth of the core at its edges. Notice, that in the case of an edge beam which is infinitely stiff and bonded to the adjacent core, see Fig. 3b, the boundary conditions consist of four geometrical conditions and three natural ones as follows:

Geometrical conditions:

$$\begin{aligned} w_t(s_e) &= w_b(s_e) = w_{c,av}(s_e) \\ \Psi_t(s_e) - \Psi_b(s_e) &= 0 \\ \Psi_t(s_e) + \frac{u_{ot}(s_e) - u_{ob}(s_e)}{z_{tcg} + z_{bcg}} &= 0 \end{aligned} \quad (26)$$

where $w_{c,av}(s_e)$ is the average vertical displacement of the core to be defined ahead, z_{jcg} ($j = t, b$) are the vertical distances of the centroid of the edge beam from the centroidal lines of the face sheets, see Fig. 3b.

Natural conditions:

$$\begin{aligned} -N_{xeg}(s_e) + \alpha(\lambda_{t1}(s_e) + \lambda_{b1}(s_e)) &= 0 \quad \text{or} \quad u_{odg}(s_e) - u_{oeg}(s_e) = 0 \\ (\lambda_{t1}(s_e)z_{tcg} - M_{sst}(s_e) - M_{ssb}(s_e) - \lambda_{b1}(s_e)z_{bcg})\alpha + M_{eg}(s_e) &= 0 \quad \text{or} \quad \Psi_t(s_e) - \Psi_{eg}(s_e) = 0 \\ -P_{zeg}(s_e) + \alpha(\lambda_{t2}(s_e) + \lambda_{b2}(s_e)) &= 0 \quad \text{or} \quad w_t(s_e) - w_{eg}(s_e) = 0 \end{aligned} \quad (27)$$

where $u_{odg}(s_e) = \frac{u_{ob}(s_e)z_{tcg}}{c + \frac{1}{2}d_t + \frac{1}{2}d_b} + u_{ot}(s_e)\left(1 - \frac{z_{bcg}}{c + \frac{1}{2}d_t + \frac{1}{2}d_b}\right) - s_e$ is the horizontal displacements at the edge beam support; $N_{xeg}(s_e)$, $P_{zeg}(s_e)$ and $M_{eg}(s_e)$ are the external horizontal and vertical external loads and the external global bending moments exerted at the edge beam support and $u_{oeg}(s_e)$, $w_{eg}(s_e)$ and $\Psi_{eg}(s_e)$ are the prescribed horizontal and vertical displacements in the global coordinate and the rotation of the edge beam support, respectively. For details see Fig. 3b.

In order to derive the governing equations of the sandwich panel the stress and the deformation fields of the core must be defined first. The core used here is isotropic with the following constitutive relations:

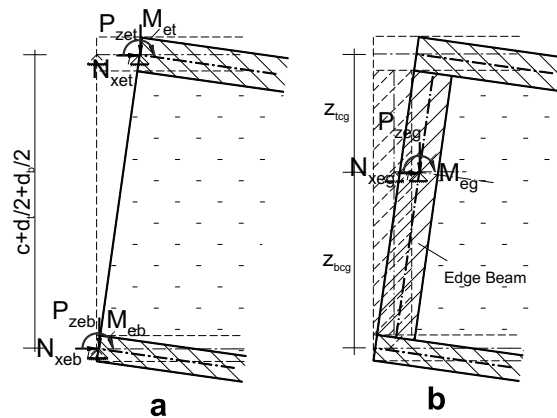


Fig. 3. Edge conditions of a sandwich (a) ordinary edge; (b) reinforced with an edge beam.

$$\epsilon_{zzc}(s, z_c) = \frac{\sigma_{zzc}(s, z_c)}{E_{zc}}, \quad \gamma_{szc}(s, z_c) = \frac{\tau(s, z_c)}{G_{xzc}} \quad (28)$$

where E_{zc} and G_{xzc} are the modulus of elasticity and the shear modulus of the core, respectively.

The displacement and the stress fields of the core follow the results of the linear high-order model and are presented briefly for completeness. For details see Frostig et al. (1992). Thus the displacements field read:

$$\begin{aligned} w_c(s, z_c) &= -\frac{1}{2} \frac{\tau(s) z_c^2}{E_{zc}} + \frac{C_{w1}(s) z_c}{E_{zc}} + C_{w2}(s) \\ u_c(s, z_c) &= \frac{1}{2} \frac{\frac{1}{3} \left(\frac{d}{ds} \tau(s) \right) z_c^3 - \left(\frac{d}{ds} C_{w1}(s) \right) z_c^2}{E_{zc}} \\ &\quad + \left(-\left(\frac{d}{ds} C_{w2}(s) \right) + \frac{\tau(s)}{G_{xzc}} \right) z_c + C_u(s) \end{aligned} \quad (29)$$

where $C_{w2}(s)$ and $C_u(s)$ are additional constants of integration in addition to $C_{w1}(s)$, see Eq. (23). Hence, in order to define these constants of integration three compatibility conditions out of four of the bonding between the face sheets and the core are considered, see Eq. (10) and the second equation of Eq. (11). Hence, after some algebraic manipulation the explicit vertical and longitudinal displacement fields read:

$$\begin{aligned} w_c(s, z_c) &= \left(1 - \frac{z_c}{c} \right) w_t(s) + \frac{z_c w_b(s)}{c} + \frac{\left(\frac{1}{2} c z_c - \frac{1}{2} z_c^2 \right) \left(\frac{d}{ds} \tau(s) \right)}{E_{zc}} \\ u_c(s, z_c) &= \left(\frac{1}{2} \frac{z_c^2}{c} - z_c \right) \left(\frac{d}{ds} w_t(s) \right) + \frac{z_c \tau(s)}{G_{xzc}} - \frac{1}{2} \frac{z_c^2 \left(\frac{d}{ds} w_b(s) \right)}{c} + \left(\frac{1}{6} \frac{z_c^3}{E_{zc}} - \frac{1}{4} \frac{c z_c^2}{E_{zc}} \right) \left(\frac{d^2}{ds^2} \tau(s) \right) + u_{ot}(s) - \frac{1}{2} d_t \Psi_t(s) \end{aligned} \quad (30)$$

The vertical normal stress field is determined through substitution of the vertical displacements, see first equations in Eq. (30), into the first equation of Eq. (28) and using the small displacement kinematic relations, see first equations of Eq. (9). Hence they read:

$$\sigma_{zzc}(s, z_c) = \left(-\frac{w_t(s)}{c} + \frac{w_b(s)}{c} \right) E_{zc} + \left(\frac{1}{2} c - z_c \right) \left(\frac{d}{ds} \tau(s) \right) \quad (31)$$

Thus, the interfacial normal stresses, at the upper and the lower face-core interfaces, at $z_c = 0, c$, read:

$$\begin{aligned} \sigma_{zzct}(s) &= \sigma_{zzc}(s, 0) = \left(-\frac{w_t(s)}{c} + \frac{w_b(s)}{c} \right) E_{zc} + \frac{1}{2} c \left(\frac{d}{ds} \tau(s) \right) \\ \sigma_{zzcb}(s) &= \sigma_{zzc}(s, c) = \left(-\frac{w_t(s)}{c} + \frac{w_b(s)}{c} \right) E_{zc} - \frac{1}{2} c \left(\frac{d}{ds} \tau(s) \right) \end{aligned} \quad (32)$$

Notice that the core fields have been determined using only three compatibility conditions, out of the four, at the face-core interfaces, which are: the longitudinal and vertical conditions of bond at the upper face-core interface (Eq. (10) and the vertical compatibility condition at the lower face-core interface (second equation in Eq. (11)). In addition, the boundary condition of the vertical displacement of the core, see second equation in Eq. (25), may be replaced by the vertical displacement distribution of the core, see first equation in Eq. (30). Hence, the displacement boundary condition of the core should be replaced by the slope of the shear stress. Thus the boundary condition of the core are either imposed on the shear stress or on its slope.

