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Legendre modified moments for Euler's constant

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Dedicated to Claude Brezinski at the occasion of his retirement.

Abstract

Polynomial moments are often used for the computation of Gauss quadrature to stabilize the numerical calculation of the orthogonal polynomials, see [W. Gautschi, Computational aspects of orthogonal polynomials, in: P. Nevai (Ed.), Orthogonal Polynomials-Theory and Practice, NATO ASI Series, Series C: Mathematical and Physical Sciences, vol. 294. Kluwer, Dordrecht, 1990, pp. 181–216 [6]; W. Gautschi, On the sensitivity of orthogonal polynomials to perturbations in the moments, Numer. Math. 48(4) (1986) 369–382 [5]; W. Gautschi, On generating orthogonal polynomials, SIAM J. Sci. Statist. Comput. 3(3) (1982) 289–317 [4]] or numerical resolution of linear systems [C. Brezinski, Padé-type approximation and general orthogonal polynomials, ISNM, vol. 50, Basel, Boston, Stuttgart, Birkhäuser, 1980 [3]]. These modified moments can also be used to accelerate the convergence of sequences to a real or complex numbers if the error satisfies some properties as done in [C. Brezinski, Accélération de la convergence en analyse numérique, Lecture Notes in Mathematics, vol. 584. Springer, Berlin, New York, 1977; M. Prévost, Padé-type approximants with orthogonal generating polynomials, J. Comput. Appl. Math. 9(4) (1983) 333–346]. In this paper, we use Legendre modified moments to accelerate the convergence of the sequence $H_n - \log(n+1)$ to the Euler's constant γ. A formula for the error is given. It is proved that it is a totally monotonic sequence. At last, we give applications to the arithmetic property of γ.

MSC: 41A21; 65B10

Keywords: Legendre moments; Euler's constant; Padé approximations

1. Introduction

The Euler constant $\gamma = 0.577215\ldots$ is the limit of the sequence

$$H_n - \log(n+1),$$

where H_n is the harmonic number defined by

$$\sum_{k=1}^{n} \frac{1}{k}.$$

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An integral representation for Euler's constant is

$$\gamma = \int_0^1 \left(\frac{1}{\ln u} + \frac{1}{1-u}\right) \mathrm{d}u. \tag{1}$$

The sequence (S_n)

$$\begin{aligned} S_n &:= \int_0^1 \left(\frac{1-u^n}{\log u} + \frac{1-u^n}{1-u}\right) \mathrm{d}u \\ &= H_n - \log(n+1) \end{aligned}$$

converges to γ but very slowly, as $\mathcal{O}(1/n)$.

The error $\gamma - S_n$ satisfies

$$\gamma - S_n = \int_0^1 u^n \left(\frac{1}{\log u} + \frac{1}{1-u}\right) \mathrm{d}u \tag{2}$$

and can be considered as moments with respect to the weight function $w(u) = (1/\log u + 1/(1-u))$ on the interval [0, 1].

2. Legendre modified moments

Suppose we are given a sequence of real or complex numbers $(x_n)_n$ converging to l and satisfying the property

$$x_n - l = \int_0^1 u^n \, \mathrm{d}\mu(u)$$

where $\mathrm{d}\mu$ is a positive measure on the interval [0, 1].

If the error of a sequence is of this form, then a way to decrease this error and so accelerate the convergence is to use modified moments, i.e. by replacing the monomial u^n by some suitable polynomials P_n normalized by $P_n(1) = 1$.

For

$$P_n(u) = \sum_{k=0}^{n} \alpha_k^{(n)} u^k,$$

$\int_0^1 P_n(u)\,\mathrm{d}\mu(u) = \sum_{k=0}^n \alpha_k^{(n)} \int_0^1 u^k \,\mathrm{d}\mu(u) = \sum_{k=0}^n \alpha_k^{(n)}(x_k - l) = \sum_{k=0}^n \alpha_k^{(n)} x_k - l$. In that case, the error between the limit l and the transformed sequence (y_n) defined by $y_n := \sum_{k=0}^n \alpha_k^{(n)} x_k$ becomes

$$y_n - l = \int_0^1 P_n(u)\,\mathrm{d}\mu(u).$$

To improve the convergence, the polynomial P_n can be chosen to be orthogonal with respect to some weight. In our case, because of the behavior of the weight function around 1 ($w(u) \sim \mathcal{O}(1)$, $u = 1$), a good choice will be the shifted Legendre polynomials which are orthogonal on [0, 1]

