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Inverse images of polynomial mappings and polynomials orthogonal on them

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Abstract

Let $\mathcal{T}$ be a polynomial with complex coefficients. First, we study the inverse images of the real and imaginary axes under a polynomial mapping $\mathcal{T}$ in detail. Then for an arbitrary polynomial ρ and a sequence (p_n) of orthogonal polynomials the orthogonality behaviour of the sequence of polynomials $(\rho(p_n \circ \mathcal{T}))_{n\in\mathbb{N}}$ is investigated. In particular necessary and sufficient conditions are given such that $(\rho(p_n \circ \mathcal{T}))_{n\in\mathbb{N}}$ is a subsequence of polynomials orthogonal with respect to a positive measure supported on a compact subset of the real line.

Keywords: Orthogonal polynomials; Polynomial mappings; Inverse images; Functionals; Positive measure; Positive definite

1. Introduction

Let $K \subset \mathbb{C}$ be a compact set and let $\mathcal{T}$ be a polynomial with complex coefficients. Let p be a polynomial which is extremal on K in some sense, for instance a polynomial which deviates least from zero with respect to a weight function and the L_p-norm among all polynomials of degree n with leading coefficient one. Then it is natural to expect that this extremal property is inherited to $p \circ \mathcal{T}$ on the set $\mathcal{T}^{-1}(K)$ with respect to the transformed weight function, i.e., in the case of the L_2-norm that $p \circ \mathcal{T}$ is orthogonal on $\mathcal{T}^{-1}(K)$ with respect to the transformed weight function. In fact its known nowadays that such an inherity property holds (see [4,5,8,11,14,20,26]). In most cases special polynomial mappings, i.e., real polynomial mappings with properties which guarantee that all inverse images are real intervals, have been considered. With respect to orthogonal polynomials such mappings have been considered in [5,11] and, interesting enough, arose at about the same time

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in a completely different way as the so-called T-polynomials in the study of extremal polynomials on several intervals with respect to the max-norm by the author [20,23,22]. A polynomial $\mathcal{T}$ is called a T-polynomial on a set E of real intervals if $\mathcal{T}$ has simple real zeros only and $E \subseteq \{x \in \mathbb{R}: |\mathcal{T}(x)| \leqslant M\}$, where $M = \min\{|\mathcal{T}(y)|: \mathcal{T}'(y) = 0\}$. The notion T-polynomial on E, which is an abbreviation for Chebyshev polynomial on E, has been chosen since it shares so many properties with the classical Chebyshev polynomial $T_n(x) = \cos(n \arccos x)$. Recently, it has been proved in three different ways [6,24,28] that a set of real intervals can be approximated arbitrary well by the inverse images of $[-1, 1]$ of a T-polynomial or in other words by a special polynomial mapping. Therefore, it is not astonishing that certain properties are inherited not only to inverse images but also to arbitrary sets of real intervals, see [27,28].

But let us return to orthogonal polynomials associated with a polynomial mapping. Marcellan and his collaborators [15–17] considered orthogonal polynomials associated with polynomial mappings of degree less or equal three. More precisely, among others they studied and solved for such polynomial mappings the following question: Let (p_n) be a sequence of polynomials orthogonal with respect to a positive measure μ with $\text{supp}(\mu) \subset \mathbb{R}$ and let ρ be a polynomial whose degree is less than the degree of $\mathcal{T}$. Under which conditions on ρ and the polynomial mapping $\mathcal{T}$ is the subsequence of polynomials $(\rho(p_n \circ \mathcal{T}))_{n \in \mathbb{N}}$ orthogonal to a positive measure supported on a subset of $\mathbb{R}$. In particular orthogonal polynomials whose recurrence coefficients are asymptotically periodic are closely related to polynomial mappings; for instance this is reflected by the fact, see [12] and also [18,2,3] for the complex case, that the essential spectrum of an Jacobi operator associated with a symmetric tridiagonal matrix whose elements are asymptotically periodic is the inverse image of the interval $[-1, 1]$ under a polynomial mapping.

In this paper, we investigate first the inverse images of the real and imaginary axes resp. of real and imaginary intervals under a polynomial mapping $\mathcal{T}$ with complex coefficients in detail which is of interest also in its own. Then sequences of polynomials $(\rho(p_n \circ \mathcal{T}))_{n \in \mathbb{N}}$ where ρ is an arbitrary polynomial and (p_n) is a sequence of polynomials orthogonal with respect to a definite functional, are studied with respect to their orthogonality behaviour. Finally, it is shown that the sequence $(\rho(p_n \circ \mathcal{T}))_{n \in \mathbb{N}}$, where (p_n) is a sequence of polynomials orthogonal with respect to a positive measure with support $[-1, 1]$, is orthogonal with respect to a positive measure supported on a compact subset of the real line if and only if $\mathcal{T}$ is a T-polynomial and ρ satisfies certain conditions.

2. Inverse polynomial images of real and imaginary intervals

We first investigate the following sets

$$Z_I := \{z \in \mathbb{C}: \ \text{Im}\, \mathcal{T}(z) = 0\} = \bigcup_{j=1}^{N} \mathcal{T}_j^{-1}(\mathbb{R}) = \mathcal{T}^{-1}(\mathbb{R}) \tag{2.1}$$

and

$$Z_R := \{z \in \mathbb{C}: \ \text{Re}\, \mathcal{T}(z) = 0\} = \bigcup_{j=1}^{N} \mathcal{T}_j^{-1}(\text{i}\mathbb{R}) = \mathcal{T}^{-1}(\text{i}\mathbb{R}), \tag{2.2}$$

which have already been studied by Gauß [9] in proving that every polynomial of degree N has N zeros in the complex plane, sometimes called the *Main Theorem in Algebra*, and then thoroughly by Ostrovski [19]. Let us divide the complex plane in $4N$ angle sections by the half-rays $se^{ij\pi/N}$, $j=-1,1,3,\ldots,8N-3$, i.e., let us set for $k=1,\ldots,4N$

$$I_k = \left(\frac{2k-3}{4N}\,\pi, \frac{2k-1}{4N}\,\pi\right)$$

and for $r>0$

$$\Gamma_{r,k} = \{se^{i\varphi}\colon\ s \geqslant r, \varphi \in I_k\}.$$

Considering the polar coordinate representation of $\mathscr{T}(x,y)$, i.e.,

$$\begin{aligned}\mathscr{T}(r,\varphi) &:= \mathscr{T}(r\cos\varphi + ir\sin\varphi)\\ &= r^N(\cos N\varphi + i\sin N\varphi) + \sum_{j=0}^{N-1} r^j(a_j\cos j\varphi + ib_j\sin j\varphi),\end{aligned}$$

