Available online at www.sciencedirect.com

ScienceDirect

International Journal of Solids and Structures 45 (2008) 1788–1803

INTERNATIONAL JOURNAL OF SOLIDS AND STRUCTURES

www.elsevier.com/locate/ijsolstr

# Orthogonality of 3D guided waves in viscoelastic laminates and far field evaluation to a local acoustic source

D.D. Zakharov

*FESBE, London South Bank University, 103 Borough Road, London SE1 0AA, UK*

Received 21 February 2007; received in revised form 22 September 2007
Available online 4 November 2007

**Abstract**

The fundamental properties of guided waves in a laminate with any homogeneous boundary conditions on its faces are considered. As shown, the waves satisfy orthogonality relations whose physical meaning is related to the additivity of the average power flow. The applications of this orthogonality for solving some particular boundary value problems are discussed. A method for exact calculation of the far field caused by an acoustic source of a finite size is suggested. The only restriction is that the distance required must exceed the longitudinal radius of the source. The obtained results can be used for evaluating the fields radiated by ultrasonic transducers of arbitrary aperture and by other realistic sources.


*Keywords:* Guided waves; Viscoelastic laminate; Orthogonality relations; Far field

## 1. Introduction

With the increasing use of composite materials in modern devices the guided waves in plates, both homogeneous and layered attract more and more attention of the research community. In the literature one can find monographs (Viktorov, 1967; Brekhovskikh, 1980; Auld, 1990; Nayfeh, 1995), reviews (e.g., paper by Chimenti, 1997 with four hundred references) and numerous original articles. As known, the guided waves in plates are generally not orthogonal like trigonometrical Fourier series, but they possess the orthogonality relations (OR) with respect to the power flow. These OR were deduced in the 70s by Auld and Kino (1971), Bobrovnitskii (1973), Fedoryuk (1974), Fraser (1976), Prakash (1978), Zilbergleit and Nuller (1977) and Slepyan (1979) for an elastic strip with various homogeneous boundary conditions on its faces. The relations for 3D guided waves in an elastic layer were derived by Zakharov (1988). Other considerations of non Sturm–Liouville systems which possess OR can be found in Lawrie and Abrahams (1999). Such OR can be used to construct the linear algebraic system of equations with respect to the unknown coefficients when using mode decomposition similarly to the various plane problems, e.g., the contact interaction between strips and a half-space (Pelts and Shikman, 1987), wave diffraction by a crack (Kasatkin, 1981; Shkerdin and Glorieux, 2004,

*E-mail address:* dd_zakh@mail.ru

0020-7683/$ - see front matter © 2007 Elsevier Ltd. All rights reserved.
doi:10.1016/j.ijsolstr.2007.10.025

2006; Flores-Lopez and Gregory, 2006) or wave reflection from the strip edge (Gregory and Gladwell, 1983; Scandrett and Vasudevan, 1991). A particular case of OR for one elastic layer was introduced and applied to expanding the Green tensor into series of Lamb's waves by Achenbach (1998, 2000) and Achenbach and Xu (1998, 1999).

In this paper, the 3D guided waves are considered in a laminate with homogeneous boundary conditions on its faces (HBCF) including stress free faces, fixed faces or any other combinations of zero displacements or zero stresses providing the total energy reflection by the faces. The viscoelasticity is taken into account in the form of Kelvin–Voigt model (see, e.g., Christensen, 1971). Then the general 3D waves are introduced and their spectra and orthogonality are deduced and discussed. The main motivation for this study is to generalise the results obtained earlier for one layer and pure elasticity, to elucidate the physics and to work out a method for exact calculation of the field, radiated by a realistic acoustic source into viscoelastic laminate. Since the numerical methods for 3D problems are time consuming the analytical and semi-analytical methods are still of interest for NDT needs when modelling the far-field and near-field.

The paper is organised as follows: in Section 2 the problem is formulated and in Section 3 the general representation of waves is introduced. Section 4 is devoted to HBCF and respective frequency equations. The orthogonality relations are derived in Section 5 and their physical meaning is discussed in Section 6. The formulation of the radiation condition in case of pure elasticity is presented in Section 7. Some applications are considered in the three last sections, namely, how to obtain the exact solutions to some particular boundary value problems (Section 8) and how to calculate the field, radiated by an acoustic source of a finite size (Section 9 and 10). A method based on the OR and the standing waves permits one to evaluate the total field at the distance which is greater than the source radius regardless to the shape of source and distribution. The paper is concluded by a few final remarks in Section 11.

## 2. Formulation

Consider a laminate composed of $N$ plies where each $j$th layer occupies a region $-\infty < x_1,x_2 < \infty$, $z_j \leqslant x_3 \leqslant z_{j+1}$ (see Fig. 1a) and subjected to the time-harmonic load. To be brief the factor $\mathrm{e}^{-\mathrm{i}\omega t}$ is omitted in what follows and the load is specified in Section 9. The layer displacements $u^j_\alpha$ satisfy the equations of motion

$$\partial_\beta \sigma^j_{\alpha\beta} + \rho_j \omega^2 u^j_\alpha + f^j_\alpha = 0, \quad (\alpha, \beta = 1, 2, 3), \tag{1}$$

where $\rho_j$ are mass densities and $f^j_\alpha$ are body forces to be specified further. The stresses $\sigma^j_{\alpha\beta}$ and strains $\varepsilon^j_{\alpha\beta}$ satisfy Hook's law and Kelvin–Voigt model of linear viscoelasticity

$$\sigma^j_{\alpha\beta} = c'^j_{\alpha\beta\gamma\delta}\varepsilon^j_{\gamma\delta} + c''^j_{\beta\beta\gamma\delta}\dot{\varepsilon}^j_{\gamma\delta}, \quad c''^j_{\beta\beta\gamma\delta} \ll 1, \tag{2}$$

$$\varepsilon^j_{\alpha\beta} = \frac{1}{2}\{\partial_\beta u^j_\alpha + \partial_\alpha u^j_\beta\}, \quad \dot{\varepsilon}^j_{\alpha\beta} = -\mathrm{i}\omega\varepsilon^j_{\alpha\beta}. \tag{3}$$

An isotropic material yields the complex-valued representation of Lame constants, wave speeds and wavenumbers

$$\lambda_j = \lambda'_j - \mathrm{i}\omega\lambda''_j, \quad \mu_j = \mu'_j - \mathrm{i}\omega\mu''_j, \tag{4}$$

$$\{c^j_P\}^2 = (\lambda_j + 2\mu_j)/\rho_j, \quad \{c^j_S\}^2 = \mu_j/\rho_j, \quad k^j_P = \omega/c^j_P, \quad k^j_S = \omega/c^j_S. \tag{5}$$

On the interfaces $x_3 = z_j$, $j = 2,3,\ldots,N$ the conditions of the full contact are assumed

$$\sigma^{j-1}_{\alpha 3} = \sigma^j_{\alpha 3}, \quad u^{j-1}_\alpha = u^j_\alpha. \tag{6}$$

In addition the field may satisfy the conditions on the faces $z^- = z_1$ and $z^+ = z_{N+1}$ in the form of given stresses $\sigma^\mp_{\alpha 3}$ or displacements $u^\mp_\alpha$ or by their combinations.

Our first task is to investigate the homogeneous solutions of the Eq. (1), i.e., the waves propagating in the longitudinal direction at the absence of body forces with various homogeneous boundary conditions on the faces. Second, their OR have to be derived.

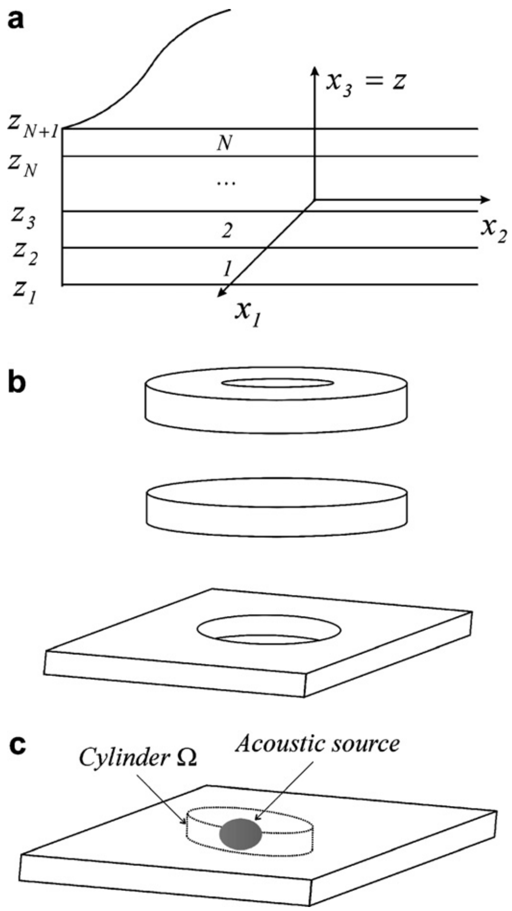


Fig. 1. (a) Sketch of laminate. (b) Regions with cylindrical geometry. (c) Acoustic source embedded into finite cylinder Ω.

