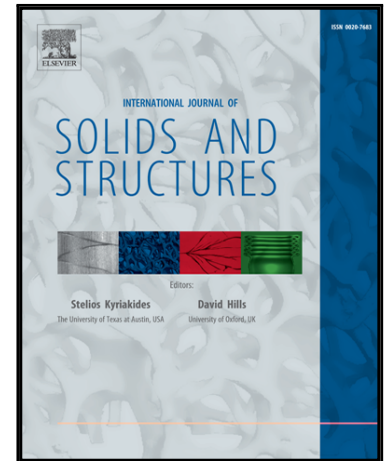


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A fractional-order model for aging materials: an application to concrete

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Abstract

In this paper, the hereditariness of aging materials is modeled within the framework of fractional calculus of variable order. A relevant application is made for the long-term behavior of concrete, for which the creep function is evaluated with the aid of Model B3. The corresponding relaxation function is derived through the Volterra iterated kernels and a comparison with the numerically-obtained relaxation function of Model B3 is also reported. The proposed fractional hereditary aging model (FHAM) for concretes leads to a relaxation function that fully agrees with the well-established Model B3. Furthermore, the FHAM takes full advantage of the formalism of fractional-order calculus to yield semi-analytic expressions in terms of material parameters.

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1. Introduction

The study of the time-dependent behavior of materials is a crucial issue in structural engineering. Such a feature is often encountered since one of the main assumptions of the structural analysis, i.e. the elasticity of materials, is not fulfilled in nature. Indeed, real-type materials show a mechanical behavior in terms of strain/stress that depends on the previously experienced stress/strain histories, and this behavior is usually observed by means of creep and relaxation tests [1]. Both phenomena, i.e. creep and relaxation, are exhibited by several materials, like rubbers, polymers, collagens, concretes, but over different durations, i.e. some hours for polymers, a few minutes for collagen structures, and many years for concretes.

In the context of linear viscoelasticity, the most common approach used to describe the mechanical behavior of materials consists in providing the so-called rheological models [2], where each model element represents a basic feature of the material, e.g. the spring indicates the elastic response, the dashpot describes the viscous response, and so on. Hence, the time-dependent behavior of materials is illustrated by a specific arrangement of the aforementioned elements, like the well-known Kelvin-Voigt, Maxwell, and Zener models. Furthermore, this approach leads to a time-dependent behavior expressed as a linear combination of exponential functions, which are characterized by parameters obtained through a best-fitting procedure with experimental data [3]. However, the major drawback

of using exponential functions is that the obtained time-dependent model often yields negative values of the relevant physical parameters that, instead, must be intrinsically positive. This problem was solved at the middle of the last century by introducing the so-called spring-pot element [4], which is a two-parameters element that can be reverted to a spring or to a viscous element by assigning specific values to the relevant parameters [5]. In a greater detail, the time-dependent behavior of the spring-pot corresponds to a power law characterized by a real exponent $0 \leq \beta < 1$ and a dimensional-force constant.

Power laws describe in an excellent way the time-dependent behavior of polymers [6, 7], biocomposites [8], and biological tissues [9, 10]. In addition, the use of power laws in combination with the linear superposition theorem [11, 12] leads to the description of the mechanical behavior of several materials by the so-called fractional order integrals and derivatives [13, 14]. Basically, these operators represent the generalization of the classical integer-order operators to the case of the real-order differentiation/integration. Another important aspect of fractional-order operators is that they are mathematically well defined [15]; moreover, they can be defined as a non-euclidean measure of Sobolev spaces [16, 17]. Some relationships between the order of differentiation/integration and the topological dimension of the underlying set were also provided [18, 19, 20, 21].

Fractional-order calculus was also introduced to capture the time-dependent behavior of non-aging concrete beams [22]. However, since the mechanical properties of concrete significantly change after many decades from its casting, for an accurate prediction of the long-term behavior of this material, the aging phenomenon cannot be neglected.

The study of the time-dependent phenomena exhibited by concrete structures

represents one of the most challenging issue in the field of linear viscoelasticity; indeed, the time-dependent behavior of concrete is influenced by several factors, such as shrinkage, drying, temperature, humidity, pore sizes [23], and additional long-term effects involved in presence of prestress [24, 25, 26]. Thus, a precise prediction of the concrete time-dependent behavior is still a matter of debate among researchers [27]. As a consequence, technical codes propose different models on this matter, which find their experimental bases on the RILEM database for data of concrete specimens and structures [28, 29]. The theoretical assumptions are instead borrowed from the so-called Model B3 [30], which consists of five parameters, mainly expressed as functions of the cement content, the age when drying begins, the material strength, the environmental conditions, and the geometrical properties. Furthermore, Model B3 accounts for creep and drying simultaneously that represents a crucial aspect for a correct prediction of the material long-term behavior, and several contributions have been provided in the relevant scientific literature [31, 32]. In some simplified approaches the so-called aging coefficient is introduced, such as the Age-Adjusted Effective Modulus method (AAEM), which is widely used by researchers and practitioners due to its satisfactory results [33, 34]. However, the main drawback related to approximate approaches, like the AAEM, is the lack of thermodynamical consistency: in particular, no relations between stiffness and compliance exist. In this regard, a possible approach to relate aging-dependent creep and relaxation was presented in [35] by resorting to approximate formulations and reducing the functional class of the aging relaxation. The proposed relaxation function, however, is not related to the relevant creep function through the fundamental relationship of linear viscoelasticity [1]. Hence, the mathematical consistency of the approach in [35] is

not guaranteed.

In this paper, a constitutive model within the framework of fractional calculus is proposed, which turns out to be thermodynamically and mathematically consistent. In other words, the obtained material response can uniquely describe both creep and relaxation phenomena and satisfies the fundamental Volterra's relationship of linear viscoelasticity. Moreover, the proposed constitutive model is conceived for any aging material and, since the constitutive behavior of concrete is quite complex among such solids, it is here applied to the study of the long-term response of concrete. More precisely, we introduce a specific form of material aging by means of a three-parameter model, which corresponds to a viscoelastic model characterized by fractional operators of variable order [36]. The aging functions of the model are derived for concretes through a best-fitting procedure with the RILEM database data. The obtained creep and relaxation functions fully comply with the aforementioned mathematical and thermodynamical consistencies and are compared against the five-parameters Model B3 functions, thereby resulting in an effective model for the description of the time-dependent behavior of aging materials.

This paper is organized as follows: we report some general definitions of material aging hereditariness in Section 2; in Section 3, we introduce the fractional-order aging hereditariness; and, in Section 4, we propose a fractional-hereditary aging model (FHAM) with a relevant application to concrete structures. Main conclusions are summarized in Section 5 with further developments.

2. Preliminary remarks on material aging hereditariness

The constitutive behavior of materials in long-standing mechanical tests is described by means of the well-known creep and relaxation functions, dubbed $J(t)$ and $G(t)$, respectively. The linear superposition applied to a generic stress/strain history, namely $\sigma(\tau)$ and $\epsilon(\tau)$ with $\tau \leq t$, yields,

$$\epsilon(t) = \int_0^t J(t-\tau) \cdot d\sigma(\tau) = \int_0^t J(t-\tau) \cdot \dot{\sigma}(\tau) d\tau \quad (1a)$$

$$\sigma(t) = \int_0^t G(t-\tau) \cdot d\epsilon(\tau) = \int_0^t G(t-\tau) \cdot \dot{\epsilon}(\tau) d\tau. \quad (1b)$$

Both expressions indicate the Boltzman superposition, where with the following notation $\dot{f}(t) = df(t)/dt$ we denote the increment of the generic function $f(t)$ and, in the case under investigation (1), the increment of the history of stress $\sigma(t)$ and strain $\epsilon(t)$.

The convolution integrals in Equations (1) are completely described as we introduce the functional class of creep and relaxation functions with the aid of experimental data. Creep and relaxation functions characterize the material behavior and they must satisfy the conjugation relation $\hat{J}(s)\hat{G}(s) = 1/s^2$, where s indicates the Laplace parameter and $[\hat{f}(s)] = \mathcal{L}[f(t)]$ the Laplace transform of the generic function $f(t)$.