$c_j = a_j + ib_j$, and observing that for $r > r_0$, where $r_0 := \max\{1, \sqrt{2}\sum_{j=0}^{N-1}|c_j|\}$, the following inequalities hold

$$|\mathrm{Re}\,\mathscr{T}(r,\varphi) - r^N\cos N\varphi| \leqslant r^{N-1}\sum_{j=0}^{N-1}|c_j| < \frac{r^N}{\sqrt{2}} \tag{2.3}$$

and

$$\left|\frac{\partial}{\partial\varphi}(\mathrm{Im}\,\mathscr{T}(r,\varphi)) - Nr^N\cos N\varphi\right| \leqslant Nr^{N-1}\sum_{j=0}^{N-1}|c_j| < N\,\frac{r^N}{\sqrt{2}}, \tag{2.4}$$

we conclude, since on I_{4k+1} (I_{4k+3}), $k=0,\ldots,N-1$, $\cos N\varphi > 0$ (<0) and $|\cos N\varphi| \geqslant 1/\sqrt{2}$, that for $r > r_0$

$$\mathrm{Re}\,\mathscr{T}(r,\varphi) = \begin{cases} >0 & \text{on } I_{4k+1},\\ <0 & \text{on } I_{4k+3}\end{cases} \tag{2.5}$$

and

$$\frac{\partial}{\partial\varphi}(\mathrm{Im}\,\mathscr{T}(r,\varphi)) = \begin{cases} >0 & \text{on } I_{4k+1},\\ <0 & \text{on } I_{4k+3}.\end{cases} \tag{2.6}$$

For the second relation we have taken into consideration the facts that $\sin N\varphi$ is strictly increasing (decreasing) on I_{4k+1} (I_{4k+3}) and has exactly one zero there. Analogously, it follows, since on I_{4k+2} (I_{4k}) $\sin N\varphi > 0$ (<0) and $|\sin N\varphi| \geqslant 1/\sqrt{2}$, that for $r \geqslant r_0$

$$\mathrm{Im}\,\mathscr{T}(r,\varphi) = \begin{cases} >0 & \text{on } I_{4k+2},\\ <0 & \text{on } I_{4k}\end{cases} \tag{2.7}$$

and

$$\frac{\partial}{\partial\varphi}(\mathrm{Re}\,\mathscr{T}(r,\varphi)) = \begin{cases} <0 & \text{on } I_{4k+2}, \\ >0 & \text{on } I_{4k}. \end{cases} \tag{2.8}$$

Hence, for every $r \geqslant r_0$ there are $\varphi_k(r) \in I_{4k+1}$, $k = 1,\dots,N$, and $\psi_k(r) \in I_{4k+3}$, $k = 1,\dots,N$, such that $\mathrm{Im}\,\mathscr{T}(r,\varphi_k(r)) = 0$ and $\mathrm{Im}\,\mathscr{T}(r,\psi_k(r)) = 0$ for $k = 1,\dots,N$. By continuity-arguments or by the Implicit Function Theorem it follows that $Z_I \cap \{z \in \mathbb{C}\colon |z| \geqslant r_0\}$ consists of $2N$ arcs C_k and $\tilde{C}_k$ with $C_k \subset \Gamma_{r_0,4k+1}$ and $\tilde{C}_k \subset \Gamma_{r_0,4k+3}$ for $k = 1,\dots,N$. Now it can be shown (see [19]) that these $2N$ arcs continue in the interior of $|z| \leqslant r_0$ in such a way that two arcs C_k and $\tilde{C}_k$ join each other to one arc. Thus, Z_I consists of N arcs $\mathscr{C}_j$, $j = 1,\dots,N$. More precisely, it is always possible to choose the N arcs $\mathscr{C}_j$ in such a way that at the right-hand side of $\mathscr{C}_j$, $j = 1,\dots,N$, there is always $\mathrm{Im}\,\mathscr{T}(x,y) > 0$ if one moves along $\mathscr{C}_j$, where at a point at which some arcs $\mathscr{C}_j$ cross each other one has to take the next arc at the right-hand side. Thus, every arc $\mathscr{C}_j$, $j = 1,\dots,N$, comes from ∞, enters the circle $|z| = r_0$ in the sector $\Gamma_{r_0,4j+1}$, continues to the interior of $|z| \leqslant r_0$, leaves then the circle $|z| = r_0$ through $\Gamma_{r_0,4j+3}$ and continues to ∞. By the way, with the help of this approach Gauß [9] proved his famous theorem using the fact that $\mathscr{T}$ has obviously a zero on each $\mathscr{C}_j$.

Next, let us summing up the above facts and let us show that $\mathrm{Re}\,\mathscr{T}(x,y)$ is strictly monotone decreasing on each $\mathscr{C}_j$ which is important in what follows.

Theorem 2.1. *Z_I consists of N arcs $\mathscr{C}_j$, $j = 1,\dots,N$, running from infinity to infinity, which can be chosen such that at the right-hand side of each $\mathscr{C}_j$, $j = 1,\dots,N$,* $\mathrm{Im}\,\mathscr{T}(x,y) > 0$. *If the $\mathscr{C}_j$ are chosen in this way then* $\mathrm{Re}\,\mathscr{T}(x,y)$ *is strictly decreasing from ∞ to $-\infty$ on each $\mathscr{C}_j$, $j = 1,\dots,N$.*

Proof. In view of what had been said above only the statement on the monotonicity of $\mathrm{Re}\,\mathscr{T}(x,y)$ remains to be shown. For abbreviation let

$$u(x,y) = \mathrm{Re}\,\mathscr{T}(x,y) \quad \text{and} \quad v(x,y) = \mathrm{Im}\,\mathscr{T}(x,y)$$

and let $(x(s),y(s))$, $s \in (-\infty,\infty)$, be a parametrization of $\mathscr{C}_j$, $j = 1,\dots,N$.

If $(\mathrm{d}/\mathrm{d}s)(x(s),y(s))$ has no zero on $(-\infty,\infty)$ then the assertion follows immediately by recalling the fact that $\lim_{r\to\infty} u(r,\varphi)$ tends to $+\infty$ $(-\infty)$ if $\varphi \in I_{4k+1}$ (I_{4k+3}).