## 3. General representation of the guided waves with cylindrical geometry

Introduce the displacement field supporting periodicity with respect to the polar angle in the plane $x_1$,$x_2$ and proceed to the cylindrical coordinates $r,\theta,z$: $x_1 = r\cos\theta$, $x_2 = r\sin\theta$, $x_3 = z$. Using Lame potentials and separation of variables at the absence of body forces, the waves propagating in $r$-direction in $j$th layer result as follows

$$u_r^j = \left[-u^j B_n' + w^j \frac{n}{sr} B_n\right] \begin{Bmatrix} \cos n\theta \\ -\sin n\theta \end{Bmatrix}, \tag{7}$$

$$u_\theta^j = \left[u^j \frac{n}{sr} B_n - w^j B_n'\right] \begin{Bmatrix} \sin n\theta \\ \cos n\theta \end{Bmatrix}, \tag{8}$$

$$u_z^j = v^j B_n \begin{Bmatrix} \cos n\theta \\ -\sin n\theta \end{Bmatrix}. \tag{9}$$

In the formulae (7)–(9) the first or second term could be chosen in the French brackets, so they represent the terms in the trigonometrical Fourier series wrt $\theta$. The terms $B_n \equiv B_n(sr)$ are any of the appropriate Bessel function or Hankel function of the first or second kind and $B_n' \equiv \mathrm{d}B_n/\mathrm{d}\{sr\}$. Functions $u^j(z)$, $v^j(z)$ and $w^j(z)$ satisfy the system of equations

$$\left[\frac{d^2}{dz^2}+\alpha_j\{q_P^j\}^2\right]u^j-\gamma_j s\frac{dv^j}{dz}=0, \tag{10}$$

$$\left[\alpha_j\frac{d^2}{dz^2}+\{q_S^j\}^2\right]v^j+\gamma_j s\frac{du^j}{dz}=0, \tag{11}$$

$$\frac{d^2w^j}{dz^2}+\{q_S^j\}^2w^j=0, \tag{12}$$

$$\alpha_j\equiv 2+\beta_j,\quad \beta^j\equiv\lambda_j/\mu_j,\quad \gamma^j\equiv\beta^j+1,\quad \{q_S^j\}^2\equiv\{k_S^j\}^2-s^2,\quad \{q_P^j\}^2\equiv\{k_P^j\}^2-s^2. \tag{13}$$

In the particular case of pure elasticity coefficients depend on Poisson's ratios $v_j$

$$\alpha_j=(2-2v_j)/(1-2v_j),\quad \beta_j=2v_j/(1-2v_j),\quad \gamma_j=1/(1-2v_j). \tag{14}$$

So, Eqs. (7)–(13) permit one to describe the guided waves of the wavenumber $s$ within constant factors. Indeed, the three second order linear differential equations (10)–(12) yield a simple so

$$\begin{aligned}\begin{bmatrix}u^j\\ v^j\end{bmatrix}&=A_P^{+j}\begin{bmatrix}\cos q_P^jz\\ \frac{q_P^j}{s}\sin q_P^jz\end{bmatrix}+A_P^{-j}\begin{bmatrix}\sin q_P^jz\\ -\frac{q_P^j}{s}\cos q_P^jz\end{bmatrix}\\ &\quad+A_S^{+j}\begin{bmatrix}-\frac{q_S^j}{s}\cos q_S^jz\\ \sin q_S^jz\end{bmatrix}+A_S^{-j}\begin{bmatrix}\frac{q_S^j}{s}\sin q_S^jz\\ \cos q_S^jz\end{bmatrix},\quad A_{P,S}^{\pm j}=\text{const},\end{aligned} \tag{15}$$

$$w^j=B_S^{+j}\cos q_s^jz+B_S^{-j}\sin q_s^jz,\quad B_S^{\pm j}=\text{const}. \tag{16}$$

The stresses look as follows (not to sum over $j$)

$$\sigma_{rr}^j=\mu_j\left\{\chi^jB_n-\frac{u^j}{r}[(n+1)B_{n+1}+(n-1)B_{n-1}]-\frac{sw^j}{2}[B_{n+2}-B_{n-2}]\right\}\begin{Bmatrix}\cos n\theta\\ -\sin n\theta\end{Bmatrix}, \tag{17}$$

$$\sigma_{r\theta}^j=\mu_j\left\{\frac{su^j}{2}[B_{n-2}-B_{n+2}]-\frac{sw^j}{2}[B_{n+2}+B_{n-2}]\right\}\begin{Bmatrix}\sin n\theta\\ \cos n\theta\end{Bmatrix}, \tag{18}$$

$$\sigma_{rz}^j=\mu_j\left\{-\tau^jB_n'+\frac{dw^j}{dz}\frac{n}{sr}B_n\right\}\begin{Bmatrix}\cos n\theta\\ -\sin n\theta\end{Bmatrix}, \tag{19}$$

$$\sigma_{\theta\theta}^j=\mu_j\left\{p^jB_n+\frac{su^j}{2}[B_{n-2}+B_{n+2}]+\frac{sw^j}{2}[B_{n+2}-B_{n-2}]\right\}\begin{Bmatrix}\cos n\theta\\ -\sin n\theta\end{Bmatrix}, \tag{20}$$

$$\sigma_{\theta z}^j=\mu_j\left\{\tau^j\frac{n}{sr}B_n-\frac{dw^j}{dz}B_n'\right\}\begin{Bmatrix}\sin n\theta\\ \cos n\theta\end{Bmatrix}, \tag{21}$$

$$\sigma_{zz}^j=\mu_j\sigma^jB_n\begin{Bmatrix}\cos n\theta\\ -\sin n\theta\end{Bmatrix}, \tag{22}$$

$$\chi^j\equiv\beta_j\frac{dv^j}{dz}+\alpha_jsu^j,\quad \tau^j\equiv\frac{du^j}{dz}-sv^j,\quad p^j\equiv\beta_j\frac{dv^j}{dz}+\gamma_jsu^j,\quad \sigma^j\equiv\alpha_j\frac{dv^j}{dz}+\beta_jsu^j, \tag{23}$$

$$\begin{aligned}\begin{bmatrix}\sigma^j\\ \tau^j\end{bmatrix}&=A_P^{+j}\begin{bmatrix}\frac{\{q_S^j\}^2-s^2}{s}\cos q_P^jz\\ -2q_P^j\sin q_P^jz\end{bmatrix}+A_P^{-j}\begin{bmatrix}\frac{\{q_S^j\}^2-s^2}{s}\sin q_P^jz\\ 2q_P^j\cos q_P^jz\end{bmatrix}\\ &\quad+A_S^{+j}\begin{bmatrix}2q_S^j\cos q_S^jz\\ \frac{\{q_S^j\}^2-s^2}{s}\sin q_S^jz\end{bmatrix}+A_S^{-j}\begin{bmatrix}-2q_S^j\sin q_S^jz\\ \frac{\{q_S^j\}^2-s^2}{s}\cos q_S^jz\end{bmatrix},\end{aligned} \tag{24}$$

$$\frac{dw^j}{dz}=q_s^j\{-B_S^{+j}\sin q_s^jz+B_S^{-j}\cos q_s^jz\}. \tag{25}$$

The equations (6) of the interface contact acquire the form

$$u^{j-1}(z_j) = u^j(z_j), \quad v^{j-1}(z_j) = v^j(z_j), \quad w^{j-1}(z_j) = w^j(z_j), \tag{26}$$

$$\mu_{j-1}\sigma^{j-1}(z_j) = \mu_j\sigma^j(z_j), \quad \mu_{j-1}\tau^{j-1}(z_j) = \mu_j\tau^j(z_j), \quad \mu_{j-1}\frac{\mathrm{d}w^{j-1}(z_j)}{\mathrm{d}z} = \mu_j\frac{\mathrm{d}w^j(z_j)}{\mathrm{d}z}, \tag{27}$$

with $j = 2,3,\ldots,N$ and give the system of $6N - 6$ linear algebraic equations with respect to $6N$ unknown coefficients in (15), (16).

## 4. Facial conditions and frequency equations

To close the system (26), (27) let us specify the facial conditions. According to the previous authors we call "pure-face" conditions the following cases: the *stress free* faces

$$\sigma^1(z^-) = \tau^1(z^-) = \frac{\mathrm{d}w^1(z^-)}{\mathrm{d}z} = 0, \tag{28}$$

$$\sigma^N(z^+) = \tau^N(z^+) = \frac{\mathrm{d}w^N(z^+)}{\mathrm{d}z} = 0, \tag{29}$$

the *fixed* faces

$$u^1(z^-) = v^1(z^-) = w^1(z^-) = 0, \tag{30}$$

$$u^N(z^+) = v^N(z^+) = w^N(z^+) = 0, \tag{31}$$

and the case when *one* face is *stress free* and *another* is *fixed* (Eqs. (28), (31) or (29), (30)). The "mixed-face" boundary conditions occur when the stresses are zero in some directions and the displacements equal zero in the complementary directions, e.g., no normal displacement nor tangent stresses

$$v^1(z^-) = 0, \quad \tau^1(z^-) = \frac{\mathrm{d}w^1(z^-)}{\mathrm{d}z} = 0, \tag{32}$$

$$v^N(z^+) = 0, \quad \tau^N(z^+) = \frac{\mathrm{d}w^N(z^+)}{\mathrm{d}z} = 0, \tag{33}$$

which correspond to a contact with a frictionless stamp. Another situation is

$$\sigma^1(z^-) = 0, \quad u^1(z^-) = v^1(z^-) = 0 \text{ and/or } \sigma^N(z^+) = 0, \quad u^N(z^+) = v^N(z^+) = 0. \tag{34}$$

Let us call homogeneous boundary conditions on the faces (HBCF) all possible combinations of the "pure-face" and "mixed-face" conditions. These HBCF give six linear algebraic equations, additional to $6N - 6$ Eqs. (26) and (27). So, the final system of equations has the size $6N \times 6N$. Denote the global matrix of this system by $\mathbf{L}_*$ with the determinant $d_* = \det\mathbf{L}_*$. As seen from relations (15), (16) and (24)–(34) the important point is that $4N$ equations with respect to $A^{\pm j}_{P,S}$ are separable from $2N$ equations with respect to $B^{\pm j}_S$. Thus, the matrix $\mathbf{L}_*$ consists of two blocks: $\mathbf{M}_*(4N \times 4N)$, $\Delta_* = \det\mathbf{M}_*$ and $\mathbf{N}_*(2N \times 2N)$, $\delta_* = \det\mathbf{N}_*$ which coincide with those of the in-plane and out-of-plane problems, respectively. The explicit form, detailed analysis and calculation algorithms for $\mathbf{M}_*$ and $\mathbf{N}_*$ can by found in Knopoff (1964), Schwab and Knopoff (1971), Fahmi and Adler (1973) and Lowe (1995). We just focus our attention on the corollary

$$d_* = \Delta_*\delta_*, \quad S_d = \{s_l : d_*(s_l) = 0\} = S_\Delta \cup S_\delta, \tag{35}$$

$$s \in S_\Delta \equiv \{s : \Delta_*(s) = 0\} : \quad w^j = 0, \quad u^j, v^j \neq 0, \tag{37}$$

$$s \in S_\delta \equiv \{s : \delta_*(s) = 0\} : \quad u^j = v^j = 0, \quad w^j \neq 0. \tag{38}$$

corresponding to the "in-plane" $(r,z)$-polarisation and "out-of-plane" $\theta$-polarisation of the eigenwaves. Their frequency equations are independent of number $n$. In general the roots $s$ are complex valued, and the mathematical problem to find out the spectra $S_\Delta$ and $S_d$ is equivalent to the study of the quadratic operator pencil. Thus, each solution (37) or (38), satisfying Eqs. (10)–(12), conditions on the interfaces (26), (27) and any

HBCF is determined within a constant factor. For example $A_P^{+1}$, $s \in S_\Delta$ (or $B_S^{+1}$, $s \in S_\delta$) can be chosen as such a factor and other coefficients are expressed through it using linear algebraic system with matrix $\mathbf{M}_*$ (or $\mathbf{N}_*$).