However, Equations (1) refer to hereditary materials, which do not show any degradation/improvement of their mechanical characteristics during their service life. Thus, the behavior of hereditary materials only depends on the loading duration, $(t - t_0)$, but does not depend on the material age, when the constant stress/strain is applied, t_0 . Conversely, several materials, like structural concretes, rubbers and polymers under photo-degradation, show a significant change of their

mechanical properties over time. In such a context, i.e. aging-dependent materials, the structural behavior depends on both the material age, t_0 , when a stress/strain history begins, and the time occurred from the stress/strain application, $(t - t_0)$. This composed behavior is defined in literature as the aging hereditariness of materials [37] and, under the assumption of the linear superposition, it may be characterized by the so-called creep function, $J_c(t, t_0)$, as well as by the relaxation function, $G_s(t, t_0)$. The creep function represents the strain undergone by the specimen due to an unitary stress applied at the time instant t_0 , i.e. $\sigma(t_0) = 1$. Furthermore, $J_c(t, t_0)$ is a function of two separate variables, t and t_0 , with $t \geq t_0$. A similar consideration holds true for the relaxation function, which measures the stress at the generic time instant t due to an applied unitary strain at t_0 , $\epsilon(t_0) = 1$, with $t \geq t_0$.

The constitutive equations for aging materials read,

$$\epsilon(t) = \int_0^t J_c(t, \tau) \cdot d\sigma(\tau) = \int_0^t J_c(t, \tau) \cdot \dot{\sigma}(\tau) d\tau \quad (2a)$$

$$\sigma(t) = \int_0^t G_s(t, \tau) \cdot d\epsilon(\tau) = \int_0^t G_s(t, \tau) \cdot \dot{\epsilon}(\tau) d\tau, \quad (2b)$$

where we assume that both stress (2a) and strain (2b) histories are sufficiently smooth functions so that the Stieltjes integrals in (2) revert to the standard Riemann integrals in terms of functions $\dot{\sigma}(t)$ and $\dot{\epsilon}(t)$.

Creep-compliance and stiffness-relaxation functions are related by means of (2), yielding the first theorem of the linear aging hereditariness [33] as,

$$J_c(t, t_0)G_s(t_0, t_0) + \int_{t_0}^t J_c(t, \tau) \frac{\partial G_s(\tau, t_0)}{\partial \tau} d\tau = 1. \quad (3)$$

This latter relationship is a Volterra integral equation of the second kind for the creep function $J_c(t, t_0)$. More precisely, Equation (3) yields the creep function as

we measured the relaxation function, $G_s(t, t_0)$, for different couples of time instants (t, t_0) . However, the knowledge of the relaxation function is not so common in structural engineering, since the relaxation function should be obtained through field experiments on existing structures.

On the other hand, field measures allow for the estimation of the creep function, $J_c(t, t_0)$, so that the corresponding relaxation function, $G_s(t, t_0)$, can be obtained from the second theorem of the linear aging hereditariness as,

$$G_s(t, t_0)J_c(t_0, t_0) + \int_{t_0}^t G_s(t, \tau) \frac{\partial J_c(\tau, t_0)}{\partial \tau} d\tau = 1. \quad (4)$$

This represents a Volterra integral equation of the second kind for the relaxation function, as we specify the time instant t_0 corresponding to the application of the strain history $\epsilon(t) = U(t - t_0)$, with $U(\cdot)$ the unit-step function.

In order to obtain an analytical expression of the relaxation function, we solve Equation (4) by means of the iterated-kernels method [38], yielding the following sequence,

$$G_s^{(1)}(t, t_0) = \frac{1}{J_c(t_0, t_0)}, \quad (5a)$$

$$G_s^{(2)}(t, t_0) = \frac{1}{J_c(t_0, t_0)} \left(1 - \int_{t_0}^t \frac{\partial J_c(\tau, t_0)}{\partial \tau} G_s^{(1)}(t, \tau) d\tau \right), \quad (5b)$$

.....

$$G_s^{(k)}(t, t_0) = \frac{1}{J_c(t_0, t_0)} \left(1 - \int_{t_0}^t \frac{\partial J_c(\tau, t_0)}{\partial \tau} G_s^{(k-1)}(t, \tau) d\tau \right), \quad (5c)$$

.....

In passing, we observe that both the first and second theorem of the linear aging hereditariness revert to the well-known equation in the Laplace domain, i.e.

$\hat{J}(s)\hat{G}(s) = 1/s^2$, as hereditary materials are considered, $J_c(t, t_0) = J(t - t_0)$ and $G_s(t, t_0) = G(t - t_0)$.

3. The fractional-hereditary aging materials

The study of the aging hereditariness of materials is highly influenced by the choice of the creep function, $J_c(t, t_0)$, as well as the choice of the relaxation function, $G_s(t, t_0)$, as shown in Equations (2a) and (2b).

In the context of the non-aging hereditariness of materials, a power-law representation of creep and relaxation functions, i.e. $J(t)$ and $G(t)$, were introduced at the beginning of the last century [6],

$$J(t) = \frac{1}{C_\beta \Gamma(\beta + 1)} t^\beta \quad (6a)$$

$$G(t) = \frac{C_\beta}{\Gamma(1 - \beta)} t^{-\beta}, \quad (6b)$$

where $\Gamma(\cdot)$ is the well-known Euler-Gamma function, $0 \leq \beta < 1$ and C_β , a positive real number, are material parameters that may be estimated through a best-fitting procedure with experimental data [39, 40]. Straightforward manipulations show that the power-law functional class chosen for the definition of both creep and relaxation functions, as in Equations (6), satisfies the conjugation relation. By substituting Equations (6) into (1) we obtain,

$$\begin{aligned} \epsilon(t) &= \frac{1}{C_\beta \Gamma(\beta + 1)} \int_0^t (t - \tau)^\beta \dot{\sigma}(\tau) d\tau = \frac{1}{C_\beta \Gamma(\beta)} \int_0^t (t - \tau)^{\beta-1} \sigma(\tau) d\tau = \\ &= \frac{1}{C_\beta} \left(I_{0+}^\beta \sigma \right) (t) = \left(I_{0+}^\beta \tilde{\epsilon} \right) (t) \end{aligned} \quad (7a)$$

$$\sigma(t) = \frac{C_\beta}{\Gamma(1 - \beta)} \int_0^t (t - \tau)^{-\beta} \dot{\epsilon}(\tau) d\tau = C_\beta \left(D_{0+}^\beta \epsilon \right) (t) = \left(D_{0+}^\beta \tilde{\sigma} \right) (t) \quad (7b)$$

that represent the generalization of the well-known Riemann integral and Leibniz derivative of non-integer orders [14]. Moreover, $\tilde{\epsilon}(t) = \sigma(t)/C_\beta$ and $\tilde{\sigma}(t) = \epsilon(t)C_\beta$ are sets of instantaneous values of strain and stress, involved in the absence of the material hereditariness.

The use of power laws and, as a consequence, of fractional-order operators is usually referred, in a rheological context, to the spring-pot element. The spring-pot is a one-dimensional element that is defined by means of two parameters, i.e. C_β and β , with C_β a positive real number and $0 \leq \beta < 1$, where β is representative of the differentiation order. More precisely, a simple spring corresponds to $\beta = 0$ and $\frac{d^\beta f}{dt^\beta} = \frac{d^0 f}{dt^0} = f$; whilst, case of $\beta = 1$ corresponds to a first order derivative, i.e. $\frac{d^\beta f}{dt^\beta} = \frac{df}{dt} = f'$, which is a Newtonian rheological element. The three cases of a spring, a spring-pot and a Newtonian element are depicted in Figure 1, where E_c indicates the Young's modulus and μ the viscous coefficient.

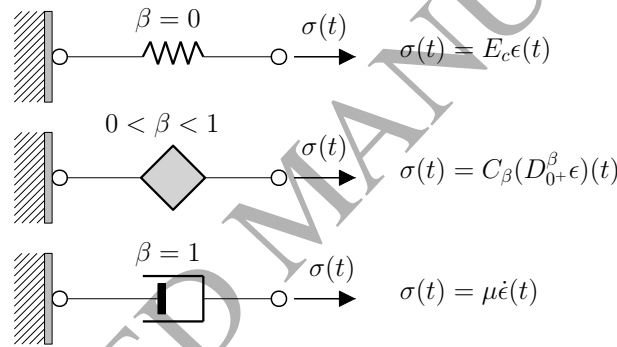


Figure 1: Representation of a spring element $\beta = 0$, a spring-pot element $0 < \beta < 1$, and a Newtonian element $\beta = 1$.