Next we claim: If

$$\left.\frac{\mathrm{d}}{\mathrm{d}s}u(x(s),y(s))\right|_{s=s_0} = 0$$

then $\mathscr{T}'(z_0) = 0$, where $z_0 = (x_0,y_0) := (x(s_0),y(s_0))$. Since

$$\frac{\mathrm{d}}{\mathrm{d}s}u(x(s),y(s)) = \frac{\partial u}{\partial x}(x(s),y(s))x'(s) + \frac{\partial u}{\partial y}(x(s),y(s))y'(s) = 0 \quad \text{at } s = s_0 \tag{2.9}$$

and since $v(x(s),y(s)) = 0$ we obtain by differentiation and by using the Cauchy–Riemann differential equation that for $s \in (-\infty,\infty)$

$$-\frac{\partial u}{\partial y}(x(s),y(s))x'(s) + \frac{\partial u}{\partial x}(x(s),y(s))y'(s) = 0. \tag{2.10}$$

Now assume that

$$\mathcal{T}'(z_0) = \frac{\partial u}{\partial x}(x(s_0), y(s_0)) - \mathrm{i}\,\frac{\partial u}{\partial y}(x(s_0), y(s_0)) \neq 0,$$

i.e. $(\partial u/\partial x)(x(s_0), y(s_0)) \neq 0$, or $(\partial u/\partial y)(x(s_0), y(s_0)) \neq 0$, then by (2.9) and (2.10)

$$\left(\frac{\partial u}{\partial x}(x(s_0), y(s_0))\right)^2 + \left(\frac{\partial u}{\partial y}(x(s_0), y(s_0))\right)^2 = 0,$$

which is a contradiction.

Now let us suppose that $\mathcal{T}^{(j)}(z_0)=0$ for $j=1,\ldots,m$, and $\mathcal{T}^{(m+1)}(z_0) \neq 0$. Then it is well known that in a neighbourhood of z_0 the locus $v(x, y) = v(x(s_0), y(s_0)) = 0$ consists of $m+1$ analytic arcs through z_0 cutting each other at z_0 in successive angles $\pi/(m+1)$ and bisecting the angles between the successive arcs of the locus $u(x, y) = u(x(s_0), y(s_0))$.

Furthermore, the neighbourhood of z_0 is divided in $u(x, y) > u(x_0, y_0)$ and $u(x, y) < u(x_0, y_0)$, respectively. Thus, (x_0, y_0) is a saddle point of v and u. Moreover, neither v nor u can have a local maximum or minimum at (x_0, y_0). Thus, since the $\mathscr{C}_j$'s have been chosen such that if we move along $\mathscr{C}_j$ to z_0 at z_0 we have to take the next arc at the right-hand side, i.e., we do not cross the saddle point z_0, $u(x, y)$ does not change its monotonicity behaviour along the curve $\mathscr{C}_j$. □

Corollary 2.2. *Let $\mathcal{T}(z) = c_N z^N + \cdots$, $c_N \in \mathbb{C}\backslash\{0\}$, be an arbitrary complex polynomial of degree N and let $\mathcal{T}_j^{-1}$, $j = 1,\ldots,N$, be the inverse functions. Then the following propositions hold*:

(a)

$$\bigcup_{j=1}^{N} \mathcal{T}_j^{-1}([-1,1]) = \{z \in \mathbb{C}\colon\ \operatorname{Im}\mathcal{T}(z) = 0 \ \textit{and}\ |\operatorname{Im}\mathcal{T}(z)| \leqslant 1\}$$

consists of N arcs $\mathfrak{C}_j$, $j = 1,\ldots,N$, of finite length which can be chosen such that moving along $\mathfrak{C}_j$ we have at the right-hand side of each $\mathfrak{C}_j$ that $\operatorname{Im}\mathcal{T}(x, y) > 0$.

(b) *Suppose that $\mathcal{T}^2(z) - 1$ and $\mathcal{T}'(z)$ have no common zero and let w_j, $j = 1,\ldots,m$, be the zeros of $\mathcal{T}'$ of multiplicity m_j which are in $\bigcup_{j=1}^{n} \mathcal{T}_j^{-1}([-1,1])$. Then for sufficiently small ρ, $\rho > 0$, the set of level lines*

$$A(\mathcal{T}, \rho) = \{z \in \mathbb{C}\colon\ \log|\mathcal{T}(z) + \sqrt{\mathcal{T}^2(z) - 1}| = \rho\}$$

consists of at most $n - \sum_{j=1}^{m} m_j$ simple closed curves $\gamma_v(\rho)$ and if $\gamma_v(\rho)$ surrounds exactly one zero w_j of $\mathcal{T}'$ of multiplicity m_j, then moving along $\gamma_v(\rho)$ $\mathcal{T}$ takes on successively in the neighbourhood of the endpoints of the arcs the values $1 + \varepsilon_1(\rho), -1 + \varepsilon_2(\rho), 1 + \varepsilon_3(\rho), \ldots, -1 + \varepsilon_{m_j+1}(\rho)$ with $\lim_{\rho\to 1} \varepsilon_k(\rho) = 0$.

Proof. Part (a) follows immediately from Theorem 2.1.

(b) Obviously, from each arc $\mathfrak{C}_v$ which does not contain a zero of $\mathcal{T}'$ there arises a simple closed curve $\gamma_v(\rho)$ with $\lim_{\rho\to 0}\gamma_v(\rho) = \mathfrak{C}_v$. At a zero w_j of $\mathcal{T}'$ of multiplicity m_j, $m_j + 1$ of the arcs $\mathfrak{C}_v$ touch each other (recall the choice of the arcs $\mathfrak{C}_v$), where w_j is contained in the interior of each $\mathfrak{C}_v$, since by assumption $|\operatorname{Re}\mathcal{T}(w_j)| < 1$. Hence, if $\bigcup_{j=1}^{m_j+1} \mathfrak{C}_{v_j}$ does not contain another zero w_j in its

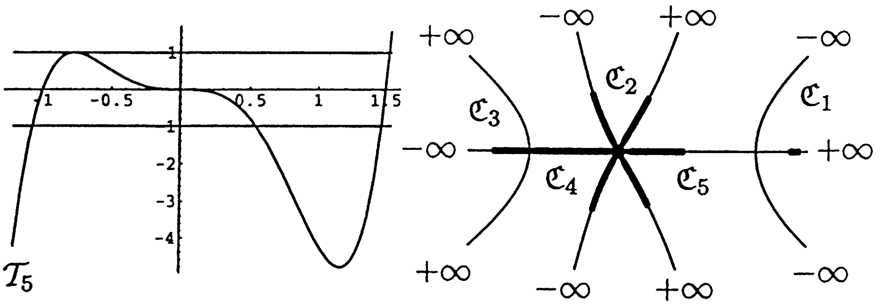

Fig. 1. The above pictures show the polynomial $\mathcal{T}_5(z)=az^3(z-3/2)(z+1)$, with $a=25{,}000/(1786\sqrt{94}-11{,}357)$, and the arcs $\mathfrak{C}_j$ from Corollary 2.2, i.e., on which $\mathcal{T}_5$ is real and $|\mathcal{T}_5|\leqslant 1$, are marked boldface. $\pm\infty$ gives the value to which the polynomial tends if we move along $\mathfrak{C}_j$.

