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A monotone method for constructing extremal solutions to second-order periodic boundary value problems ☆

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Abstract

In this paper we describe a constructive method which yields two monotone sequences that converge uniformly to extremal solutions to the periodic boundary value problem

$$u''(t) = f(t, u, u'(t)), \quad t \in [0, 2\pi],$$
$$u(0) = u(2\pi), \quad u'(0) = u'(2\pi)$$

in the presence of an upper solution β and a lower solution α with $\beta \leqslant \alpha$. © 2001 Elsevier Science B.V. All rights reserved.

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1. Introduction and main result

In this paper, we study a second-order periodic boundary value problem by means of a monotone iterative technique. In order to stress its particularity, we consider the periodic boundary value problem of the form

$$\begin{aligned} &u''(t) = f(t, u, u'(t)), \quad t \in [0, 2\pi], \\ &u(0) = u(2\pi), \quad u'(0) = u'(2\pi), \end{aligned} \tag{1.1}$$

where $f(t, u, v)$ is a Caratheodory function.

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A function $f:[0,2\pi]\times\mathbb{R}^2\to\mathbb{R}$ is said to be a Caratheodory function if it possesses the following three properties

(i) For all $(u,v)\in\mathbb{R}^2$, the function $t\to f(t,u,v)$ is measurable on $[0,2\pi]$.
(ii) For almost all $t\in[0,2\pi]$, the function $(u,v)\to f(t,u,v)$ is continuous on $\mathbb{R}^2$.
(iii) For any given $N>0$, there exists $g_N(t)$, a Lebesgue integrable function defined on $[0,2\pi]$ such that

$$|f(t,u,v)|\leqslant g_N(t) \quad \text{for a.e. } t\in[0,2\pi],$$

whenever $|u|,|v|\leqslant N$.

To develop a monotone method, we need the concepts of upper and lower solutions. We say that $\beta\in W^{2,1}[0,2\pi]$ is an upper solution to problem (1.1), if it satisfies

$$\begin{aligned}&\beta''(t)\leqslant f(t,\beta(t),\beta'(t)), \quad t\in[0,2\pi],\\ &\beta(0)=\beta(2\pi), \quad \beta'(0)\leqslant\beta'(2\pi).\end{aligned} \tag{1.2}$$

Similarly, a function $\alpha\in W^{2,1}[0,2\pi]$ is said to be a lower solution to (1.1), if it satisfies

$$\begin{aligned}&\alpha''(t)\geqslant f(t,\alpha(t),\alpha'(t)), \quad t\in[0,2\pi],\\ &\alpha(0)=\alpha(2\pi), \quad \alpha'(0)\geqslant\alpha'(2\pi).\end{aligned} \tag{1.3}$$

We call a function $u\in W^{2,1}[0,2\pi]$ a solution to problem (1.1), if it is an upper and a lower solution to (1.1).

Under the classical assumption that $\alpha(t)\leqslant\beta(t)$, a number of authors have studied the existence of the methods of lower and upper solutions or the monotone iterative technique (see [1,3–6, 8–11,14,15]). Only a few have dealt with the case where $\alpha(t),\beta(t)$ satisfy the opposite ordering condition $\beta(t)\leqslant\alpha(t)$ (see [1,2,7,12,13,16,17]). Recently, Wang [16] investigated a special case of (1.1) (where $f(t,u,v)=-kv+F(t,u)$ and $F(t,u)$ is increasing with respect to u) in the presence of a lower solution $\alpha(t)$ and an upper solution $\beta(t)$ with $\beta(t)\leqslant\alpha(t)$. Moreover, Rachunkova [17] recently proved that problem (1.1) has at least one solution $u(t)$ under the case $\beta(t)\leqslant\alpha(t)$. However, the proof of the result in [17] is not constructive and is not able to guarantee that $u(t)$ satisfies $\beta(t)\leqslant u(t)\leqslant\alpha(t)$.