The governing equations of a unidirectional sandwich panel with isotropic face sheets and compliant core are derived using the following force-displacement relations ($j = t, b$):

$$N_{ssj}(s) = EA_j \epsilon_j(s), \quad V_{snj}(s) = k_j GA_j \gamma_j(s), \quad M_{ssj}(s) = -EI_j \left(\frac{d}{ds} \Psi_j(s) \right) \quad (33)$$

where EA_j , GA_j and EI_j ($j = t, b$) are the axial, shear and the flexural rigidity of each face sheet, respectively and k_j is the shear correction coefficient of the various face sheets.

The governing equations are described in terms of the displacements and the Lagrange multipliers. They are defined through the description of the in-plane strain and the shear angle, using Eq. (33), in terms of the axial and shear stress resultants that are described by the Lagrange multipliers, see Eqs. (17) and (18) and the stress fields of the core, see Eqs. (23) and (32). Hence, after some algebraic manipulation the governing equations for the face sheets ($j = t, b$) read:

$$\frac{d}{ds} \lambda_{j1}(s) = -\alpha \tau(s) b_w - n_{xj}(s) \quad (34)$$

$$\frac{d}{ds} \lambda_{j2}(s) = -\alpha \left(\left(-\frac{w_t(s)}{c} + \frac{w_b(s)}{c} \right) E_{zc} + \frac{1}{2} \alpha_c \left(\frac{d}{dx} \tau(s) \right) \right) b_w - q_{zj}(s) \quad (35)$$

$$\begin{aligned} \frac{d}{ds} M_{ssj}(s) = & \left(-\frac{\cos(\Psi_j(s)) \sin(\Psi_j(s))}{EA_j} + \frac{\cos(\Psi_j(s)) \sin(\Psi_j(s))}{k_j GA_j} \right) \lambda_{j1}(s)^2 \\ & + \left(\left(-\frac{\sin(\Psi_j(s))^2}{EA_j} - \frac{\cos(\Psi_j(s))^2}{k_j GA_j} + \frac{\cos(\Psi_j(s))^2}{EA_j} + \frac{\sin(\Psi_j(s))^2}{k_j GA_j} \right) \lambda_{j2}(s) \right. \\ & \left. - \sin(\Psi_j(s)) \lambda_{j1}(s) + \left(\frac{\cos(\Psi_j(s)) \sin(\Psi_j(s))}{EA_j} - \frac{\cos(\Psi_j(s)) \sin(\Psi_j(s))}{k_j GA_j} \right) \lambda_{j2}(s)^2 \right. \\ & \left. + \lambda_{j2}(s) \cos(\Psi_j(s)) - \frac{1}{2} \tau(s) b_w d_j + m_j(s) \right) \end{aligned} \quad (36)$$

$$\frac{d}{ds} \Psi_j(s) = -\frac{M_{ssj}(s)}{EI_j} \quad (37)$$

And the relation between the displacements and Lagrange multiplier are defined by substitution of the in-plane strain and the shear angle expressed in terms of stress resultants in Eqs. (15) and (16) as follows:

$$\begin{aligned} \frac{d}{ds} u_{oj}(s) = & \left(\frac{\cos(\Psi_j(s))^2}{EA_j} + \frac{\sin(\Psi_j(s))^2}{k_j GA_j} \right) \lambda_{j1}(s) + \left(\frac{\cos(\Psi_j(s)) \sin(\Psi_j(s))}{EA_j} \right. \\ & \left. - \frac{\cos(\Psi_j(s)) \sin(\Psi_j(s))}{k_j GA_j} \right) \lambda_{j2}(s) + \cos(\Psi_j(s)) \end{aligned} \quad (38)$$

$$\begin{aligned} \frac{d}{ds} w_j(s) = & \left(\frac{\cos(\Psi_j(s)) \sin(\Psi_j(s))}{EA_j} - \frac{\cos(\Psi_j(s)) \sin(\Psi_j(s))}{k_j GA_j} \right) \lambda_{j1}(s) \\ & + \left(\frac{\sin(\Psi_j(s))^2}{EA_j} + \frac{\cos(\Psi_j(s))^2}{k_j GA_j} \right) \lambda_{j2}(s) + \sin(\Psi_j(s)) \end{aligned} \quad (39)$$

Notice, that when the Lagrange multipliers are isolated using Eqs. (38) and (39) and substituted in the governing equations, Eqs. (34) and (35), they yield two differential equations of the order of two in terms of the global in-plane and vertical displacements.

The last governing equation, denoted as the compatibility equation, is that of the bonding compatibility condition at the lower face-core interface in the longitudinal direction, see first equation of Eq. (11), along with the longitudinal displacement defined by second equation of Eq. (30). Hence, after some algebraic manipulation it reads:

$$-\frac{1}{2} c \left(\frac{d}{ds} w_t(s) \right) + \frac{c \tau(s)}{G_{xzc}} - \frac{1}{2} c \left(\frac{d}{ds} w_b(s) \right) - \frac{1}{12} \frac{c^3 \left(\frac{d^2}{ds^2} \tau(s) \right)}{E_{zc}} + u_{ot}(s) - \frac{1}{2} d_t \Psi_t(s) - u_{ob}(s) - \frac{1}{2} d_b \Psi_b(s) = 0 \quad (40)$$

The governing equations consist of five ordinary differential equations, Eqs. (34)–(37) and Eq. (40), with an order of 14 which corresponds to the number of boundary conditions that have been defined through the variational calculation process, see Eqs. (24) and (25).