Hence, the "in-plane" guided waves ($s \in S_\Delta$) and "out-of-plane" waves ($s \in S_\delta$) are given by formulas (7)–(9) and (37), (38) (see also Zakharov, 1988; Achenbach and Xu, 1998). For $r \gg 1$ the curvature of the cylindrical wave front can be neglected and the asymptotics of (7)–(9) lead to quasi plane waves of the magnitude order $r^{-\frac{1}{2}}$ with the leading parts $u^j$,$v^j$ for the in-plane and $w^j$ for the out of plane polarisation.

Note that the dispersion relations $\Delta_* = 0$ and $\delta_* = 0$ remain the same for $s$ and $-s$ and the normalisation can be chosen in a such a way that

$$u_l^j = -u_m^j, \quad v_l^j = v_m^j \text{ and } w_l^j = w_m^j \text{ for } s_l = -s_m, \quad j = 1, 2, \ldots, N. \tag{39}$$

In the case of pure elasticity the complex roots appear in conjugated pairs since the left hand sides $\Delta_*$ and $\delta_*$ of the frequency equations can be expanded in series with real coefficients. In addition we may set (over bar means complex conjugation)

$$u_l^j = \bar{u}_m^j, \quad v_l^j = -\bar{v}_m^j, \quad w_l^j = \bar{w}_m^j \text{ for } s_l = -\bar{s}_m \text{ and } u_l^j, v_l^j, w_l^j \in R \text{ for } s_l \in R. \tag{40}$$

To sum up let us represent the final form of the mode decomposition

$$\begin{bmatrix} u_r^j \\ u_\theta^j \\ u_z^j \end{bmatrix} = \sum_{n=0}^{+\infty} \sum_{\{s_l \in S_\Delta, S_\delta\}} \begin{bmatrix} \left(-u_l^j(z) B_n'(s_l r) + w_l^j(z) \frac{n}{s_l r} B_n(s_l r)\right)\left(M_n^{l,c} \cos n\theta - M_n^{l,s} \sin n\theta\right) \\ \left(u_l^j(z) \frac{n}{s_l r} B_n(s_l r) - w_l^j(z) B_n'(s_l r)\right)\left(M_n^{l,c} \sin n\theta + M_n^{l,s} \cos n\theta\right) \\ v_l^j(z) B_n(s_l r)\left(M_n^{l,c} \cos n\theta - M_n^{l,s} \sin n\theta\right) \end{bmatrix}. \tag{41}$$

## 5. Orthogonality relations

Introduce the scalar products of any functions $f_l^j$ and $g_m^j$ related to the wavenumbers $s_l$ and $s_m$, and to $j$th layer by standard formula

$$(f_l^j, g_m^j) \equiv \int_{z_j}^{z_{j+1}} f_l^j g_m^j \mathrm{d}z. \tag{42}$$

For the sake of simplicity assume that the wavenumbers are single roots of the frequency equations (37) and (38). For any $s_l$,$s_m \in$ $S_\Delta$ combine the products of $z$-components of the displacements (7)–(9) and stresses (17)–(23) into the expressions below

$$U_{lm}^j \equiv s_l s_m (v_l^j, v_m^j) + \{k_S^j\}^2 (u_l^j, u_m^j) - \left(\frac{\mathrm{d}}{\mathrm{d}z} u_l^j, \frac{\mathrm{d}}{\mathrm{d}z} u_m^j\right), \tag{43}$$

$$V_{lm}^j \equiv s_l s_m (u_l^j, u_m^j) + \{k_P^j\}^2 (v_l^j, v_m^j) - \left(\frac{\mathrm{d}}{\mathrm{d}z} v_l^j, \frac{\mathrm{d}}{\mathrm{d}z} v_m^j\right), \tag{44}$$

$$W_{lm}^j \equiv (\chi_l^j, v_m^j) - (\tau_m^j, v_l^j). \tag{45}$$

For $s_l \in S_\Delta$ and $s_m \in S_\delta$ introduce the additional combinations

$$H_{lm}^j \equiv \left[\alpha_j s_m^2 - s_l^2 - \gamma_j \{k_S^j\}^2\right] (v_l^j, w_m^j) - \beta_j s_l \left(u_l^j, \frac{\mathrm{d}}{\mathrm{d}z} w_m^j\right) + s_l \left(\frac{\mathrm{d}}{\mathrm{d}z} u_l^j, w_m^j\right), \tag{46}$$

$$G_{lm}^j \equiv (p_l^j, w_m^j) - \left(v_l^j, \frac{\mathrm{d}}{\mathrm{d}z} w_m^j\right) + \frac{s_l^2 - s_m^2}{s_l} (u_l^j, w_m^j), \tag{47}$$

and for $s_l$,$s_m \in S_\delta$—the combination

$$T_{lm}^j \equiv (w_l^j, v_m^j). \tag{48}$$

Statement 1. For any $s_l$,$s_m \in S_\Delta$ such that $s_l^2 \neq s_m^2$ the following equations hold

$$U_{lm}^* \equiv \sum_j \mu_j U_{lm}^j = 0, \quad V_{lm}^* \equiv \sum_j \alpha_j \mu_j V_{lm}^j = 0, \tag{49}$$

$$W^*_{lm} \equiv \sum_j \mu_j W^j_{lm} = 0. \tag{50}$$

Indeed, the integration by parts of the equations (10)–(12) yields

$$\int_{z_j}^{z_{j+1}} \left\{ \{ s_m u^j_m (\text{Eq. (10)})|_{s=s_l} - s_l v^j_l (\text{Eq. (11)})|_{s=s_m} \} \mathrm{d}z \right\} = s_m U^j_{lm} - \alpha_j s_l V^j_{lm} + \left[ s_m u^j_m \tau^j_l - s_l v^j_l \sigma^j_m \right]|_{z_j}^{z_{j+1}} = 0. \tag{51}$$

and by virtue of the conditions (26), (27) and any HBCF (28)–(35) we obtain

$$\sum_j \mu_j [s_l v^j_l \sigma^j_m - s_m u^j_m \tau^j_l]|_{z_j}^{z_{j+1}} = s_l \sum_j \{ v^j_l(z_{j+1}) \mu_j \sigma^j_m(z_{j+1}) - v^j_l(z_j) \mu_j \sigma^j_m(z_j) \} \\ - s_m \sum_j \{ u^j_m(z_{j+1}) \mu_j \tau^j_l(z_{j+1}) - u^j_m(z_j) \mu_j \tau^j_l(z_j) \} = 0, \tag{52}$$

$$\sum_j \mu_j \left( s_m U^j_{lm} - \alpha_j s_l V^j_{lm} \right) = s_m U^*_{lm} - s_l V^*_{lm} = 0. \tag{53}$$

Changing indices $l \leftrightarrow m$ with $U^*_{lm}, V^*_{lm} = U^*_{ml}, V^*_{ml}$ in (53) we also arrive at the equations $s_l U^*_{lm} - s_m V^*_{lm} = 0$ and then at the Eq. (49). Similarly we obtain

$$\sum_j \mu_j \int_{z_j}^{z_{j+1}} u^j_m (\text{Eq.(10)})|_{s=s_l} \mathrm{d}z = U^*_{lm} - s_l W^*_{lm} + \sum_j \mu_j u^j_m \tau^j_l |_{z_j}^{z_{j+1}} = U^*_{lm} - s_l W^*_{lm} = 0, \tag{54}$$

$$\sum_j \mu_j \int_{z_j}^{z_{j+1}} v^j_l (\text{Eq.(11)})|_{s=s_m} \mathrm{d}z = V^*_{lm} - s_m W^*_{lm} + \sum_j \mu_j v^j_l \sigma^j_m |_{z_j}^{z_{j+1}} = V^*_{lm} - s_m W^*_{lm} = 0, \tag{55}$$

and two symmetrical equation (49) are equivalent to one non-symmetrical equation (50). Note that the Eq. (50) can be rewritten using functions $p^j$ and $d^j$ as follows

$$\mathrm{d}^j \equiv \frac{1}{3}\{\sigma^j + 2p^j\} = \frac{1}{\tilde{\kappa}_j}\left\{ \chi^j + 2\frac{\mathrm{d}v^j}{\mathrm{d}z} \right\}, \quad \tilde{\kappa}_j \equiv \frac{3\alpha_j}{\beta_j + 2\gamma_j}, \tag{56}$$

$$Y^*_{lm} \equiv \sum_j \mu_j \left\{ \tilde{\alpha}_j (p^j_l, u^j_m) - \tilde{\beta}_j \left( \frac{\mathrm{d}v^j_l}{\mathrm{d}z}, u^j_m \right) - (\tau^j_m, v^j_l) \right\} = 0, \quad \tilde{\alpha}_j \equiv \frac{\alpha_j}{\gamma_j}, \quad \tilde{\beta}_j \equiv \frac{\beta_j}{\gamma_j}, \tag{57}$$

$$Z^*_{lm} \equiv \sum_j \mu_j \left\{ \tilde{\kappa}_j (d^j_l, u^j_m) - 2\left( \frac{\mathrm{d}v^j_l}{\mathrm{d}z}, u^j_m \right) - (\tau^j_m, v^j_l) \right\} = 0. \tag{58}$$

Statement 2. The Eq. (12) for the in-plane waves yields the following OR

$$T^*_{lm} = \sum_j \mu_j T^j_{lm} = 0, \quad \sum_j \left\{ \mu_j \left( \frac{\mathrm{d}w^j_l}{\mathrm{d}z}, \frac{\mathrm{d}w^j_m}{\mathrm{d}z} \right) - \rho_j \omega^2 T^j_{lm} \right\} = 0 \text{ for } s_l, s_m \in S_\delta, s_l^2 \neq s_m^2. \tag{59}$$

The result follows from the integration of Eq. (12)

$$\int_{z_j}^{z_{j+1}} w^j_l (\text{Eq. 12})|_{s=s_m} \mathrm{d}z = \left[ w^j_l \frac{\mathrm{d}w^j_m}{\mathrm{d}z} \right] \Bigg|_{z_j}^{z_{j+1}} - \left( \frac{\mathrm{d}w^j_l}{\mathrm{d}z}, \frac{\mathrm{d}w^j_m}{\mathrm{d}z} \right) + \left[ \frac{\rho_j \omega^2}{\mu_j} - s_m^2 \right] T^j_{lm} = 0, \tag{60}$$

with taking into account equations (26), (27) and HBCF.

