



Nonlinear strain wave localization in periodic composites

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ABSTRACT

Nonlinear strain wave propagation along the lamina of a periodic two-component composite was studied. A nonlinear model was developed to describe the strain dynamics. The model asymptotically satisfies the boundary conditions between the lamina, in contrast to previously developed models. Our model reduces an initial two-dimensional problem into a single one-dimensional nonlinear governing equation for longitudinal strains in the form of the Boussinesq equation. The width of the lamina may control the propagation of either compression or tensile localized strain waves, independent of the elastic constants of the materials of the composite.

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1. Introduction

The study of strain wave propagation in composites has attracted considerable attention, e.g., Nayfeh (1995), Reddy (2004) and Sun et al. (1968). A prominent application of strain waves is estimating the elastic constants of a composite material (Liu et al., 2002; Roy et al., 2011; Wang et al., 2002). All of these works utilize linear waves. Less work has been devoted to the nonlinear description of dynamical processes in composites (Benveniste and Aboudi, 1977; Fares, 2000). Finite amplitude strains should be considered nonlinear. Nonlinearity provides new important phenomena, one of them is the localization of a strain wave. Localized strain waves with a permanent shape usually exist due to a balance between nonlinearity and other factors, such as dispersion. Composites are a type of material with microstructure, and their features differ from those of classic elastic materials. The microstructure of materials gives rise to the dispersion of bulk strain waves (Erofeyev, 2002; Erofeyev and Potapov, 1993; Savin et al., 1973). In these works the dispersion terms in the governing equations appear due to the gradient terms (caused in turn by microstructural effects) in the potential energy expression (Eringen, 1968; Mindlin, 1964). When nonlinearity is modeled as in classic elasticity, the governing equation contains both nonlinear and dispersion terms whose balance gives rise to a bell-shaped localized wave or a strain solitary wave in a medium, as shown in Erofeyev and Potapov (1993). Porubov (2000) demonstrated that the microstructure also affects strain wave localization in elastic rods; how-

ever, now the usual dispersion caused by the finite radius of the rod, dominates. The same situation is expected for composites, whose dispersion is caused by the finite width of the lamina.

Modeling strain waves in composites depends on the reference direction of lamination and strain wave propagation. When the wave propagates across the lamina, homogenization methods are widely used, especially for periodic composites in the linear case (Charalamboukis, 2010; Nemat-Nasser et al., 2011; Kalamkarov and Kolpakov, 1997). The extension of this approach to a nonlinear problem has been studied in Andrianov et al. (2011) to account for longitudinal strain waves, and a Boussinesq-like equation has been obtained. Other methods have been employed to describe wave propagation along the lamina in composites. Physically reasonable approximations for displacements are used in Chitnis et al. (2003), Fu and Brookes (2006) and Reddy (2004), and these are generally applicable to all lamina. The governing equation is obtained by averaging the equation coefficients across the composite. Being rather universal and applicable to any composite (including periodic composites), this method does not satisfy all of the inter-layer boundary conditions. Another method is based on the consideration of the unit cell of a composite (Benveniste and Aboudi, 1976; Clements et al., 1996; Faidi and Nayfeh, 2000). The approach involves the boundary conditions. Thus, power series approximations for the displacements are used in Faidi and Nayfeh (2000) to satisfy the boundary conditions in the linear case. Similar approximations have been employed for geometrically nonlinear composite in Benveniste and Aboudi (1977).

In this paper we modeled nonlinear strain waves propagating along the lamina of a periodic two-component composite and con-

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sidered all of the boundary conditions between the lamina. For simplicity, both of the materials comprising the layers were assumed to be isotropically nonlinear, in contrast to the widely used orthotropic materials. We extended the ideas of Benveniste and Aboudi (1977) and Faidi and Nayfeh (2000) to include physical and geometrical nonlinearities. However, our asymptotic technique is different from that employed in Benveniste and Aboudi (1977). Previously, this technique was developed by Porubov and Pastrone (2004) for materials with microstructures. Our method allowed us to reduce asymptotically an initial two-dimensional (2D) problem to a single one-dimensional (1D) nonlinear governing equation for longitudinal strain waves. Additionally, averaging procedures were not employed, in contrast to Andrianov et al. (2011) and Benveniste and Aboudi (1977). The analytical study of the dynamic processes in composites is possible due to the reduction of the initial equations to a single nonlinear 1D equation. Thus, the presence of nonlinear and dispersion terms in the equation gives rise to a balance between them, and this results in the localization of a strain wave with a permanent shape. The relationships between the wave parameters established a connection between the sign of the amplitude and the lamina elastic properties and widths that allowed us to predict the propagation of either tensile or compression localized strain waves. This effect is not described within the model of geometrically nonlinear materials studied in Benveniste and Aboudi (1977), which illustrates the importance of previously identified physical nonlinearity, e.g., Engelbrecht (1983).