2.2. Equivalent single layer (ESL) sandwich panel

The ESL approach replaces the actual layered unidirectional sandwich panel with an equivalent single layer panel that has the same rigidities and follows the well known first-order shear deformable model (FOSDT), see Mindlin (1951). Thus, the axial and the flexural rigidity of the equivalent panel corresponds to that of the face sheet only while the equivalent shear rigidity is mainly that of the core only, see Fig. 4, and they yield the following constitutive relations:

$$N_{ss}(s) = EA_g \varepsilon(s), \quad V_{sn}(s) = k_g GA_g \gamma(s), \quad M_{ss}(s) = -EI_g \left(\frac{d}{ds} \Psi(s) \right) \quad (41)$$

where $\varepsilon(s)$ and $\gamma(s)$ are the mid-plane strains and the shear angle of the equivalent panel, see Fig. 2, and Eq. (7); EA_g , EI_g and $k_g GA_g$ are the equivalent axial, flexural and shear rigidity of the ESL model and they are related to the rigidities of the sandwich panel through the following relations:

$$\begin{aligned} EA_g &= EA_t + EA_b \\ EI_g &= EI_t + EI_b + EA_t z_{cgt}^2 + EA_b z_{cgb}^2 \\ k_g GA_g &= G_c b_w \left(c + \frac{1}{2} d_t + \frac{1}{2} d_b \right), \quad k_g = 1.0 \end{aligned} \quad (42)$$

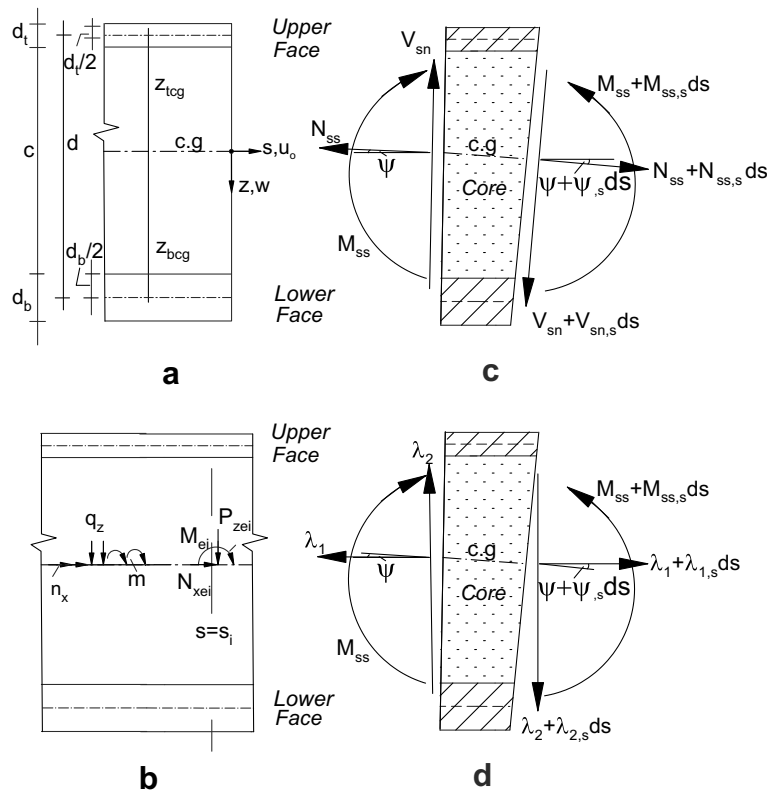


Fig. 4. Dimensions and signs conventions of an equivalent single layer (ESL) sandwich panel. (a) Geometry; (b) loads at face sheets. Internal stress resultants in (c) local coordinate directions and (d) in global coordinate directions.

Notice that z_{cgj} ($j = t, b$) denotes the distance between the centroid of the sandwich panel and the centroid of the upper and the lower face sheet respectively, see Fig. 4a.

The elastica response of the sandwich panel that uses the HSAPT approach takes into account the shear deformations of the face sheets using the FOSDT model with Reissners' moderate non-linear strains assumption, see Eq. (20) in previous chapter. Hence, the governing equations of the ESL model that correspond to those of the face sheets of the sandwich panel, see Eqs. (34)–(39), with some modifications as a result of the free upper and lower surfaces of the equivalent panel. Thus, when substituting $\tau_{cj}(s) = \sigma_{zzcj}(s) = \tau(s) = 0$ ($j = t, b$) into the original equations, Eqs. (34)–(37), of the governing equations of the equivalent panel with the shear deformations read:

$$-\left(\frac{d}{ds}\lambda_1(s)\right) - n_x(s) = 0 \quad (43)$$

$$-\left(\frac{d}{ds}\lambda_2(s)\right) - q_z(s) = 0 \quad (44)$$

$$\begin{aligned} \frac{d}{ds}M_{ss}(s) = & \left(-\frac{\cos(\Psi(s))\sin(\Psi(s))}{EA} + \frac{\cos(\Psi(s))\sin(\Psi(s))}{kGa}\right)\lambda_1(s)^2 \\ & + \left(\left(-\frac{\sin(\Psi(s))^2}{EA} - \frac{\cos(\Psi(s))^2}{kGA} + \frac{\cos(\Psi(s))^2}{EA} + \frac{\sin(\Psi(s))^2}{kGa}\right)\lambda_2(s) - \sin(\Psi(s))\right)\lambda_1(s) \\ & + \left(\frac{\cos(\Psi(s))\sin(\Psi(s))}{EA} - \frac{\cos(\Psi(s))\sin(\Psi(s))}{kGA}\right)\lambda_2(s)^2 \\ & + \lambda_2(s)\cos(\Psi(s)) + m(s) \end{aligned} \quad (45)$$

$$\frac{d}{ds}\Psi(s) = -\frac{M_{ss}(s)}{EI} \quad (46)$$

where λ_1 , λ_2 and M_{ss} and the projections of the axial and shear stress resultants in the longitudinal and vertical directions of the global coordinate system and the bending moment of the equivalent panel, respectively, see Fig. 4c and d, and the relation between the Lagrange multiplier and the in-plane and vertical stress resultants follow Eqs. (17) and (18) but without the j subscript. Notice, that the mid-plane strain and the shear angle of the equivalent panel are those that appear in Eq. (20) but

without the subscript j and the same is true for the boundary conditions that follows those of face sheets, see Eq. (24). In addition, the relation between the external loads applied at the centroid of the equivalent single layer panel, see Fig. 4b, and the loads applied to the face sheet, see Fig. 1b, equals:

$$n_x(s) = n_{xt}(s) + n_{xb}(s), \quad q_z(s) = q_{zt}(s) + q_{zb}(s), \quad m(s) = m_t(s) + m_b(s) + n_{xt}(s)z_{tcg} - n_{xb}(s)z_{bcg} \quad (47)$$

The elastica response is described ahead through the numerical solution of the non-linear set of differential equations that can be solved using numerical schemes such as the multiple-shooting points method, see Stoer and Bulirsch (1980), or the finite-difference (FD) approach using trapezoid or mid-point methods with Richardson extrapolation or deferred corrections, see Ascher and Petzold (1998), as implemented in Maple, see Char et al. (1991), along with parametric or arc-length continuation methods, see Keller (1992). Here, the FD approach has been used.

3. Numerical study

The numerical study presents the elastica response of a sandwich panel with a “soft” core when subjected to in-plane compressive loads and compares it with the elastica response of an equivalent single layer (ESL) model of the real panel with shear deformation. The results include description of the response along the panel and deformed shape at various load levels and equilibrium curves of load versus extreme values of some structural quantities. The main purpose of this study is to demonstrate the differences between the responses of the real sandwich structure and its ESL counterpart.