$$\int_0^1 P_n^*(t) P_m^*(t)\,\mathrm{d}t = 0, \quad n \neq m.$$

These polynomials can be expressed in different bases

$$\begin{aligned} P_n^*(t) &= \sum_{k=0}^{n} \binom{n}{k}^2 t^{n-k}(t-1)^k \qquad (3) \\ &= \sum_{k=0}^{n} \binom{n}{k}\binom{n+k}{k} (-1)^{n+k} t^k. \qquad (4) \end{aligned}$$

Theorem 1. *For $0 \leqslant m \leqslant n$, let us define*

$$J_{n,m} := \int_0^1 u^{n-m} P_n^*(u) \left(\frac{1}{\ln u} + \frac{1}{1-u} \right) \mathrm{d}u, \tag{5}$$

$$L_{n,m} := -\int_0^1 \left(\frac{1 - u^{n-m} P_n^*(u)}{\ln u} \right) \mathrm{d}u, \tag{6}$$

$$A_{n,m} := \int_0^1 \left(\frac{1 - u^{n-m} P_n^*(u)}{1-u} \right) \mathrm{d}u \tag{7}$$

then

$$\gamma = A_{n,m} - L_{n,m} + J_{n,m}, \tag{8}$$

$$L_{n,m} = \sum_{k=0}^{n} \binom{n}{k} \binom{n+k}{k} (-1)^{n+k} \ln(n-m+k+1), \tag{9}$$

$$A_{n,m} = 2H_n. \tag{10}$$

Proof. We first prove the identity (8) linking Euler's constant γ, the linear combination of logarithms numbers $L_{n,m}$, the rational numbers $A_{n,m}$ and the integrals $J_{n,m}$. From formula (1), one substitutes the integrand $w(u)=(1/\ln u+1/(1-u))$ by an approximation involving Legendre Polynomials as follows:

$$\begin{aligned}\gamma &= \int_0^1 \left(\frac{1}{\ln u} + \frac{1}{1-u} \right) \mathrm{d}u \\ &= \int_0^1 \left(\frac{1 - u^{n-m} P_n^*(u)}{\ln u} + \frac{1 - u^{n-m} P_n^*(u)}{1-u} \right) \mathrm{d}u + \int_0^1 u^{n-m} P_n^*(u) w(u)\, \mathrm{d}u.\end{aligned}$$

The expression (4) of P_n^* leads to analogous expressions $L_{n,m}$.

By linearity

$$\begin{aligned}L_{n,m} &= -\sum_{k=0}^{n} \binom{n}{k} \binom{n+k}{k} (-1)^{n+k} \int_0^1 \left(\frac{1 - u^{k+n-m}}{\ln u} \right) \mathrm{d}u & (11)\\ &= \sum_{k=0}^{n} \binom{n}{k} \binom{n+k}{k} (-1)^{n+k} \ln(n-m+k+1). & (12)\end{aligned}$$

$A_{n,m}$ is treated quite differently: from the orthogonality relation between two polynomials P_n^*

$$\int_0^1 P_n^*(u) q(u)\, \mathrm{d}u = 0 \quad \text{for all polynomial } q \text{ of degree less than } n \tag{13}$$

by taking $q(u) = (1 - u^{n-m})/(1-u)$, another expression for $A_{n,m}$ is

$$A_{n,m} = \int_0^1 P_n^*(u) q(u)\, \mathrm{d}u + \int_0^1 \frac{1 - P_n^*(u)}{1-u} \mathrm{d}u = \int_0^1 \frac{P_n^*(1) - P_n^*(u)}{1-u} \mathrm{d}u \tag{14}$$

and so $A_{n,m}$ is independent of $0 \leqslant m \leqslant n$.

Let us now compute the integral in (14).