The fractional-order hereditariness of materials has been related to appropriate mechanical models [41, 42, 43], as well as to the fractal distribution of the material microstructure [9]. Despite of the fractional-calculus success in describing the constitutive behavior of hereditary materials [25], the presence of the material aging has never been accounted for.

In order to fill this gap, we assume that the material aging is described through the variation over time of the coefficients $C_\beta \rightarrow C_\beta(t_0)$ and $\beta \rightarrow \beta(t_0)$. In this

setting, we assume that the creep function is expressed as,

$$J_c(t, t_0) = \frac{(t - t_0)^{\beta(t_0)}}{C_\beta(t_0)\Gamma(\beta(t_0) + 1)} \quad (8)$$

and by substituting (8) into Equation (2a) we obtain,

$$\begin{aligned} \epsilon(t) &= \int_0^t \frac{(t - \tau)^{\beta(\tau)}}{C_\beta(\tau)\Gamma(\beta(\tau) + 1)} \dot{\sigma}(\tau) d\tau = \\ &= \int_0^t \frac{(t - \tau)^{\beta(\tau)}}{\Gamma(\beta(\tau) + 1)} \dot{\tilde{\epsilon}}(\tau) d\tau + \int_0^t \frac{(t - \tau)^{\beta(\tau)}}{\Gamma(\beta(\tau) + 1)} \frac{\dot{C}_\beta(\tau)}{C_\beta(\tau)} \tilde{\epsilon}(\tau) d\tau = \\ &= - \int_0^t \left[\frac{(t - \tau)^{\beta(\tau)}}{\Gamma(\beta(\tau) + 1)^2} (\ln(t - \tau)\Gamma(1 + \beta(\tau)) - \Gamma'(1 + \beta(\tau))) \right] \tilde{\epsilon}(\tau) d\tau + \\ &\quad + \int_0^t \left[\frac{(t - \tau)^{\beta(\tau)-1}}{\Gamma(\beta(\tau))} + \frac{(t - \tau)^{\beta(\tau)}}{\Gamma(\beta(\tau) + 1)} \frac{\dot{C}_\beta(\tau)}{C_\beta(\tau)} \right] \tilde{\epsilon}(\tau) d\tau = \left(\mathbf{I}_{0+}^{\beta(t)} \tilde{\epsilon} \right)(t) + \tilde{\epsilon}_r(t), \end{aligned} \quad (9)$$

where we use the usual chain rule of differentiation and we introduce a rate-dependent field $\tilde{\epsilon}_r(t)$, described by the latter term in Equation (9). The first term in Equation (9) is the Riemann-Liouville fractional integral of order $\beta = \beta(\tau)$ [36, 44], as reported in Appendix B. Incidentally, we observe that, as we assign constant values to β and C_β , the well-know fractional hereditary model is obtained as in Equation (6a).

The relaxation function corresponding to the presented fractional-order creep function (8) is provided by the iterated-kernels method (5).

A constitutive equation, formally similar to (9), may also be provided by assuming that the fractional-hereditary aging material is characterized by a relaxation function with the following form,

$$G_s(t, t_0) = \frac{(t - t_0)^{-\beta(t_0)}}{\Gamma(\beta(t_0))} C_\beta(t_0). \quad (10)$$

The latter yields the following expression of the stress,

$$\begin{aligned}\sigma(t) &= \int_0^t \frac{(t-\tau)^{-\beta(\tau)}}{\Gamma(\beta(\tau))} C_\beta(\tau) \dot{\epsilon}(\tau) d\tau = \int_0^t \frac{(t-\tau)^{-\beta(\tau)}}{\Gamma(\beta(\tau))} \dot{\sigma}(\tau) d\tau + \\ &\quad - \int_0^t \frac{(t-\tau)^{-\beta(\tau)}}{\Gamma(\beta(\tau))} \frac{\dot{C}_\beta(\tau)}{C_\beta(\tau)} \tilde{\sigma}(\tau) d\tau = \left(D_{0+}^{\beta(t)} \tilde{\sigma} \right)(t) - \tilde{\sigma}_r(t),\end{aligned}\tag{11}$$

where the additional contribution relevant to the evaluation of the material parameters up to the time instant t is actually involved.

In the next section, variable-order fractional calculus is used to deal with the aging hereditariness of concrete by means of a three-parameters model.

4. The fractional-aging hereditariness of concrete

In this section, we aim to formulate a constitutive model for the aging hereditariness of concrete within the formalism of the fractional-order calculus presented in Section 3. In this regard, we observe that a fractional-order constitutive model for the hereditariness of concrete was recently proposed by [22]. It was shown, indeed, that, for fixed values of the aging time t_0 , a precise representation of the creep function and the relaxation function is achieved by means of power laws as t^β and $t^{-\beta}$, respectively.

In order to derive the Fractional-Hereditary Aging Model (FHAM), we first present Model B3, one of the most widely used viscoelastic models for prestressed concrete structures. Then, in Subsection 4.2 we illustrate the FHAM creep function and we define its three aging terms through a best-fitting procedure with Model B3. A numerical procedure for the evaluation of the FHAM relaxation function is also proposed in Subsection 4.2 and interesting comparisons between Model B3 and the proposed FHAM are reported in Subsection 4.3.

4.1. The aging hereditariness of concrete: Model B3

Several experimental studies reported since the beginning of eighties [45, 46] show that a significant evolution of the material hereditariness is involved for concrete specimens and structures. These latter considerations led to the introduction of the Model B3 [30] conceived to capture the aging hereditariness of concrete as,

$$J_B(t, t_0) = q_1 + C_0(t, t_0) + C_d(t, t_0, t'_0), \quad (12)$$

where $J_B(t, t_0)$ is the creep function of Model B3, in which q_1 is the instantaneous strain due to a unitary stress, $C_0(t, t_0)$ is the compliance function for basic creep (creep at constant moisture content and no moisture movement through the material) and $C_d(t, t_0, t')$ is an additional compliance function due to the simultaneous drying starting at time t' . Then, the full expressions of $C_0(t, t_0)$ and $C_d(t, t_0, t')$ read as follows:

$$C_0(t, t_0) = q_2 Q(t, t_0) + q_3 \ln[1 + (t - t_0)^n] + q_4 \ln(t/t_0) \quad (13a)$$

$$C_d(t, t_0, t') = q_5 [\exp\{-8H(t)\} - \exp\{-8H(t'_0)\}]^{1/2}, \quad (13b)$$

where q_2 , q_3 , and q_4 represent the aging viscoelastic compliance, the non-aging viscoelastic compliance and the flow compliance, respectively [47]. Moreover, $Q(t, t_0)$ can be calculated either using an approximate explicit formula or by means of a numerical integration of its time derivative, whereas $t'_0 = \max(t_0, t')$ is the time at which drying and loading first act simultaneously. It is worth noting that the expression of C_d (13) is valid if $t \geq t'_0$, otherwise $C_d = 0$. With regard to the five parameters, they can be determined by means of a statistical analysis or through the following expressions: $q_1 = 0.6 \cdot 10^6 / E_{28}$, $q_2 = 185.4c^{0.5}\bar{f}_c^{-0.9}$, $q_3 = 0.29(w/c)^4 q_2$, $q_4 = 20.3(a/c)^{-0.7}$, $q_5 = 7.57 \cdot 10^5 \bar{f}_c^{-1} |\epsilon_{sh\infty}|^{-0.6}$. All the formulas above are given in

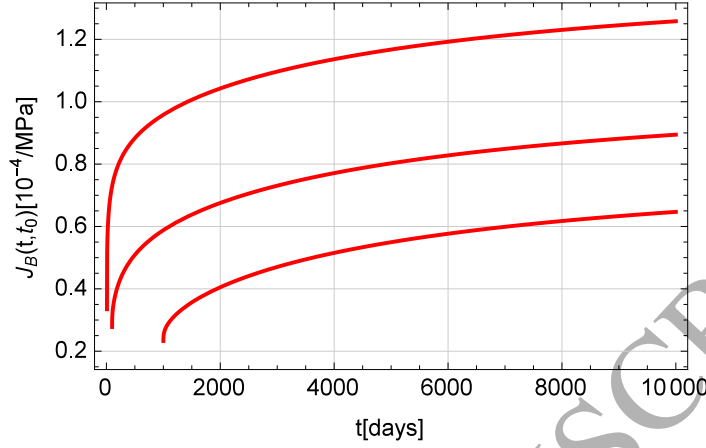


Figure 2: Creep function of Model B3 for different values of t_0 .