The panel consists of two face sheets made of Kevlar with an equivalent modulus of elasticity of 27.4 GPa and a shear modulus of 10.55 GPa, and a lightweight, low strength core of Rohacell 50 with $E_{zc} = 70.0$ MPa and $G_{xzc} = 19.0$ MPa. The shear deformations of the face sheets, in this case, have been neglected as a result of their large shear moduli. Hence, the equations used here are the governing ones, see Eqs. (34)–(40), modified by a null shear angle, $\gamma_j(s)$ ($j = t, b$) in the face sheets along with the assumption that the rotations of the face sheets are moderate. The edges of the sandwich panel are clamped on the left edge and simply supported and the right one with immovable conditions. The edges have been reinforced with special beams to make it comparable with the conditions of the ESL model. The in-plane compressive load, applied at the right edge of the panel, has been induced through end-shortening (horizontal movements), denoted by u_{oe} , of the right pinned support, see Fig. 5. The geometry, material properties and boundary conditions of the sandwich panel appear in Fig. 5a and of the ESL

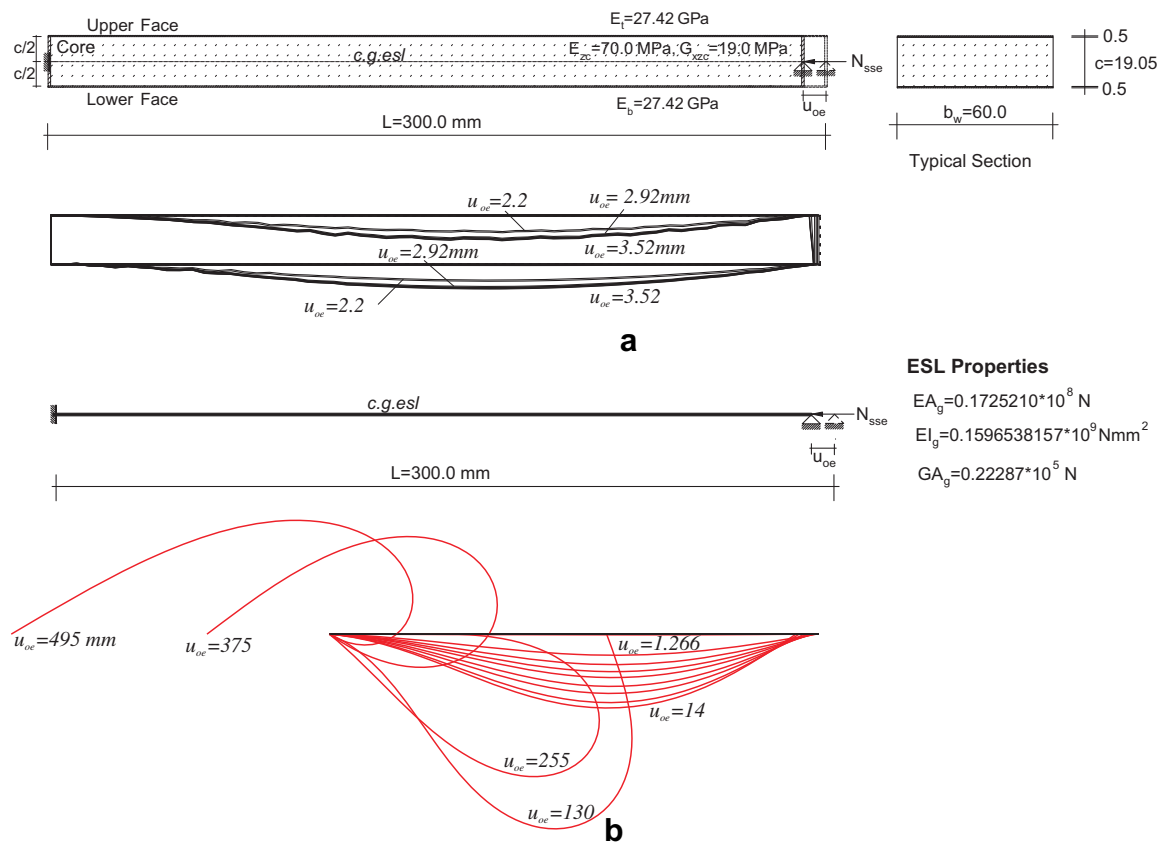


Fig. 5. Geometry, mechanical properties and deformed shape of (a) sandwich panel; (b) equivalent single layer panel.

