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On non-singular crack fields in Helmholtz type enriched elasticity theories

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ABSTRACT

Recently, simple non-singular stress fields of cracks of mode I and mode III have been published by Aifantis (2009, 2011), Isaksson and Häggglund (2013) and Isaksson et al. (2012). In this work we investigate the physical meaning and interpretation of those solutions and if they satisfy important physical conditions (equilibrium, boundary and compatibility conditions).

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1. Introduction

During the last years, some non-singular crack fields have been published in the literature (Aifantis, 2009, 2011; Isaksson and Häggglund, 2013; Isaksson et al., 2012) neglecting equilibrium, boundary and compatibility conditions. The aim of that research was the regularization of the classical singular crack fields. In fact, the non-singular crack fields are zero at the crack tip. However, not any equilibrium condition was used by Aifantis (2011), Isaksson and Häggglund (2013) and Isaksson et al. (2012). Therefore, it is doubtful if their results are correct from the point of view of fracture mechanics.

On the other hand, Ari and Eringen (1983), Eringen and Suresh (1983) and Eringen (1984) (see also Eringen, 2002) investigated cracks in the framework of nonlocal elasticity of Helmholtz type in the 80s. Eringen (1984, 2002) found a non-singular stress of a mode III crack zero at the crack tip. For the mode I crack problem, using appropriate boundary conditions, Ari and Eringen (1983) (see also Eringen, 2002) found a non-singular stress finite at the crack tip and becoming zero inside the crack. A regularization procedure was also discussed for crack curving in Eringen and Suresh (1983).

The aim of this paper is to show that the recent crack solutions given by Aifantis (2009, 2011), Isaksson et al. (2012) and Isaksson and Häggglund (2013) cannot be the correct solutions of a nonlocal or gradient compatible elastic fracture mechanics problem.

Therefore, the modest goal of the present paper is not to give new solutions but rather to discuss existing and recently given crack solutions using gradient enhanced elasticity theories.

The paper is organized as follows: Section 2 provides the basics of the theories of nonlocal elasticity and strain gradient elasticity. Next, Section 3 explains why the non-singular mode III crack solution given by Aifantis (2009, 2011) cannot be considered as a solution of a nonlocal and gradient elastic fracture mechanics problem. The same is explained in Section 4 for the mode I crack solution given by Aifantis (2011), Isaksson et al. (2012) and Isaksson and Häggglund (2013). Finally, in Section 5 a possible way-out for the physical interpretation of the non-singular solutions is discussed.

2. Theoretical framework

In this section we outline the basics of the theories of nonlocal elasticity and gradient elasticity.

2.1. Theory of nonlocal elasticity of Helmholtz type

In the theory of nonlocal elasticity (e.g., Eringen, 2002, 1983), the so-called nonlocal stress tensor t_{ij} is defined at any point \mathbf{x} of the analyzed domain of volume V as

$$t_{ij}(\mathbf{x}) = \int_V \alpha(|\mathbf{x} - \mathbf{y}|) \sigma_{ij}^0(\mathbf{y}) dV(\mathbf{y}), \quad (1)$$

where $\alpha(|\mathbf{x} - \mathbf{y}|)$ is a nonlocal kernel and σ_{ij}^0 is the stress tensor of classical elasticity defined at the point $\mathbf{y} \in V$ as

$$\sigma_{ij}^0(\mathbf{y}) = \lambda \delta_{ij} e_{kk}^0(\mathbf{y}) + 2\mu e_{ij}^0(\mathbf{y}), \quad (2)$$

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with λ, μ are the Lamé constants, δ_{ij} is the Kronecker delta and e_{ij}^0 denotes the classical strain tensor, which is the symmetric part of the classical distortion tensor

$$e_{ij}^0 = \frac{1}{2} (\rho_{ij}^0 + \rho_{ji}^0). \quad (3)$$

We employ a comma to indicate partial derivative with respect to rectangular coordinates x_j , i.e. $t_{ij,j} = \frac{\partial t_{ij}}{\partial x_j}$. As usual, repeated indices indicate summation.

In absence of body forces, the nonlocal stress tensor satisfies the equilibrium condition

$$t_{ij,j} = 0, \quad (4)$$

which means that the stress is self-equilibrated. In addition, the classical stress tensor fulfills the equilibrium equation of classical elasticity

$$\sigma_{ij,j}^0 = 0. \quad (5)$$

If the nonlocal kernel function $\alpha(|\mathbf{x} - \mathbf{y}|)$ is the Green function (fundamental solution) of the differential operator $L = 1 - \ell^2 \Delta$, i.e.