SI (metrics) units (MPa,m). In addition, E_{28} is the Young's modulus at 28 days, c is the cement content, $\bar{f}_c$ is the standard cylinder compression strength of concrete at 28 days, w/c is the water-cement ratio, a/c is the aggregate-cement ratio, and $\epsilon_{sh\infty}$ is the mean shrinkage strain.

An attentive exam of (12) entails that the creep function of Model B3 depends on both the loading time t_0 and the elapsed time $(t - t_0)$. Thus, the aging nature of Model B3 is also evident in Figure 2, where the value of $J_B(10000, t_0)$ decreases by an increase of t_0 . For further details about Model B3, the interested readers may refer to [30].

Regarding the relaxation function of Model B3, [35] proposes an approximate formula that prevents any violation of the thermodynamic requirement of negativeness. Furthermore, this formula can be utilized to describe long-term relaxation phenomena even in concretes loaded at a young age. The expression of

the aforementioned approximate formula follows, i.e.,

$$G_B(t, t_0) = \frac{1}{J_B(t, t_0)} \cdot \left[1 + \frac{c_1 \alpha(t, t_0) J_B(t, t_0)}{q J_B(t, t - \eta)} \right]^{-q}, \quad (14)$$

where $c_1 = 0.0119 \cdot \ln(t_0) + 0.08$, $\alpha(t, t_0) = J_B(\frac{t+t_0}{2}, t_0) / J_B(t, \frac{t+t_0}{2}) - 1$, with $q = 10$ and $\eta = 1$. Thus, under the linear viscoelasticity hypothesis, the relaxation function of Model B3 can be accurately estimated by the explicit formula (14), leading to an analytical expression of the stress $\sigma(t)$ as function of the strain history and material properties. Conversely, no analytical expression for $\sigma(t)$ can be derived by numerically solving the Volterra equation [33].

4.2. The fractional-aging hereditariness of concrete

In this subsection, we aim to formulate the constitutive behavior of aging concretes in the framework of variable-order fractional calculus. In this regard, we assume that the creep function belongs to the same fractional class of (8) yielding the following expression.

$$J_F(t, t_0) = \frac{1}{G_{\beta}(t_0)} + \frac{(t - t_0)^{\beta(t_0)}}{C_{\beta}(t_0) \Gamma(\beta(t_0) + 1)}, \quad (15)$$

where the creep function $J_F(t, t_0)$ depends on both the elapsed time $(t - t_0)$ and the material age t_0 , as expected from a model describing the concrete aging hereditariness. Moreover, it corresponds to the definition of a rheological model consisting of a time-varying spring and a variable-order fractional dashpot, as depicted in Figure 3. The quantities $C_{\beta}(t_0)$, $G_{\beta}(t_0)$ and $\beta(t_0)$ are aging functions representative of the material mechanical characteristics and environmental conditions. In particular, $1/G_{\beta}(t_0)$ is the time-dependent compliance of concrete, whereas, $\beta(t_0)$

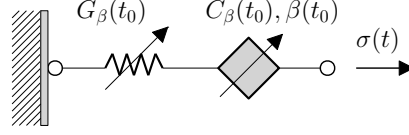


Figure 3: Variable-order fractional Maxwell model.

and $C_\beta(t_0)$ are related to the change of pore shapes and to the rearranging of the material microstructure during the closed phase. The explicit expressions of functions $C_\beta(t_0)$, $\beta(t_0)$ and $G_\beta(t_0)$ can be derived through a best-fitting procedure with experimental data or in terms of an existing creep function.

From the creep function in Equation (15), we can then obtain the analytical expression of the relaxation function $G_F(t, t_0)$ by means of the iterated-kernels method, as illustrated in Section 2. In this regard, we rewrite the Volterra integral equation (4) as follows:

$$G_F(t, t_0) = \frac{1}{J_F(t_0, t_0)} - \frac{1}{J_F(t_0, t_0)} \int_{t_0}^t G_F(t, \tau) \frac{\partial J_F(\tau, t_0)}{\partial \tau} d\tau, \quad (16)$$

where, according to Equation (15), the first partial derivative of the creep function takes the following form:

$$\frac{\partial J_F(\tau, t_0)}{\partial \tau} = \frac{(\tau - t_0)^{\beta(t_0)-1}}{\Gamma(\beta(t_0))C_\beta(t_0)}. \quad (17)$$

Then, through the application of the iterated-kernels method, we obtain the following approximations for the relaxation function, and similar expressions, herein omitted for brevity, can be derived for the successive kernels, i.e.:

$$G_F^{(1)}(t, t_0) = \frac{1}{J_F(t_0, t_0)} = G_\beta(t_0), \quad (18a)$$

$$G_F^{(2)}(t, t_0) = G_\beta(t_0) - G_\beta(t_0) \int_{t_0}^t \frac{(\tau - t_0)^{\beta(t_0)-1}}{\Gamma(\beta(t_0))C_\beta(t_0)} G_\beta(\tau) d\tau, \quad (18b)$$

$$G_F^{(3)}(t, t_0) = G_\beta(t_0) - G_\beta(t_0) \left[\int_{t_0}^t \frac{(\tau - t_0)^{\beta(t_0)-1}}{\Gamma(\beta(t_0))C_\beta(t_0)} G_\beta(\tau) d\tau - \int_{t_0}^t \frac{(\tau - t_0)^{\beta(t_0)-1}}{\Gamma(\beta(t_0))C_\beta(t_0)} \cdot \left(\int_\tau^t \frac{(\xi - \tau)^{\beta(\tau)-1}}{\Gamma(\beta(\tau))C_\beta(\tau)} G_\beta(\tau) G_\beta(\xi) d\xi \right) d\tau \right], \quad (18c)$$

.....

It is then important to highlight that, to accurately estimate the relaxation phenomenon, a large number of iterations is required, and this number increases with the size of the time interval $[t, t_0]$. Moreover, due to the variable-order nature of Riemann-Liouville integrals in (18), the composition property, illustrated in Appendix B and used to reduce the dimension of multiple integrals within the framework of fractional calculus, cannot be applied anymore. In order to solve this problem, we simplify the multiple integrals involved into the iterated-kernels procedure by introducing the Grünwald-Letnikov approximation [14]. Thus, Equations (18b) and (18c) become:

$$G_F^{(2)}(t, t_0) = G_\beta(t_0) - \frac{G_\beta(t_0)h_1^{\beta(t_0)}}{C_\beta(t_0)} \left(\sum_{r_1=0}^{\bar{n}_1} (-1)^{r_1} \binom{-\beta(t_0)}{r_1} G_\beta(t_0 + r_1 h_1) \right) \quad (19a)$$

$$G_F^{(3)}(t, t_0) = G_\beta(t_0) - \frac{G_\beta(t_0)h_1^{\beta(t_0)}}{C_\beta(t_0)} \left[\left(\sum_{r_1=0}^{\bar{n}_1} (-1)^{r_1} \binom{-\beta(t_0)}{r_1} G_\beta(t_0 + r_1 h_1) \right) - \left(\sum_{r_1=0}^{\bar{n}_1} (-1)^{r_1} \binom{-\beta(t_0)}{r_1} \frac{G_\beta(t_0 + r_1 h_1)}{C_\beta(t_0 + r_1 h_1)} h_2^{\beta(t_0 + r_1 h_1)} \left(\sum_{r_2=0}^{\bar{n}_2} (-1)^{r_2} \binom{-\beta(t_0 + r_1 h_1)}{r_2} G_\beta(t_0 + r_1 h_1 + r_2 h_2) \right) \right) \right], \quad (19b)$$

where $\bar{n}_i$ indicates the number of subintervals used to approximate the generic fractional integral, and h_i represents the interval amplitude. For the sake of simplicity, we set a unique number of subintervals, i.e. $\bar{n}$; thus, $h_1 = h_2 = \dots = h_i = (t - t_0)/\bar{n} = h$, $\bar{n}_2 = \bar{n} - r_1$, $\bar{n}_3 = \bar{n} - r_1 - r_2$, $\bar{n}_4 = \bar{n} - r_1 - r_2 - r_3$, and so on.