interior, the level set which arise from $\bigcup_{j=1}^{m_j+1}\mathfrak{C}_{v_j}$ is a simple closed curve. Since by Theorem 2.1 $\operatorname{Re}\mathcal{T}$ decreases from 1 to -1 along each arc $\mathfrak{C}_v$ the assertion follows by continuity arguments. $\square$

In Fig. 1, Theorem 2.1 and Corollary 2.2 are demonstrated for the polynomial $\mathcal{T}_5(z)=az^3(z-3/2)(z+1)$, $a=25{,}000/(1786\sqrt{94}-11{,}357)$.

To get the link with [23] let us observe that $\bigcup_{j=1}^{N}\mathcal{T}_j^{-1}([-1,1])$ consists of l arcs with endpoints a_j, $j=1,\dots,2l$, if and only if there exists a polynomial $\mathcal{U}$ such that

$$\mathcal{T}^2(z)-H(z)\mathcal{U}^2(z)=1, \tag{2.11}$$

where $H(z)=\prod_{j=1}^{2l}(z-a_j)$. A point a_j is called an endpoint if there exists an $m\in\{0,1,2,\dots\}$ such that $(\mathcal{T}\pm 1)^{(k)}(a_j)=0$ for $k=0,\dots,2m$. Let us mention that in [25] the polynomial mappings whose inverse images of $[-1,1]$ consist of two arcs only have been characterized with the help of elliptic functions. Next let us turn to the description of the polynomial mappings whose inverse images are real intervals only.

Corollary 2.3. *Let $\mathcal{T}(x)=c_Nx^N+\cdots$, $c_N\in\mathbb{C}\setminus\{0\}$, be a polynomial. Then*

$$\bigcup_{j=1}^{N}\mathcal{T}_j^{-1}([-1,1])\subset\mathbb{R}$$

if and only if all coefficients of $\mathcal{T}$ are real, $\mathcal{T}$ has N simple real zeros and $\min\{|\mathcal{T}(y)|\colon \mathcal{T}'(y)=0\}\geqslant 1$.

Proof. *Necessity*: Since

$$\bigcup_{j=1}^{N}\mathcal{T}_j^{-1}([-1,1])=\{z\in\mathbb{C}\colon \mathcal{T}(z)\in[-1,1]\}\subseteq\mathbb{R},$$

it follows that for every $\kappa\in(-1,1)$ the polynomial $\mathcal{T}-\kappa$ has N real zeros. Furthermore, we claim that each of these zeros is simple. Let us assume to the contrary that $\mathcal{T}-\kappa$ has a zero at x_0 of multiplicity $m+1, m \geqslant 1$, i.e., $\mathscr{T}^{(k)}(x_0)=0$ for $k=1,\dots,m$ and $\mathscr{T}^{(m+1)}(x_0) \neq 0$: Then we have seen that there are $m+1$ arcs $\mathscr{C}_{j_v}$, $\mathscr{C}_{j_v} \subset Z_I$ given in Theorem 2.1, which touch each other at x_0. Since $|\mathrm{Re}\,\mathscr{T}(x_0)| = |\mathscr{T}(x_0)| = |\kappa| < 1$, there exists an $\varepsilon > 0$ such that $|\mathrm{Re}\,\mathscr{T}(z)| < 1$ for $z \in K(x_0,\varepsilon) := \{z \in \mathbb{C}\colon |z-x_0| < \varepsilon\}$ and thus for any $z \in K(x_0,\varepsilon) \cap \mathscr{C}_{j_v}$, $v \in \{1,\dots,m+1\}$, we have $\mathscr{T}(z) \in [-1,1]$. Since $\mathscr{C}_j$, $j \in \{1,\dots,N\}$, have been chosen such that $\mathrm{Im}\,\mathscr{T}(x,y) > 0$ at the right-hand side it follows, since $m \geqslant 1$, that

$$(K(z_0,\varepsilon) \cap \mathscr{C}_{j_v}) \cap (\mathbb{C}\setminus\mathbb{R}) \neq \emptyset \quad \text{for } v \in \{1,\dots,m+1\},$$

which is a contradiction to $\{z \in \mathbb{C}\colon \mathscr{T}(z) \in [-1,1]\} \subset \mathbb{R}$. Hence, for every $\kappa \in (-1,1)$, $\mathscr{T}-\kappa$, and in particular $\mathscr{T}$, has N simple real zeros which implies that $\mathscr{T}(x)/a_N$ has real coefficients and thus, because of $\mathscr{T}(z) \in [-1,1]$, $\mathscr{T}$ has real coefficients. Furthermore, it follows that $\mathscr{T}'$ has $N-1$ simple real zeros.

What remains to be shown is that $\min\{|\mathscr{T}(y)|\colon \mathscr{T}'(y)=0\} \geqslant 1$. Let us assume that this statement does not hold true. Then there exists a $y_0 \in \mathbb{R}$ such that $\mathscr{T}'(y_0)=0$ and $|\mathrm{Re}\,\mathscr{T}(y_0)| < 1$. As above, then there are two arcs C_{j_v}, $v=1,2$, $C_{j_v} \subset Z_I$ given as in Theorem 2.1, which touch each other at y_0 such that for $\varepsilon > 0$ sufficiently small $|\mathrm{Re}\,\mathscr{T}(z)| < 1$ on $K(y_0,\varepsilon) \cap \mathscr{C}_{j_v}$, $v=1,2$, which is again a contradiction to $\{z \in \mathbb{C}\colon \mathscr{T}(z) \in [-1,1]\} \subset \mathbb{R}$.