$$(1 - \ell^2 \Delta) \alpha(|\mathbf{x} - \mathbf{y}|) = \delta(\mathbf{x} - \mathbf{y}), \quad (6)$$

with ℓ, Δ, δ being a characteristic length scale ($\ell \geq 0$), the Laplacian and the Dirac delta function, respectively, then the integral relation (1) reduces to the inhomogeneous Helmholtz equation

$$(1 - \ell^2 \Delta) t_{ij} = \sigma_{ij}^0, \quad (7)$$

where the classical stress is the source for the nonlocal stress. The natural boundary condition reads

$$t_{ij} n_j = \hat{t}_i, \quad (8)$$

where n_j and \hat{t}_i represent the normal to the external boundary and the prescribed boundary tractions, respectively. In nonlocal elasticity, no nonlocal strain exists. Thus, using a nonlocal kernel, being a Green function, yields a differential equation for t_{ij} instead of an integrodifferential equation in the 'strongly' nonlocal theory with seemingly and physically equivalent solution at the output. In such a 'weakly' nonlocal elasticity the concept of a gradient theory might be used (Maugin, 1979, 2012). The 'weakly' nonlocal theory of elasticity represented by Eqs. (4)–(7) is called of Helmholtz type because the Helmholtz operator, $L = 1 - \ell^2 \Delta$, enters in the form of Eqs. (6) and (7).

It was pointed out by Eringen and Suresh (1983) that the stress field t_{ij} of a crack is obtained by solving Eq. (7), subject to regularity conditions, i.e., t_{ij} must be bounded at the crack tip and at infinity. This is borne out from the problems of non-singular dislocations (Eringen, 2002). At large distance from the crack tip, the classical solution will approximate the stress field well, namely if $\ell \rightarrow 0$, Eq. (7) gives $t_{ij} \rightarrow \sigma_{ij}^0$. This also suggests that one may obtain a full solution of Eq. (7) and match it to the outer solution σ_{ij}^0 in order to obtain a non-singular solution.

2.2. Theory of gradient elasticity of Helmholtz type

In the theory of gradient elasticity (see, e.g., Mindlin, 1964; Mindlin and Eshel, 1968; Eshel and Rosenfeld, 1970; Jaunzemis, 1967), the equilibrium condition is given by

$$\tau_{ij,j} - \tau_{ijk,jk} = 0, \quad (9)$$

where τ_{ij} is the Cauchy-like stress tensor and τ_{ijk} is the so-called double-stress tensor. It can be seen in Eq. (9) that the Cauchy-like stress tensor τ_{ij} is, in general, not self-equilibrated. The natural boundary conditions in strain gradient elasticity are much more complicated than the corresponding ones in nonlocal elas-

ticity; they read (see, e.g., Mindlin and Eshel, 1968; Jaunzemis, 1967)

$$\left. \begin{aligned} (\tau_{ij} - \partial_k \tau_{ijk}) n_j - \partial_j (\tau_{ijk} n_k) + n_j \partial_l (\tau_{ijk} n_k n_l) &= \bar{t}_i \\ \tau_{ijk} n_j n_k &= \bar{q}_i \end{aligned} \right\} \quad \text{on } \partial\Omega, \quad (10)$$

where \bar{t}_i and \bar{q}_i are the prescribed Cauchy traction vector and the prescribed double stress traction vector, respectively. Moreover, $\partial\Omega$ is the smooth boundary surface of the domain Ω occupied by the body.

In a simplified version of strain gradient elasticity, called gradient elasticity of Helmholtz type (e.g., Lazar and Maugin, 2005; Lazar, 2013; Polyzos et al., 2003), the double stress tensor is nothing but the gradient of the Cauchy-like stress tensor multiplied by ℓ^2

$$\tau_{ijk} = \ell^2 \tau_{ij,k} \quad (11)$$

and the Cauchy-like stress tensor reads

$$\tau_{ij} = C_{ijkl} \beta_{kl}, \quad (12)$$

where β_{ij} denotes the elastic distortion tensor and C_{ijkl} is the tensor of the elastic moduli given by

$$C_{ijkl} = \mu (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}) + \lambda \delta_{ij} \delta_{kl}. \quad (13)$$

Substituting Eq. (11) into (9), Eq. (9) simplifies to the following partial differential equation (pde) of 3rd order

$$(1 - \ell^2 \Delta) \tau_{ij,j} = 0 \quad (14)$$

and, using Eq. (12), Eq. (14) reads in terms of the elastic distortion tensor

$$(1 - \ell^2 \Delta) C_{ijkl} \beta_{kl,j} = 0. \quad (15)$$

Following Jaunzemis (1967), the polarization of the Cauchy-like stress, sometimes called 'total stress tensor', is defined by

$$\sigma_{ij} := (1 - \ell^2 \Delta) \tau_{ij}. \quad (16)$$

Then the equilibrium condition (14) reads in terms of the total stress tensor

$$\sigma_{ij,j} = 0. \quad (17)$$

On the other hand, using the so-called 'Ru-Aifantis theorem' (Ru and Aifantis, 1993) in terms of stresses, Eq. (14) can be written as an equivalent system of pdes of 1st order and of 2nd order, namely