2. Statement of the problem

We considered the propagation of plane nonlinear strain waves along the lamina of a periodically laminated composite. In this case, one can assume that the zero component of the displacement vector in the direction along the plane and the other components do not vary in this direction. This assumption allows us to reduce the initial three-dimensional (3D) problem to a 2D problem (Fig. 1). Both of the materials that constitute lamina are assumed to be isotropic and nonlinearly elastic, and their potential energy is accounted for by the five-constant model developed by Murnaghan (1951).

$$\Pi_i = \frac{\lambda_i + 2\mu_i}{2} (I_1^{(i)})^2 - 2\mu_i I_2^{(i)} + \frac{l_i + 2m_i}{3} (I_1^{(i)})^3 - 2m_i I_1^{(i)} I_2^{(i)} + n_i I_3^{(i)}, \quad (1)$$

where for each layer, $i = 1, 2$, $I_k^{(i)}$, $k = 1, 2, 3$ are the invariants of the Cauchy–Green deformation tensor $\mathbf{C}^{(i)}$:

$$I_1^{(i)}(\mathbf{C}^{(i)}) = \text{tr} \mathbf{C}^{(i)}, \quad I_2^{(i)}(\mathbf{C}^{(i)}) = [(\text{tr} \mathbf{C}^{(i)})^2 - \text{tr} \mathbf{C}^{(i)^2}] / 2, \quad I_3^{(i)}(\mathbf{C}^{(i)}) = \det \mathbf{C}^{(i)}. \quad (2)$$

The Lamé coefficients (λ_i, μ_i), and the third order elastic moduli, or the Murnaghan moduli (l_i, m_i, n_i) are introduced. The geometrical nonlinearity is described by the tensor $\mathbf{C}^{(i)}$, whereas the Murnaghan model (1) accounts for a physical nonlinearity. In contrast to the Lamé coefficients, the Murnaghan moduli can be either positive or negative. Assume that the displacement vector is $\vec{V}^{(i)} = (u^{(i)}(x, y, t), v^{(i)}(x, y, t))$, where x is directed along the layers, and y is directed across the layers. Using the familiar relationship between energy and stress, the following relationships for the components of the Piola–Kirchhoff stress tensor $\mathbf{P}^{(i)}$ hold (subscripts denote differentiation):

$$\begin{aligned} P_{xx}^{(i)} = & (\lambda_i + 2\mu_i) u_x^{(i)} + \lambda_i v_y^{(i)} + \frac{3\lambda_i + 6\mu_i + 2l_i + 4m_i}{2} (u_x^{(i)})^2 \\ & + \frac{\lambda_i + 2\mu_i + m_i}{2} ((v_y^{(i)})^2 + (u_y^{(i)})^2) + \frac{\lambda_i + 2l_i}{2} ((v_y^{(i)})^2 + 2v_y^{(i)} u_x^{(i)}) \\ & + (\mu_i + m_i) v_x^{(i)} u_y^{(i)}, \end{aligned} \quad (3)$$

$$\begin{aligned} P_{yy}^{(i)} = & (\lambda_i + 2\mu_i) v_y^{(i)} + \lambda_i u_x^{(i)} + \frac{3\lambda_i + 6\mu_i + 2l_i + 4m_i}{2} (v_y^{(i)})^2 \\ & + \frac{\lambda_i + 2\mu_i + m_i}{2} ((u_y^{(i)})^2 + (v_x^{(i)})^2) + \frac{\lambda_i + 2l_i}{2} ((u_x^{(i)})^2 + 2u_x^{(i)} v_y^{(i)}) \\ & + (\mu_i + m_i) v_x^{(i)} u_y^{(i)}, \end{aligned} \quad (4)$$

$$\begin{aligned} P_{yx}^{(i)} = P_{xy}^{(i)} = & \mu_i (u_y^{(i)} + v_x^{(i)}) + (\lambda_i + 2\mu_i + m_i) (u_y^{(i)} v_y^{(i)} + u_x^{(i)} u_y^{(i)}) \\ & + (\mu_i + m_i) v_x^{(i)} (u_x^{(i)} + v_y^{(i)}). \end{aligned} \quad (5)$$

Then, the non-linear equations of motion are

$$\rho_i \frac{\partial^2 u^{(i)}}{\partial t^2} = \frac{\partial P_{xx}^{(i)}}{\partial x} + \frac{\partial P_{yx}^{(i)}}{\partial y}, \quad (6)$$

$$\rho_i \frac{\partial^2 v^{(i)}}{\partial t^2} = \frac{\partial P_{yy}^{(i)}}{\partial y} + \frac{\partial P_{yx}^{(i)}}{\partial x}, \quad (7)$$

where ρ_i is the density. Perfect contact between the lamina is assumed, which leads to the continuity conditions for the displace-

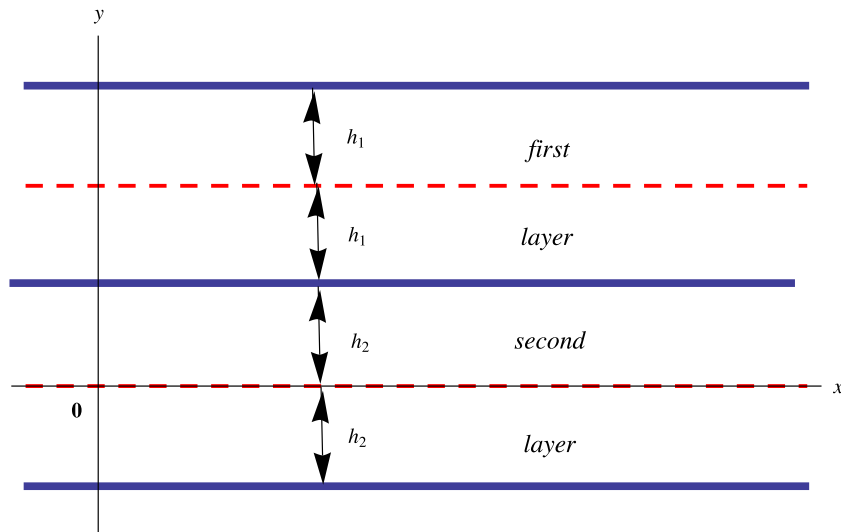


Fig. 1. Repeating unit cell $0 < y < h_1 + h_2$ (bounded by two dashed lines) of a two-component periodic composite. Solid lines mark the boundaries of the lamina.

ments and the stresses $P_{yy}^{(i)}$ and $P_{yx}^{(i)}$. Symmetric, $u^{(i)}$, and antisymmetric, $v^{(i)}$, displacements are assumed to occur near the center of each layer (Sun et al., 1968; Benveniste and Aboudi, 1977; Faidi and Nayfeh, 2000) to consider. This assumption results in zero shear stresses, $P_{yx}^{(i)}$, and transverse displacements, $v^{(i)}$, at the center of each layer, which allows us to isolate the repeating unit cell of the composite with two layers (see Fig. 1). The considered problem then reduces to the problem within the unit cell. The solution repeats above and below the unit cell.