Sufficiency: Since $\mathscr{T}$ has N simple real zeros and thus $\mathscr{T}'$ has $N-1$ simple real zeros $y_1,\dots,y_{N-1} \in \mathbb{R}$ it follows that $\mathscr{T}$ is strictly monotone in (y_j,y_{j+1}), $y_0 := -\infty$ and $y_N := \infty$. Now by assumption $|\mathscr{T}(y_j)| \geqslant 1$ for $j=1,\dots,N-1$, which implies, taking a look at the graph of such a polynomial, that for every $\kappa \in [-1,1]$ the polynomial $\mathscr{T}-\kappa$ has all its N zeros real, including multiplicity, which proves the assertion. $\square$

3. Orthogonal polynomials associated with polynomials mappings

First let us introduce polynomials orthogonal with respect to a functional. Let (c_k) be a sequence of complex numbers and $\mathscr{L}: \mathbb{P} \to \mathbb{C}$ be a linear functional on the space $\mathbb{P}$ of polynomials given by

$$\mathscr{L}(z^k) = c_k \quad \text{for } k \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}, \tag{3.1}$$

which is assumed to be extendable to the space of continuous functions on a compact set $K \subset \mathbb{C}$. Furthermore assume that the associated function

$$Q_0(y) := \sum_{k=0}^{\infty} c_k y^{-(k+1)} = \mathscr{L}\left(\frac{1}{y-z}\right) \tag{3.2}$$

converges in a neighbourhood of $y=\infty$. Here, $\mathscr{L}$ acts on z. It is known (see [7, Chapter 1.3]) that for a given function Q_0 of form (3.2), which is not a rational function, there exists an infinite unique sequence of the so-called basic integers (n_v), $n_0 := 0 < n_1 < n_2 < \cdots$, and a unique sequence of monic polynomials (p_{n_v}) (p_{n_v} of degree n_v) such that

$$\mathscr{L}(z^j p_{n_v}) = 0 \quad \text{for } j=0,\dots,n_{v+1}-2$$

and

$$\mathscr{L}(z^{n_{v+1}-1} p_{n_v}) \neq 0.$$

The polynomials (p_{n_ν}) satisfy a recurrence relation of the form

$$p_{n_{\nu+1}}(z)=d_{n_{\nu+1}}(z)p_{n_\nu}(z)-\lambda_{n_{\nu+1}}p_{n_{\nu-1}}(z) \quad \text{for } \nu\in\mathbb{N}_0, \tag{3.3}$$

where $d_{n_{\nu+1}}\in\mathbb{P}_{n_{\nu+1}-n_\nu}$, $\lambda_{n_{\nu+1}}\in\mathbb{C}\setminus\{0\}$, $p_0(z)=1$ and $p_{-1}=0$. If $n_\nu=n$, i.e., if $\mathscr{L}$ is definite, then (3.3) becomes the usual form

$$p_{\nu+1}(z)=(z-\alpha_{\nu+1})p_\nu(z)-\lambda_{\nu+1}p_{\nu-1}(z).$$

The monic associated polynomials $(p_n^{(k)})$ of order k, $k\in\mathbb{N}_0$, with respect to the definite functional $\mathscr{L}$ are defined by the shifted recurrence relation

$$p_{\nu+1}^{(k)}(z)=(z-\alpha_{\nu+1+k})p_\nu^{(k)}(z)-\lambda_{\nu+1+k}p_{\nu-1}^{(k)}(z),$$

where $p_0^{(k)}(z)=1$ and $p_{-1}^{(k)}(z)=0$. Let us mention that the associated polynomials (of order one) of (p_n), also called polynomials of the second kind, have a representation of the form, $n\in\mathbb{N}$,

$$\lambda_1 p_{n-1}^{(1)}(z)=\mathscr{L}\left(\frac{p_n(z)-p_n(x)}{z-x}\right),$$

where $\lambda_1=\mathscr{L}(1)$, with the help of which the Padé-approximation property

$$\frac{\lambda_1 p_{n-1}^{(1)}(z)}{p_n(z)}=\mathscr{L}\left(\frac{1}{z-x}\right)+\mathcal{O}\left(\frac{1}{z^{2n+1}}\right) \quad \text{as } |z|\to\infty$$

follows.

Certainly of foremost interest is the case that the moments c_k have a representation of the form

$$c_k=\int_K z^k\,\mathrm{d}\mu(z), \tag{3.4}$$

where μ is a complex (not necessarily real and/or positive) measure on the curve or arc K. Then the polynomials orthogonal with respect to $\mathscr{L}$ become the polynomials orthogonal with respect to μ on K and the function Q_0 becomes the so-called Stieltjes-function

$$Q_0(y)=\int_K\frac{1}{y-z}\,\mathrm{d}\mu(z). \tag{3.5}$$

A functional $\mathscr{L}$ is called positive definite if $\mathscr{L}$ has a representation of the form (3.5) with $K\subset\mathbb{R}$ and μ a positive real measure on K.

Following Bessis and Moussa [5], see also [10], let us show how a new functional is generated in a natural way by the functional $\mathscr{L}$ and a polynomial (mapping) $\mathscr{T}$.

Definition 3.1. For given $\mathscr{T}\in\mathbb{P}_N\setminus\mathbb{P}_{N-1}$ and $S\in\mathbb{P}_m$, $m\leqslant N-1$, define

$$\mathscr{L}^{\mathscr{T},S}(f(z)):=\mathscr{L}\left(\sum_{j=1}^{N}\frac{S(\mathscr{T}_j^{-1}(z))}{\mathscr{T}'(\mathscr{T}_j^{-1}(z))}f(\mathscr{T}_j^{-1}(z))\right), \tag{3.6}$$

where it is assumed that the right-hand side is well defined. Here, $\{\mathscr{T}_j^{-1}\colon j=1,\ldots,N\}$ denotes the complete assignment of branches of $\mathscr{T}^{-1}$. The definition (3.6) of the linear functional $\mathscr{L}^{\mathscr{T},S}$ is quite natural and can be understood in the following way: By partial fraction expansion we have

$$\frac{S(y)}{\mathcal{T}(y)-z}=\sum_{j=1}^{N}\frac{S(\mathcal{T}_j^{-1}(z))}{\mathcal{T}'(\mathcal{T}_j^{-1}(z))}\cdot\frac{1}{y-\mathcal{T}_j^{-1}(z)}, \tag{3.7}$$

from which we get for large $y\in\mathbb{C}$ and for all $k\in\mathbb{N}_0$

$$\begin{aligned}\mathcal{L}^{\mathcal{T},S}\left(\frac{\mathcal{T}(z)}{y-z}\right)&=\sum_{v=0}^{\infty}y^{-(v+1)}\mathcal{L}^{\mathcal{T},S}(z^v\mathcal{T}(z))\\&=\sum_{v=0}^{\infty}y^{-(v+1)}\mathcal{L}\left(\sum_{j=1}^{N}\frac{S(\mathcal{T}_j^{-1}(z))}{\mathcal{T}'(\mathcal{T}_j^{-1}(z))}[\mathcal{T}_j^{-1}(z)]^v z^k\right)\\&=\mathcal{L}\left(z^k\sum_{j=1}^{N}\frac{S(\mathcal{T}_j^{-1}(z))}{\mathcal{T}'(\mathcal{T}_j^{-1}(z))}\frac{1}{y-\mathcal{T}_j^{-1}(z)}\right)=S(y)\mathcal{L}\left(\frac{z^k}{\mathcal{T}(y)-z}\right).\end{aligned} \tag{3.8}$$