$$\sigma_{ij,j}^0 = 0, \quad (18)$$

$$(1 - \ell^2 \Delta) \tau_{ij} = \sigma_{ij}^0, \quad (19)$$

where σ_{ij}^0 is the classical stress tensor. Eqs. (18) and (19) also play the role of the basic equations in Aifantis' version of gradient elasticity (see, e.g., Askes and Aifantis, 2011). Using the 'Ru-Aifantis theorem', the total stress tensor σ_{ij} is identified with the classical stress tensor σ_{ij}^0 :

$$\sigma_{ij} \equiv \sigma_{ij}^0. \quad (20)$$

The so-called 'Ru-Aifantis theorem' is a special case of a more general technique well-known in the theory of partial differential equations (see, e.g., Vekua, 1967). Moreover, the 'Ru-Aifantis theorem' is restricted only to situations involving a body of infinite extent (with no need to enforce boundary conditions). In the presence of boundary conditions, the 'Ru-Aifantis theorem' is no longer valid and can lead to erroneous solutions. Therefore, it is questionable if the 'Ru-Aifantis theorem' should be used in the construction of crack solutions in gradient elasticity. In physics, such a method of the reduction of the order of higher order field equations is known and used in the so-called Bopp-Podolsky theory (Bopp, 1940; Podolsky, 1942), which is the gradient theory

of electrodynamics. In the Bopp–Podolsky theory, the (linear) field equation is of fourth order and can be decomposed into two partial differential equations of second order (e.g., Davis, 1970).

Now, comparing the theory of ‘weakly’ nonlocal elasticity and the theory of gradient elasticity of Helmholtz type one can say the following:

- (i) The ‘weakly’ nonlocal theory of elasticity is described by Eqs. (4), (5) and (7), while the gradient elastic one by Eq. (14) and considering the ‘Ru–Aifantis theorem’ by Eqs. (18) and (19).
- (ii) Eqs. (7) and (19) indicate that the nonlocal stresses t_{ij} and the Cauchy-like stresses of gradient elasticity theory τ_{ij} are identical, i.e.

$$t_{ij} \equiv \tau_{ij}, \quad (21)$$

when the ‘Ru–Aifantis theorem’ is used.

- (iii) The nonlocal stress tensor t_{ij} is different to the total stress tensor σ_{ij} , as it is evident from (7) and (16).

- (iv) The natural boundary conditions in nonlocal elasticity theory are referred to prescribed values of tractions taken from the nonlocal stresses t_{ij} , as it is illustrated in Eq. (8). In other words the natural boundary conditions are as simple as in the classical case. On the contrary, in gradient elasticity theory natural boundary conditions are complicated expressions of double stresses τ_{ijk} , which in the Helmholtz version of the theory are expressed in terms of the Cauchy-like stresses τ_{ij} according to the relation (11). More details can be found in Polyzos et al. (2003). Aifantis and co-workers, adopting the relation (21) and ignoring the double stresses in (10), have considered that natural boundary conditions in gradient elasticity are similar to nonlocal ones, i.e.

$$\tau_{ij}n_j = \hat{\tau}_i. \quad (22)$$

However, such an assumption is arbitrary since it cannot be supported by a variational consideration (10).

In addition, in gradient elasticity of Helmholtz type the following inhomogeneous Helmholtz equation can be found for the elastic distortion β_{ij} (e.g., Lazar and Maugin, 2005, 2006; Lazar, 2013)

$$(1 - \ell^2 \Delta) \beta_{ij} = \beta_{ij}^0, \quad (23)$$

where β_{ij}^0 denotes the classical elastic distortion tensor.

3. Non-singular mode III crack solutions provided by Aifantis (2009, 2011)

The classical solution of the stress produced by a crack of mode III obtained in classical fracture mechanics is of the form

$$\sigma_{zx}^0 = -\frac{K_{III}}{\sqrt{2\pi r}} \sin \frac{\theta}{2}, \quad (24)$$

$$\sigma_{zy}^0 = \frac{K_{III}}{\sqrt{2\pi r}} \cos \frac{\theta}{2}, \quad (25)$$

where K_{III} is the mode III stress intensity factor, and $r = \sqrt{x^2 + y^2}$, $\theta = \arctan y/x$ are the polar coordinates centered at the crack tip. The classical elastic distortion tensor of a mode III crack reads

$$\beta_{zx}^0 = -\frac{K_{III}}{\mu\sqrt{2\pi r}} \sin \frac{\theta}{2}, \quad (26)$$

$$\beta_{zy}^0 = \frac{K_{III}}{\mu\sqrt{2\pi r}} \cos \frac{\theta}{2}. \quad (27)$$

The fields (24)–(27) possess a $1/\sqrt{r}$ -singularity at the crack tip, while the stresses (24) and (25) fulfill the equilibrium condition

$$\sigma_{zx,x}^0 + \sigma_{zy,y}^0 = 0 \quad (28)$$

and the elastic distortions (26) and (27) satisfy the compatibility condition

$$\beta_{zy,x}^0 - \beta_{zx,y}^0 = 0. \quad (29)$$

Moreover, on the free of stresses surface of the crack ($\theta = \pi$) the boundary condition $\sigma_{zy}^0(r, \pi) = 0$ is fulfilled.