Assume that the width of the first layer is equal to $2h_1$ and that the width of the second layer is equal to $2h_2$. Let the second layer be located below the first layer. The central line of the second layer corresponds to $y = 0$, while the central line of the first layer lies at $y = h_1 + h_2$. The interface is accounted for at $y = h_2$, as shown in Fig. 1. The considerations of symmetry then yield the conditions at $y = 0$ in the form

$$v^{(2)} = 0, \quad P_{yx}^{(2)} = 0. \quad (8)$$

The boundary conditions at $y = h_2$ are

$$u^{(1)} = u^{(2)}, \quad (9)$$

$$v^{(1)} = v^{(2)}, \quad (10)$$

$$P_{yy}^{(1)} = P_{yy}^{(2)}, \quad (11)$$

$$P_{yx}^{(1)} = P_{yx}^{(2)}. \quad (12)$$

Similar to Eq. (8), we obtain at $y = h_1 + h_2$

$$v^{(1)} = 0, \quad P_{yx}^{(1)} = 0. \quad (13)$$

Here, the superscripts (1), (2) denote layer 1 and layer 2, respectively.

3. Derivation of the governing equation

The Murnaghan model (1) is a truncated power series expansion and is not an exact representation of the energy. Higher order nonlinearities are neglected since finite, but small, strains are considered. An asymptotic procedure was developed to obtain simplified governing equations with the needed precision. The widths of the layers or lamina in the composite are small, and this allowed us to use the natural power series approximations for the displacement components. Only two terms were used,

$$u^{(1)} = U_0(x, t) + (y - h_1 - h_2)^2 U_1(x, t), \quad (14)$$

$$u^{(2)} = Q_0(x, t) + y^2 Q_1(x, t), \quad (15)$$

$$v^{(1)} = (y - h_1 - h_2) V_0(x, t) + (y - h_1 - h_2)^3 V_1(x, t), \quad (16)$$

$$v^{(2)} = y W_0(x, t) + y^3 W_1(x, t). \quad (17)$$

The problem is then reduced to a 1D problem. As shown in the following, the number of terms in the approximations is sufficient for deriving the final nonlinear governing equation within the accuracy of the Murnaghan model (1).

These approximations automatically satisfy the boundary conditions at the center line of the layers, (8) and (13). We have eight Eqs. (6), (7), (9)–(12) for determining eight unknown functions U_0 , U_1 , Q_0 , Q_1 , V_0 , V_1 , W_0 , and W_1 . Eqs. (6) and (7) become polynomials of $y - h_1 - h_2$ and y , respectively. We only assign zero to the coefficients at the lower order of $y - h_1 - h_2$ and y due to the chosen order of accuracy. Therefore, Eqs. (6) give rise to two equations whose solutions give the relationships for the functions U_1 and Q_1 ,

respectively. Additionally, we have to avoid negligibly small nonlinear terms. Following the method of Porubov and Pastrone (2004), only the linear part of the coefficient at $(y - h_1 - h_2)^0$ in Eq. (6) with $i = 1$, is considered:

$$2\mu_1 U_1 - \rho_1 U_{0,tt} + (\lambda_1 + 2\mu_1) U_{0,xx} + (\lambda_1 + \mu_1) V_{0,x} = 0. \quad (18)$$

This simplification gives rise to the expression for the intermediate solution for the function U_1 , which is further substituted into the nonlinear part of the coefficient at $(y - h_1 - h_2)^0$ in Eq. (6). The complete coefficient becomes an algebraic equation for U_1 , with the solution

$$U_1 = \frac{1}{2\mu_1} (\rho_1 U_{0,tt} - (\lambda_1 + 2\mu_1) U_{0,xx} - (\lambda_1 + \mu_1) V_{0,x}) - \rho_1 a_1 (V_0 + U_{0,x}) U_{0,tt} + (a_1 \lambda_1 + a_2) (V_0 + U_{0,x}) V_{0,x} + (a_1 (\lambda_1 + 2\mu_1) + a_2 - a_3) V_0 U_{0,xx} + (a_1 (\lambda_1 - 2\mu_1) + a_2 + a_3) U_{0,x} U_{0,xx}, \quad (19)$$

where

$$a_1 = \frac{(m_1 + \lambda_1 + 2\mu_1)}{2\mu_1^2}, \quad a_2 = \frac{1}{2} - \frac{l_1}{\mu_1}, \quad a_3 = \frac{1}{2} \left(1 + \frac{\lambda_1}{\mu_1} \right). \quad (20)$$

Similarly the solution for Q_1 is obtained by equating the coefficients at y^0 in Eq. (6) with $i = 2$ to zero:

$$Q_1 = \frac{1}{2\mu_2} (\rho_2 Q_{0,tt} - (\lambda_2 + 2\mu_2) Q_{0,xx} - (\lambda_2 + \mu_2) W_{0,x}) - \rho_2 b_1 (W_0 + Q_{0,x}) Q_{0,tt} + (b_1 \lambda_2 + b_2) (W_0 + Q_{0,x}) W_{0,x} + (b_1 (\lambda_2 + 2\mu_2) + b_2 - b_3) W_0 Q_{0,xx} + (b_1 (\lambda_2 - 2\mu_2) + b_2 + b_3) Q_{0,x} Q_{0,xx}, \quad (21)$$

where

$$b_1 = \frac{(m_2 + \lambda_2 + 2\mu_2)}{2\mu_2^2}, \quad b_2 = \frac{1}{2} - \frac{l_2}{\mu_2}, \quad b_3 = \frac{1}{2} \left(1 + \frac{\lambda_2}{\mu_2} \right). \quad (22)$$