To develop a monotone method, we also need the following hypotheses:

(H1) For any given $\beta,\ \alpha\in C[0,2\pi]$ with $\beta(t)\leqslant\alpha(t)$ on $[0,2\pi]$, there exist $0<A\leqslant B$ such that

$$A(v_2-v_1)\leqslant f(t,u,v_2)-f(t,u,v_1)\leqslant B(v_2-v_1) \tag{1.4}$$

or

$$-B(v_2-v_1)\leqslant f(t,u,v_2)-f(t,u,v_1)\leqslant -A(v_2-v_1) \tag{1.4$'$}$$

for a.e. $t\in[0,2\pi]$ whenever $\beta(t)\leqslant u\leqslant\alpha(t),\ v_1,v_2\in R$, and $v_1\leqslant v_2$.

(H2) Inequality

$$f(t,u_2,v)-f(t,u_1,v)\geqslant -\frac{A^2}{4}(u_2-u_1)$$

holds for a.e. $t\in[0,2\pi]$, whenever $\beta(t)\leqslant u_1\leqslant u_2\leqslant\alpha(t),\ v\in\mathbb{R}$.

The purpose of this paper is to give the existence result of solution of (1.1) under the assumption that there exist a lower solution $\alpha(t)$ and an upper solution $\beta(t)$ of (1.1) with $\beta(t)\leqslant\alpha(t)$. We

develop the monotone iterative method to approximate the extremal solution of (1.1) and prove that the solution $u(t)$ of (1.1) satisfies $\beta(t) \leqslant u(t) \leqslant \alpha(t)$. Our result extends and complements those in [16,17].

The main result of this paper is stated as follows.

Theorem 1. *Suppose that there exists a lower solution $\alpha(t)$ and an upper solution $\beta(t)$ of* (1.1) *such that $\beta(t) \leqslant \alpha(t)$ on $[0, 2\pi]$, and $f(t,u,v)$ is a Caratheodory function satisfying hypotheses* (H1) *and* (H2). *Then there exist two sequences $\{\beta_j\}$ and $\{\alpha_j\}$, nondecreasing and nonincreasing, respectively, with $\beta_0 = \beta$ and $\alpha_0 = \alpha$, which converge uniformly and monotonically to the extremal solution to problem* (1.1) *in the segment*

$$[\beta, \alpha] := \{u \in C[0, 2\pi]: \beta(t) \leqslant u(t) \leqslant \alpha(t) \text{ on } [0, 2\pi]\}.$$

Section 2 is devoted to a maximum–minimum principle, which is the key to developing the monotone iterative technique. The proof of Theorem 1 is given in Section 3. An example for the forced Duffing equation is given in Section 4.

2. Maximum–minimum principle

To prove the validity of the monotone iterative technique, we use the following maximum–minimum principle.

Lemma 2. *Let $y \in W^{1,1}[0, 2\pi]$, and it satisfies*

$$y'(t) + My(t) + L|y(t)| \geqslant 0 \quad \text{for a.e. } t \in [0, 2\pi],$$

$$y(0) \geqslant y(2\pi),$$

where $|M| > L \geqslant 0$. Then $My(t) \geqslant 0$ on $[0, 2\pi]$, i.e., when $M > 0$ the minimum of $y(t)$ is nonnegative; when $M < 0$ the maximum of $y(t)$ is nonpositive.

Proof. Let $M > 0$. Suppose to the contrary that $y(t) < 0$ for some $t \in [0, 2\pi]$. It is enough to consider the following three cases.

Case (i): $y(t) < 0$ on $[0, 2\pi]$. In this case, we have

$$y'(t) \geqslant (L - M)y(t) > 0 \quad \text{for a.e. } t \in [0, 2\pi]$$

and hence, $y(0) < y(2\pi)$, which contradicts the fact that $y(0) \geqslant y(2\pi)$.

Case (ii): $y(2\pi) \geqslant 0$ and $y(t) < 0$ for some $t \in [0, 2\pi]$. Since $y(0) \geqslant y(2\pi \geqslant 0$, there exists an interval (a, b), $0 \leqslant a < b \leqslant 2\pi$, such that $y(t) < 0$ in (a, b) and $y(a) = y(b) = 0$. Therefore, we have

$$y'(t) \geqslant (L - M)y(t) > 0 \quad \text{for a.e. } t \in [a, b]$$

and hence, $y(a) < y(b)$, which is a contradiction.