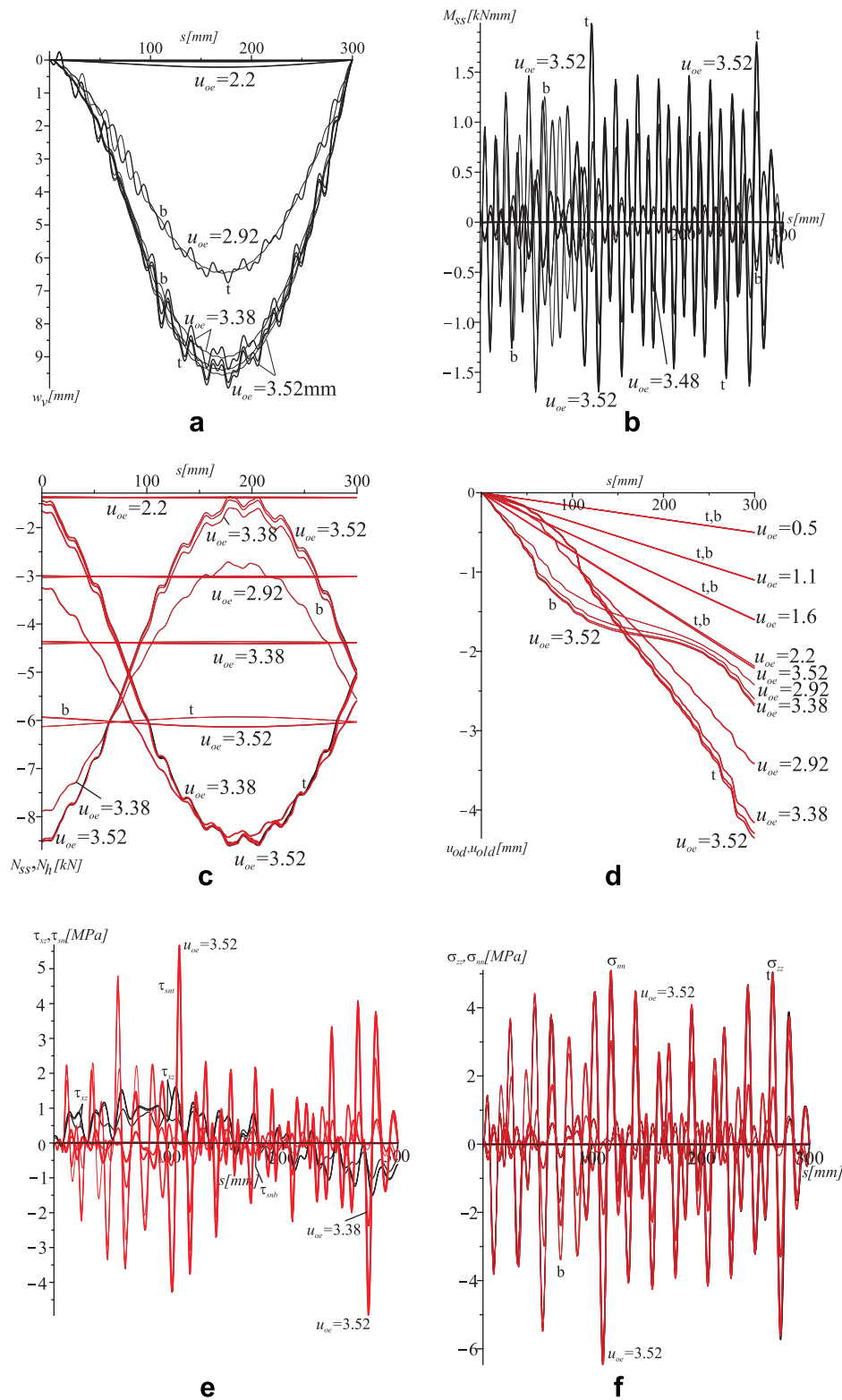


Fig. 6. Sandwich panel results along its length in global and local coordinate directions. At upper and lower face sheets (a) vertical displacements; (b) bending moments; and in core (c) in-plane stress resultant; (d) in-plane displacements; and in core (e) shear stress (global and interfacial locally); (f) interfacial vertical stresses at face-core interfaces. Legend: — (thick) upper face/interface, — (thin) lower face/interface, black: global direction; red: local direction. (For interpretation of the references to colours in this figure legend, the reader is referred to the web version of this paper.)

one in Fig. 5b. The ESL model is described by the centroidal line of the sandwich panel, denoted by $c.g.esl$, see Fig. 5a. In order to achieve a non-trivial solution an imperfection of a small distributed load has been applied to the face sheets of $q_{zt} = q_{zb} = 0.01$ N/mm and $q = 0.00001$ N/mm for the ESL model.

The results of the sandwich panel corresponds to a maximum end-shortening of $u_{oe} = 3.52$ mm due to numerical difficulties while with the ESL model there is no limit and the maximum end-shortening considered reached a value of 495 mm. Numerically, the elastica problem of the sandwich panel is much more sensitive as compared with that of the ESL model when the continuation method used for the two models are identical.

The deformed shape of the two panels appears in Fig. 5. The deformed shape of the sandwich panel appears in Fig. 5a and it describes the deformed shape of the panel at various compressive load levels that correspond to small ($u_{oe} = 2.2$ mm) and large ($u_{oe} = 3.52$ mm) end-shortening. At low load levels the panel exhibits overall buckling, see curve of $u_{oe} = 2.2$ mm, where the two face sheets almost move the same. As the end-displacement increases and the corresponding compressive stress resultant decreases, see Fig. 7a ahead, around mid-span the upper face sheet wrinkles in addition to an overall buckling, see curves of $u_{oe} = 2.92$ – 3.52 mm while the lower face sheet maintain almost a smooth curve with mild wrinkles. In the vicinity of the support the pattern changes and the lower face sheets wrinkles while the upper face sheet curve is smooth

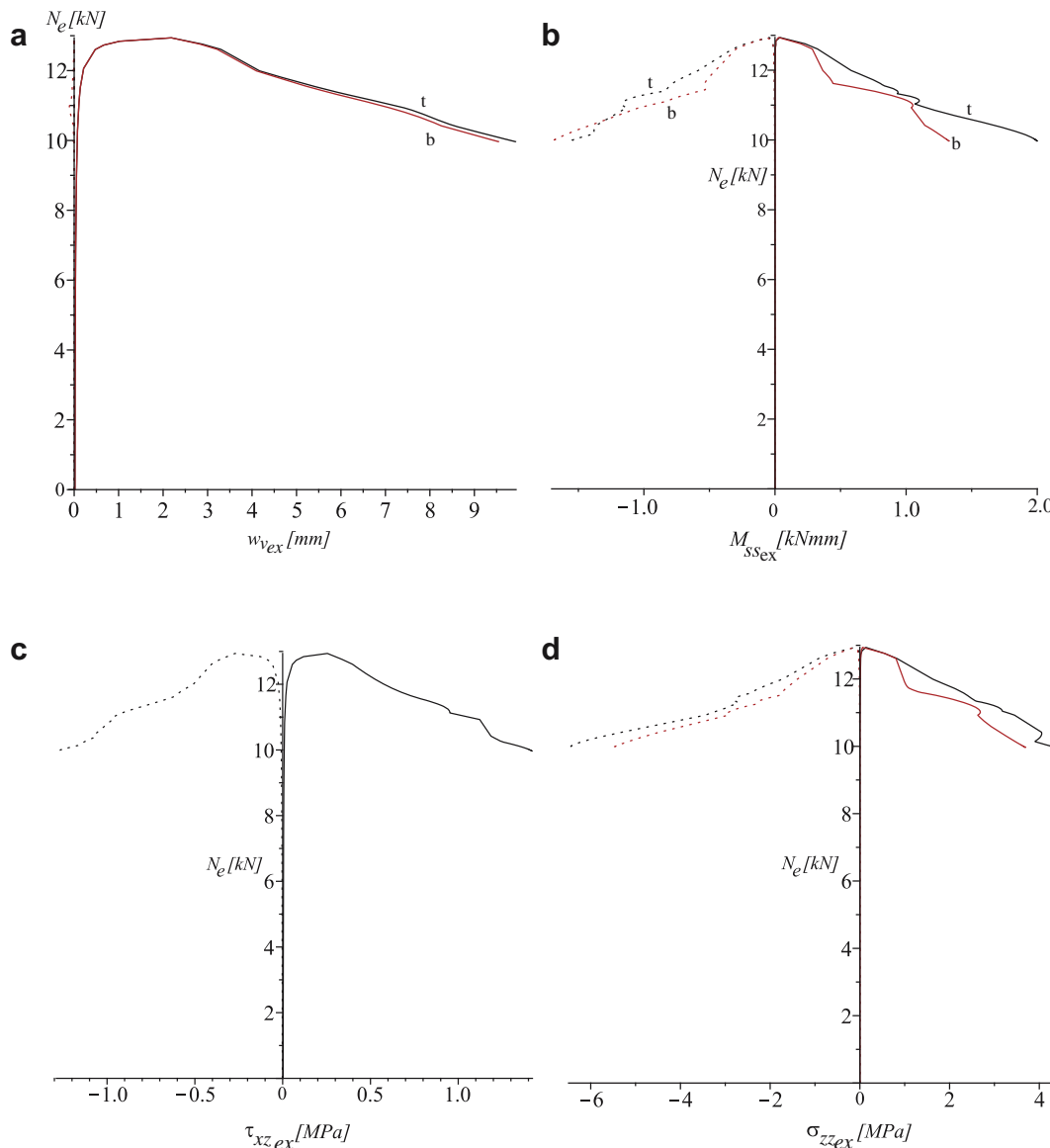


Fig. 7. Equilibrium curves of a sandwich pane of load versus extreme values, in global coordinate system direction of (a) vertical displacements of faces sheets; (b) bending moments in faces; (c) shear stress in core; (d) interfacial vertical stresses at face-core interfaces. Legend: — positive value, negative value, black: upper face sheet; red: lower face sheet. (For interpretation of the references to colours in this figure legend, the reader is referred to the web version of this paper.)

with mild wrinkles. The deformed shape of the ESL model appears in Fig. 5b and it exhibits overall buckling which is totally different then that of the corresponding sandwich panel. The ESL model exhibits overall buckling with large deformations and large end-shortening displacements, $u_{oe} = 14$ –495 mm, that correspond to compressive loads that decrease and increases, see Fig. 9a ahead. Notice that when the end-shortening reaches values that are in the vicinity of the length of the panel it moves upwards.