Legendre polynomials satisfy a three term recurrence relation which is

$$(n+1) P_{n+1}^*(u) = (2n+1)(2u-1) P_n^*(u) - n P_{n-1}^*(u), \tag{15}$$

$$P_0^*(u) = 1, \quad P_1^*(u) = 2u - 1. \tag{16}$$

Thus, $A_{n,m}$'s also satisfy a similar recurrence relation

$$(n+1)A_{n+1,m} = (2n+1)A_{n,m} - nA_{n-1,m}, \tag{17}$$

$$A_{0,m} = 0, \quad A_{1,m} = 2. \tag{18}$$

With (17) and (18), it is not difficult to prove that

$$A_{n,m} = 2H_n, \quad 0 \leqslant m \leqslant n. \qquad \square \tag{19}$$

In Section 4 , we will prove the asymptotic formula

$$\gamma = A_{n,m} - L_{n,m} + \mathcal{O}(4^{-n}). \tag{20}$$

Actually, we will prove that the error term $J_{n,m}$ is a totally monotone sequence (i.e. a sequence of monomial moments with respect a positive measure), converging to 0 as 4^{-n}.

Before, we need to investigate the analytic property of the weight function w.

3. An interesting integral representation for Euler's constant

Euler constant can be written as sum of series or with integral representation. (See http://numbers.computation.free.fr/Constants/Gamma/gamma.html).

A complete study (with more than 130 references) can be found in [7].

In this section, we give an integral representation of the weight w in Lemma 2, which leads to a formula for γ first proved by Schlömlich [12] in 1880 and rediscovered by Krämer [7, p. 129]. We give another proof of this formula.

Lemma 2. *The function* $1/\ln(1-u) + 1/u$ *is a Markov–Stieltjes function. More precisely*,

$$\frac{1}{\ln(1-u)} + \frac{1}{u} = \int_0^1 \frac{1}{1-ut}\mu(t)\,\mathrm{d}t, \tag{21}$$

where the weight function μ *is*

$$\mu(t) := \frac{1}{t(\ln^2(1/t-1)+\pi^2)}.$$

Proof. After a change of variable ($u \to (1-u)$ and $x = 1/t - 1$), formula (21) is equivalent to

$$\frac{1}{\ln(u)} + \frac{1}{1-u} = \int_0^\infty \frac{1}{x+u}\frac{1}{\ln^2 x + \pi^2}\,\mathrm{d}x. \tag{22}$$

The weight function μ can be found with the Stieltjes inversion formula (see [16]). Another way to prove formula (22) is to apply residue theorem to the function

$$f(x) := \frac{1}{x+u}\,\frac{1}{\ln x + \mathrm{i}\pi}.$$

Taking the determination of $\ln x$ on the complex plane cut along the positive real axis, the poles of f are $x = -u$ and $x = -1$.

Let us define γ_r a small semi-circle $z = r\mathrm{e}^{\mathrm{i}\theta}$, $-\pi/2 \leqslant \theta \leqslant \pi/2$, $r > 0$. D_r^+ the line $z = x + \mathrm{i}r$, x running from 0 to R, Γ_R the circle $z = R\mathrm{e}^{\mathrm{i}\theta}$, $0 \leqslant \theta \leqslant 2\pi$ and D_r^- the line $z = x - \mathrm{i}r$, for x from R to 0.

Now, we compute $\int_{\mathscr{C}} f(x)\,\mathrm{d}x$ where $\mathscr{C}$ is the union of D_r^+, Γ_R, D_r^- and γ_r, with the theorem of residue to obtain

$$\int_0^\infty \frac{1}{x+u}\left(\frac{1}{\ln x + \mathrm{i}\pi} + \frac{-1}{\ln x - \mathrm{i}\pi}\right)\mathrm{d}x = \int_0^\infty \frac{1}{x+u}\left(\frac{-2\mathrm{i}\pi}{\ln^2 x + \pi^2}\right)\mathrm{d}x \tag{23}$$

$$= -2\mathrm{i}\pi\left(\frac{1}{\ln u} + \frac{1}{1-u}\right). \qquad \square \tag{24}$$

Now, we are in position to prove the integral of Schlömlich for Euler's constant γ with a new method.

Theorem 3. *The Euler's constant γ satisfies*

$$\gamma = \int_{-\infty}^{+\infty} \frac{\ln(1+\mathrm{e}^{-z})\mathrm{e}^{z}}{z^2+\pi^2}\,\mathrm{d}z \tag{25}$$

Proof. In the integral representation (1) of γ, let us substitute the integrand by the expression (21). This leads to

$$\gamma = \int_0^1 -\frac{\ln(1-t)}{t}\frac{1}{t(\ln^2(1/t-1)+\pi^2)}\,\mathrm{d}t \tag{26}$$

$$= \int_{-\infty}^{\infty} \frac{\ln(1+\mathrm{e}^{-z})\mathrm{e}^{z}}{z^2+\pi^2}\,\mathrm{d}z \tag{27}$$

with the change of variable $t=(1+\mathrm{e}^{z})^{-1}$. □

With the expression of the weight function as a Markov Stieltjes function, it is possible to deduce some interesting properties for the error, for example to show that it is a sequence of moments.