In particular, we have

$$\mathcal{L}^{\mathcal{T},1}(z^k)=0\quad\text{for } k=0,\ldots,N-2,\text{ and thus } S(y)\mathcal{L}\left(\frac{1}{\mathcal{T}(y)-z}\right)=\mathcal{L}\left(\frac{S(z)}{\mathcal{T}(y)-z}\right). \tag{3.9}$$

Thus $\mathcal{T}$ compositions have the following orthogonality property, see e.g., [26, Theorem 2(b)].

Proposition 3.2. *Let $\mathcal{T}$ and S be polynomials of degree N and $m\leqslant N-1$, respectively. Suppose that $\mathcal{L}(z^j f(z))=0$ for $j=0,1,\ldots,n-1$. Then*

$$\mathcal{L}^{\mathcal{T},S}(z^j(f\circ\mathcal{T})(z))=0\quad \textit{for } j=0,1,\ldots,(n+1)N-m-2.$$

Let us mention that an analog orthogonality property holds with respect to the Hermitian inner product (see [26, Theorem 2(a) and also Theorem 3]).

Now of special interest are functionals with an integral representation (3.4). Then transformation (3.6) defines a measure $\mathrm{d}\mu^{\mathcal{T},S}$ on the inverse image $\mathcal{T}^{-1}(K)$ by

$$\begin{aligned}\mathcal{L}^{\mathcal{T},S}(f(z))&=\sum_{j=1}^{N}\int_K f(\mathcal{T}_j^{-1}(z))\frac{S(\mathcal{T}_j^{-1}(z))}{\mathcal{T}'(\mathcal{T}_j^{-1}(z))}\,\mathrm{d}\mu(z)\\&=:\int_{\mathcal{T}^{-1}(K)}f(z)\,\mathrm{d}\mu^{\mathcal{T},S}(z).\end{aligned} \tag{3.10}$$

For the following we need an extension of the functional $\mathcal{L}^{\mathcal{T},S}$ leading to important classes orthogonal polynomials as such ones with periodic recurrence coefficients. Let $\rho\in\mathbb{P}_v$ be a polynomial of degree v and suppose that Q_0 from (3.2) converges at the zeros of ρ. By the way, note that Q_0

certainly converges at the zeros of ρ if the moments have a representation of the form (3.4) and the zeros of ρ are outside of the support of the measure μ. Now let us define a new functional $\mathscr{L}^{1/\rho}$ by

$$\rho(y)\mathscr{L}^{1/\rho}\left(\frac{1}{y-z}\right) := Y(y) + \mathscr{L}\left(\frac{1}{y-z}\right), \tag{3.11}$$

where $Y \in \mathbb{P}_{v-1}$ is that unique interpolatory-polynomial which satisfies at the zeros w_i of ρ

$$Y(w_i) = -\mathscr{L}\left(\frac{1}{w_i - z}\right). \tag{3.12}$$

We claim that

$$Y(y) = \mathscr{L}\left(\frac{\rho(y)-\rho(z)}{y-z}\,\frac{1}{\rho(z)}\right). \tag{3.13}$$

Indeed, obviously the expression at the right-hand side is a polynomial of degree $\leqslant v-1$ and has the interpolatory property (3.12) which, by uniqueness, proves the claim. Furthermore, we have

$$\mathscr{L}\left(\frac{1}{y-z}\right) = \mathscr{L}\left(\frac{\rho(z)-\rho(y)}{y-z}\,\frac{1}{\rho(z)}\right) + \rho(y)\mathscr{L}\left(\frac{1}{y-z}\,\frac{1}{\rho(z)}\right)$$

and hence, by (3.13),

$$\mathscr{L}^{1/\rho}\left(\frac{1}{y-z}\right) = \mathscr{L}\left(\frac{1}{y-z}\,\frac{1}{\rho(z)}\right), \tag{3.14}$$

i.e., the moments $c_k^{1/\rho}$ of the new linear functional $\mathscr{L}^{1/\rho}$ are given by

$$\mathscr{L}^{1/\rho}(z^k) =: c_k^{1/\rho} = \mathscr{L}\left(\frac{z^k}{\rho(z)}\right).$$

Now, let $\mathscr{L}$ be the functional $\mathscr{L}^{\mathscr{T},S}$, introduced above then by (3.14) we have

$$\mathscr{L}^{\mathscr{T},S/\rho}\left(\frac{1}{y-z}\right) := (\mathscr{L}^{\mathscr{T},S})^{1/\rho}\left(\frac{1}{y-z}\right) = \mathscr{L}^{\mathscr{T},S}\left(\frac{1}{y-z}\,\frac{1}{\rho(z)}\right). \tag{3.15}$$

The following special case is of particular interest: Let K be a complex curve, let $w : K \to \mathbb{C}$ an integrable function, and put

$$\mathrm{d}\mu(z) := w(z)\,\mathrm{d}z + \sum_{v=1}^{n} \mu_v \delta(z - y_v),$$

where $\mu_1, \ldots, \mu_n, y_1, \ldots, y_n \in \mathbb{C}$ and where δ denotes the Dirac-measure. Suppose that $\mathscr{L}$ has an integral representation (3.5). Then applying (3.10) to $f(z) = 1/(y-z)\rho(z)$ and by changing the variable of integration we get in view of (3.15)

$$\mathscr{L}^{\mathscr{T},S/\rho}\left(\frac{1}{y-z}\right)=\int_{\mathscr{T}^{-1}(K)}\frac{1}{y-z}\,\mathrm{d}\mu^{\mathscr{T},S/\rho}(z)$$

with

$$\mathrm{d}\mu^{\mathscr{T},S/\rho}(z)=\frac{S(z)}{\rho(z)}(w\circ\mathscr{T})(z)\,\mathrm{d}z + \sum_{v=1}^{n}\mu_v\sum_{j=1}^{N}\frac{S(\mathscr{T}_j^{-1}(y_v))}{\mathscr{T}'(\mathscr{T}_j^{-1}(y_v))\cdot\rho(\mathscr{T}_j^{-1}(y_v))}\,\delta(z-\mathscr{T}_j^{-1}(y_v)), \tag{3.16}$$

where the orientation of integration on the subcurves of $\mathscr{T}^{-1}(K)$ is given by the orientation of integration on K. Note, if $K=[-1,1]$ then the absolute continuous part in (3.16) can be written in the convenient form $(S(z)/\rho(z))(w\circ\mathscr{T})(z)\,\mathrm{sgn}\,\overline{\mathscr{T}'(z)}\,|\mathrm{d}z|$. The orthogonality property with respect to $\mathscr{L}$ is inherited to $\mathscr{L}^{\mathscr{T},S/\rho}$ as the following theorem shows.