The linearized Eq. (7) are sufficient for determining the expressions for the functions V_1 and W_1 . The solution for V_1 is obtained by equating the terms at $y - h_1 - h_2$ in the linearized Eq. (7) with $i = 1$ to zero,

$$V_1 = \frac{\rho_1 V_{0,tt}}{6(\lambda_1 + 2\mu_1)} + \frac{\lambda_1 V_{0,xx}}{6\mu_1} - \frac{(\lambda_1 + \mu_1) \rho_1 U_{0,xtt}}{6\mu_1 (\lambda_1 + 2\mu_1)} + \frac{(\lambda_1 + \mu_1) U_{0,xxx}}{6\mu_1}. \quad (23)$$

Similarly the solution for W_1 is obtained by equating the terms at y in the linearized Eq. (7) with $i = 2$ to zero,

$$W_1 = \frac{\mu_2 \rho_2 W_{0,tt}}{6\mu_2 (\lambda_2 + 2\mu_2)} + \frac{\lambda_2 W_{0,xx}}{6\mu_2} - \frac{(\lambda_2 + \mu_2) \rho_2 Q_{0,xtt}}{6\mu_2 (\lambda_2 + 2\mu_2)} + \frac{(\lambda_2 + \mu_2) Q_{0,xxx}}{6\mu_2}. \quad (24)$$

Substitution of the obtained solutions into the boundary conditions (9)–(12) allows us to determine the solutions for the functions Q_0 , W_0 and V_0 .

The linearized Eqs. (9) and (10) are then used to obtain the intermediate solutions Q_0 , W_0 , which are subsequently substituted into the nonlinear parts of Eqs. (9) and (10). The complete boundary conditions (9) and (10) become algebraic equations for Q_0 and W_0 , respectively. The solution for Q_0 is

$$Q_0 = U_0 + B_0 U_{0,tt} - B_1 V_{0,x} + B_2 U_{0,xx}, \quad (25)$$

where

$$B_0 = \frac{(h_1^2 \mu_2 \rho_1 - h_2^2 \mu_1 \rho_2)}{2\mu_1 \mu_2}, \quad B_1 = \frac{h_1(h_1 \mu_2 [\lambda_1 + \mu_1] + h_2 \mu_1 [\lambda_2 + \mu_2])}{2\mu_1 \mu_2}, \quad (26)$$

$$B_2 = \frac{h_2^2 \mu_1 (\lambda_2 + 2\mu_2) - h_1^2 \mu_2 (\lambda_1 + 2\mu_1)}{2\mu_1 \mu_2}, \quad (27)$$

while the solution of Eq. (10) for W_0 has the form

$$W_0 = -\frac{h_1 V_0}{h_2} - A_0 V_{0,tt} - A_1 V_{0,xx} + A_2 U_{0,xtt} - A_3 U_{0,xxx}, \quad (28)$$

where

$$A_0 = \frac{h_1(h_1^2 \rho_1 [\lambda_2 + 2\mu_2] - h_2^2 \rho_2 [\lambda_1 + 2\mu_1])}{6h_2(\lambda_1 + 2\mu_1)(\lambda_2 + 2\mu_2)},$$

$$A_1 = \frac{h_1(h_1^2 \lambda_1 \mu_2 - h_2^2 \lambda_2 \mu_1)}{6h_2 \mu_1 \mu_2}, \quad (29)$$

$$A_2 = \frac{h_1^3 \mu_2 \rho_1 [\lambda_1 + \mu_1] [\lambda_2 + 2\mu_2] + h_2^3 \mu_1 \rho_2 [\lambda_1 + 2\mu_1] [\lambda_2 + \mu_2]}{6h_2 \mu_1 \mu_2 (\lambda_1 + 2\mu_1)(\lambda_2 + 2\mu_2)}, \quad (30)$$

$$A_3 = \frac{h_1^3 \mu_2 (\lambda_1 + \mu_1) + h_2^3 \mu_1 (\lambda_2 + \mu_2)}{6h_2 \mu_1 \mu_2}. \quad (31)$$

With these solutions, Eq. (11) becomes an algebraic equation for the function V_0 yielding a solution of the form

$$V_0 = \frac{h_2(\lambda_2 - \lambda_1)U_{0,x}}{h_2(\lambda_1 + 2h_2\mu_1) + h_1(\lambda_2 + 2\mu_2)} + C_0 U_{0,x}^2 + C_1 U_{0,xtt} - C_2 U_{0,xxx}, \quad (32)$$

where the coefficients C_j are given in Appendix 1.

The substitution of all of the previously found solutions into the boundary condition (12) yields the final single nonlinear governing equation for the function U_0 .

4. Governing equation for longitudinal strain waves

The equation following from Eq. (12) is

$$U_{0,tt} - c^2 U_{0,xx} - \alpha U_{0,x} U_{0,xx} - \beta_1 U_{0,xxxx} - \beta_2 U_{0,xtt} - \beta_3 U_{0,ttt} = 0. \quad (33)$$

Introducing the width ratio, $\gamma = h_2/h_1$, we have

$$c^2 = \frac{(1 + \gamma^2)(\lambda_1 + 2\mu_1)(\lambda_2 + 2\mu_2) + 2\gamma[2\mu_1(\lambda_1 + \mu_1) + \lambda_1 \lambda_2 + 2\mu_2(\lambda_2 + \mu_2)]}{[\gamma(\lambda_1 + 2\mu_1) + (\lambda_2 + 2\mu_2)](\rho_1 + \gamma\rho_2)}. \quad (34)$$

The other coefficients are given in Appendix 2.