Statement 3. For $s_l \in S_\Delta$ and $s_m \in S_\delta$ the following orthogonality relations hold

$$H^*_{lm} \equiv \sum_j \mu_j H^j_{lm} = 0, \quad G^*_{lm} \equiv \sum_j \mu_j G^j_{lm} = 0. \tag{61}$$

The proof can be obtained as above

$$\sum_j \left\{ \mu_j \int_{z_j}^{z_{j+1}} \{ w_m^j (\text{Eq. 11})|_{s=s_l} - \alpha_j v_l^j (\text{Eq. 12})|_{s=s_m} \} \mathrm{d}z \right\} = H_{lm}^* + \sum_j \mu_j \left[ \sigma_l^j w_m^j - \alpha_j v_l^j \frac{\mathrm{d}}{\mathrm{d}z} w_m^j \right] \Bigg|_{z_j}^{z_{j+1}} = H_{lm}^* = 0, \tag{62}$$

$$\sum_j \left\{ \mu_j \int_{z_j}^{z_{j+1}} \{ w_m^j (\text{Eq. 10})|_{s=s_l} - u_l^j (\text{Eq. 12})|_{s=s_m} \} \mathrm{d}z \right\} = -s_l G_{lm}^* + \sum_j \mu_j \left[ \tau_l^j w_m^j - u_l^j \frac{\mathrm{d}}{\mathrm{d}z} w_m^j \right] \Bigg|_{z_j}^{z_{j+1}} = -s_l G_{lm}^* = 0. \tag{63}$$

## 6. Physical meaning of the orthogonality relations

Now consider the obtained OR from the viewpoint of energy. To this end multiply the Eq. (1) with $f_\alpha^j = 0$ and $s = s_l$ by a speed of particle $\dot{u}_{\alpha m}^j = -\mathrm{i}\omega u_{\alpha m}^j$ for the wavenumber $s = s_m$

$$(\text{Eq. 1})|_{s=s_l} \dot{u}_{\alpha m}^j = \partial_\beta \{ \sigma_{\alpha\beta l}^j \dot{u}_{\alpha m}^j \} - \sigma_{\alpha\beta l}^j \{ \dot{\varepsilon}_{\alpha\beta m}^j + \dot{\omega}_{\alpha\beta m}^j \} - \rho_j \ddot{u}_{\alpha l}^j \dot{u}_{\alpha m}^j = 0, \tag{64}$$

where $\omega_{\alpha\beta}^j = \frac{1}{2} \{ \partial_\beta u_\alpha^j - \partial_\alpha u_\beta^j \}$ is a rotation tensor. Using the symmetry of stiffness and viscosity tensors $c_{\alpha\beta\gamma\delta}^{\prime j}, c_{\alpha\beta\gamma\delta}^{\prime\prime j}$ and antisymmetry of the rotation tensor we obtain

$$\sigma_{\alpha\beta l}^j \left\{ \dot{\varepsilon}_{\alpha\beta m}^j + \dot{\omega}_{\alpha\beta m}^j \right\} = \sigma_{\alpha\beta l}^j \dot{\varepsilon}_{\alpha\beta m}^j = c_{\alpha\beta\gamma\delta}^{\prime j} \dot{\varepsilon}_{\alpha\beta m}^j \varepsilon_{\gamma\delta l}^j + c_{\alpha\beta\gamma\delta}^{\prime\prime j} \dot{\varepsilon}_{\alpha\beta m}^j \dot{\varepsilon}_{\gamma\delta l}^j, \tag{65}$$

$$c_{\alpha\beta\gamma\delta}^{\prime j} \dot{\varepsilon}_{\alpha\beta m}^j \varepsilon_{\gamma\delta l}^j = c_{\alpha\beta\gamma\delta}^{\prime j} \varepsilon_{\alpha\beta m}^j \dot{\varepsilon}_{\gamma\delta l}^j, \tag{66}$$

$$\partial_\beta \left\{ \sigma_{\alpha\beta l}^j \dot{u}_{\alpha m}^j \right\} = \sigma_{\alpha\beta l}^j \dot{\varepsilon}_{\alpha\beta m}^j + \rho \ddot{u}_{\alpha l}^j \dot{u}_{\alpha m}^j = \partial_\beta \{ \sigma_{\alpha\beta m}^j \dot{u}_{\alpha l}^j \}. \tag{67}$$

Consider a cylinder $\Omega_j = \{ r \leqslant R, z_j \leqslant z \leqslant z_{j+1} \}$ with the upper surface $\Omega_+^j = \{ r \leqslant R, z = z_{j+1} \}$, lower surface $\Omega_-^j = \{ r \leqslant R, z = z_j \}$ and the lateral surface $\Omega_R^j = \{ r = R, z_j \leqslant z \leqslant z_{j+1} \}$. Then

$$\begin{aligned} \iiint_{\Omega_j} \{ \partial_\beta \{ \sigma_{\alpha\beta l}^j \dot{u}_{\alpha m}^j - \sigma_{\alpha\beta m}^j \dot{u}_{\alpha l}^j \} \} \mathrm{d}\Omega &= \iint_{\Omega_R^j} \{ \sigma_{\alpha\beta l}^j \dot{u}_{\alpha m}^j - \sigma_{\alpha\beta m}^j \dot{u}_{\alpha l}^j \} n_\beta \mathrm{d}A \\ &+ \left\{ \iint_{\Omega_+^j} - \iint_{\Omega_-^j} \right\} \{ \sigma_{\alpha z l}^j \dot{u}_{\alpha m}^j - \sigma_{\alpha z m}^j \dot{u}_{\alpha l}^j \} \mathrm{d}A = 0, \end{aligned} \tag{68}$$

where $n_\beta$ are coordinates of the outer unit normal to $\Omega_R^j$. The conditions on the interfaces and HBCF yield

$$\sum_j \left\{ \iint_{\Omega_+^j} - \iint_{\Omega_-^j} \right\} \{ \sigma_{\alpha z l}^j \dot{u}_{\alpha m}^j - \sigma_{\alpha z m}^j \dot{u}_{\alpha l}^j \} \mathrm{d}A = 0, \text{ or} \tag{69}$$

$$\langle \sigma_{rr}^l, \mathbf{u}_r^m \rangle + \langle \sigma_{r\theta}^l, \mathbf{u}_\theta^m \rangle + \langle \sigma_{rz}^l, \mathbf{u}_z^m \rangle - \langle \sigma_{rr}^m, \mathbf{u}_r^l \rangle - \langle \sigma_{r\theta}^m, \mathbf{u}_\theta^l \rangle - \langle \sigma_{rz}^m, \mathbf{u}_z^l \rangle = 0, \tag{70}$$

$$\langle \mathbf{f}^l, \mathbf{g}^m \rangle \equiv \sum_j \iint_{\Omega_R^j} f_l^j g_m^j \mathrm{d}A = R \sum_j \int_0^{2\pi} (f_l^j, g_m^j) \mathrm{d}\theta. \tag{71}$$

The left hand side of Eq. (70) for $s_l, s_m \in S_\Delta$ is reduced to the form

$$\begin{aligned} &\langle \sigma_{rr}^l, \mathbf{u}_r^m \rangle + \langle \sigma_{r\theta}^l, \mathbf{u}_\theta^m \rangle + \langle \sigma_{rz}^l, \mathbf{u}_z^m \rangle - \langle \sigma_{rr}^m, \mathbf{u}_r^l \rangle - \langle \sigma_{r\theta}^m, \mathbf{u}_\theta^l \rangle - \langle \sigma_{rz}^m, \mathbf{u}_z^l \rangle \\ &\quad = \pi R \xi_n \left\{ W_{ml}^* B_n(s_m R) B_n'(s_l R) - W_{lm}^* B_n'(s_m R) B_n(s_l R) \right\} \end{aligned}, \quad \xi_n \equiv \begin{cases} 1, & n \geqslant 1 \\ 2, & n = 0. \end{cases} \tag{72}$$

For $s_l \in S_\Delta$ and $s_m \in S_\delta$ it is rewritten as follows

$$\langle \sigma_{rr}^l, \mathbf{u}_r^m \rangle + \langle \sigma_{r\theta}^l, \mathbf{u}_\theta^m \rangle + \langle \sigma_{rz}^l, \mathbf{u}_z^m \rangle - \langle \sigma_{rr}^m, \mathbf{u}_r^l \rangle - \langle \sigma_{r\theta}^m, \mathbf{u}_\theta^l \rangle - \langle \sigma_{rz}^m, \mathbf{u}_z^l \rangle = \pi \xi_n G_{lm}^* \frac{n}{s_m} B_n(s_m R) B_n(s_l R), \tag{73}$$

and for the horizontally polarised waves $s_l, s_m \in S_\delta$ it acquires the form

$$\langle \sigma_{rr}^l, u_r^m \rangle + \langle \sigma_{r\theta}^l, \mathbf{u}_\theta^m \rangle + \langle \sigma_{rz}^l, \mathbf{u}_z^m \rangle - \langle \sigma_{rr}^m, \mathbf{u}_r^l \rangle - \langle \sigma_{r\theta}^m, \mathbf{u}_\theta^l \rangle - \langle \sigma_{rz}^m, \mathbf{u}_z^l \rangle = \pi R \xi_n T_{lm}^* (E_{lm} - E_{ml}), \tag{74}$$

where $E_{lm} = \frac{1}{2} s_l \{ B_{n-1}(s_m R) B_{n-2}(s_l R) - B_{n+1}(s_m R) B_{n+2}(s_l R) \}$.

Thus, the obtained OR (50), (59) and (61) are in fact the reciprocity relations which hold for a linearly viscoelastic laminate due to the energy symmetry.