Case (iii): $y(2\pi) < 0$ and $y(t) \geqslant 0$ for some $t < 2\pi$. In this case, there exists a point $a \in [0, 2\pi)$ such that $y(t) < 0$ in $(a, 2\pi]$ and $y(a) = 0$. As a result, we have

$$y'(t) \geqslant (L - M)y(t) > 0 \quad \text{for a.e. } t \in (a, 2\pi]$$

and hence, $y(a) < y(2\pi) < 0$, which is also a contradiction. This proves that $y(t) \geqslant 0$ on $[0, 2\pi]$ when $M > 0$.

In the same way, we can prove that $y(t) \leqslant 0$ on $[0, 2\pi]$ when $M < 0$. The proof of the lemma is complete. □

Lemma 2 is an improvement and extension of Lemma 1.2.2 of [6].

3. Proof of Theorem 1

Throughout this section, we assume that $\beta(t)$ and $\alpha(t)$ are upper and lower solutions to problem (1.1), respectively, $\beta(t) \leqslant \alpha(t)$ on $[0, 2\pi]$, and $f(t, u, v)$ is a Caratheodory function satisfying hypotheses (H1) and (H2).

We will only consider the case that (1.4′) holds, since the other case that (1.4) holds can be treated in a similar way.

For each given $\eta \in [\beta, \alpha]$, we consider a periodic boundary value problem of the form

$$\begin{aligned} &u''(t) + \frac{A+B}{2}u'(t) + \frac{AB}{4}u(t) = g\left(t, \eta(t), u'(t) + \frac{A}{2}u(t) - \frac{A}{2}\eta(t)\right) + \frac{AB}{4}\eta(t), \quad t \in [0, 2\pi], \\ &u(0) = u(2\pi), \quad u'(0) = u'(2\pi), \end{aligned} \tag{3.1}$$

where

$$g(t, u, v) = f(t, u, v) + \frac{A+B}{2}v. \tag{3.2}$$

Let $u(t)$ be a solution to (3.1) and $v(t) := u'(t) + (A/2)u(t)$. Then we get

$$\begin{aligned} &v'(t) + \frac{B}{2}v(t) = g\left(t, \eta(t), v(t) - \frac{A}{2}\eta(t)\right) + \frac{AB}{4}\eta(t), \quad t \in [0, 2\pi], \\ &v(0) = v(2\pi). \end{aligned} \tag{3.3}$$

It is easy to see that the above problem is equivalent to the integral equation

$$v(t) = \int_0^{2\pi} G_B(t, s)\left[g(s, \eta(s), v(s) - \frac{A}{2}\eta(s)) + \frac{AB}{4}\eta(s)\right] \mathrm{d}s, \tag{3.4}$$

where

$$G_B(t, s) := \begin{cases} \dfrac{\exp\{(B/2)(2\pi + s - t)\}}{\exp\{B\pi\} - 1}, & 0 \leqslant s \leqslant t \leqslant 2\pi, \\ \dfrac{\exp\{(B/2)(s - t)\}}{\exp\{B\pi\} - 1}, & 0 \leqslant t \leqslant s \leqslant 2\pi. \end{cases}$$

Let $v(t)$ be a solution to problem (3.4). Then

$$u(t) := \int_0^{2\pi} G_A(t, s)v(s)\,\mathrm{d}s, \quad t \in [0, 2\pi], \tag{3.5}$$

is certainly a solution to problem (3.1).

Concerning problem (3.1), we can prove the following statement.

Lemma 3. *For each fixed* $\eta \in [\beta, \alpha]$, *problem* (3.1) *has a unique solution* $u \in W^{2,1}[0, 2\pi]$.

Proof. Define the mapping $T : C[0, 2\pi] \to C[0, 2\pi]$ by

$$(Tv)(t) := \int_0^{2\pi} G_B(t,s) \left[g\left(s, \eta(s), v(s) - \frac{A}{2}\eta(s)\right) + \frac{AB}{4}\eta(s) \right] \mathrm{d}s.$$

For $v \in C[0, 2\pi]$, its norm is defined by

$$\|v\| := \sup\{|v(t)|: 0 \leqslant t \leqslant 2\pi\}.$$

By virtue of (H1) and (3.2), we have, for any $v_1, v_2 \in C[0, 2\pi]$

$$|g(t, \eta, v_2) - g(t, \eta, v_1)| \leqslant \frac{B - A}{2}|v_2 - v_1|. \tag{3.6}$$