Substituting Eqs. (24) and (25) into the inhomogeneous Helmholtz equation (7) or (19), the non-singular stress is obtained as

$$t_{zx} = \tau_{zx} = -\frac{K_{III}}{\sqrt{2\pi}} \sin \frac{\theta}{2} f_1(r), \quad (30)$$

$$t_{zy} = \tau_{zy} = \frac{K_{III}}{\sqrt{2\pi}} \cos \frac{\theta}{2} f_1(r), \quad (31)$$

where $f_1(r)$ is given by (see Eq. (A.4))

$$f_1(r) = \frac{1}{\sqrt{r}} (1 - e^{-r/\ell}). \quad (32)$$

By construction, Eqs. (30) and (31) satisfy Eq. (14). In nonlocal elasticity, on the free of stresses surface of the crack ($\theta = \pi$) the boundary condition $t_{zy}(r, \pi) = 0$ is satisfied.

In the same way, substituting Eqs. (26) and (27) into the inhomogeneous Helmholtz equation (23), the non-singular elastic distortion is found as

$$\beta_{zx} = -\frac{K_{III}}{\mu\sqrt{2\pi}} \sin \frac{\theta}{2} f_1(r), \quad (33)$$

$$\beta_{zy} = \frac{K_{III}}{\mu\sqrt{2\pi}} \cos \frac{\theta}{2} f_1(r). \quad (34)$$

Eqs. (30)–(34) have been first published¹ by Aifantis (2009, 2011). Aifantis (2009, 2011) has adopted the regularization conditions at $r = 0$ and $r = \infty$ proposed by Eringen and Suresh (1983). It is obvious that these fields are non-singular and zero at the crack tip. Extremum stress and distortion occur near the crack tip.

However, although the stresses and elastic distortions provided by (30)–(34) are non-singular at the crack tip, they cannot be considered as a solution of the nonlocal and gradient mode III fracture mechanics problem. Eqs. (30) and (31) are not a solution for a mode III crack in nonlocal elasticity since they do not satisfy the equilibrium condition (4):

- (i) Indeed, inserting Eqs. (30) and (31) into (4) one obtains instead of zero the nonzero line force at the crack tip

$$F_z = t_{zx,x} + t_{zy,y} = \frac{K_{III}}{\sqrt{2\pi r}} \sin \frac{\theta}{2} \frac{e^{-r/\ell}}{\ell}, \quad (35)$$

which, in the limit to classical elasticity this line force becomes zero, i.e.

$$\lim_{\ell \rightarrow 0} F_z = 0. \quad (36)$$

Also Eqs. (30)–(34) cannot be considered as solution of the gradient elastic mode III fracture mechanics problem for the following reasons:

- (i) The corresponding elastic distortions given by Eqs. (33) and (34) do not satisfy the compatibility condition of gradient elasticity: $\beta_{zy,x} - \beta_{zx,y} = 0$. Indeed, a nonzero dislocation density of screw dislocations α_{zz} appears at the crack tip

¹ It is noted that also Lazar (2003) found these non-singular crack fields of mode III. However, Lazar (2003) did not publish his results.

$$\alpha_{zz} = \beta_{zy,x} - \beta_{zx,y} = \frac{K_{III}}{\mu\sqrt{2\pi r}} \cos \frac{\theta}{2} \frac{e^{-r/\ell}}{\ell}, \quad (37)$$

which, in the limit to classical elasticity, this dislocation density becomes zero

$$\lim_{\ell \rightarrow 0} \alpha_{zz} = 0. \quad (38)$$

(ii) It is easy to see that due to Eq. (37), the elastic distortions β_{zx} and β_{zy} are not anymore a displacement gradient. In other words, there is not a displacement field that supports Eqs. (33) and (34).

(iii) Eqs. (30) and (31) describe the Cauchy-like stresses τ_{ij} and not the total stresses σ_{ij} . Not using the ‘Ru–Aifantis theorem’, Georgiadis (2003) obtained that near the tip of a gradient elastic mode III crack, the total stresses are more singular than in the classical case appearing a singularity of order $r^{-3/2}$ and, therefore, $\sigma_{ij} \neq \sigma_{ij}^0$.

Thus, the conclusion here is that the stresses provided by Eqs. (30) and (31) and the elastic distortion (33) and (34) are just formal solutions of the inhomogeneous Helmholtz equations (19) and (23) neglecting the equilibrium and compatibility conditions of the nonlocal and gradient boundary value problems, respectively. They produce line forces at the crack tip in nonlocal elasticity. In addition, since the compatibility condition is not fulfilled in gradient elasticity they produce a certain distribution of screw dislocations at the crack tip.