Note that the Eq. (67) for $s_l = s_m$ can be easily rewritten in terms of the density of kinetic energy $K_l^j$, elastic energy $E_l^j$, Rayleigh function $R_l^j$ and Pointing's vector $\mathbf{P}_l^j$:

$$\partial_t\{K_l^j + E_l^j\} + 2R_l^j + \mathrm{div}\mathbf{P}_l^j = 0, \tag{75}$$

$$K_l^j \equiv \frac{1}{2}\rho_j \dot{u}_{\alpha l}^j \dot{u}_{\alpha l}^j, \quad E_l^j \equiv \frac{1}{2} c'^j_{\alpha\beta\gamma\delta}\varepsilon^j_{\alpha\beta l}\varepsilon^j_{\gamma\delta l}, \tag{76}$$

$$R_l^j \equiv \frac{1}{2} c''^j_{\alpha\beta\gamma\delta}\dot{\varepsilon}^j_{\alpha\beta l}\dot{\varepsilon}^j_{\gamma\delta l}, \quad \mathbf{P}_l^j = \begin{bmatrix} \mathrm{P}_{1l}^j & \mathrm{P}_{2l}^j & \mathrm{P}_{3l}^j \end{bmatrix}, \quad \mathrm{P}_{\alpha l}^j = -\sigma^j_{\alpha\beta l}\dot{u}^j_{\beta l}. \tag{77}$$

## 7. Radiation conditions in case of pure elasticity

Substituting $\mathrm{Re}\{u_{\alpha l}^j \mathrm{e}^{-\mathrm{i}\omega t}\}$ instead of $u_{\alpha l}^j$ into Eq. (75) we arrive at the energy relation with positively determined quadratic forms $K_l^j$, $E_l^j$ and $R_l^j$. Introduce also the average power flow $\mathrm{P}_{rl}^*$ across the lateral surface $\Omega_R$

$$\mathrm{P}_{rl}^* \equiv \frac{\omega}{2\pi}\int_0^{2\pi/\omega} \sum_j \iint_{\Omega_R^j} \mathrm{P}_{rl}^j \mathrm{d}A\mathrm{d}t, \quad \mathrm{P}_{rl}^j \equiv \mathrm{P}_{1l}^j\cos\theta + \mathrm{P}_{2l}^j \sin\theta. \tag{78}$$

It is easily to show that

$$\mathrm{P}_{rl}^j = \omega\left[\mathrm{Re}\{\sigma^j_{rrl}\mathrm{e}^{-\mathrm{i}\omega t}\}\mathrm{Re}\{\mathrm{i}u^j_{rl}\mathrm{e}^{-\mathrm{i}\omega t}\} + \mathrm{Re}\{\sigma^j_{r\theta l}\mathrm{e}^{-\mathrm{i}\omega t}\}\mathrm{Re}\{\mathrm{i}u^j_{\theta l}\mathrm{e}^{-\mathrm{i}\omega t}\} + \mathrm{Re}\{\sigma^j_{rzl}\mathrm{e}^{-\mathrm{i}\omega t}\}\mathrm{Re}\{\mathrm{i}u^j_{zl}\mathrm{e}^{-\mathrm{i}\omega t}\}\right], \tag{79}$$

$$\frac{\omega}{2\pi}\int_0^{2\pi/\omega} \mathrm{P}_{rl}^j \mathrm{d}t = -\frac{1}{2}\mathrm{Re}\left\{\sigma^j_{rrl}\dot{\bar{u}}^j_{rl} + \sigma^j_{r\theta l}\dot{\bar{u}}^j_{\theta} + \sigma^j_{rzl}\dot{\bar{u}}^j_{zl}\right\}, \tag{80}$$

where $\bar{u}^j_{rl}$ denotes the complex conjugation of $u^j_{rl}$. For the real positive $s_l$ the integration of the Eq. (80) over $\Omega_R^j$ and summation over $j$ yields (see formulae (72) and (74))

$$\mathrm{P}_r^* = \frac{\pi\xi_n R}{2}\mathrm{Re}\left\{\sum_j \mu_j[(\chi_l^j, \bar{u}_l^j) - (\bar{\tau}_l^j, v_l^j)]\mathrm{i}\omega \overline{B}'_n(s_l R)B_n(s_l R)\right\}, \quad s_l \in S_\Delta, \tag{81}$$

$$\mathrm{P}_r^* = \frac{\xi_n \pi}{2}\mathrm{Re}\left\{\sum_j \mu_j(w_l^j, \bar{w}_l^j)\mathrm{i}\omega\bar{s}_l[B_{n-1}(s_l R)\overline{B}_{n-2}(s_l R) - \overline{B}_{n+2}(s_l R)B_{n+1}(s_l R)]\right\}, \quad s_l \in S_\delta. \tag{82}$$

For the propagating wave with the real wavenumber the cylindrical function $B_n$ should be replaced by the Hankel function $H_n^{(1,2)}$ of the first or second kind.

Statement 4. For the propagating wave with $s_l > 0$, $s_l \in S_\Delta$ in elastic laminate the average power flow acquires the form

$$\mathrm{P}_r^* = \pm\xi_n c_l W_{ll}^*, \tag{83}$$

where $c_l = \omega/s_l$ is the phase speed and the sign + or − corresponds to the Hankel function of the first or second kind, respectively.

The result follows from the formula (81) with taking into account the property (45) and the identities for the cylindrical functions (see Abramovitz and Stegun, 1972)

$$\mathrm{Re}\{\mathrm{i}\overline{H}'_n(s_l R)H_n(s_l R)\} = \pm\frac{2}{\pi R s_l}, \quad H_n = H_n^{(1,2)} \equiv J_n \pm \mathrm{i}N_n, \tag{84}$$

$$J_{n+1}(s_l R)N_n(s_l R) - J_n(s_l R)N_{n+1}(s_l R) = \frac{2}{\pi R s_l}. \tag{85}$$

Statement 5. The average power flow of the propagating wave with $s_l > 0$, $s_l \in S_\delta$ in elastic laminate acquires the form

$$\mathrm{P}_r^* = \pm\xi_n \omega T_{ll}^*. \tag{86}$$

The proof follows from the formulae (40), (82) and (85).

The formulae (83) and (86) can be used for selecting waves satisfying the radiation condition. In the case of viscoelastic materials and complex-valued roots $s_l$ the selection is based on the decay of function $H_n^{(1,2)}(s_l R)$ provided by $\mathrm{Im} s_l$ in the asymptotic formula [see Abramovitz and Stegun (1972)]

$$H_n^{(1,2)}(s_l r) = \sqrt{\frac{2}{\pi s_l r}}\{1 + \mathrm{O}(s_l r)^{-1}\}\mathrm{e}^{\pm \mathrm{i}\left[s_l r - \frac{2n+1}{4}\pi\right]}, \quad |s_l r| \gg 1. \tag{87}$$

For the pure elasticity and the real wavenumber the analogue of the Leontovich–Lighthill theorem can be proven.

**Theorem.** *For $s_l > 0$ and $s_l \in S_\Delta$ or $s_l \in S_\delta$*

$$\mathrm{P}_{rl}^* \sim \pm c_g^l \{K_l + E_l\}^*, \quad R \to +\infty, \quad c_g^l \equiv \left.\frac{\mathrm{d}\omega}{\mathrm{d}s}\right|_{s=s_l}. \tag{88}$$

*The sign $\pm$ is chosen accordingly to the choice of Hankel's function as previously.*

Consider again the Eq. (75) with $R_l^j = 0$. For the real functions the kinetic energy $K_l^j$ yields

$$K_l^j = \frac{1}{2}\rho_j \mathrm{Re}\{\dot{u}_\alpha^j\}\mathrm{Re}\{\dot{u}_\alpha^j\} = \frac{1}{8}\{\dot{u}_\alpha^j \dot{u}_\alpha^j + 2\dot{u}_\alpha^j \dot{\bar{u}}_\alpha^j + \dot{\bar{u}}_\alpha^j \dot{\bar{u}}_\alpha^j\}, \tag{89}$$

and its time averaging involves terms like (80)

$$\frac{\omega}{2\pi}\int_0^{\frac{2\pi}{\omega}} K_l^j \mathrm{d}t = \frac{1}{4}\rho_j \mathrm{Re}\{\dot{u}_\alpha^j \dot{\bar{u}}_\alpha^j\}. \tag{90}$$

The terms $\dot{u}_\alpha^j \dot{u}_\alpha^j$ and $\dot{\bar{u}}_\alpha^j \dot{\bar{u}}_\alpha^j$ vanish due to zero average value of $\mathrm{e}^{\mp \mathrm{i}2\omega t}$. The similar rule holds for the contributions into the energy $E_l^j$. Introduce the frequency variation $\delta\omega$: $\omega_0 = \omega + \mathrm{i}\delta\omega$ and the respective wavenumber variation involving the group velocity $c_g^l$: $s_0 = s_l + \mathrm{i}\delta s$, $\delta\omega = c_g^l \delta s$. Then, using representation (89) we obtain

$$\frac{\omega}{2\pi}\int_0^{\frac{2\pi}{\omega}} \partial_t\{K_l^j + E_l^j\}\mathrm{d}t = 2\delta\omega\Phi(\omega,\delta\omega)\{K_l^j + E_l^j\}, \tag{91}$$

$$\frac{\omega}{2\pi}\int_0^{\frac{2\pi}{\omega}} \partial_t \mathrm{e}^{-\mathrm{i}(\omega_0 - \bar{\omega}_0)}\mathrm{d}t = 2\delta\omega\Phi(\omega,\delta\omega), \quad \Phi(\omega,\delta\omega) \equiv \frac{\omega}{4\pi}\frac{\mathrm{e}^{\frac{4\pi\delta\omega}{\omega}} - 1}{\delta\omega} \underset{\delta\omega\to 0}{\to} 1. \tag{92}$$

For the Pointing vector we have

$$\mathrm{div}\mathbf{P}_l^j = \mathrm{D}_r \mathrm{P}_{rl}^j + r^{-1}\partial_\theta \mathrm{P}_{\theta l}^j + \partial_z \mathrm{P}_{zl}^j, \quad \mathrm{D}_r \equiv \partial_r + r^{-1}, \tag{93}$$

$$\iint_{\Omega_R^j}\{r^{-1}\partial_\theta P_{\theta l}^j + \partial_z \mathrm{P}_{zl}^j\}\mathrm{d}A = \int_{z_{j-1}}^{z_j} P_{\theta l}^j\big|_0^{2\pi}\mathrm{d}z + R\int_0^{2\pi} \mathrm{P}_{zl}^j\big|_{z_{j-1}}^{z_j}\mathrm{d}\theta, \tag{94}$$

and $P_{\theta l}^j|_0^{2\pi} = P_{\theta l}^j|_{\theta=2\pi} - P_{\theta l}^j|_{\theta=0} = 0$ due to the periodicity wrt $\theta$. Additionally, the interface conditions (6) and HBCF yield

$$\sum_j R\int_0^{2\pi} \mathrm{P}_{zl}^j\big|_{z_{j-1}}^{z_j}\mathrm{d}\theta = R\int_0^{2\pi}\sum_j\{\mathrm{P}_{zl}^j|_{z=z_j} - \mathrm{P}_{zl}^j|_{z=z_{j-1}}\}\mathrm{d}\theta = 0. \tag{95}$$

So, only $\mathrm{D}_r \mathrm{P}_{rl}^j$ remains in the sum of integrals over $\Omega_R^j$.