Hence, we have

$$\begin{aligned} |(Tv_1)(t) - (Tv_2)(t)| &\leqslant \int_0^{2\pi} G_B(t,s) \left| g\left(s, \eta(s), v_1(s) - \frac{A}{2}\eta(s)\right) \right. \\ &\quad \left. - g\left(s, \eta(s), v_2(s) - \frac{A}{2}\eta(s)\right) \right| \mathrm{d}s \\ &\leqslant \frac{B - A}{2}\|v_1 - v_2\| \int_0^{2\pi} |G_B(t,s)| \, \mathrm{d}s \\ &= \frac{B - A}{B}\|v_1 - v_2\| \quad \text{for all } t \in [0, 2\pi], \end{aligned}$$

i.e.,

$$\|Tv_1 - Tv_2\| \leqslant \frac{B - A}{B}\|v_1 - v_2\|,$$

which shows that T is a contraction mapping. The Banach contraction principle tells us that T has a unique fixed point in $C[0, 2\pi]$. Let $v(t)$ be the unique fixed point. Then it satisfies the integral equation (3.4). Put

$$u(t) := \int_0^{2\pi} G_A(t,s) v(s) \, \mathrm{d}s, \quad t \in [0, 2\pi]. \tag{3.7}$$

It is easy to show that the function $u(t)$ defined by (3.7) is a unique solution to (3.1). The lemma is proved. □

We now define a mapping $\Phi : [\beta, \alpha] \to W^{2,1}[0, 2\pi]$ by setting $(\Phi\eta) := u(t)$, where $u(t)$ is the unique solution to problem (3.1) with given $\eta \in [\beta, \alpha]$. It then follows by Lemma 3 that the mapping Φ is well defined.

Concerning the mapping Φ, the following statement holds.

Lemma 4. *The mapping* Φ *has the following properties*:

(i) $\beta(t) \leqslant (\Phi\beta)(t), (\Phi\alpha)(t) \leqslant \alpha(t)$ *on* $[0, 2\pi]$;
(ii) Φ *is monotone increasing mapping on the segment* $[\beta, \alpha]$, *namely*, $(\Phi\eta_1)(t) \leqslant (\Phi\eta_2)(t)$ *on* $[0, 2\pi]$ *when* $\eta_1, \eta_2 \in [\beta, \alpha]$ *and* $\eta_1(t) \leqslant \eta_2(t)$ *on* $[0, 2\pi]$.

Proof. We now prove assertion (i). Set

$$\beta_1(t) := (\Phi\beta)(t),$$
$$x(t) := \beta_1(t) - \beta(t),$$
$$y(t) := x'(t) + \frac{A}{2}x(t).$$

As follows from (3.6), we have

$$\begin{aligned} x''(t) + \frac{A+B}{2}x'(t) + \frac{AB}{4}x(t) &\geqslant g\left(t, \beta(t), \beta_1'(t) + \frac{A}{2}\beta_1(t) - \frac{A}{2}\beta(t)\right) - g(t, \beta(t), \beta'(t)) \\ &\geqslant -\frac{B-A}{2}\left|x'(t) + \frac{A}{2}x(t)\right|, \\ x(0) = x(2\pi), \quad x'(0) &\geqslant x'(2\pi), \end{aligned}$$

i.e.,

$$y'(t) + \frac{B}{2}y(t) + \frac{B-A}{2}|y(t)| \geqslant 0 \quad \text{for a.e. } t \in [0, 2\pi],$$
$$y(0) \geqslant y(2\pi).$$

(Here we have used (1.2).) Applying Lemma 2, we conclude that $y(t) \geqslant 0$ on $[0, 2\pi]$. Thus, we have

$$x'(t) + \frac{A}{2}x(t) \geqslant 0 \quad \text{on } [0, 2\pi],$$
$$x(0) \geqslant x(2\pi).$$

Again applying Lemma 2, we get $x(t) \geqslant 0$ on $[0, 2\pi]$, namely, $(\Phi\beta)(t) \geqslant \beta(t)$ on $[0, 2\pi]$. A similar argument shows that $\alpha(t) \geqslant (\Phi\alpha)(t)$ on $[0, 2\pi]$. This proves assertion (i).