The first two terms in Eq. (33) account for the linear wave operator, and c is the velocity of the linear strain waves. Recently, special attention has been given to the form of the dispersion terms in equations, such as Eq. (33) (Christov et al., 2007; Engelbrecht et al., 2005, 2011; Porubov and Pastrone, 2004). Sometimes, the spatial and mixed derivative terms must be retained in the equation. A careful dispersion analysis of the equation corresponding to the linearized Eq. (33) in Engelbrecht et al. (2005) reveals the importance of taking into account all of the terms, particularly, for the co-existence of “optical” and “acoustical” branches. At the same time, “acoustical” branches, obtained for various sets of dispersion terms, do not differ much for small wave numbers, or for long waves. Recently, long strain solitary waves in an elastic rod were described using an equation similar to Eq. (33), but excluding the dispersion term $U_{0,ttt}$; see Samsonov (2001). However, the experimental observation of strain solitary waves did not reveal the

influence of the mixed derivative dispersion term, $U_{0,xtt}$ (Dreiden et al., 1995; Samsonov, 2001). Lastly, an exact localized traveling wave solution to Eq. (33) may be obtained, but the relationships for its parameters will be complicated for using them for the predictions of wave localization.

We are interested in the study of long localized nonlinear waves, and our aim was to simplify the dispersion terms in Eq. (33) due to the aforementioned reasons mentioned. The dispersion terms in Eq. (33) are smaller than the first two terms, U_{tt} and U_{xx} , in the long wave approximation. Therefore, one can replace the terms $U_{0,xtt}$ and $U_{0,ttt}$ with the spatial derivative dispersion term $U_{0,xxxx}$ as $U_{0,ttt} = c^2 U_{0,xxxx}$, $U_{0,ttt} = c^4 U_{0,xxxx}$. (35)

The governing Eq. (33) is then transformed into the celebrated Boussinesq equation for the new variable $w = U_{0,x}$, which describes the main part of the longitudinal strain:

$$w_{tt} - c^2 w_{xx} - \alpha(w^2)_{xx} - qw_{xxxx} = 0, \quad (36)$$

where $q = \beta_1 + \beta_2 c^2 + c^4 \beta_3$.

An important limit arises at $\lambda_2 \rightarrow 0$, $\mu_2 \rightarrow 0$, $\rho_2 \rightarrow 0$, $h_2 \rightarrow 0$, $m_2 \rightarrow 0$ when we arrive at the problem of the layer with free lateral boundaries. The velocity of the linear strain waves transforms to the familiar expression

$$c_0^2 = \frac{4\mu_1(\lambda_1 + \mu_1)}{(\lambda_1 + 2\mu_1)\rho_1}, \quad (37)$$

while the coefficient of nonlinear term, α , reduces to α_0 , which has the form

$$\alpha_0 = \frac{4\mu_1}{(\lambda_1 + 2\mu_1)^3 \rho_1} (3\lambda_1^2(2m_1 + \lambda_1) + 3\lambda_1(4m_1 + 5\lambda_1)\mu_1 + 4(l_1 + 2m_1 + 6\lambda_1)\mu_1^2 + 12\mu_1^3). \quad (38)$$

Similarly, the dispersion term coefficient q reduces to q_0 ,

$$q_0 = \frac{h_1^2 \lambda_1 \mu_1 (13\lambda_1^2 + 18\lambda_1 \mu_1 + 4\mu_1^2)}{6(\lambda_1 + 2\mu_1)^3 \rho_1}. \quad (39)$$

Another interesting limit appears at $h_2 \rightarrow 0$, which corresponds to the absence of layering, and the velocity tends to that of the linear longitudinal waves in a medium,

$$c_l^2 = \frac{\lambda_1 + 2\mu_1}{\rho_1}. \quad (40)$$

The nonlinear term coefficient is

$$\alpha_l = \frac{2l_1 + 4m_1 + 3\lambda_1 + 6\mu_1}{\rho_1}, \quad (41)$$

while the dispersion disappears, $\beta_1 + \beta_2 c^2 + c^4 \beta_3 = 0$, in agreement with the classic elastic theory in which strain waves in a medium do not possess dispersion.

5. Localization of strain waves

The localization of strain waves occurs due to a balance between nonlinearity and dispersion. As we see from Eq. (36), composites possess dispersion due to the finite size of their layers, $\gamma \neq 0$. Nonlinearity, $\alpha \neq 0$, is caused by geometrical and physical reasons, which are described by the Cauchy-Green deformation tensor $\mathbf{C}^{(i)}$ and the Murnaghan model (1) respectively.