For $s_l, s_0 \in S_\Delta$ after application of the recurrent formulae for Hankel's functions and asymptotics (87) the last factor in formula (81) acquires the form

$$\mathrm{D}_r\{\overline{B}_n' B_n\} = \bar{s}_0 \overline{B}_n' B_n + s_0|B_n'|^2 + r^{-1}\overline{B}_n' B_n = s_0|B_n'|^2 - \bar{s}_0|B_n|^2 + n^2 r^{-2}\bar{s}^{-1}|B_n|^2, \tag{96}$$

$$s_0|H_n'|^2 = s_0\left|\frac{2}{\pi s_0 r}\right|\mathrm{e}^{\mp 2r\delta s} + \mathrm{O}(r^{-2}), \quad \bar{s}_0|H_n|^2 = \bar{s}_0\left|\frac{2}{\pi s_0 r}\right|\mathrm{e}^{\mp 2r\delta s} + \mathrm{O}(r^{-2}), \tag{97}$$

and

$$\mathrm{D}_r\{\overline{B}'_n B_n\} = \pm 2\mathrm{i}\delta s\left|\frac{2}{\pi s_0 r}\right|\mathrm{e}^{\mp 2r\delta s} + \mathrm{O}(r^{-2}) \Rightarrow \mathrm{D}_r\{\mathrm{P}^*_{rl}\} = \mp 2\delta s \mathrm{P}^*_{rl}\mathrm{e}^{\mp 2R\delta s} + \mathrm{O}(R^{-2}). \tag{98}$$

Finally the Eqs. (91) and (98) result in the following

$$\{K_l + E_l + \mathrm{div}\mathrm{P}_{rl}\}^* = 2\delta\omega\{K_l + E_l\}^* \mp 2\delta s \mathrm{P}^*_{rl} + \mathrm{O}(\delta s)^2 + \mathrm{O}(R^{-1}) = 0. \tag{99}$$

Tending $R \to +\infty$ and dividing by $2\delta\ s \to +0$ we derive the formulation (88).

For the wave of another kind $s_l \in S_\delta$ the proof is similar and uses formulae (82) and (87).

## 8. Exact solutions for some particular boundary value problems

Let us reformulate OR in terms of the total displacements and stresses. For $s_l, s_m \in S_\delta$ the field structure (7), (9), (18), (19) with properties (59) results in the relation

$$\langle \sigma^l_{rr}, \mathbf{u}^m_r\rangle = \langle \sigma^l_{r\theta}, \mathbf{u}^m_\theta\rangle = 0, \quad s_l^2 \neq s_m^2. \tag{100}$$

Introduce additional combinations of the displacements and stresses

$$\psi^j_l \equiv \mu_j\tilde{\beta}_j\partial_z u^j_{zl} - \tilde{\alpha}_j\{\sigma^j_{rrl} + \sigma^j_{\theta\theta l}\} = -\mu_j\chi^j_l B_n(s_l r)\begin{Bmatrix}\cos n\theta \\ -\sin n\theta\end{Bmatrix}, \tag{101}$$

$$\phi^j_l \equiv 2\mu_j\partial_z u^j_{zl} - \tilde{\alpha}_j\{\sigma^j_{rrl} + \sigma^j_{\theta\theta l} + \sigma^j_{zzl}\} = -\mu_j\chi^j_l B_n(s_l r)\begin{Bmatrix}\cos n\theta \\ -\sin n\theta\end{Bmatrix}. \tag{102}$$

Then for $s_l, s_m \in S_\Delta$, $s_l^2 \neq s_m^2$ the relations (57) and (58) can be rewritten in the form

$$\left\langle \psi^l, \mathbf{u}^m_r\right\rangle + \left\langle \sigma^m_{rz}, \mathbf{u}^l_z\right\rangle = 0, \tag{103}$$

$$\left\langle \phi^l, \mathbf{u}^m_r\right\rangle + \left\langle \sigma^m_{rz}, \mathbf{u}^l_z\right\rangle = 0, \tag{104}$$

$$\left\langle \sigma^l, \mp\mathbf{u}^m_r + \frac{1}{n}\partial_\theta\mathbf{u}^m_\theta\right\rangle + \left\langle \mp\sigma^m_{rz} + \frac{1}{n}\partial_\theta\sigma^m_{\theta z}, \mathbf{u}^l_z\right\rangle = 0, \tag{105}$$

$$\left\langle \phi^l, \mp\mathbf{u}^m_r + \frac{1}{n}\partial_\theta\mathbf{u}^m_\theta\right\rangle + \left\langle \mp\sigma^m_{rz} + \frac{1}{n}\partial_\theta\sigma^m_{\theta z}, \mathbf{u}^l_z\right\rangle = 0. \tag{106}$$

The general case of the arbitrary $s_l$ and $s_m$, $s_l^2 \neq s_m^2$ satisfies the relations (70).

Relations (100), (103)–(106) can be used to find the exact or approximate solution to the boundary value problem for a laminate occupying a region with cylindrical geometry (see Fig. 1b). First of all it concerns the axisymmetrical problem for an infinite laminate with a cylindrical opening $\Omega = \cup_j\Omega_j$ of radius $R$. Assume HBCF are satisfied on $\Omega^+_N$ and $\Omega^-_1$ and the lateral surface $\Omega_R = \bigcup_j\Omega^j_R$ is loaded. Consider the axisymmetrical torsion with the boundary conditions $\sigma^j_{r\theta} = \Theta^j$ or $u^j_\theta = W^j$ on $\Omega^j_R$ and seek the exact solution using the mode decomposition (41) for $n = 0$

$$\mathbf{u}_\theta = \sum_l M^l\mathbf{u}^l_\theta, \quad M^l \equiv M^{l,s}_0, s_l \in S_\delta. \tag{107}$$

Thus, coefficients $M^l$ follow from the formulae (100) and (74) in a closed form

$$M^l = -\sum_j(\Theta^j, w^j_l)/s_l T^*_{ll}B_2(s_l R) \text{ or } M^l = -\sum_j(W^j, \mu_j w^j_l)/T^*_{ll}B'_0(s_l R). \tag{108}$$

For an infinite laminate with a cylindrical opening we set $Ims_l \geqslant\ 0$, $B_n = H^{(1)}_n$ and select positive real roots $s_l$ in case of pure elasticity. But if the respective $c^l_g < 0$ choose $B_n = H^{(2)}_n$. The laminate occupying a finite cylindrical region $\Omega$ is considered similarly and for this case the formulae (108) remain in force with $B_n = J_n$. Assume now that the laminate occupies a finite cylindrical region $\Omega_2$ of radius $R_2$ with a coaxial cylindrical opening $\Omega_1$ of radius $R_1 < R_2$. Denote the boundary conditions on the lateral surfaces $\Omega_{R_2}$ and $\Omega_{R_1}$ by

$\Theta_2^j, \Theta_1^j$ or $W_2^j, W_1^j$, respectively. The mode decomposition (107) remains in force with the Bessel and Neumann functions

$$B_n(s_l r) = M_J^l J_n(s_l r) + M_N^l N_n(s_l r), \tag{109}$$

whose coefficients $M_J^l$ and $M_N^l$ in the field representation (7)–(9) and (19)–(23) are given by expressions

$$M_J^l = -\sum_j \frac{\left(N_2(s_l R_2)\Theta_1^j - N_2(s_l R_1)\Theta_2^j, w_l^j\right)}{s_l T_{ll}^* \{J_2(s_l R_1)N_2(s_l R_2) - J_2(s_l R_2)N_2(s_l R_1)\}}, \tag{110}$$

$$M_N^l = -\sum_j \frac{\left(-J_2(s_l R_2)\Theta_1^j + J_2(s_l R_1)\Theta_2^j, w_l^j\right)}{s_l T_{ll}^* \{J_2(s_l R_1)N_2(s_l R_2) - J_2(s_l R_2)N_2(s_l R_1)\}}, \tag{111}$$

or

$$M_J^l = -\sum_j \frac{\left(N_0'(s_l R_2)W_1^j - N_0'(s_l R_1)W_2^j, \mu_j w_l^j\right)}{T_{ll}^* \{J_0'(s_l R_1)N_0'(s_l R_2) - J_0'(s_l R_2)N_0'(s_l R_1)\}}, \tag{112}$$

$$M_N^l = -\sum_j \frac{\left(-J_0'(s_l R_2)W_1^j + J_0'(s_l R_1)W_2^j, \mu_j w_l^j\right)}{T_{ll}^* \{J_0'(s_l R_1)N_0'(s_l R_2) - J_0'(s_l R_2)N_0'(s_l R_1)\}}. \tag{113}$$

The relations (103) and (104) can be used for solving some axisymmetrical problems with the “in-plane” polarisation using mode decomposition (41) for $n = 0$

$$\begin{bmatrix} \mathbf{u}_r \\ \mathbf{u}_z \end{bmatrix} = \sum_l M^l \begin{bmatrix} \mathbf{u}_r^l \\ \mathbf{u}_z^l \end{bmatrix}, \quad M^l \equiv M_0^{l,c}, \quad s_l \in S_\Delta. \tag{114}$$

As above let us begin with the cylindrical opening $\Omega$ in the infinite laminate with a few variants of boundary conditions on $\Omega_R$: $\sigma_{rz}^j = \mathrm{T}^j$, $u_r^j = \mathrm{U}^j$ or $\frac{1}{2}\{\sigma_{rr}^j + \sigma_{\theta\theta}^j\} \equiv \mathrm{P}^j$, $u_z^j \equiv \mathrm{V}^j$ or $\frac{1}{3}\{\sigma_{rr}^j + \sigma_{\theta\theta}^j + \sigma_{zz}^j\} \equiv \mathrm{D}^j, \mathrm{V}^j$. Using relations (103) for the first couple of boundary conditions we obtain

$$M^l = \frac{F(\mathrm{T}^j, v_l^j, \chi_l^j, \mathrm{U}^j)}{B_0'(s_l R)}, \quad F(\mathrm{T}^j, v_l^j, \chi_l^j, \mathrm{U}^j) \equiv \sum_j \frac{(\mathrm{T}^j, v_l^j) - \mu_j(\chi_l^j, \mathrm{U}^j)}{W_{ll}^*}. \tag{115}$$

Other couples of boundary conditions result in the following

$$M^l = F(2\tilde{\alpha}_j \mathrm{P}^j - \mu_j \tilde{\beta}_j \partial_z \mathrm{V}^j, u_l^j, \tau_l^j, \mathrm{V}^j)/B_0(s_l R), \tag{116}$$

$$M^l = F(3\tilde{\alpha}_j \mathrm{D}^j - 2\mu_j \partial_z \mathrm{V}^j, u_l^j, \tau_l^j, \mathrm{V}^j)/B_0(s_l R). \tag{117}$$