Next, we prove assertion (ii). Let,

$$u_j(t) = (\Phi\eta_j)(t), \quad j = 1, 2,$$
$$x(t) := u_2(t) - u_1(t),$$
$$y(t) := x'(t) + \frac{A}{2}x(t),$$

where $\eta_1, \eta_2 \in [\beta, \alpha]$ and $\eta_1(t) \leqslant \eta_2(t)$ on $[0, 2\pi]$. As follows from (3.6) and (H2), we have

$$\begin{aligned} x''(t) + \frac{A+B}{2}x'(t) + \frac{AB}{4}x(t) &= g\left(t, \eta_2(t), u_2'(t) + \frac{A}{2}u_2(t) - \frac{A}{2}\eta_2(t)\right) \\ &\quad - g\left(t, \eta_1(t), u_1'(t) + \frac{A}{2}u_1(t) - \frac{A}{2}\eta_1(t)\right) + \frac{AB}{4}(\eta_2(t) - \eta_1(t)) \\ &\geqslant -\frac{A^2}{4}(\eta_2(t) - \eta_1(t)) + \frac{AB}{4}(\eta_2(t) - \eta_1(t)) \\ &\quad - \frac{B-A}{2}\left|x'(t) + \frac{A}{2}x(t) - \frac{A}{2}(\eta_2(t) - \eta_1(t))\right| \end{aligned}$$

$$\geqslant \left(\frac{AB}{4} - \frac{A^2}{4}\right)(\eta_2 - \eta_1) - \frac{A}{2}\frac{B-A}{2}(\eta_2 - \eta_1)$$
$$-\frac{B-A}{2}\left|x'(t) + \frac{A}{2}x(t)\right|$$
$$\geqslant -\frac{B-A}{2}\left|x'(t) + \frac{A}{2}x(t)\right| \quad \text{for a.e. } t \in [0, 2\pi],$$

that is

$$y'(t) + \frac{B}{2}y(t) + \frac{B-A}{2}|y(t)| \geqslant 0 \quad \text{for a.e. } t \in [0, 2\pi],$$
$$y(0) = y(2\pi).$$

Applying Lemma 2 again as before, we get $x(t) \geqslant 0$ on $[0, 2\pi]$. This shows that $(\Phi\eta_2)(t) \geqslant (\Phi\eta_1)(t)$ on $[0, 2\pi]$ when $\eta_1, \eta_2 \in [\beta, \alpha]$ and $\eta_2(t) \geqslant \eta_1(t)$ on $[0, 2\pi]$. Assertion (ii) is thus proved. □

Let us define sequences $\{\beta_j\}$ and $\{\alpha_j\}$ such that

$$\beta_{j+1}(t) := (\Phi\beta_j)(t) \quad \text{and} \quad \alpha_{j+1}(t) := (\Phi\alpha_j)(t)$$

with $\beta_0 = \beta$, $\alpha_0 = \alpha$. From Lemma 4, we conclude that $\{\beta_j\}$ is nondecreasing and $\{\alpha_j\}$ nonincreasing.

From (3.1), we know that

$$\beta''_{n+1}(t) + \frac{A+B}{2}\beta'_{n+1}(t) + \frac{AB}{4}\beta_{n+1}(t) = g\left(t, \beta_n(t), \beta'_{n+1}(t) + \frac{A}{2}\beta_{n+1}(t) - \frac{A}{2}\beta_n(t)\right)$$
$$+\frac{AB}{4}\beta_n(t), \quad t \in [0, 2\pi],$$

$$\beta_{n+1}(0) = \beta_{n+1}(2\pi), \quad \beta'_{n+1}(0) = \beta'_{n+1}(2\pi). \tag{3.8}$$

Let $v_n(t) := \beta'_n(t) + (A/2)\beta_n(t)$. Then we get

$$v_{n+1}(t) = \int_0^{2\pi} G_B(t,s)\left[g\left(s, \beta_n(s), v_{n+1}(s) - \frac{A}{2}\beta_n(s)\right) + \frac{AB}{4}\beta_n(s)\right] \mathrm{d}s. \tag{3.9}$$