The Eq. (36) is integrable by the Inverse Scattering Transform (IST) method. However, its known single solitary wave solution is easy to obtain by direct integration (see Ablowitz and Segur (1981)), yielding

$$w = \frac{6k^2 q}{\alpha} \text{sech}^2(k(x - Vt)), \quad (42)$$

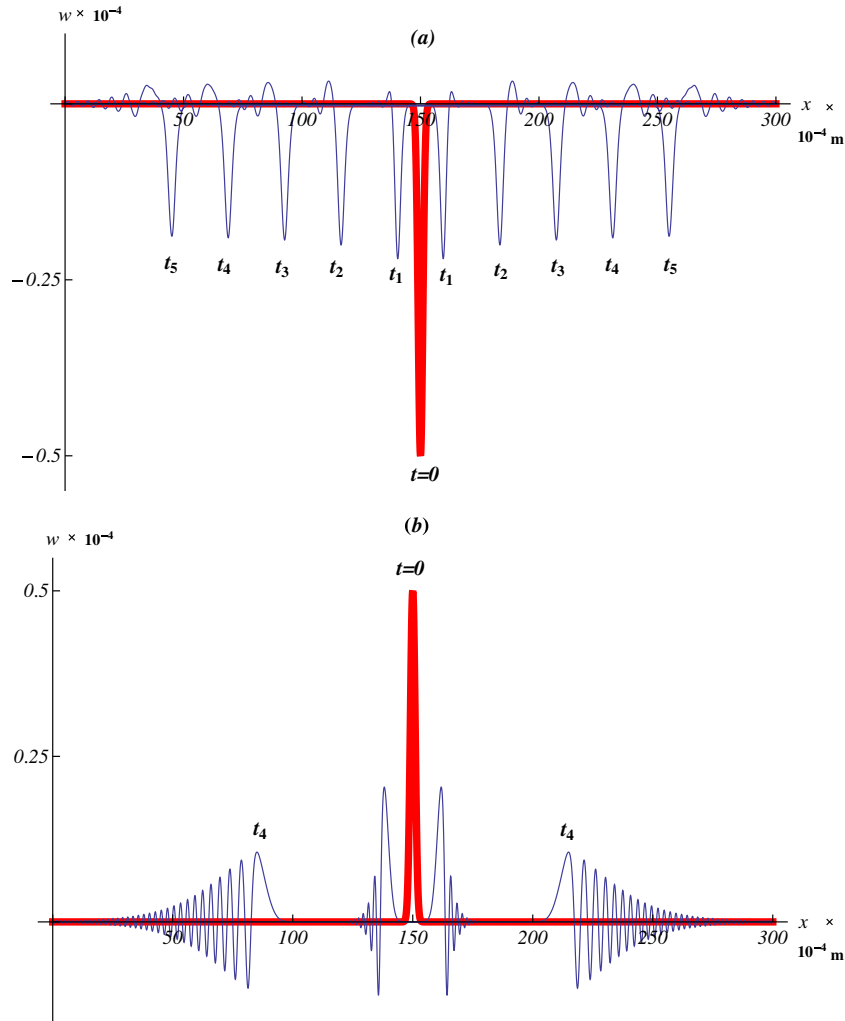


Fig. 2. Typical evolution for $\alpha < 0$, $q > 0$, $\gamma > \gamma_0$: (a) Generation of localized compression strain waves. (b) Delocalization of the tensile input. The times are $t_1 < t_2 < t_3 < t_4 < t_5$.

where $V^2 = c^2 + 4qk^2$. In the limit of the absence of the second layer the solution reads

$$w_0 = \frac{6k^2 q_0}{\alpha_0} \text{sech}^2(k(x - V_0 t)), \quad (43)$$

where $V_0^2 = c^2 + 4q_0 k^2$. The sign of the amplitude of the wave (43) strongly depends on the sign of the coefficient of the nonlinear term, α_0 , because the coefficient of the dispersion term, q_0 , is always positive due to Eq. (39). The sign of α_0 is fixed by the values of the elastic constants according to Eq. (38). Therefore only the material properties (not the width of the layer) define whether compression, $\alpha_0 < 0$, or tensile, $\alpha_0 > 0$, localized waves may propagate. In the case of only a geometrically nonlinear material, $l_1 = 0$, $m_1 = 0$, α_0 is always positive, and compression waves are not described. At the same time, compression localized strain waves are experimentally observed in Dreiden et al. (1995), that demonstrates that physical nonlinearity is important.

Determining the sign of the complicated expression for q in the general case is not straightforward, however, calculations confirm that it is positive for many materials. Positive and negative values for α may be achieved only if the geometrical and physical nonlinearities are taken into account. In contrast to α_0 , the coefficient α also depends on γ . Therefore, variations in the width of the lamina can change the sign of the amplitude of the wave (42) in the composite.

To check this possibility, consider a composite consisting of SiC and the aluminum alloy 8091. The constants are taken from Chen and Jiang (1993). For aluminum, we have

$$\rho = 2.7 \times 10^3 \frac{\text{kg}}{\text{m}^3}, \quad \lambda = 44.93 \text{ GPa}, \quad \mu = 31.0 \text{ GPa}, \quad (44)$$

$$l = -218 \text{ GPa}, \quad m = -378 \text{ GPa}. \quad (45)$$

For SiC the constants are

$$\rho = 3.21 \times 10^3 \frac{\text{kg}}{\text{m}^3}, \quad \lambda = 97.7 \text{ GPa}, \quad \mu = 188 \text{ GPa}, \quad (46)$$

$$l = -82.1 \text{ GPa}, \quad m = -310 \text{ GPa}. \quad (47)$$

First, consider single layers for both materials. The calculations give positive values of q_0 for both sets of parameters. The nonlinear parameter is $\alpha_0 < 0$ for aluminum, and it is positive for SiC. This result means that only compression localized waves (43) propagate in the single aluminum layer, while only tensile waves propagate in the single layer made of SiC. Now, assume that the width, h_1 , of one layer in the composite is fixed, while another width, h_2 , varies. Comparable widths of the layers made of aluminum and SiC, give rise to the negative sign for α . The same occurs when the SiC layer is thinner than the aluminum layer. In these cases, only compression localized waves (42) exist in the composite. However, a threshold value, $\gamma_0 = 0.0066$, separates the positive and negative

α values for the case corresponding to a thin layer 2 made of aluminum and to a layer 1 made of SiC. Hence, for a very thin aluminum layer, tensile localized waves propagate the same as in a single SiC layer. However, with wider aluminum layers, the composite will support only compression localized strain waves.