The results along the sandwich panel at various levels of end-shortening displacements appear in Fig. 6. The vertical displacements along the panel appear in Fig. 6a. It reveals that at low end-shortening displacements overall buckling is observed and as these prescribed displacements increase wrinkling of the upper face sheet around mid-span and lower face sheet in support vicinity, in addition to the overall buckling is detected. Notice that also the face sheet that does not buckle has only small wrinkles. The bending moments at each of the face sheets, see Fig. 6b, reveal extremely high values at the upper face sheet and small ones at the lower one at large values of end-shortening at zones of positive overall bending mo-

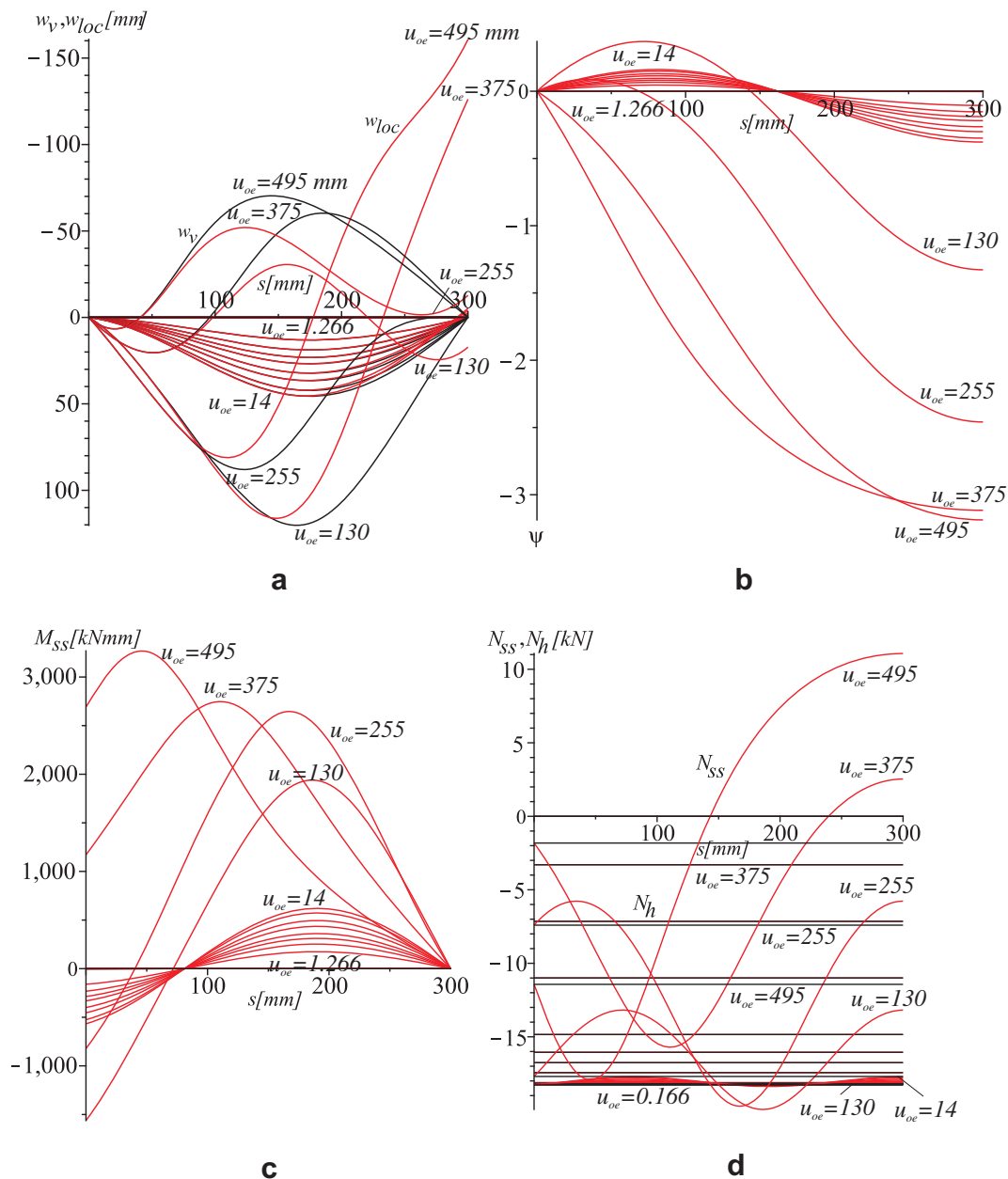


Fig. 8. ESL panel results along its length in global and local coordinate directions. (a) Vertical displacements; (b) slope of section; (c) bending moments; (d) in-plane stress resultant; (e) in-plane displacements; (f) shear angle. Legend: — (black), global direction; — (red), local direction. (For interpretation of the references to colours in this figure legend, the reader is referred to the web version of this paper.)

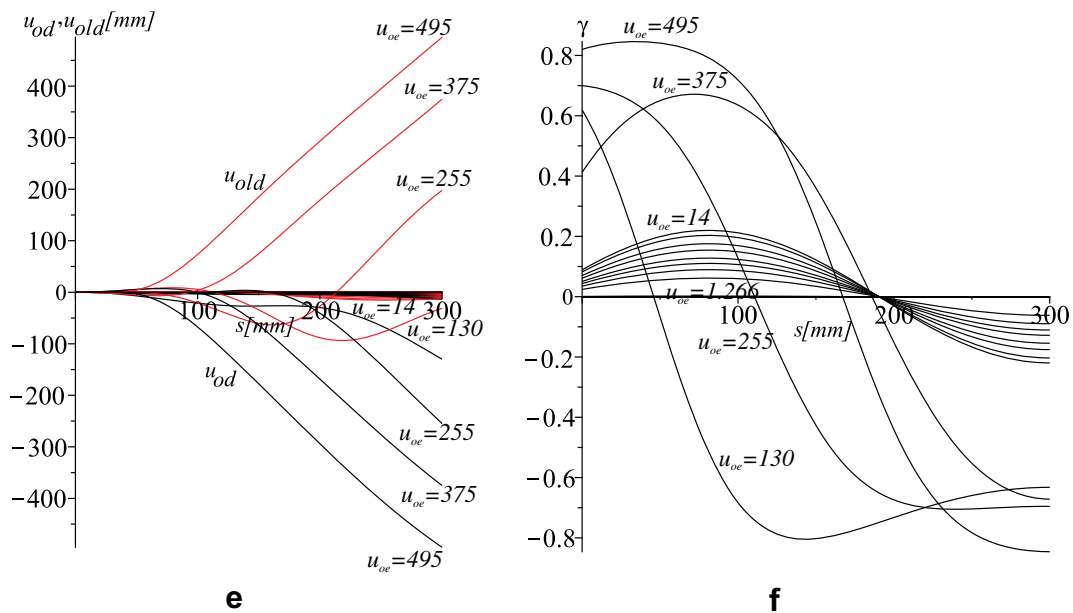


Fig. 8 (continued)