By (3.6), we have

$$|v_{n+1}(t)| \leqslant \int_0^{2\pi} G_B(t,s)\left[\left|g\left(s, \beta_n(s), v_{n+1}(s) - \frac{A}{2}\beta_n(s)\right)\right.\right.$$
$$\left.\left. -g\left(s, \beta_n(s), -\frac{A}{2}\beta_n(s)\right)\right| + \left|g\left(s, \beta_n(s), -\frac{A}{2}\beta_n(s)\right)\right| + \left|\frac{AB}{4}\beta_n(s)\right|\right] \mathrm{d}s$$
$$\leqslant \frac{B-A}{2}\|v_{n+1}\| \int_0^{2\pi} |G_B(t,s)|\,\mathrm{d}s + C_1$$
$$= \frac{B-A}{B}\|v_{n+1}\| + C_1 \quad \text{for all } t \in [0, 2\pi]. \tag{3.10}$$

Here we have used the facts that f is a Caratheodory function and

$$|\beta_n(t)| \leqslant \max\{|\alpha_0(t) + |\beta_0(t)|: t \in [0, 2\pi]\}.$$

So, we get

$$\|v_{n+1}\| \leqslant C_2. \tag{3.11}$$

It shows that

$$|\beta'_{n+1}(t)| \leqslant C_3. \tag{3.12}$$

In a similar way, we have

$$|\alpha'_{n+1}(t)| \leqslant C_4. \tag{3.13}$$

It then follows by a standard argument (see, e.g., [6]) that

$$\lim_{j\to\infty} \beta_j(t) := \beta^*(t) \quad \text{and} \quad \lim_{j\to\infty} \alpha_j(t) := \alpha^*(t)$$

uniformly and monotonically on $[0, 2\pi]$. From the integral representations (3.4) and (3.5), we concluded that $\beta^*(t)$ and $\alpha^*(t)$ are both solutions to problem (3.1), and hence to problem (1.1).

Furthermore, if $u \in [\beta, \alpha]$ is a solution to the problem (1.1), then, by induction, $\beta_j(t) \leqslant u(t) \leqslant \alpha_j(t)$ on $[0, 2\pi]$, $j = 0, 1, 2, \ldots$, and hence, $u \in [\beta^*, \alpha^*]$. This shows that $\beta^*(t)$ and $\alpha^*(t)$ are, respectively, minimal and maximal solutions to problem (1.1) in the segment $[\beta, \alpha]$.

The proof of Theorem 1 is complete. □

4. An example

In this section, we consider the periodic boundary value problem for the forced Duffing equation

$$u''(t) + ku'(t) = F(t, u), \quad t \in [0, 2\pi],$$

$$u(0) = u(2\pi), \quad u'(0) = u'(2\pi), \tag{4.1}$$

where $F(t, u)$ is a Caratheodory function, $k > 0$ or $k < 0$.

We say that $\beta \in W^{2,1}[0, 2\pi]$ is an upper solution to problem (4.1), if it satisfies

$$\beta''(t) + k\beta'(t) \leqslant F(t, \beta(t)), \quad t \in [0, 2\pi],$$

$$\beta(0) = \beta(2\pi), \quad \beta'(0) \leqslant \beta'(2\pi). \tag{4.2}$$

Similarly, a function $\alpha \in W^{2,1}[0, 2\pi]$ is said to be a lower solution to (4.1), if it satisfies

$$\alpha''(t) + k\alpha'(t) \geqslant F(t, \alpha(t)), \quad t \in [0, 2\pi],$$

$$\alpha(0) = \alpha(2\pi), \quad \alpha'(0) \geqslant \alpha'(2\pi). \tag{4.3}$$

To develop a monotone method, we also need the following hypothesis:

(H): For any given β, $\alpha \in C[0, 2\pi]$ with $\beta(t) \leqslant \alpha(t)$ on $[0, 2\pi]$, inequality

$$F(t, u_2) - F(t, u_1) \geqslant -\frac{k^2}{4}(u_2 - u_1)$$

holds for a.e. $t \in [0, 2\pi]$, whenever $\beta(t) \leqslant u_1 \leqslant u_2 \leqslant \alpha(t)$.

Let $A = B = k$, then (H1) and (H2) hold. Hence the conclusions of Theorem 1 hold.

Our results generalize and improve the results obtained in [16].

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