4. Non-singular mode I crack solutions provided by Aifantis (2011), Isaksson and Häggglund (2013) and Isaksson et al. (2012)

For classical mode I cracks, the stresses are singular at the crack tip and have the form

$$\sigma_{xx}^0 = \frac{K_I}{\sqrt{2\pi r}} \left[\frac{3}{4} \cos \frac{\theta}{2} + \frac{1}{4} \cos \frac{5\theta}{2} \right], \quad (39)$$

$$\sigma_{yy}^0 = \frac{K_I}{\sqrt{2\pi r}} \left[\frac{5}{4} \cos \frac{\theta}{2} - \frac{1}{4} \cos \frac{5\theta}{2} \right], \quad (40)$$

$$\sigma_{xy}^0 = \frac{K_I}{\sqrt{2\pi r}} \left[-\frac{1}{4} \sin \frac{\theta}{2} + \frac{1}{4} \sin \frac{5\theta}{2} \right], \quad (41)$$

where K_I denotes the mode I stress intensity factor. The fields (39)–(41) possess a $1/\sqrt{r}$ -singularity. Eqs. (39)–(41) satisfy the equilibrium conditions

$$\sigma_{xx,x}^0 + \sigma_{xy,y}^0 = 0, \quad (42)$$

$$\sigma_{yx,x}^0 + \sigma_{yy,y}^0 = 0, \quad (43)$$

the boundary conditions

$$\sigma_{xy}^0(r, \pi) = 0, \quad (44)$$

$$\sigma_{yy}^0(r, \pi) = 0 \quad (45)$$

and the stress compatibility condition

$$\Delta \sigma_{ij}^0 + \frac{1}{1+\nu} (\sigma_{kk,ij}^0 - \delta_{ij} \Delta \sigma_{kk}^0) = 0. \quad (46)$$

Substituting Eqs. (39)–(41) into Eqs. (7) or (19), non-singular fields are obtained and they have the form (see also Appendix A)

$$t_{xx} = \tau_{xx} = \frac{K_I}{\sqrt{2\pi}} \left[\frac{3}{4} \cos \frac{\theta}{2} f_1(r) + \frac{1}{4} \cos \frac{5\theta}{2} f_5(r) \right], \quad (47)$$

$$t_{yy} = \tau_{yy} = \frac{K_I}{\sqrt{2\pi}} \left[\frac{5}{4} \cos \frac{\theta}{2} f_1(r) - \frac{1}{4} \cos \frac{5\theta}{2} f_5(r) \right], \quad (48)$$

$$t_{xy} = \tau_{xy} = \frac{K_I}{\sqrt{2\pi}} \left[-\frac{1}{4} \sin \frac{\theta}{2} f_1(r) + \frac{1}{4} \sin \frac{5\theta}{2} f_5(r) \right], \quad (49)$$

where again $f_1(r)$ is given by Eqs. (32) and (A.4), and $f_5(r)$ reads (see Eq. (A.5)):

$$f_5(r) = \frac{1}{\sqrt{r}} \left(1 - \frac{6\ell^2}{r^2} + 2 \left(1 + \frac{3\ell}{r} + \frac{3\ell^2}{r^2} \right) e^{-r/\ell} \right). \quad (50)$$

The fields (47)–(49) are non-singular and zero at the crack tip.

The non-singular stresses² (47)–(49) were first published by Isaksson et al. (2012) and Isaksson and Häggglund (2013). The components (47) and (48) were also published by Aifantis (2011). In finding non-singular crack fields Aifantis (2011), Isaksson et al. (2012) and Isaksson and Häggglund (2013) used regularity conditions at $r = 0$ and $r = \infty$ (see Appendix A). Isaksson and Häggglund (2013) and Isaksson et al. (2012) have claimed that they found the stresses in nonlocal elasticity of Helmholtz type and Aifantis (2011) has claimed that he found the stresses in gradient elasticity of Helmholtz type. However, the non-singular stresses (47)–(49) do not satisfy the equilibrium condition (4). In nonlocal elasticity, they produce line forces at the crack tip

$$F_x = t_{xx,x} + t_{xy,y} = \frac{K_I}{\sqrt{2\pi r}} \frac{1}{4} \cos \frac{\theta}{2} \frac{e^{-r/\ell}}{\ell}, \quad (51)$$

$$F_y = t_{yx,x} + t_{yy,y} = \frac{K_I}{\sqrt{2\pi r}} \frac{3}{4} \sin \frac{\theta}{2} \frac{e^{-r/\ell}}{\ell}. \quad (52)$$

In the limit to classical elasticity these line forces become zero:

$$\lim_{\ell \rightarrow 0} F_x = 0, \quad (53)$$

$$\lim_{\ell \rightarrow 0} F_y = 0. \quad (54)$$

In nonlocal elasticity, the boundary condition $t_{xy}(r, \pi) = 0$ is not satisfied. Indeed, from Eq. (49) one obtains

$$t_{xy}(r, \pi) = \frac{K_I}{\sqrt{2\pi r}} \left[-\frac{1}{4} f_1(r) + \frac{1}{4} f_5(r) \right] \neq 0. \quad (55)$$