ments around mid-span and an opposite trend at the negative bending moment, in the vicinity of the clamped support. In addition, notice that there are also bending moments at the simply-supported right edge although it is a moment free edge, due to the existence of an edge beam that the face sheets are fixed to. The overall bending moment at this edge is null when the in-plane compressive stress resultants in the face sheets are considered. The in-plane stress resultants, in local and global coordinate directions in the various face sheets, see Fig. 6c, are affected by the wrinkles as they deepens, see curves with $u_{oe} = 2.2$ – 3.52 mm. The longitudinal displacements of the face sheets appears in Fig. 6d and reveal non-smooth curves as a result of the wrinkles of the upper face sheets when the end-shortening displacement increases. The effect of the wrinkling of the face sheets on the shear stresses of the core (in the global vertical direction and in the and local direction at the upper and the lower face-core interface, τ_{snj} ($j = t, b$)), see Fig. 6e, is significant and is associated with extremely large values for the interfacial shear stresses in the local coordinate as a result of the large vertical interfacial normal stresses, see Fig. 6f. The magnitude of the shear stress in the global coordinate is much smaller. The interfacial normal stresses at the upper and the lower face sheets in the global and the local vertical directions appear in Fig. 6f. Notice, that also here the effects of the wrinkling of the upper face sheets are extremely large and yields an erratic behavior along the panel with large values and in addition the magnitude of the stresses in local the global coordinate are almost identical. In addition, notice that in Fig. 6c, d and f the results in the global coordinates system and the local one, in red, almost coincides. In general, the non-regular wrinkling of the face sheets is associated with extremely large stresses that are quite erratic and are presented, especially Fig. 6b, e and f, for the sake of visualization and completeness rather than for exact values.

The equilibrium curves of the compressive load in the horizontal direction that is induced at the right support as a result of the prescribed end-shortening for various structural quantities appear in Fig. 7. Load versus the extreme vertical displacement along the panel appears in Fig. 7a. The curves reveal that up to the buckling load, at about 12.7 kN, the extreme vertical displacement is almost null and there is an abrupt change in the displacement as this load is reached. In the post-buckling range the displacement increases while the load decreases which reflects a shell type of post-buckling behavior. Notice that the curves of the two face sheets are almost identical. The same trends are observed for the extreme bending moment in the face sheets, see Fig. 7b, the shear stress in the core in the global vertical direction and in the vertical interfacial stresses, in the vertical direction, at the upper and the lower face core interfaces, see Fig. 7d.

The results of the ESL model along the panel appear in Fig. 8. The vertical displacements in the global and local coordinate system directions appear in Fig. 8a. They reveal that there is a global buckling response with a continuous decrease as the end-shortening displacement increases and they change from a downward vertical displacement as the end-shortening reaches values that are in the vicinity of the length of the panel. The same trends are observed for the slope of the section, see Fig. 8b. The bending moments exhibits similar trends, see Fig. 8c, and they reach large values as the prescribed end-shortening increases. The in-plane stress resultants in the local and global directions appear in Fig. 8d. The stress resultants in the horizontal global direction are uniform through the entire length of the panel. The compressive stress resultants in the local direction even change to tensile ones near the right edge of the panel. The in-plane displacements, in the global and local directions, appear in Fig. 8e. The global ones are negative and correspond to the end-shortening at right support while the local ones change from negative to positive as the prescribed displacements increases. The shear angle appears in

Fig. 8f and is very similar to the slope of the section, see Fig. 8b. Notice that the shear angle are quite large and at the high end-shortening values they exceeds the range of moderate strains.

The equilibrium curves of the ESL model for various extreme structural quantities appear in Fig. 9. Fig. 9a describes the horizontal compressive load exerted at the right support of the panel through the prescribed end-shortening versus the extreme vertical (in global direction) deflection within the panel length. The curve reveals a bifurcation point, at about 18.5 kN. Notice that up to the bifurcation load there is an increase in the load with very small displacement. Beyond the bifurcation point any increase in the end-shortening is associated with an increase in the vertical displacement without any increase in the load (horizontal reaction at end-shortening location). At a certain point when the end-shortening is about the length of the panel there is a continuous decline in the load and the displacements up to a very small load and the load increases while the vertical displacement decreases. A similar pattern of the load as the end-shortening increases appears in Wang (1997) for a non-extensional clamped-simply-supported beam. In the case of the load versus the extreme bending moment, see Fig. 9b, the bifurcation point is clearly observed and again the bending moment even drops as the end-shortening increases. The load versus the in-plane stress resultants (in global, denoted by N_{hex} , and local, N_{sex} , directions) appear in Fig. 9c. The load versus the compressive stress resultants, in the global direction, is linear where the external loads and the in-plane stress resultants