This prediction is based on a particular single traveling wave solution (42). However, localization in a more general case occurs according to this prediction. This finding may be checked numerically following the evolution of the input in the form of the Gaussian distribution and choosing zero as the initial velocity. The numerical tools within the Mathematica 8 software package were employed. Consider the case in which the coefficient of the nonlinear term is negative. The initial conditions do not satisfy the exact traveling wave solution. Nevertheless, an initial compression input separates into counter-propagating localized compression waves, as shown in Fig. 2(a). Over time, the amplitudes of the waves achieve a constant value, and both solitary waves propagate with a permanent velocity according to the exact solution (42). At the same time, the initial Gaussian tensile input does not give rise to localized waves with a permanent shape. Instead, the input disperses, as shown in Fig. 2(b). For the composite considered in this study, this evolution of the strain waves, w , corresponds to the case $\gamma > \gamma_0$. Otherwise, negative amplitude input disperses. A positive amplitude input gives rise to the formation of two counter-propagating localized tensile waves.

Unfortunately, the values for the third-order constant are known for a few materials. Other composites with available third-order constants for their components do not demonstrate opposite sign for the amplitudes of the solitary wave (43) for each material. Nevertheless, the lamina cause an increase or decrease in the amplitude of the localized wave in comparison with a single layer. This behavior of the localized waves in a composite is similar to the behavior of strain waves in an elastic rod surrounded by an external elastic medium considered in Porubov et al. (1998). The external medium affects the localization, similar to the layers in the composite. The description of the amplification of the strain wave due to the influence of the lamina in composites seems important in terms of durability.

6. Conclusions

We used our approximations for displacements to obtain primary nonlinear and dispersion terms, whose balance supports localized strain waves with a permanent shape. The correct consideration of the role of higher order dispersion and nonlinear terms requires that the Murnaghan truncated expansion (1) be extended to account for a higher order nonlinearity. This modification is achieved using the so-called nine-constant model of Murnaghan (1951) that considers fourth-order elastic constants. Physical reasons are needed to perform such an advanced analysis.

Further progress may be achieved by considering imperfect bonding between the layers up to the breaking point. Recently, a similar problem has been studied by Khusnutdinova and Samsonov (2008) to account for the splitting of localized waves in a rod with a cut. This splitting was experimentally observed in Dreiden et al. (2010). This work emphasizes why the nonlinear study of imperfect bonding appears promising.

Another extension of our work concerns piezoelectric effects, similar to Faidi and Nayfeh (2000) for the linear case. The influence of piezoelectricity on the localization of nonlinear strain waves may introduce new effects of localization or delocalization.

Finally, other composites should be examined for nonlinear strain wave localization. Sometimes, third-order elastic constants are measured, but not within the isotropic model, e.g., the epoxy-graphite composites (Prosser, 1990; Prosser and Green, 1990). For these cases, the re-calculation of the moduli is required.

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Appendix 1. The coefficients of Eq. (32)

$$C_0 = \frac{h_2}{2(h_2(\lambda_1 + 2\mu_1) + h_1(\lambda_2 + 2\mu_2))^3} \left(h_1^2 [4m_2(\lambda_1 - \lambda_2)^2 + 2l_2(\lambda_1 + 2\mu_2)^2 - (\lambda_2 + 2\mu_2)(2l_1(\lambda_2 + 2\mu_2) - (\lambda_1 - \lambda_2)[3\lambda_1 - 2(\lambda_2 + \mu_2)])] + 2h_1h_2[\lambda_2(\lambda_1^2 - \lambda_1\lambda_2 - 2l_1(\lambda_2 + 2\mu_1)) - 4(l_1\lambda_2 + (2l_1 + \lambda_1 - \lambda_2)\mu_1)\mu_2 + 2l_2(\lambda_1 + 2\mu_1)(\lambda_1 + 2\mu_2)] - h_2^2[2\lambda_1^3 + 4m_1(\lambda_1 - \lambda_2)^2 - \lambda_2[5\lambda_1^2 - \lambda_2(2l_1 + 3\lambda_1)] + 2(3(\lambda_1 - \lambda_2)^2 + 4l_1\lambda_2)\mu_1 - 4(2l_1 + \lambda_1 - \lambda_2)\mu_1^2 + 2l_2(\lambda_1 + 2\mu_1)^2] \right), \quad (48)$$

$$C_1 = \frac{h_2}{2(h_2(\lambda_1 + 2\mu_1) + h_1(\lambda_2 + 2\mu_2))^2} \left(h_1(\lambda_2 + 2\mu_2)(2A_2\lambda_2 + 4A_2\mu_2 + h_1^2\rho_1) + h_2(2[(A_0 + A_2)\lambda_1 - A_0\lambda_2 + 2A_2\mu_1](\lambda_2 + 2\mu_2) + h_1^2\rho_1(2\lambda_1 - \lambda_2 + 2\mu_1)) - h_2^3\rho_2(\lambda_1 + 2\mu_1) + h_1h_2^2\rho_2[\lambda_1 - 2(\lambda_2 + \mu_2)] \right), \quad (49)$$

$$C_2 = \frac{h_2}{2(h_2(\lambda_1 + 2\mu_1) + h_1(\lambda_2 + 2\mu_2))^2} \left(h_1^2h_2[4\mu_1^2 + \lambda_1(\lambda_2 + 4\mu_1)] + h_1^2(\lambda_1 + 2\mu_1) \times (\lambda_2 + 2\mu_2) - h_2(\lambda_2 + 2\mu_2)[(2A_1 - 2A_3 + h_2^2)\lambda_1 - 2A_1\lambda_2 - 2\mu_1(2A_3 - h_2^2)] + h_1[2A_3(\lambda_2 + 2\mu_2)^2 - h_2^2(\lambda_1\lambda_2 + 4\mu_2(\lambda_2 + \mu_2))] \right) \quad (50)$$