A further simplification can be reached with the introduction of the following column vector $\mathbf{G}_\beta$ with dimension $(\bar{n} + 1)$:

$$\mathbf{G}_\beta = \begin{bmatrix} G_\beta(t_0) & G_\beta(t_0 + h) & \cdots & G_\beta(t_0 + \bar{n}h) \end{bmatrix}^T = [g_1 \quad g_2 \quad \cdots \quad g_{\bar{n}+1}]^T \quad (20)$$

and the upper-diagonal matrix $\mathbf{L}$ with dimension $(\bar{n} + 1) \times (\bar{n} + 1)$:

$$\mathbf{L} = \begin{bmatrix} \frac{h^{\beta(t_0)} G_\beta(t_0)}{C_\beta(t_0)} (-1)^0 \binom{-\beta(t_0)}{0} & \cdots & \frac{h^{\beta(t_0)} G_\beta(t_0)}{C_\beta(t_0)} (-1)^{\bar{n}} \binom{-\beta(t_0)}{\bar{n}} \\ & \ddots & \vdots \\ & & \frac{h^{\beta(t_0 + \bar{n}h)} G_\beta(t_0 + \bar{n}h)}{C_\beta(t_0 + \bar{n}h)} (-1)^0 \binom{-\beta(t_0 + \bar{n}h)}{0} \end{bmatrix}. \quad (21)$$

Moreover, we indicate the matrix product between the generic j^{th} power of $\mathbf{L}$ and $\mathbf{G}_\beta$ with the following notation:

$$\mathbf{L}^j \cdot \mathbf{G}_\beta = \begin{bmatrix} a_1^j & a_2^j & \cdots & a_{\bar{n}+1}^j \end{bmatrix}^T. \quad (22)$$

The notation introduced in (20), (21), and (22) yields a compact definition of iterated kernels in (18) as follows:

$$G_F^{(1)}(t, t_0) = g_1 \quad (23a)$$

$$G_F^{(2)}(t, t_0) = g_1 - a_1^1 \quad (23b)$$

$$G_F^{(3)}(t, t_0) = g_1 - [a_1^1 - a_1^2] \quad (23c)$$

.....

$$G_F^{(k)}(t, t_0) = g_1 - \left[\sum_{j=1}^{k-1} (-1)^{j-1} a_1^j \right], \quad (23d)$$

where with k we indicate the k^{th} approximation for the FHAM relaxation function that we assume can accurately describe the relaxation phenomena.

4.3. Derivation of the aging functions and model comparisons

In what follows, we illustrate the best-fitting procedure performed between the proposed FHAM and the existing aging Model B3 by considering as free parameters β , C_β and G_β . The best fitting was referred to a specific concrete specimen and environmental conditions. In particular, the mechanical characteristics, the relative humidity and the geometry of the cross-section are summarized in Table 1. Hence, the five parameters of Model B3 can be now computed: $q_1 = 19.33 \cdot 10^{-6}/\text{MPa}$; $q_2 = 116.32 \cdot 10^{-6}/\text{MPa}$; $q_3 = 2.87 \cdot 10^{-6}/\text{MPa}$; $q_4 = 6.49 \cdot 10^{-6}/\text{MPa}$; $q_5 = 323.50 \cdot 10^{-6}/\text{MPa}$.

We perform the best-fitting procedure for a time window extended from t_0 to 10000 days and by fulfilling the following physical conditions for a given t_0 : C_β a positive real number, G_β a positive real number, and $0 \leq \beta < 1$. Table 2 collects the results of the best-fitting procedure, where a careful reader can observe that both β and C_β increase until $t_0 = 8000$ days, after that they start to decrease. Likewise, the parameter G_β increases until $t_0 = 400$ days, after that it is almost constant. From the results in Table 2, the corresponding trends of β , G_β , and C_β can be easily expressed as function of t_0 . As a result, the expressions of $\beta(t_0)$, $G_\beta(t_0)$, and $C_\beta(t_0)$ are depicted in Figures 4(a), 4(b), 4(c) and listed in (24a) (24b) (24c).

$$\begin{aligned} \beta(t_0) = & \Theta(T_2 - t_0) \cdot f_1^\beta(t_0) + \Theta(t_0 - T_2) \cdot f_2^\beta(t_0) + \Theta(t_0 - T_3) \cdot f_3^\beta(t_0) \\ & - \Theta(t_0 - T_3) \cdot f_2^\beta(t_0) \end{aligned} \quad (24a)$$

$$\begin{aligned} G_\beta(t_0) = & \Theta(T_2 - t_0) \cdot f_1^{G_\beta}(t_0) + \Theta(t_0 - T_2) \cdot f_2^{G_\beta}(t_0) + \Theta(t_0 - T_3) \cdot f_3^{G_\beta}(t_0) \\ & - \Theta(t_0 - T_3) \cdot f_2^{G_\beta}(t_0) \end{aligned} \quad (24b)$$

$$C_\beta(t_0) = \Theta(T_4 - t_0) \cdot f_1^{C_\beta}(t_0) + \Theta(t_0 - T_4) \cdot f_2^{C_\beta}(t_0) + \Theta(t_0 - T_3) \cdot f_3^{C_\beta}(t_0)$$

Table 1: Mechanical and environmental conditions of a specimen chosen for the best-fitting procedure.

$\bar{f}_c$ - cylinder compression strength at 28 days [MPa]	43
E_{28} - Young modulus at 28 days [MPa]	31043
w/c - water-cement ratio	0.54
a/c - aggregate-cement ratio	5.10
c - cement content [kg/m ³]	343.04
m - empirical parameter	0.5
n - empirical parameter	0.1
t' - drying time [days]	7
RH - relative humidity [%]	60
k_s - cross-section shape factor	1.25
V/S - volume to surface ratio [cm]	15

Table 2: Values of β , G_β , and C_β obtained through the best-fitting procedure.

t_0	β	G_β	C_β
10	0.199	27483.0	74753.76
100	0.247	47511.9	152988.32
200	0.282	47832.5	238684.18
400	0.317	48061.9	373440.96
600	0.337	47994.2	484741.42
800	0.350	47829.6	582564.53
1000	0.361	47636.2	670241.33
3000	0.406	46132.2	1354441.39
5000	0.427	45429.0	1986155.88
7000	0.437	45181.1	2565382.86
9500	0.392	45656.2	2266373.77

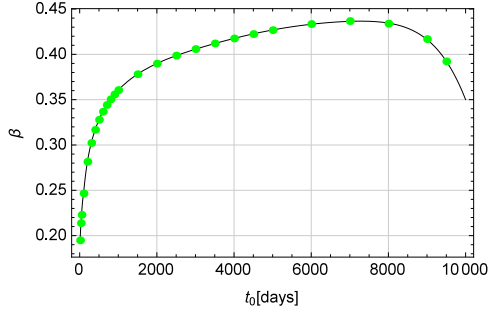
$$- \Theta(t_0 - T_3) \cdot f_2^{C_\beta}(t_0), \quad (24c)$$

where $\Theta(\cdot)$ indicates the Heaviside step function and $T_1 = 10$ days, $T_2 = 100$ days, $T_3 = 1000$ days, $T_4 = 200$ days.

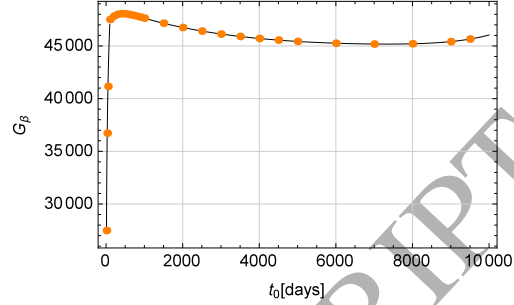
In order to reduce percentage errors between approximate and real trends of the three parameters, three functions active in different time intervals are used to describe each parameter. In particular, a second-degree function $f_1^\beta(t_0)$ is the best choice to fit the values of β within the time interval $[10, 100]$ days. Later on, two sixth-degree polynomials, i.e. $f_2^\beta(t_0)$ and $f_3^\beta(t_0)$, are selected to perform the best-fitting within $[100, 1000]$ and $[1000, 10000]$ days, respectively. Then, a logarithmic function $f_1^{G_\beta}(t_0)$ is chosen to fit the values of G_β within the interval $[10, 100]$ days; afterwards two sixth-degree polynomials, i.e. $f_2^{G_\beta}(t_0)$ and $f_3^{G_\beta}(t_0)$, are selected for $[100, 1000]$ and $[1000, 10000]$ days. Regarding the last parameter, C_β , a second-degree function $f_1^{C_\beta}(t_0)$ is selected for the best-fitting within $[10, 200]$ days, whilst two sixth-degree polynomials, i.e. $f_2^{C_\beta}(t_0)$ and $f_3^{C_\beta}(t_0)$, are chosen for $[200, 1000]$ and $[1000, 10000]$ days. All functions used to describe $\beta(t_0)$, $G_\beta(t_0)$, and $C_\beta(t_0)$ are reported in Appendix C.