Proposition 3.3. *Let $\mathscr{L}$ be a linear functional given by (3.1) and let $(p_{n_k})_{k\in\mathbb{N}_0}$ be a sequence of polynomials of exact degree n_k satisfying the orthogonality condition*

$$\mathscr{L}(z^j p_{n_k})=0 \quad \textit{for } j=0,\dots,n_k-1.$$

Furthermore, let $\mathscr{T}_N$, S_m and ρ_v be an arbitrary complex polynomials of degree N, m, and v, respectively, with $m+v\leqslant N-1$. Then the sequence of polynomials $(\rho_v(p_{n_k}\circ\mathscr{T}_N))_{k\in\mathbb{N}_0}$ is orthogonal to $\mathbb{P}_{(n_k+1)N-m-2}$ with respect to the linear functional $\mathscr{L}^{\mathscr{T}_N,S_m/\rho_v}$.

Proof. The assertion follows immediately from Proposition 3.2 since

$$\mathscr{L}^{\mathscr{T}_N,S_m/\rho_v}(z^j\rho_v(z)(p_{n_k}\circ\mathscr{T}_N)(z))=\mathscr{L}^{\mathscr{T}_N,S_m}(z^j(p_{n_k}\circ\mathscr{T}_N)(z)). \qquad \square$$

For example $\{\rho_v(z)p_n(z^N)\}_{n\in\mathbb{N}}$, $N\in\mathbb{N}$, is orthogonal on the star $\mathscr{T}^{-1}([-1,1])=\{r\exp 2k\pi\mathrm{i}/N\colon r\in[0,1],\ k=0,\dots,2N-1\}$ with respect to the weight function $(S_m(z)/\rho_v(z))w(z^N)\exp(2k\pi\mathrm{i}/N)|\mathrm{d}z|$ on the kth ray, $k=0,\dots,2N-1$ if (p_n) is orthogonal on $[-1,1]$ with respect to $w(x)$. Next let us give some more informations on the polynomials orthogonal with respect to $\mathscr{L}^{\mathscr{T}_N,S_m/\rho_v}$.

Theorem 3.4. *Suppose that the polynomials (p_n) orthogonal with respect to the definite functional $\mathscr{L}$ satisfy the recurrence relation*

$$p_n(z)=(z-\beta_n)p_{n-1}(z)-\gamma_n p_{n-2}(z), \quad p_0(z)=1, \quad p_{-1}(z)=0.$$

Furthermore assume that the functional $\mathscr{L}^{\mathscr{T}_N,S_m/\rho_v}$ is definite, where $\mathscr{T}_N$ has the leading coefficient $1/L$ and S_m and ρ_v are monic with $m+v=N-1$. Suppose that the polynomials (P_n) orthogonal *with respect to* $\mathscr{L}^{\mathscr{T}_N, S_m/\rho_v}$ *satisfy the recurrence relation*

$$P_n(z) = (z - \alpha_n)P_{n-1}(z) - \lambda_n P_{n-2}(z), \quad P_0(z) = 1, \quad P_{-1}(z) = 0. \tag{3.17}$$

Then the following relations hold:

(1) $P_{kN+v}(z) = \rho_v(z) p_k(\widehat{\mathscr{T}_N(z)})$ *for* $k \in \mathbb{N}_0$
(2) $P^{(1)}_{kN+v-1}(z) = P^{(1)}_{v-1}(z) p_k(\widehat{\mathscr{T}_N(z)}) + \text{const.} \cdot S_m(z) p^{(1)}_{k-1}(\widehat{\mathscr{T}_N(z)})$ *for* $k \in \mathbb{N}_0$
(3) $\prod_{\kappa=(k-1)N+v+2}^{kN+v+1} \lambda_\kappa = L^2 \gamma_{k+1}$
(4) $P_N^{(kN+v)}(z) - \lambda_{kN+v+1} P_{N-2}^{((k-1)N+v+1)}(z) = \hat{\mathscr{T}}_N(z) - L\beta_{k+1}$ *for* $k \in \mathbb{N}$
(5) $P_{N-1}^{(kN+v+1)}(z) = \rho_v(z) \tilde{S}_{m,k}(z)$ *for* $k \in \mathbb{N}_0$.

Proof. Relation (1) follows immediately from Proposition 3.3. By the recurrence relation of (p_k) and (1) we have

$$P_{(k+1)N+v}(z) = (\hat{\mathscr{T}}_N(z) - L\beta_{k+1})P_{kN+v}(z) - L^2\gamma_{k+1}P_{(k-1)N+v}(z). \tag{3.18}$$

Since by Lemma 3.1(b) from [21]

$$b_{kN+v} P_{(k-1)N+v}(z) = P_{N-1}^{((k-1)N+v+1)}(z) P_{kN+v-1}(z) - P_{N-2}^{((k-1)N+v+1)}(z) P_{kN+v}(z),$$

where

$$b_{kN+v} = \prod_{\kappa=(k-1)N+v+2}^{kN+v} \lambda_\kappa,$$

it follows from (3.18) that

$$P_{(k+1)N+v}(z) = \left(\hat{\mathscr{T}}_N(z) - L\beta_{k+1} + (L^2\gamma_{k+1}/b_{kN+v}) P_{N-2}^{((k-1)N+v+1)}(z)\right) P_{kN+v}(z)$$
$$- (L^2\gamma_{k+1}/b_{kN+v}) P_{N-1}^{((k-1)N+v+1)}(z) P_{kN+v-1}(z). \tag{3.19}$$

On the other hand, we know by Lemma 3.1(c) from [21] that $P_{(k+1)N+v}$ has a unique representation of the form

$$P_{(k+1)N+v} = P_N^{(kN+v)} P_{kN+v} - \lambda_{kN+v+1} P_{N-1}^{(kN+v+1)} P_{kN+v-1},$$

which gives by (3.19) the relations (3)–(5).