Appendix 2. The coefficients of Eq. (33)

$$\alpha = \frac{1}{(\gamma[\lambda_1 + 2\mu_1] + \lambda_2 + 2\mu_2)^2(\rho_1 + \gamma\rho_2)} \times \left([2(l_1 + 2m_1 + C_0[\lambda_1 - \lambda_2]) + 3(\lambda_1 + 2\mu_1)](\lambda_2 + 2\mu_2)^2 + \gamma[2\lambda_2^2(2l_1 + 2m_2 + \lambda_2) + 24\mu_1^2(\lambda_2 + 2\mu_2) + 2\lambda_2(4l_1 + 8m_2 + 7\lambda_2)\mu_2 + 4(2l_2 + 4m_2 + 9\lambda_2)\mu_2^2 + 24\mu_2^3 + \lambda_1^2(2l_2 + 5\lambda_2 + 4C_0\lambda_2 + 8\mu_2 + 8C_0\mu_2) + 2\lambda_1(4m_1\lambda_2 + \lambda_2^2 - 2C_0\lambda_2^2 + 4l_2\mu_2 + 8m_1\mu_2 + 4\lambda_2\mu_2 - 4C_0\lambda_2\mu_2) + 8(l_1 + 2m_1 + 3\lambda_1 + C_0(\lambda_1 - \lambda_2))\mu_1(\lambda_2 + 2\mu_2)]\gamma^2[2(1 + C_0)\lambda_1^3 + 2l_1\lambda_2^2 + 2\lambda_1^2(2l_2 + 2m_1 + \lambda_2 - C_0\lambda_2) + 4(2l_1 + 4m_1 + 9\lambda_1 + 2C_0[\lambda_1 - \lambda_2])\mu_1^2 + 24\mu_1^3 + 2\mu_1(l_2 + 2m_1)\lambda_1 + 7\lambda_1^2 + 4C_0\lambda_1[\lambda_1 - \lambda_2] + 4\lambda_2(l_1 + 2m_2 + \lambda_1 + \lambda_2) + 8\mu_2(l_2 + 2m_2 + 3\lambda_2 + 3\mu_2)] + \lambda_1(\lambda_2(8m_2 + 5\lambda_2) + 8(l_2 + 2m_2)\mu_2 + 24\mu_2(\lambda_2 + \mu_2))] \gamma^3(\lambda_1 + 2\mu_1)^2(2l_2 + 4m_2 + 3\lambda_2 + 6\mu_2) \right) \quad (51)$$

$$\beta_1 = \frac{1}{6(\gamma[\lambda_1 + 2\mu_1] + [\lambda_2 + 2\mu_2])(\rho_1 + \gamma\rho_2)} \left(\gamma^4 h_1^2(\lambda_1 + 2\mu_1)(\lambda_2 + \mu_2) + \gamma^3 h_1^2[\lambda_2(\lambda_1 + 2\mu_2) + \mu_2(\lambda_2 + 2\mu_2)] - \gamma^2[6A_1(\lambda_1 - \lambda_2)\lambda_2 - 6A_3\lambda_2(\lambda_1 + 2\mu_1)6B_1(\lambda_1 - \lambda_2)(\lambda_2 + 2\mu_2) + 6B_2(\lambda_1 + 2\mu_1)(\lambda_2 + 2\mu_2)] + \gamma[h_1^2(\lambda_1(\lambda_2 + 2\mu_1) + \mu_1(\lambda_1 + 2\mu_1))] - 6C_2(2\mu_1 + \lambda_1)(\lambda_1 - \lambda_2) + 6A_3\lambda_2(\lambda_2 + 2\mu_2) - 6B_2(\lambda_2 + 2\mu_2)^2] + [h_1^2(\lambda_1 + \mu_1) - 6C_2(\lambda_1 - \lambda_2)](\lambda_2 + 2\mu_2) \right), \quad (52)$$

$$\beta_2 = \frac{1}{6(\rho_1 + \gamma\rho_2)} \left(\frac{\rho_1 h_1^2 [\gamma((\lambda_1 + 2\mu_1)^2 - \mu_1(\lambda_2 + 2\mu_1)) + (\lambda_1 + \mu_1)(\lambda_2 + 2\mu_2)]}{(\lambda_1 + 2\mu_1)[\gamma(\lambda_1 + 2\mu_1) + (\lambda_2 + 2\mu_2)]} - \frac{\gamma^3 h_1^2 \mu_2 \rho_2 (\lambda_1 - \lambda_2)}{(\lambda_2 + 2\mu_2)[\gamma(\lambda_1 + 2\mu_1) + (\lambda_2 + 2\mu_2)]} \right. \\ \left. + \frac{\gamma^3 h_1^2 \rho_2 (\lambda_2 + \mu_2)}{\lambda_2 + 2\mu_2} + 6C_1(\lambda_1 - \lambda_2) - \frac{6\gamma^2(\lambda_2 - \lambda_1)[A_0\lambda_2 - B_1\rho_2]}{\gamma(\lambda_1 + 2\mu_1) + (\lambda_2 + 2\mu_2)} - 6\gamma(B_2 - B_0(\lambda_2 + 2\mu_2) - A_2\lambda_2) \right), \quad (53)$$

$$\beta_3 = \frac{h_1^2 \gamma \rho_2 (\gamma^2 \mu_1 \rho_2 - \mu_2 \rho_1)}{2\mu_1 \mu_2 (\rho_1 + \gamma\rho_2)}. \quad (54)$$

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