Now we have to investigate if the stresses (47)–(49) possess geometric incompatibilities caused by a distribution of dislocations. In order to analyze the incompatibility condition of gradient elasticity in terms of stresses, we start from the so-called incompatibility tensor η_{ij} which is defined in terms of the elastic strain tensor (Kröner, 1958; Kröner and Balian, 1981; Teodosiu, 1982)

$$\eta_{ij} = -\epsilon_{ikl} \epsilon_{jmn} e_{ln,km}, \quad (56)$$

where ϵ_{ikl} is the Levi–Civita tensor. Taking into account $\tau_{ijj} = F_i$ and the inverse of the Hooke law, we may rewrite (56) as (see, e.g., Kröner, 1958; Teodosiu, 1982)

$$\Delta \tau_{ij} + \frac{1}{1+\nu} (\tau_{kk,ij} - \delta_{ij} \Delta \tau_{kk}) - F_{ij} - F_{j,i} + \delta_{ij} F_{k,k} = 2\mu \eta_{ij}. \quad (57)$$

If the incompatibility tensor is zero, Eq. (57) reduces to the Beltrami–Michell stress compatibility condition. On the other hand, the incompatibility tensor η_{ij} can be given (Kröner, 1958, 1981; Teodosiu, 1982)

$$\eta_{ij} = -\frac{1}{2} (\epsilon_{ikl} \alpha_{lj,k} + \epsilon_{jkl} \alpha_{li,k}) \quad (58)$$

in terms of the dislocation density tensor

$$\alpha_{ij} = \epsilon_{jkl} \beta_{il,k}. \quad (59)$$

It states that if η_{ij} is nonzero, then the elastic strain is incompatible due to dislocations. On the other hand, if the dislocation density is nonzero, the plastic distortion tensor β_{ij}^p is nonvanishing:

$$\alpha_{ij} = -\epsilon_{jkl} \beta_{il,k}^p. \quad (60)$$

² Also Lazar (2010) found these non-singular solutions. Due to the properties as discussed in this section, he did not publish the result.

The physical reason is that a dislocation is the elementary carrier of plasticity (Kröner, 1958).

For plane strain, only the component η_{zz} is nonzero, namely

$$\eta_{zz} = \alpha_{xz,y} - \alpha_{yz,x}. \quad (61)$$

Using $\tau_{zz} = v(\tau_{xx} + \tau_{yy})$, Eq. (57) simplifies to

$$-(1-v)\Delta(\tau_{xx} + \tau_{yy}) + F_{k,k} = 2\mu\eta_{zz}. \quad (62)$$

From Eqs. (47) and (48) we obtain

$$\tau_{xx} + \tau_{yy} = \frac{2K_I}{\sqrt{2\pi}} \cos \frac{\theta}{2} f_1(r). \quad (63)$$

Using Eqs. (63) and (A.1), we find

$$\Delta(\tau_{xx} + \tau_{yy}) = -\frac{2K_I}{\sqrt{2\pi r}} \cos \frac{\theta}{2} \frac{e^{-r/\ell}}{\ell^2}. \quad (64)$$

In addition, $\text{div} \mathbf{F}$ is calculated for Eqs. (51) and (52) as

$$F_{k,k} = -\frac{1}{2} \frac{K_I}{\sqrt{2\pi r}} \cos \frac{\theta}{2} \frac{e^{-r/\ell}}{\ell^2} + \frac{1}{4} \frac{K_I}{\sqrt{2\pi r}} \cos \frac{3\theta}{2} \left[1 + \frac{\ell}{r} \right] \frac{e^{-r/\ell}}{\ell^2}. \quad (65)$$

Substituting Eqs. (64) and (65) into (62), the geometric incompatibility is obtained as

$$\eta_{zz} = \frac{1}{2\mu} \frac{K_I}{\sqrt{2\pi r}} \left(\frac{3-4\nu}{2} \cos \frac{\theta}{2} + \frac{1}{4} \cos \frac{3\theta}{2} \left[1 + \frac{\ell}{r} \right] \right) \frac{e^{-r/\ell}}{\ell^2}, \quad (66)$$

which is caused by a nonzero distribution of edge dislocations, α_{xz} and α_{yz} , at the crack tip, namely

$$\alpha_{xz} = \beta_{xy,x} - \beta_{yx,x}, \quad \alpha_{yz} = \beta_{yy,x} - \beta_{yx,y}. \quad (67)$$

Thus, the mode I stresses (47)–(49) produce a nonzero compatibility condition in gradient elasticity in contrast to the classical mode I stresses (39)–(41), which give zero for the incompatibility and the dislocation density. In the limit to classical elasticity the geometric incompatibility (66) becomes zero:

$$\lim_{\ell \rightarrow 0} \eta_{zz} = 0. \quad (68)$$

Consequently, the non-singular stresses (47)–(49) cannot be considered as a solution of the nonlocal elastic mode I fracture mechanics problem for the following reasons:

- (i) They do not fulfill the equilibrium condition of nonlocal elasticity (4).
- (ii) They are not compatible with a free of stresses crack, as it is evident by relation (55).