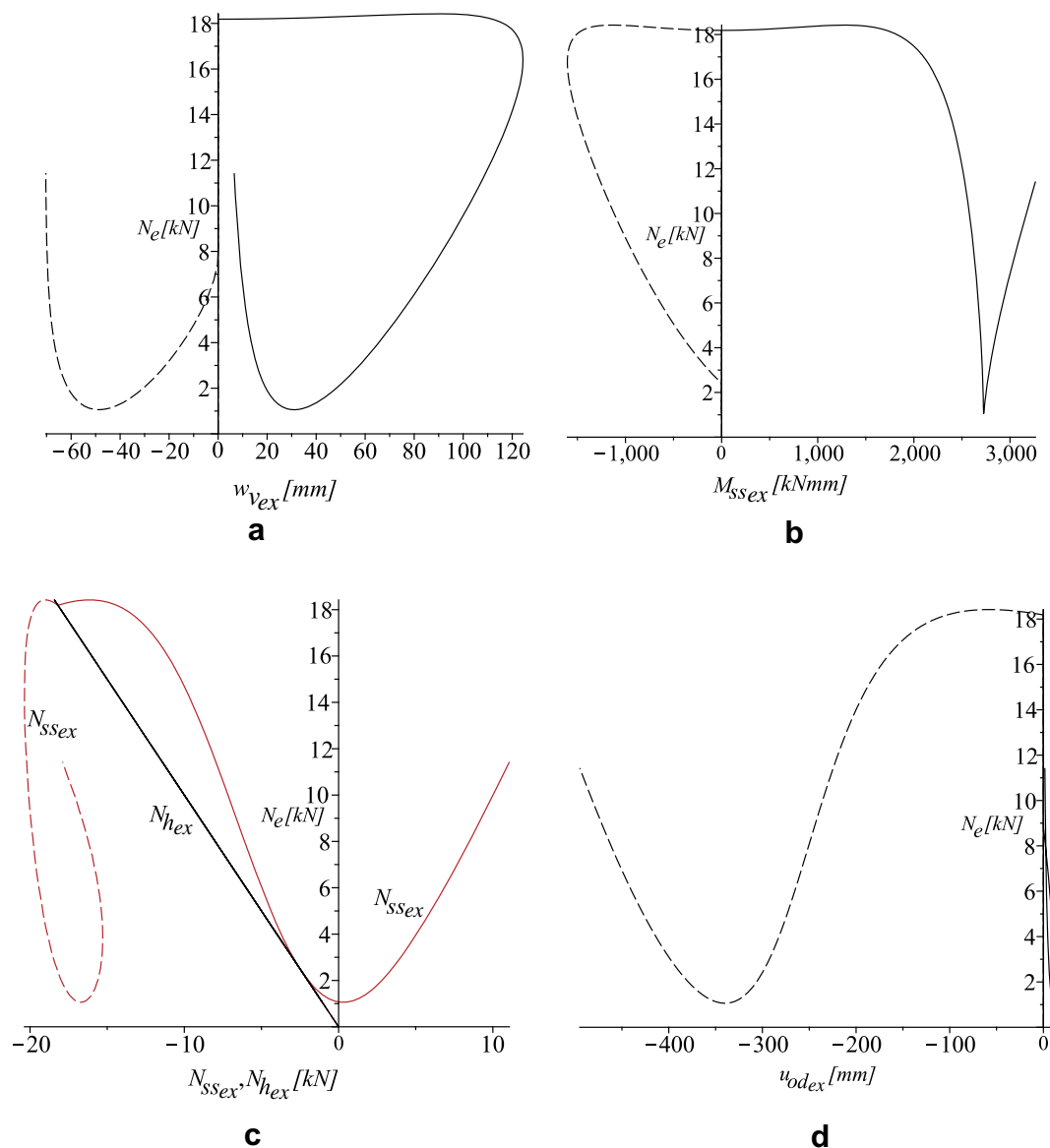


Fig. 9. Equilibrium curves of ESL panel of external load versus of extreme values in global coordinates systems of (a) vertical displacement; (b) bending moment; (c) in-plane stress resultant (local as well as global); (d) in-plane displacements. Legend: — (black), global direction; — (red), local direction; —, positive values; ---, negative values. (For interpretation of the references to colours in this figure legend, the reader is referred to the web version of this paper.)

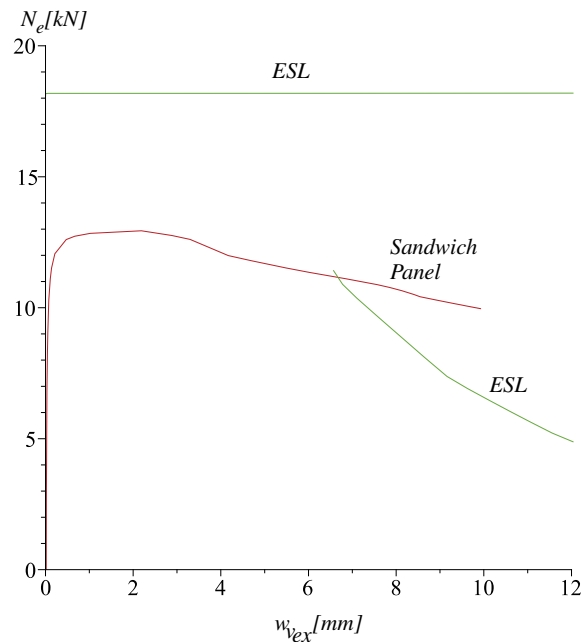


Fig. 10. Load versus extreme vertical displacement, in global direction, of sandwich panel and ESL model.

are identical. The in-plane stress resultants in the local direction change from compression to tension as the end-shortening exceeds the bifurcation point. The in-plane displacement in the global direction appears in Fig. 9d and exhibits a continuous increase as the load decreases and increases beyond the bifurcation point.

A comparison between the sandwich panel and its ESL model reveal a totally different response even for small end-shortening in all structural aspects. A detailed comparison of the equilibrium curves of load versus vertical displacement in the two model appears in Fig. 10. The curves reveal that the bifurcation load of the sandwich panel is smaller, about 12.9 kN, than that of the ESL one, of about 18.5 kN, and the post-buckling curve are quite different. The post-buckling response of the sandwich panel is associated with a drop in the load as the displacement increases while the ESL described a beam behavior where the load remains constant while the vertical displacement increases and at a certain point when the end-shortening is in the vicinity of the length of the panel, the load and the vertical displacements decrease. Hence, the elastica response of the ESL panel is totally different and should not be used to simulate the elastica behavior of a real sandwich panel.

4. Summary and conclusions

The elastica behavior of a sandwich panel with a soft/compliant core is presented. The analysis considers the shear deformations in the face sheets using the first-order shear deformation theory (FOSDT) in addition to the flexural ones and takes into account the extension of the centroid lines of the face sheets. The kinematic relations adopted are based on large displacements and large rotations with moderate strains. The strains adopted coincide with Reissner strains only when the strains and shear angles are small to moderate. The mathematical formulation is based on a variational approach and uses Lagrange coefficient to impose the special strain relations (Reissner strains). The formulation is general and can be applied to any type of structural layout. Here, it has been applied to isotropic face sheets and core for simplicity and brevity.

The elastica behavior of a single layer panel (ESL model) with shear deformations is presented for comparison with the real sandwich panel. The ESL formulation uses the basic equations of a face sheet of the sandwich panel but with the conditions that the shear and interfacial vertical normal stresses are null along with equivalent mechanical properties.

The numerical study presents the results of a sandwich panel with edge beam constraints and its equivalent one where the shear deformation of the core is considered while that of the face sheets is neglected. The numerical results are described in terms of deformed shapes, structural quantities along the panel, and equilibrium curves of load versus extreme structural quantities.

The numerical investigation reveals that the sandwich panel reaches a bifurcation point in a global buckling mode and wrinkles in addition to the global buckling within the post-buckling range. This wrinkling phenomenon is associated with a drop in the load as the imposed end-shortening increases. Thus, a shell buckling behavior with a snap through may occur when a load control test is conducted. The wrinkling waves are also associated with extremely large stresses and deforma-

tions. In addition, a numerical instability of the governing equations has been observed as a result of the loss of the physical stability deep in the post-buckling range that is associated with a reduction of the load as the displacement increases.

The ESL response follows the well known response of elastica of slender beams with an overall buckling. The numerical solution is stable due to the fact that the response is physically stable and as the displacement increases the load remain almost constant and drops only when the end-shortening reaches values that are about the length of the panel. The bifurcation load in this case is higher than that of the sandwich panel.

The comparison between the elastica response of the sandwich panel and that of the ESL model reveal a totally different behavior. Hence, the use of an ESL model to simulate the real behavior of a sandwich panel may be quite inaccurate. Thus, in order to detect the real failure patterns of sandwich panels which exceed large deformations the proposed elastica formulation must be used.

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