The function $B_n$ for infinite laminate with an opening or for laminate occupying a finite cylinder $\Omega$ or cylinder $\Omega_2$ with opening $\Omega_1$ is chosen as above. For the case (109) and the boundary conditions $\mathrm{T}_{1,2}^j$, $\mathrm{U}_{1,2}^j$ on $\Omega_{R_{1,2}}$ the respective coefficients acquire the form

$$M_J^l = \frac{F(\mathrm{T}_1^j, v_l^j, \chi_l^j, \mathrm{U}_1^j)N_0'(s_l R_2) - F(\mathrm{T}_2^j, v_l^j, \chi_l^j, \mathrm{U}_2^j)N_0'(s_l R_1)}{J_0'(s_l R_1)N_0'(s_l R_2) - J_0'(s_l R_2)N_0'(s_l R_1)}, \tag{118}$$

$$M_J^l = \frac{-F(\mathrm{T}_1^j, v_l^j, \chi_l^j, \mathrm{U}_1^j)J_0'(s_l R_2) + F(\mathrm{T}_2^j, v_l^j, \chi_l^j, \mathrm{U}_2^j)J_0'(s_l R_1)}{J_0'(s_l R_1)N_0'(s_l R_2) - J_0'(s_l R_2)N_0'(s_l R_1)}. \tag{119}$$

For other boundary conditions $\mathrm{P}_{1,2}^j$ (or $\mathrm{D}_{1,2}^j$) and $\mathrm{U}_{1,2}^j$ we have to replace $J_0'$, $N_0'$ by $J_0$, $N_0$ in formulae (118), (119) and to use the function $F$ from the relations (116) or (117).

The application of the obtained OR to other boundary conditions leads to an infinite system of algebraic equations (see Zakharov, 1988) with respect to coefficients $M^l$. The same holds for the non-axisymmetrical problem with $n \geqslant 1$.

## 9. "Far field" calculation for an acoustic source localised in a finite region

Assume that the motion of an infinite laminate is caused by an acoustic source in the form of body forces $f_\alpha^j$, distributed in a finite volume embedded into cylinder $\Omega$, or in the form of surface load distributed over a finite region on $\Omega_1^-$ or $\Omega_N^+$ (see Fig. 1c). Another part of the faces satisfies HBCF. Represent the laminate response as a function of coordinates $r,\theta,z$ and decompose it into Fourier series wrt the angle $\theta$. Upon the general theory of differential equations in partial derivatives the field inside the region $\Omega$ ($r<R$) contains two components: a particular solution according to the acoustic source and a general *homogeneous* solution. At $r>R$ the particular solution vanishes since there is no longer body force nor facial load and the field can be represented by mode decomposition

$$\begin{bmatrix}\mathbf{u}_r\\ \mathbf{u}_\theta\\ \mathbf{u}_z\end{bmatrix}=\sum_l M_n^l\begin{bmatrix}\mathbf{u}_r^l\\ \mathbf{u}_\theta^l\\ \mathbf{u}_z^l\end{bmatrix},\quad s_l\in S_\Delta,S_\delta, \tag{120}$$

with $B_n=H_n^{(1)}$ and $s_l$ satisfying radiation condition. In the series (120) the components $u_r^j,u_z^j\sim\cos n\theta$, $u_\theta^j\sim\sin n\theta$, $M_n^l\equiv M_n^{l,c}$ or $u_r^j,u_z^j\sim-\sin n\theta$, $u_\theta^j\sim\cos n\theta$ and $M_n^l\equiv M_n^{l,s}$, i.e., they run all the components in formulae (41). On the lateral surfaces $\Omega_R^j$ $\quad(r=R)$ the inner and outer solutions satisfy the continuity of $u_\alpha^j$ and $\sigma_{r\alpha}^j$.

For each propagating wave of the wavenumber $s_m$ introduce a standing wave with $B_n(s_mr)\equiv J_n(s_mr)$ and with the same components $u_m$, $v_m$ and $w_m$. Then, integrating the Eq. (1) similarly to considerations (64)–(69) we obtain

$$\sum_j\iint_{\Omega_R^j}\left\{\sigma_{\alpha\beta}^j u_{\alpha m}^j-\sigma_{\alpha\beta m}^j u_\alpha^j\right\}n_\beta\mathrm{d}A=\Gamma_{mn}, \tag{121}$$

$$\Gamma_{mn}\equiv\left\{\iint_{\Omega_1^-}+\iint_{\Omega_N^+}\right\}\{\sigma_{\alpha\beta m}u_\alpha-\sigma_{\alpha\beta}u_{\alpha m}\}n_\beta\mathrm{d}A-\sum_j\iiint_{\Omega^j}\{f_\alpha^j u_{\alpha m}^j\}\mathrm{d}V. \tag{122}$$

Here $\Gamma_{mn}$ does not contain any unknowns. For example, if the source is given by the stresses $\sigma_{\alpha z}^-$ on $\Omega_1^-$ and $\sigma_{\alpha z}^+$ on $\Omega_N^+$ the expression (122) yields

$$\Gamma_{mn}=-\iint_{\Omega_N^+}\{\sigma_{zz}^+u_{zm}^N+\sigma_{zr}^+u_{rm}^N+\sigma_{z\theta}^+u_{\theta m}^N\}\mathrm{d}A-\iint_{\Omega_1^-}\{\sigma_{zz}^-u_{zm}^1+\sigma_{zr}^-u_{rm}^1+\sigma_{z\theta}^-u_{\theta m}^1\}\mathrm{d}A. \tag{123}$$

Using the identities (85) and relations (72)–(74) for propagating waves with functions $H_n^{(1)}$ we arrive at the following closed form of coefficients

$$\begin{cases}M_n^m=-\mathrm{i}s_m\Gamma_{mn}/\{2\xi_nW_{mm}^*\}, & s_m\in S_\Delta,\\ M_n^m=-\mathrm{i}\Gamma_{mn}/\{2\xi_nT_{mm}^*\}, & s_m\in S_\delta.\end{cases} \tag{124}$$

For the function $H_n^{(2)}$ the formulae (124) are used with the opposite sign.

Hence, we suggest a general method to evaluate the "far" field—but in fact the total field at the distance $r>R$, where $2R$ is the longitudinal size of an acoustic source. The method requires the calculation of spectra $S_\Delta$ and $S_\delta$, modes (7)–(9) and exact coefficients (124) in the double series (41).

In case of pure elasticity the *far field* in its classical meaning of waves propagating to infinity is expressed by ordinary series wrt the counter $n$ because at each frequency there is a finite number of real wavenumbers.

## 10. Exact solutions for some types of loadings

Consider a few examples of calculating $\Gamma_{mn}$. Assume that the load is distributed over a circular region $\Omega_N^+$ and the surface stresses $\sigma_{zz}^+(r,\theta)$, $\sigma_{rz}^+(r,\theta)$ and $\sigma_{\theta z}^+(r,\theta)$ are expanded into the trigonometrical Fourier series wrt $\theta$. In accordance with the representations (17)–(23) let us for a moment denote coefficients of $\cos n\theta$ (or $-\sin n\theta$) for $\sigma_{zz}^+$ and $\sigma_{rz}^+$ by $\tau_{zn}^+(r)$ and $\tau_{rn}^+(r)$, respectively. For $\sigma_{\theta z}^+$ the coefficient of $\sin n\theta$ (or $\cos n\theta$) is denoted by $\tau_{\theta n}^+(r)$. The substitution into (123) yields

$$\Gamma_{mn} = -\pi\xi_n\{u_m^N(z^+)T_r^+ + w_m^N(z^+)T_\theta^+ + v_m^N(z^+)T_z^+\}, \tag{125}$$

$$T_{r,\theta}^+ = \frac{1}{2}\int_0^R \{[\tau_{rn}^+(r) + \tau_{\theta n}^+(r)]J_{n+1}(s_m r) \pm [\tau_{rn}^+(r) - \tau_{\theta n}^+(r)]J_{n-1}(s_m r)\} r\mathrm{d}r, \tag{126}$$

$$T_z^+ = \int_0^R \tau_{zn}^+(r)J_n(s_m r)r\mathrm{d}r. \tag{127}$$

Solution (125)–(127) is of practical interest for evaluating the field, radiated by a circular transducer. In particular, for a constant normal load $\tau_{z0}^+/2$ we obtain

$$\Gamma_{m0} = -\frac{\pi}{s_m}RJ_1(s_m R)\tau_{z0}^+ \times \left\{\begin{matrix} v_m^N(z^+), s_m \in S_\Delta \\ 0, s_m \in S_\delta \end{matrix}\right\}, \tag{128}$$

and for a constant tangent load $\tau_{10}^+$ in the direction $x_1$ the coefficients are

$$\Gamma_{m1} = \frac{\pi}{s_m}RJ_1(s_m R)\tau_{10}^+ \times \left\{\begin{matrix} u_m^N(z^+), s_m \in S_\Delta \\ -w_m^N(z^+), s_m \in S_\delta \end{matrix}\right\}. \tag{129}$$

Other $\Gamma_{mn} = 0$. The rough estimate of the convergence rate of series (120) can be seen from the results (125)–(129). For example, the Lamb waves in an elastic layer have the wavenumbers $s_l \in S_\Delta$ with the asymptotic behaviour $\mathrm{Re}(s_l) = \mathrm{O}(\ln l)$, $\mathrm{Im}(s_l) = \mathrm{O}(l)$ and the out-of-plane waves $s_l \in S_\delta$: $\mathrm{Re}(s_l) = 0$, $\mathrm{Im}(s_l) = \mathrm{O}(l)$ as $l \to +\infty$ (see Auld, 1990). Hence, the terms of series (120) cannot exceed the order

$$M_n^l u_\alpha^l \sim \mathrm{O}(l^{k(z)})\mathrm{e}^{(r-R)\mathrm{O}(l)}, \tag{130}$$

for a certain $k(z)$ and the convergence holds at least at $r > R$.