Figure 5(a) depicts the comparison between the Model B3 creep function and the FHAM creep function, both computed for a loading time of 10 days. The fractional-aging model provides values of the creep function very similar to the values of the reference model. For instance, at 11 days we compute the maximum percentage error of 5.6%. There is also a high degree of matching in Figure 5(c), with the maximum percentage error of 6.2% at 1001 days.

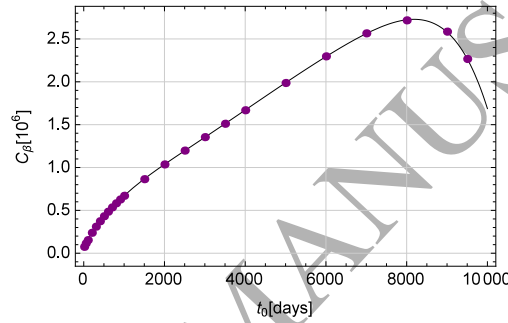
Another important comparison regards relaxation functions evaluated for different loading times. In particular, to compute the FHAM relaxation function, we follow the procedure described in Subsection 4.2 by setting $\bar{n} = 170$ and $k = 200$;



(a) Trend of the parameter β .



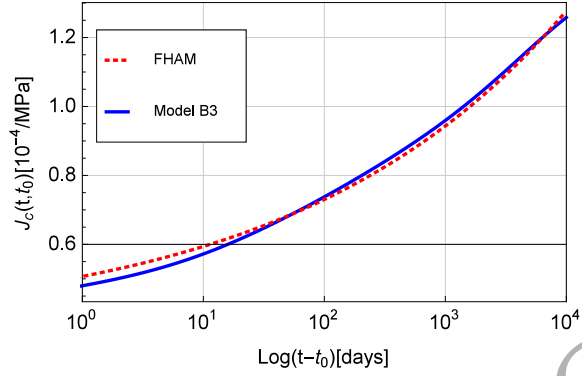
(b) Trend of the parameter G_β .



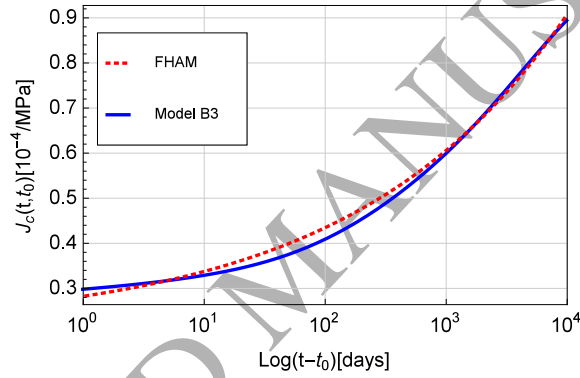
(c) Trend of the parameter C_β .

Figure 4: Results of the best-fitting procedure with Model B3 creep function and representations of functions $\beta(t_0)$, $G_\beta(t_0)$, and $C_\beta(t_0)$ by means of the solid black line.

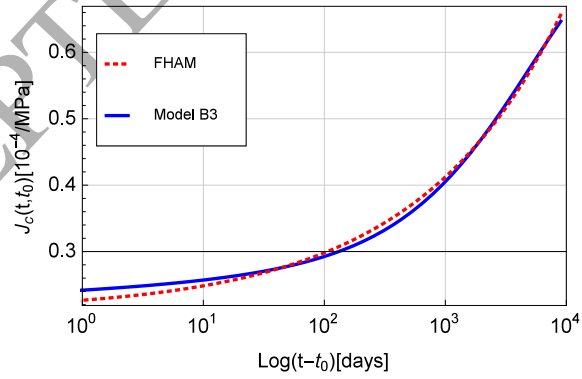
whilst, to compute the relaxation function of Model B3, we use the improved approximate formula proposed by [35] and illustrated in Subsection 4.1. Thus, Figures 6(a), 6(b), and 6(c) depict both the FHAM and Model B3 relaxation function for loading ages $t_0 = 10$, 100, and 1000 days, respectively. The fractional relaxation function is plotted as dotted red line, whereas, the Model B3 relaxation function is plotted as thick solid blue line. Moreover, the thin solid black line indicates our reference solution, i.e. the relaxation function of Model B3 obtained by numerically solving the Volterra's integral as shown in Appendix of [33]. Over-



(a) Comparison for $t_0 = 10$ days.

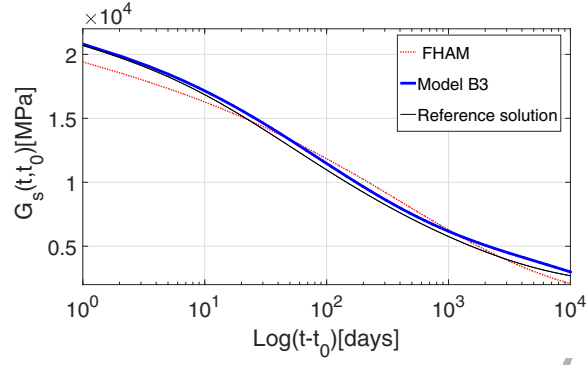


(b) Comparison for $t_0 = 100$ days.

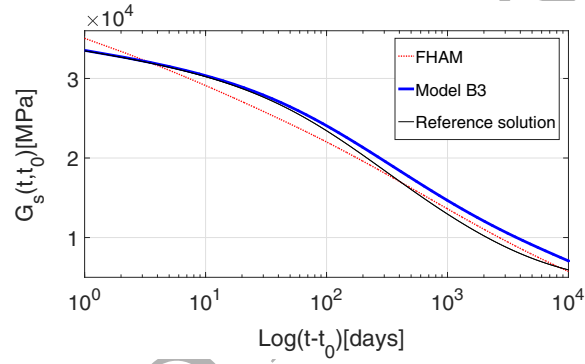


(c) Comparison for $t_0 = 1000$ days.

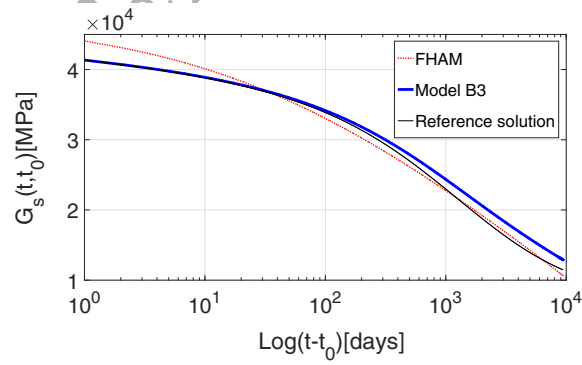
Figure 5: Comparisons between the Model B3 creep function and the FHAM creep function for different values of the loading time t_0 .



(a) Comparison for $t_0 = 10$ days.



(b) Comparison for $t_0 = 100$ days.



(c) Comparison for $t_0 = 1000$ days.

Figure 6: Comparisons among the Model B3 relaxation function, the FHAM relaxation function, and the reference solution, which was evaluated according to [33].

all, from the comparison of relaxation functions, we conclude that the procedure proposed in Subsection 4.2 leads to values of the relaxation function very close to the reference solution, especially for the case of long-term relaxation beginning at ages $t_0 = 100$ and 1000 days. Furthermore, the approximate formula suggested by [35] resulted less accurate than the proposed formula (23) for the evaluation of the structural response at the long term, i.e. $t = 10000$ days.

5. Conclusions

In this paper we propose a novel fractional-order model for the characterization of the hereditariness of aging materials.