Also, Eqs. (47)–(49) do not represent the correct stresses of a gradient elastic mode I fracture mechanics problem, because:

- (i) They do not satisfy the compatibility conditions of gradient theory of compatible elasticity, as it is evident from Eqs. (57) and (66).
- (ii) As in the case of mode III crack, the elastic distortions β_{xx} , β_{xy} , β_{yx} and β_{yy} cannot be taken as a gradient of a displacement vector.
- (iii) As it has been mentioned in the case of mode III crack, Eqs. (47)–(49) represent the Cauchy-like stresses τ_{ij} and not the total stresses σ_{ij} . As it is shown by Karlis et al. (2007) and Gourgiotis and Georgiadis (2009), ignoring the ‘Ru–Aifantis theorem’, total stresses are more singular than in the classical case appearing a singularity of order $r^{-3/2}$ near the tip of a gradient elastic mode I crack and, therefore, $\sigma_{ij} \neq \sigma_{ij}^0$.

Thus, the conclusion is that the non-singular stress components (47)–(49) are formal solutions of an inhomogeneous Helmholtz

equation. They produce line forces in nonlocal elasticity and a distribution of edge dislocations at the crack tip in gradient elasticity.

5. Discussion

The main conclusion of the present work is that the non-singular stress fields provided by Aifantis (2009, 2011), Isaksson and Häggglund (2013) and Isaksson et al. (2012) cannot be considered as the solution of a nonlocal or strain gradient elastic fracture mechanics problem. As it has been already mentioned, in the framework of nonlocal elasticity Eringen (2002) were the first who proposed non-singular stress fields near the mode I and mode III crack tips by solving rigorously the corresponding nonlocal fracture mechanics problem.

On the other hand, in classical fracture mechanics and in compatible gradient elasticity (e.g., Georgiadis, 2003; Gourgiotis and Georgiadis, 2009), the mathematical solutions of cracks that take into account the equations of elasticity, the equilibrium equations, boundary conditions, and compatibility conditions lead to nice, exact description of the high magnitude crack tip stress field (classical singularity $r^{-1/2}$ and non-classical singularity $r^{-3/2}$). Karlis et al. (2007) solving numerically the strain gradient elastic mode I crack problem provided plots for the variation of $r^{-1/2}$ - and $r^{-3/2}$ -terms as a function of the internal length scale parameter ℓ .

However, such solutions do not answer the question: Why is the stress at the crack tip so big? Weertman (1996) gave a physical answer that the stress is caused by a distribution of many dislocations near the crack tip. Also Weertman (1996) pointed out that fracture mechanics cannot be understood at a deeper level unless its study includes dislocations. In this paper, we found that the non-singular crack solutions do not fulfill the compatibility conditions. That means that dislocations are behind these non-singular crack solutions and it does not make sense anymore to require to fulfill a compatibility condition. A dislocation is the building block of a crack as mentioned by Weertman (1996). Unlike a crack, a dislocation is an elementary defect in solids able to build composite defects. For that reason the incompatibility of a crack is expressed in terms of dislocation densities.

Indeed for a mode III crack, due to Eq. (37), the elastic distortions β_{zx} and β_{zy} are not anymore a displacement gradient. In fact, the elastic distortions are incompatible and the incompatible parts are the plastic distortions β_{zx}^p and β_{zy}^p :

$$\beta_{zx} = u_{z,x} - \beta_{zx}^p, \quad \beta_{zy} = u_{z,y} - \beta_{zy}^p \quad (69)$$

and there is not a displacement field u_z which provides the elastic distortions (33) and (34) as compatible displacement gradient. In the framework of compatible elasticity, Eqs. (33) and (34) cannot be considered as a solution of the gradient elastic mode III fracture mechanics problem since they do not satisfy the compatibility condition. The only way-out is the interpretation of the incompatible elastic distortions (33) and (34) in the framework of incompatible elasticity or a dislocation-based mode III fracture problem (see also Hurtado and Weertman, 1993; Weertman, 1996). In dislocation-based fracture mechanics, a dislocation is the basic building block of the crack and fracture mechanics can be developed from its dislocation foundation. Then α_{zz} plays the role of distribution of non-redundant dislocations which are known also as ‘geometrically necessary’ dislocations. The density of non-redundant dislocations may define the ‘plastic’ zone near the crack tip (see Fig. 1). A distribution of non-redundant screw dislocations appears within the plastic zone of the crack tip. α_{zz} is singular at the crack tip and zero at the crack faces.