It is also easily to obtain the laminate response to a concentrated load. For the concentrated body forces $f_\alpha^j = T_0\delta_\alpha^\beta\delta(x_1, x_2, x_3 - z_0)$ ($z_j \leqslant z_0 \leqslant z_{j+1}$; $\delta_\alpha^\beta$ is a Kronecker delta) at any HBCF we obtain

$$\Gamma_{mn} = -\sum_j \iiint_{\Omega^j} \{f_\alpha^j u_{\alpha m}^j\}\mathrm{d}V = -T_0 u_{\beta m}^j|_{r=0,z=z_0}, \tag{131}$$

with a similar result for the concentrated surface load $\sigma_{\alpha z}^+ = \tau_0^+\delta_\alpha^\beta\delta(x_1, x_2)$:

$$\Gamma_{mn} = -\tau_0^+ u_{\beta m}^N|_{r=0,z=z^+}. \tag{132}$$

Note that formulae (131) and (132) are non singular since the dummy displacements $u_{\beta m}^j(r, \theta, z_0)$ contain Bessel's function $B_n(s_m r) \equiv J_n(s_m r)$ whose value at the origin is regular. However, the solution (120) might have singularity at the origin due to the Hankel functions involved. By the same reason for the transversal load (axisymmetrical problem, $\beta = 3$) the terms with $n \geqslant 1$ vanish and only $\Gamma_{m0} \neq 0$. For the longitudinal load ($\beta = 1,2$) only $\Gamma_{m1} \neq 0$. Formulae (132) can be also obtained from (128) and (129) replacing $\tau_{\alpha n}^+$ ($\alpha,n = z,0$ or $\alpha,n = 1,1$) by $2\tau_0^+/\pi R^2$ and proceeding to a limit

$$\frac{\pi}{s_m}RJ_1(s_m R)\tau_{\alpha 0}^+ \to \frac{\pi}{s_m}\frac{s_m R^2}{2}\frac{2\tau_0^+}{\pi R^2} = \tau_0^+ \text{ as } R \to +0. \tag{133}$$

Thus, the mode decomposition (120) with coefficients (124), (131) and (132) represents the exact Green functions of different kinds and generalises the previous results for the case of any viscoelastic laminate and any HBCF. In contrast to other studies of Bai et al. (2004) or Lih and Mal (1996) our approach does not involve the semi-analytical finite element method or complex integration with Fourier inversion using FFT.

## 11. Discussion and conclusive remarks

The obtained results can be clearly subdivided into *two* groups. *First* group includes orthogonality relations for the cylindrical guided waves satisfying homogeneous boundary conditions on the laminate faces. They correlate with the results of previous authors for an elastic layer and plane waves, which can be obtained as a limit case for large radius. The explicit expressions for reciprocity relations are obtained as well. They are valid for both elastic and linearly viscoelastic media due to the symmetry of their energy functionals. The *second* group describes solving methods using OR. Some particular boundary value problems for a finite cylinder, cylinder with an opening or infinite laminate with a cylindrical opening can be solved in this manner. For other boundary value problems OR can be used to construct a linear algebraic system of equations with respect to the mode coefficients. However, one important problem to evaluate the *far field* of an acoustic source—surface loads or body forces localised in a finite region—can be solved in a closed form. The obtained Green's functions are applicable for representing fields using convolution integrals and the solution for a circular region is of practical interest for modelling circular transducers. In particular, this approach permits one to calculate the time-harmonic field radiated into laminate by an ultrasonic transducer of arbitrary aperture and then to evaluate the pulse train using harmonic synthesis.

Another formal question is the completeness of the guided waves. Normally, the total set of eigenfunctions of the polynomial operator pencil is multiply complete accordingly to its degree (see, e.g., Keldysh, 1971). Omitting a part of this set, this multiplicity can be reduced to an ordinary completeness, namely, in our case when choosing basic functions $B_n = H_n^{(1)}$ the subset with $\mathrm{Im}s_l < 0$ is excluded. The proof of the completeness obtained for the plane waves in an elastic homogeneous isotropic strip or in a cylinder in a functional Sobolev's space on a cross-section can be found in Kostyuchenko and Orazov (1975, 1977, 1986), Orazov (1976), Kirrmann (1995), Folk and Herszynski (1986) and Herszynski and Folk (1989). The same property is expected for 2D and 3D guided waves in laminates.

## Acknowledgement

This work was carried out during the course of research sponsored by EPSRC project GR/T22339/01.

## References

Abramovitz, M., Stegun, I.A., 1972. Handbook of Mathematical Functions. National Bureau of Standards, Washington.

Achenbach, J.D., 1998. Explicit solution for carrier waves supporting surface waves and plane waves. Wave Motion 28, 89–97.

Achenbach, J.D., Xu, Y., 1998. Use of elastodinamic reciprocity to analyze point-load generated axisymmetric waves in a plate. Wave Motion 30, 57–67.

Achenbach, J.D., Xu, Y., 1999. Wave motion in an isotropic elastic layer generated by time-harmonic load of arbitrary direction. Journal of the Acoustical Society of America 106, 83–90.

Achenbach, J.D., 2000. Calculation of wave fields using elastodinamic reciprocity. International Journal of Solids and Structures 37, 7043–7053.

Auld, A., Kino, G.S., 1971. Normal mode theory for acoustic waves and its application to interdigital transducer. IEEE Transactions on Electron Devices ED-18 (10), 898–908.

Auld, B.A., 1990. In: Acoustic Fields and Waves in Solids, Vol. 1–2. Krieger Inc, Malabar.

Bai, H., Zhu, J., Shah, A.H., Popplewell, N., 2004. Three dimensional steady state Green function for a layered isotropic plate. Journal of Sound and Vibration 269, 251–271.

Brekhovskikh, L.M., 1980. Waves in Layered Media. Academic Press, New York.

Bobrovnitskii, Y.I., 1973. Orthogonality relations for Lamb waves. Soviet Acoustical Physics 18 (4), 432–433.

Chimenti, D.E., 1997. Guided waves in plates and their use in material characterisation. Applied Mechanics Review 50, 247–284.

Christensen, R.M., 1971. Theory of Viscoelasticity: An Introduction. Academic Press, New York.

Fahmi, A.H., Adler, E.D., 1973. Propagation of acoustic waves in multilayers: a matrix description. Applied Physics Letters 20, 495–497.

Fedoryuk, M.V., 1974. Orthogonality-type relations in solid waveguides. Soviet Acoustical Physics 20 (2), 188–190.

Folk, R.T., Herszynski, A., 1986. Solutions of elastodynamic slab problems using a new orthogonality condition. Journal of the Acoustical Society of America 80 (4), 1103–1110.

Fraser, W.B., 1976. Orthogonality relations for Rayleigh-Lamb modes of vibration of a plate. Journal of the Acoustical Society of America 59, 215–216.

Flores-Lopez, M.A., Gregory, R.D., 2006. Scattering of Rayleigh-Lamb waves by a surface breaking crack in an elastic plate. Journal of the Acoustical Society of America 119 (4), 2041–2049.

Gregory, R.D., Gladwell, I., 1983. reflection of a symmetric Rayleigh-Lamb wave at a fixed or free edge of a plate. Journal of Elasticity 13, 185–206.

Herszynski, A., Folk, R.T., 1989. Orthogonality conditions for the Pochhammer-Chree modes. Quarterly Journal of Mechanics and Applied Mathematics 42 (4), 523–536.

Kasatkin, B.A., 1981. Normal mode diffraction by a compliant baffle resembling segregate in a plane. Soviet Acoustical Physics 27 (2), 135–137.

Keldysh, M.V., 1971. On the completeness of eigenfunctions of some classes of non self-adjoint linear operators. Russian Mathematical Surveys 26 (4), 15–44.

Kirrmann, P., 1995. On the completeness of Lamb modes. Journal of Elasticity 37, 39–69.

Knopoff, A.L., 1964. A matrix method for elastic wave problem. Bulletin of the Seismological Society of America 54 (1), 434–438.

Kostyuchenko, A.G., Orazov, M.B., 1975. Certain properties of roots of a self-adjoint quadratic bundles. Functional Analysis and its Applications 9 (4), 295–305.

Kostyuchenko, A.G., Orazov, M.B., 1977. The completeness of root vectors in certain self-adjoint quadratic bundles. Functional Analysis and its Applications 11 (4), 317–319.

Kostyuchenko, A.G., Orazov, M.B., 1986. Problem of oscillations of an elastic half-cylinder and related self-adjoint quadratic pencil. Journal of Mathematical Sciences 33 (3), 1025–1065.

Lawrie, J.B., Abrahams, I.D., 1999. An orthogonality relation for a class of problems with high-order boundary conditions; Applications in sound-structure interaction. Quarterly Journal of Mechanics and Applied Mathematics 52 (Part 2), 161–181.

Lih, S.S., Mal, A.K., 1996. Response of multilayered composite laminates to dynamic surface loads. Composites Part B Engineering 27B, 633–641.

Lowe, M.J.S., 1995. Matrix technique for modelling ultrasonic waves in multilayered media. IEEE Transactions on Ultrasonics, Ferroelectrics and Frequency Control 42, 525–542.

Nayfeh, A.H., 1995. Wave Propagation in Layered Anisotropic Media. North Holland, Amsterdam.

Orazov, M.B., 1976. The completeness of eigenvectors and associated vectors of a self-adjoint quadratic bundles. Functional Analysis and its Applications 10 (2), 153–155.

Pelts, S.P., Shikman, V.M., 1987. Rayleigh wave scattering by elastic half space coupled on the end with elastic half-plane. Doklady-Physics 292 (2), 299–303.

Prakash, B.G., 1978. Generalized orthogonality relations for rectangular strips in elastodynamics. Mechanics Research Communications 5 (4), 251–255.

Scandrett, C., Vasudevan, N., 1991. The propagation of time harmonic Rayleigh-Lamb waves in a bimaterial plate. Journal of the Acoustical Society of America 89 (4), 1606–1614.

Schwab, F., Knopoff, A.L., 1971. Surface waves in multilayered anelastic media. Bulletin of the Seismological Society of America 61 (4), 893–912.

Shkerdin, G., Glorieux, G., 2004. Lamb mode conversion in a plate with a delamination. Journal of the Acoustical Society of America 116, 2089–2100.

Shkerdin, G., Glorieux, G., 2006. Lamb mode conversion in an absorptive bi-layer with a delamination. Journal of the Acoustical Society of America 118, 2253–2264.

Slepyan, L.I., 1979. Betti theorem and orthogonality relations for eigenfunctions. Mechanics of Solids 1, 83–87.

Viktorov, I.A., 1967. Rayleigh and Lamb Waves: Physical Theory and Applications. Plenum, New York.

Zakharov, D.D., 1988. Generalised orthogonality relations for eigenfunctions in three dimensional dynamic problem for an elastic layer. Mechanics of Solids 6, 62–68.

Zilbergleit, A.S., Nuller, B.M., 1977. Generalized orthogonality of the homogeneous solutions to the dynamic problems of elasticity. Doklady–Physics 234 (2), 333–335.