More precisely, as discussed in Sections 2 and 3, no fractional-order model available in literature is suitable to describe the degradation/improvement of mechanical characteristics, which is a phenomenon typical in aging materials, like structural concretes, rubbers and polymers under photo-degradation. Nonetheless, the use of fractional-order models to describe both creep and relaxation phenomena can bring many advantages, such as a more realistic description of creep/relaxation and computational savings for the solution of the Volterra integral equation, as proven by many studies on material hereditariness. On these bases, in Section 4, we propose a fractional-order creep function suitable for aging-hereditary materials, characterized by three aging terms, $C_\beta(t_0)$, $\beta(t_0)$ and $G_\beta(t_0)$, and an hereditary term function of the elapsed time $(t - t_0)$. Moreover, considering an aging concrete, the inverse of $G_\beta(t_0)$ describes the time-dependent compliance, and both $\beta(t_0)$ and $C_\beta(t_0)$ are related to the change of pore shapes and to the rearranging of the material microstructure. In Section 4, we also propose an effective procedure for exactly deriving the corresponding relaxation function by

starting from the well-known Volterra integral equation of the second kind. The explicit expressions of $G_\beta(t_0)$, $\beta(t_0)$ and $C_\beta(t_0)$ are derived for a specific medium-strength concrete and through a best-fitting procedure with the creep function of Model B3, one of the most widely used models for prestressed concrete. The relevant comparison between Model B3 creep function and the creep function of the proposed fractional model entails percentage errors less than 6%, which is a reasonable value in engineering practice. A good agreement is also found between the relaxation functions, proving once more that the proposed model can be effectively applied to the simulation of aging hereditariness of materials like concrete and prestressed concrete.

As further development, the best-fitting procedure illustrated in Section 4 can be used to correlate $G_\beta(t_0)$, $\beta(t_0)$ and $C_\beta(t_0)$ of the proposed fractional-order model with experimental data available in the database [29]. Finally, the influence of mechanical properties, environmental conditions and cross-section geometry on $G_\beta(t_0)$, $\beta(t_0)$ and $C_\beta(t_0)$ deserves further studies.

Appendix A. Fundamental remarks of fractional calculus

Fractional calculus may be considered the extension of the ordinary differential calculus to non-integer powers of derivation orders. In this section, we address some basic notions about this mathematical tool.

Riemann-Liouville fractional integrals and derivatives with $0 < \beta < 1$ of functions defined on the entire real axis have the following forms,

$$(\mathcal{I}_+^\beta f)(t) = \frac{1}{\Gamma(\beta)} \int_{-\infty}^t \frac{f(\tau)}{(t-\tau)^{1-\beta}} d\tau \quad (\text{A.1a})$$

$$(\mathcal{D}_+^\beta f)(t) = \frac{1}{\Gamma(1-\beta)} \frac{d}{dt} \int_{-\infty}^t \frac{f(\tau)}{(t-\tau)^\beta} d\tau. \quad (\text{A.1b})$$

Riemann-Liouville fractional integrals and derivatives, with $0 < \beta < 1$, of functions defined over intervals of the real axis, namely $f(t)$ such that $t \in [a, b] \subset \mathbb{R}$, have the following forms,

$$(\mathcal{I}_a^\beta f)(t) = \frac{1}{\Gamma(\beta)} \int_a^t \frac{f(\tau)}{(t-\tau)^{1-\beta}} d\tau \quad (\text{A.2a})$$

$$(\mathcal{D}_a^\beta f)(t) = \frac{f(a)}{\Gamma(1-\beta)(t-a)^\beta} + \frac{1}{\Gamma(1-\beta)} \int_a^t \frac{f'(\tau)}{(t-\tau)^\beta} d\tau \quad (\text{A.2b})$$

where the Euler-Gamma function $\Gamma(z)$ is the generalization of the factorial function since as z assumes integer values as $\Gamma(z+1) = z!$. The Euler-Gamma function defined as the result of the integral reads,

$$\Gamma(z) = \int_0^\infty e^{-x} x^{z-1} dx. \quad (\text{A.3})$$

In addition to the Riemann-Liouville fractional operators illustrated above, another class of fractional derivatives that is often used in the context of the fractional viscoelasticity is represented by Caputo fractional derivatives, defined as,

$$({}_c\mathcal{D}_{a^+}^\beta f)(t) := \mathcal{I}_{a^+}^{m-\beta} (\mathcal{D}_{a^+}^m f)(t) \quad m-1 < \beta < m \quad (\text{A.4})$$

and, whenever $0 < \beta < 1$, it reads,

$$({}_c\mathcal{D}_{a^+}^\beta f)(t) = \frac{1}{\Gamma(1-\beta)} \int_a^t \frac{f'(\tau)}{(t-\tau)^\beta} d\tau \quad (\text{A.5})$$

A closer analysis of (A.4) and (A.5) shows that Caputo fractional derivative coincides with the integral part of the Riemann-Liouville fractional derivative in a bounded domain. Moreover, the definition in (A.4) entails that the function $f(t)$ has to be absolutely integrable of order m (e.g. in (A.5) the order is $m = 1$). Whenever $f(a) = 0$, Caputo and Riemann-Liouville fractional derivatives coalesce.

Similar considerations hold true also for Caputo and Riemann-Liouville fractional derivatives defined on the entire real axis. Caputo fractional derivatives may

be considered as the interpolation among the well-known integer-order derivatives, operating over functions $f(\circ)$ that belong to the class of Lebegue integrable functions ($f(\circ) \in L^1$); as a consequence, they are very useful in the mathematical description of a complex system evolution. We recall that Laplace and Fourier integral transforms are defined as follows,

$$\mathcal{L}[f(t)] = \int_0^{\infty} f(t)e^{-st} dt \quad (\text{A.6a})$$

$$\mathcal{F}[f(t)] = \int_{-\infty}^{+\infty} f(t)e^{i\omega t} dt \quad (\text{A.6b})$$

It is worth introducing integral transforms for fractional operators and, similarly to classical calculus, the Laplace integral transform $\mathcal{L}(\circ)$ can be defined in the following forms:

$$\mathcal{L}[(D_{0+}^{\beta} f)(t)] = s^{\beta} \mathcal{L}[f(t)] = s^{\beta} \tilde{f}(s) \quad (\text{A.7a})$$

$$\mathcal{L}[(I_{0+}^{\beta} f)(t)] = s^{-\beta} \mathcal{L}[f(t)] = s^{-\beta} \tilde{f}(s) \quad (\text{A.7b})$$

Likewise, the Fourier integral transform $\mathcal{F}(\circ)$ assumes the following forms,

$$\mathcal{F}[(D_{+}^{\beta} f)(t)] = (-i\omega)^{\beta} \mathcal{F}[f(t)] = (-i\omega)^{\beta} \hat{f}(\omega) \quad (\text{A.8a})$$

$$\mathcal{F}[(I_{+}^{\beta} f)(t)] = (-i\omega)^{-\beta} \mathcal{F}[f(t)] = (-i\omega)^{-\beta} \hat{f}(\omega) \quad (\text{A.8b})$$

Appendix B. Fractional calculus of variable order

In this section we briefly recall the main concepts of fractional calculus, when the order of integration/differentiation is allowed to vary with the independent variable t . [48] first suggested the concept of variable-order fractional operators and several candidate definitions for variable-order fractional integrals are addressed in [36]. Moreover, the definitions for the variable-order integration

introduced by [36] may be formally extended to variable-order fractional differentiation as illustrated in [49] and [44].

Recalling Riemann-Liouville fractional integrals and derivatives with order $0 < \beta < 1$ defined in Equations (A.2), in the current case of variable-order fractional calculus, they become,

$$\left(I_a^{\beta(t)} f\right)(t) = \int_a^t \frac{(t-\tau)^{\beta(t,\tau)-1}}{\Gamma(\beta(t,\tau))} f(\tau) d\tau \quad (\text{B.1a})$$

$$\left(D_a^{\beta(t)} f\right)(t) = \frac{(t-a)^{-\beta(t,a)}}{\Gamma(1-\beta(t,a))} f(a) + \int_a^t \frac{(t-\tau)^{-\beta(t,\tau)}}{\Gamma(1-\beta(t,\tau))} f'(\tau) d\tau \quad (\text{B.1b})$$

where the order of integration/differentiation is now function of the independent variable t and the integration/differentiation variable τ .

In the discussion that follows we focus on the main implications resulting from various choices of $\beta(t, \tau)$, assuming that the arguments of $\beta(t, \tau)$ at the exponent of $(t - \tau)$ and in the Gamma function are the same. In particular, the following choices are considered: $\beta(t, \tau) = \beta(t)$; $\beta(t, \tau) = \beta(\tau)$; $\beta(t, \tau) = \beta(t - \tau)$. However, a briefly discussion is pointed out in this appendix; thus, more details about all cases can be found in [36] and [44].