The same can be shown for a mode I crack. Due to nonzero dislocation density of edge dislocations α_{xz} and α_{yz} and nonzero incompatibility η_{zz} , the corresponding elastic distortions are not a

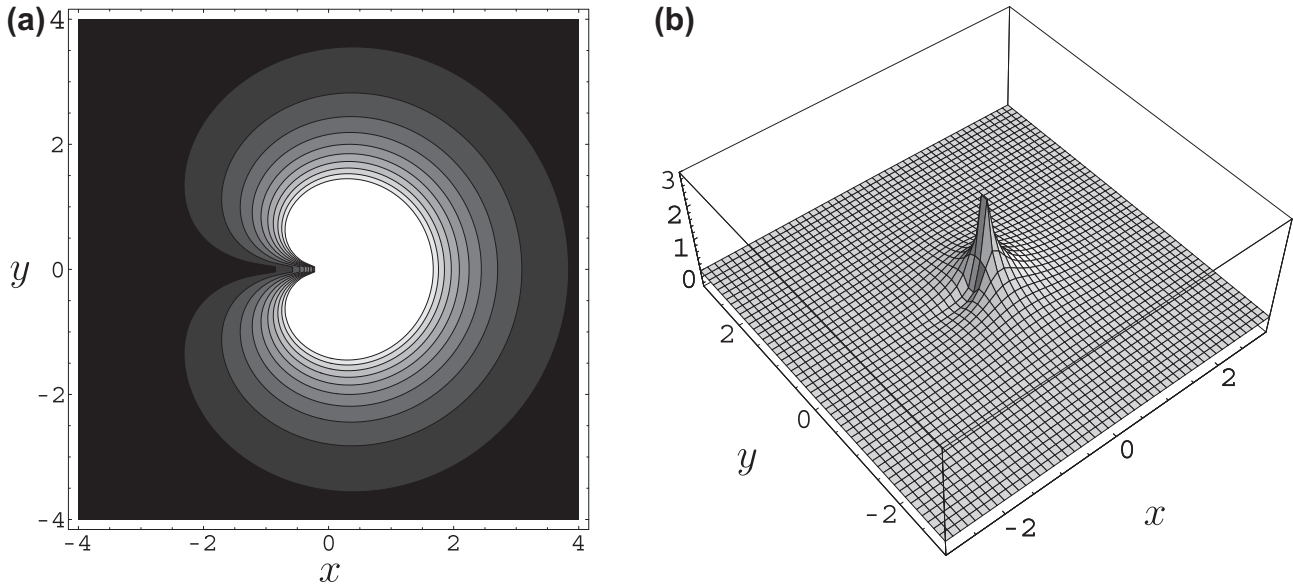


Fig. 1. Dislocation density α_{zz} of a mode III crack: (a) contour plot and (b) 3D plot.

simple gradient of the displacements u_x and u_y . They are incompatible due to nonzero plastic distortions:

$$\beta_{xx} = u_{x,x} - \beta_{xx}^p, \quad \beta_{xy} = u_{x,y} - \beta_{xy}^p, \quad (70)$$

$$\beta_{yx} = u_{y,x} - \beta_{yx}^p, \quad \beta_{yy} = u_{y,y} - \beta_{yy}^p. \quad (71)$$

As for the mode III crack problem, a possible interpretation of the incompatible elastic distortions may be a dislocation-based mode I fracture problem (see also Hurtado and Weertman, 1993; Weertman, 1996).

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Appendix A

Solving the inhomogeneous Helmholtz equations for cracks requires finding the solution of the partial differential equation of the form

$$[1 - \ell^2 \Delta] g_n(r, \theta) = \frac{1}{\sqrt{r}} e^{in\theta/2}, \quad n = \pm 1, \pm 5, \quad (A.1)$$

where the Laplacian reads in polar coordinates

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} \quad (A.2)$$

and

$$g_n(r, \theta) = f_n(r) e^{in\theta/2}, \quad (A.3)$$

where $f_n(r)$ is the radial part of the solutions. Using the regularity conditions at $r = 0$ and $r = \infty$, namely $f_n(r)$ should be bounded at $r = 0$ and $r = \infty$, we find

$$f_{\pm 1}(r) = \frac{1}{\sqrt{r}} (1 - e^{-r/\ell}) \quad (A.4)$$

and

$$f_{\pm 5}(r) = \frac{1}{\sqrt{r}} \left(1 - \frac{6\ell^2}{r^2} + 2 \left(1 + \frac{3\ell}{r} + \frac{3\ell^2}{r^2} \right) e^{-r/\ell} \right). \quad (A.5)$$

It is easy to see that $f_n(r)$ is zero at $r = 0$. The near fields are

$$f_{\pm 1}(r) \approx \frac{r^{1/2}}{\ell} - \frac{r^{3/2}}{2\ell^2} + \frac{r^{5/2}}{6\ell^3} - \frac{r^{7/2}}{24\ell^4} + \dots, \quad (A.6)$$

$$f_{\pm 5}(r) \approx \frac{r^{3/2}}{4\ell^2} - \frac{2r^{5/2}}{15\ell^3} + \frac{r^{7/2}}{24\ell^4} - \dots. \quad (A.7)$$

It is worth noting that also Eringen and Suresh (1983) derived the solution (A.4). The expression for $f_{\pm 5}(r)$ was given by Eringen and Suresh (1983) in a more sophisticated integral representation.

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