Concerning relation (2) we have in view of (1)

$$\text{const.} \cdot P^{(1)}_{kN+v-1}(y) = \mathscr{L}^{\mathscr{T}_N, S_m/\rho_v}\left(\frac{\rho_v(y) p_k(\mathscr{T}_N(y)) - \rho_v(z) p_k(\mathscr{T}_N(z))}{y - z}\right)$$
$$= p_k(\mathscr{T}_N(y)) \mathscr{L}^{\mathscr{T}_N, S_m/\rho_v}\left(\frac{\rho_v(y) - \rho_v(z)}{y - z}\right)$$
$$+ S_m(y) \mathscr{L}^{\mathscr{T}_N}\left(\frac{p_k(\mathscr{T}_N(y)) - p_k(\mathscr{T}_N(z))}{y - z}\right),$$

which is in view of

$$\mathcal{L}^{\mathcal{T}_N}\left(\frac{p_k(\mathcal{T}_N(y)) - p_k(\mathcal{T}_N(z))}{y-z}\right) = p_k(\mathcal{T}_N(y))\mathcal{L}\left(\frac{1}{\mathcal{T}_N(y)-z}\right) - \mathcal{L}\left(\frac{p_k(z)}{\mathcal{T}_N(y)-z}\right)$$

the assertion. □

For the case $\rho \equiv 1$ Theorem 3.4 has been proved by different methods in [5], see also [11, Theorem 6]. If $\rho \equiv 1$ and if $\mathcal{T}_N$ is the classical Chebyshev polynomial, i.e., $\mathcal{T}_N(x) = \cos N \arccos x$, then one obtains the so-called sieved orthogonal polynomials, see [1].

If we put in Theorem 3.4 $\mathrm{d}\mu(x) = \sqrt{1-x^2}$ then the weight function $\mathrm{d}\mu^{\mathcal{T}_N,1/\rho} = (\sqrt{1-\mathcal{T}_N^2(z)}/\rho(z))\,\mathrm{sgn}\,\overline{\mathcal{T}_N'(z)}\,|\mathrm{d}z|$ leads to orthogonal polynomials with periodic recurrence coefficients from a certain index n_0 upwards. This fact can be proved in exactly the same way as in [21, Theorem 3.1], where it is assumed that the coefficients of $\mathcal{T}_N$ are real. For orthogonal polynomials with periodic or asymptotically periodic recurrence coefficients see e.g. [10,13,21,22,2,3]. Next let us turn to the question when polynomials from Theorem 3.4 are orthogonal with respect to a positive measure supported on a subset of the real line.

4. Characterization of polynomial mappings generating positive definite functionals

Theorem 4.1. *Let μ be a positive measure with $\mathrm{supp}(\mu) = [-1,1]$. Suppose that (p_n) is orthogonal with respect to μ and that $\mathcal{T}$ is a polynomial of degree N. Then the following statements are equivalent*:

(a) *$\mathcal{T}$ has N simple real zeros and* $\min\{|\mathcal{T}(z)|: \mathcal{T}'(z) = 0\} \geqslant 1$
(b) *all zeros of $p_n \circ \mathcal{T}$, $n \in \mathbb{N}$, are real*
(c) *$(p_n \circ \mathcal{T})$ is orthogonal with respect to a positive measure supported on a compact subset of $\mathbb{R}$.*

Proof. (b) ⇒ (a). Let $x_1, \ldots, x_n \in [-1,1]$ be the zeros of p_n and let $y_{\nu,j} \in \mathcal{T}^{-1}([-1,1])$, $\nu = 1,\ldots,N$, $j = 1,\ldots,n$, be those numbers such that

$$\mathcal{T}(y_{\nu,j}) = x_j \quad \text{for } \nu = 1,\ldots,N,$$

i.e., the $y_{\nu,j}$'s are the zeros of $p_n \circ \mathcal{T}$. Now, let us assume that $\mathcal{T}$ does not satisfy the given conditions. Then it follows from Corollary 2.3 that $\mathcal{T}^{-1}([-1,1])$ contains an arc $\mathcal{C}$ which lies in $\mathbb{C}\setminus\mathbb{R}$. Recalling the well-known fact that the zeros of $\{p_n\}$ are dense in $[-1,1]$ this implies that $p_n \circ \mathcal{T}$ has complex zeros for sufficiently large n. But this is a contradiction to the fact that $(p_n \circ \mathcal{T})_{n\in\mathbb{N}}$ is orthogonal with respect to a positive definite functional.

(a) ⇒ (b) and (c). In view of Theorem 3.3 and Corollary 2.3 $(p_n \circ \mathcal{T})$ is orthogonal on the set of real intervals $\mathcal{T}^{-1}([-1,1])$ with respect to the positive measure $\mu^{\mathcal{T},\mathcal{T}'}$. Thus all zeros of $p_n \circ \mathcal{T}$ are in the convex hull of $\mathcal{T}^{-1}([-1,1])$.

(c) ⇒ (b) is well known. □

Moreover we are able to give a necessary and sufficient condition such that $\{\rho_v(p_n \circ \mathscr{T})\}_{n\in\mathbb{N}}$ is orthogonal with respect to a positive measure.

Corollary 4.2. *Suppose that $\mathscr{L}$ is positive definite and that the associated measure μ has support $[-1,1]$. Let $\mathscr{T}_N, S_m, \rho_v$ be polynomials of degree N, m and v, respectively, with $m+v=N-1$ and $\operatorname{sgn} S_m/\rho_v = \operatorname{sgn} \mathscr{T}'_N$ on $\mathscr{T}_N^{-1}([-1,1])$. Then $\mathscr{L}^{\mathscr{T}_N, S_m/\rho_v}$ is positive definite if and only if $\mathscr{T}_N$ has N simple real zeros and $\min\{|\mathscr{T}_N(z)|\colon \mathscr{T}'_N(z)=0\} \geqslant 1$.*

Proof. *Necessity*: Since $\mathscr{L}^{\mathscr{T}_N, S_m/\rho_v}$ is positive definite and $\rho_v \cdot (p_n \circ \mathscr{T}_N)$ is orthogonal with respect to $\mathscr{L}^{\mathscr{T}_N, S_m/\rho_v}$, it follows that $p_n \circ \mathscr{T}_N$ has real zeros only.

Sufficiency: By the properties of $\mathscr{T}_N$, $\mathscr{T}_N^{-1}([-1,1])$ consists of real intervals only. In view of the assumptions $\mu^{\mathscr{T}_N, S_m/\rho_v}$ is a positive measure, which proves the corollary. □

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