Appendix B.1. Case I $\beta(t, \tau) = \beta(t)$

For this case, we redefine the Riemann-Liouville fractional integral and derivative as follows,

$$\left(I_a^{\beta(t)} f\right)(t) = \int_a^t \frac{(t-\tau)^{\beta(t)-1}}{\Gamma(\beta(t))} f(\tau) d\tau \quad (\text{B.2a})$$

$$\left(D_a^{\beta(t)} f\right)(t) = \frac{(t-a)^{-\beta(t)}}{\Gamma(1-\beta(t))} f(a) + \int_a^t \frac{(t-\tau)^{-\beta(t)}}{\Gamma(1-\beta(t))} f'(\tau) d\tau \quad (\text{B.2b})$$

Since the argument of $\beta(t)$ at the exponent of $(t - \tau)$ in (B.2) is not a function of $(t - \tau)$, the fractional operator becomes time variant. We recall that an operator is said to be time invariant when the input $f(t)$ produces a response $y(t)$, then the input $f(t + \Delta t)$ produces the response $y(t + \Delta t)$ [36]. Moreover, it is noted that the operator is no more linear with respect to the order $\beta(t)$. Hence, the composition property is not applicable, i.e.,

$$I_+^{\beta(t)} I_+^{\alpha(t)} f(t) \neq \left(I_+^{\beta(t)+\alpha(t)} f \right) (t) \quad (\text{B.3})$$

Appendix B.2. Case 2 $\beta(t, \tau) = \beta(\tau)$

For this case, we redefine the Riemann-Liouville fractional integral and derivative as follows,

$$\left(I_a^{\beta(t)} f \right) (t) = \int_a^t \frac{(t - \tau)^{\beta(\tau)-1}}{\Gamma(\beta(\tau))} f(\tau) d\tau \quad (\text{B.4a})$$

$$\left(D_a^{\beta(t)} f \right) (t) = \frac{(t - a)^{-\beta(a)}}{\Gamma(1 - \beta(a))} f(a) + \int_a^t \frac{(t - \tau)^{-\beta(\tau)}}{\Gamma(1 - \beta(\tau))} f'(\tau) d\tau \quad (\text{B.4b})$$

Because of the argument of $\beta(\tau)$ in both the exponent of $(t - \tau)$ and Gamma function, the fractional operator results to be time variant. Moreover, as previously discussed for Case 1, the composition property in (B.3) is no more valid. It is important to highlight that the last implication strongly influenced the evaluation of the fractional relaxation function in Subsection (4.2). In particular, if the composition property was still valid for the case under investigation, i.e. $\beta(t, \tau) = \beta(\tau)$, the relaxation function of the proposed fractional aging model would be simpler to carry out and the final expression of the relaxation function would be a combination of Riemann-Liouville integrals with maximum order $i \cdot \beta$, where i indicates the number of iterated kernels used to approximate the solution [38].

Appendix B.3. Case 3 $\beta(t, \tau) = \beta(t - \tau)$

For the last case under investigation we consider the definitions,

$$(\mathcal{I}_a^{\beta(t)} f)(t) = \int_a^t \frac{(t - \tau)^{\beta(t-\tau)-1}}{\Gamma(\beta(t - \tau))} f(\tau) d\tau \quad (\text{B.5a})$$

$$(\mathcal{D}_a^{\beta(t)} f)(t) = \frac{(t - a)^{-\beta(t-a)}}{\Gamma(1 - \beta(t - a))} f(a) + \int_a^t \frac{(t - \tau)^{-\beta(t-\tau)}}{\Gamma(1 - \beta(t - \tau))} f'(\tau) d\tau \quad (\text{B.5b})$$

Now, since the argument of $\beta(t - \tau)$ in both the exponent of $(t - \tau)$ and Gamma function is function of $(t - \tau)$, the fractional operators are time invariant, as defined for Case 1 earlier. Moreover, also for this last case of variable-order operator the composition property in (B.3) is no more applicable.

Appendix C. FHAM functions

In the following, we list all functions needed to define $\beta(t_0)$, $C_\beta(t_0)$, and $G_\beta(t_0)$ in Subsection 4.3, i.e.

$$f_1^\beta(t_0) = -3.10 \cdot 10^{-6}(t_0 - T_1)^2 + 7.99 \cdot 10^{-4}(t_0 - T_1) + 0.199 \quad (\text{C.1a})$$

$$\begin{aligned} f_2^\beta(t_0) = & -1.11 \cdot 10^{-18}(t_0 - T_2)^6 + 3.63 \cdot 10^{-15}(t_0 - T_2)^5 - 4.81 \cdot 10^{-12}(t_0 - T_2)^4 + \\ & + 3.39 \cdot 10^{-9}(t_0 - T_2)^3 - 1.45 \cdot 10^{-6}(t_0 - T_2)^2 + 4.68 \cdot 10^{-4}(t_0 - T_2) + \\ & + f_1^\beta(T_2) \end{aligned} \quad (\text{C.1b})$$

$$\begin{aligned} f_3^\beta(t_0) = & -6.41 \cdot 10^{-24}(t_0 - T_3)^6 + 1.53 \cdot 10^{-19}(t_0 - T_3)^5 - 1.46 \cdot 10^{-15}(t_0 - T_3)^4 + \\ & + 7.17 \cdot 10^{-12}(t_0 - T_3)^3 - 2.02 \cdot 10^{-8}(t_0 - T_3)^2 + 4.36 \cdot 10^{-5}(t_0 - T_3) + \\ & + f_2^\beta(T_3) \end{aligned} \quad (\text{C.1c})$$

$$f_1^{G\beta}(t_0) = 8674.33 \cdot \ln(t_0) + 7379.72 \quad (\text{C.1d})$$

$$\begin{aligned} f_2^{G\beta}(t_0) = & -3.03 \cdot 10^{-15}(t_0 - T_2)^6 + 9.65 \cdot 10^{-11}(t_0 - T_2)^5 - 1.24 \cdot 10^{-7}(t_0 - T_2)^4 + \\ & + 8.45 \cdot 10^{-5}(t_0 - T_2)^3 - 3.41 \cdot 10^{-2}(t_0 - T_2)^2 + 7.72(t_0 - T_2) + f_1^{G\beta}(T_2) \end{aligned} \quad (\text{C.1e})$$

$$\begin{aligned} f_3^{G\beta}(t_0) = & 3.06 \cdot 10^{-20}(t_0 - T_3)^6 - 6.47 \cdot 10^{-16}(t_0 - T_3)^5 + 5.89 \cdot 10^{-12}(t_0 - T_3)^4 + \\ & - 3.50 \cdot 10^{-8}(t_0 - T_3)^3 + 2.07 \cdot 10^{-4}(t_0 - T_3)^2 - 1.06(t_0 - T_3) + f_2^{G\beta}(T_3) \end{aligned} \quad (\text{C.1f})$$

$$f_1^{C\beta}(t_0) = -0.37(t_0 - T_1)^2 + 931(t_0 - T_1) + 74753.76 \quad (\text{C.1g})$$

$$\begin{aligned} f_2^{C\beta}(t_0) = & 2.32 \cdot 10^{-13}(t_0 - T_4)^6 - 3.65 \cdot 10^{-10}(t_0 - T_4)^5 - 5.71 \cdot 10^{-8}(t_0 - T_4)^4 + \\ & + 4.9 \cdot 10^{-4}(t_0 - T_4)^3 - 0.55(t_0 - T_4)^2 + 767.75(t_0 - T_4) + f_1^{C\beta}(T_4) \end{aligned} \quad (\text{C.1h})$$

$$\begin{aligned} f_3^{C\beta}(t_0) = & -5.62 \cdot 10^{-17}(t_0 - T_3)^6 + 1.2 \cdot 10^{-12}(t_0 - T_3)^5 - 1.06 \cdot 10^{-8}(t_0 - T_3)^4 + \\ & + 4.74 \cdot 10^{-5}(t_0 - T_3)^3 - 0.11(t_0 - T_3)^2 + 437.57 \cdot 10^{-3}(t_0 - T_3) + f_2^{C\beta}(T_3) \end{aligned} \quad (\text{C.1i})$$

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