4. Behavior of the error

Theorem 4. *For each fixed integer m, the sequence $((-1)^m J_{n,m})_n$ defined in Theorem 1 is totally monotonic. More precisely*

$$(-1)^m J_{n,m} = \int_0^{1/4} v^n \rho_m(v)\,\mathrm{d}v = \mathcal{O}(4^{-n}), \tag{28}$$

where the weight function is

$$\rho_m(v) = \int_{(1-\sqrt{1-4v})/2}^{(1+\sqrt{1-4v})/2} \left(\frac{u-u^2-v}{uv}\right)^m \frac{1}{(u-u^2-v)\left(\pi^2+\ln^2\left(\frac{-uv}{u^2-u+v}\right)\right)}\,\mathrm{d}u.$$

Proof. $J_{n,m} = \int_0^1 u^{n-m} P_n^*(u)(1/\ln u + 1/(1-u))\,\mathrm{d}u$ appears as Legendre modified moments of the weight function $(1/\ln u + 1/(1-u))$.

For some particular cases of weight function, a sequence of polynomial modified moments can be itself a sequence of monomial moments, with respect to a positive measure (see [10]). Using Rodrigues formula for orthogonal polynomials, Lemma 2, Fubini's theorem and after n integrations by parts, it arises

$$J_{n,m} = \int_0^1 u^{n-m} \frac{(-1)^n}{n!}\frac{\mathrm{d}^n}{\mathrm{d}u^n}(u^n(1-u)^n)\left(\frac{1}{\ln u}+\frac{1}{1-u}\right)\mathrm{d}u \tag{29}$$

$$= \int_0^1 u^{n-m} \frac{(-1)^n}{n!}\frac{\mathrm{d}^n}{\mathrm{d}u^n}(u^n(1-u)^n)\,\mathrm{d}u \int_0^1 \frac{1}{1-(1-u)t}\mu(t)\,\mathrm{d}t \tag{30}$$

$$= \int_0^1\int_0^1 \frac{(-1)^n}{n!} u^n(1-u)^n\,\mathrm{d}u \frac{\mathrm{d}^n}{\mathrm{d}u^n}\left(\frac{u^{n-m}}{1-(1-u)t}\right)\mu(t)\,\mathrm{d}t. \tag{31}$$

The computation of $\mathrm{d}^n/\mathrm{d}u^n(u^{n-m}/(1-(1-u)t))$ needs the partial decomposition of the rational function $u^{n-m}/(1-(1-u)t) = q(u) + ((t-1)/t)^{n-m}1/(1-(1-u)t)$, where q is polynomial of degree $n-m-1$.

Another expression of $J_{n,m}$ is then

$$J_{n,m} = \int_0^1\int_0^1 u^n(1-u)^n\left(\frac{t-1}{t}\right)^{n-m}\frac{t^n}{(1-(1-u)t)^{n+1}}\mu(t)\,\mathrm{d}t\,\mathrm{d}u. \tag{32}$$

We do the following change of variable

$$v = \frac{u(1-u)(1-t)}{1-(1-u)t} \in [0, 1/4] \quad \Leftrightarrow \quad t = \phi(v) = \frac{u^2 - u + v}{(u-v)(u-1)} \in [0, 1].$$

Let $\phi_1(v)$ and $\phi_2(v)$ denote the two roots of the quadratic equation $v = u - u^2$, $\phi_1(v) = (1+\sqrt{1-4v})/2$, $\phi_2(v) = (1-\sqrt{1-4v})/2$.

$$J_{n,m} = \int_0^{1/4} v^n \, \mathrm{d}v \int_{\phi_1(v)}^{\phi_2(v)} \left(\frac{-\phi(v)}{\phi(v)-1}\right)^m \frac{u^2 \mu(\phi(v))}{(u-1)(u-v)^2} \frac{(-1)^m \, \mathrm{d}u}{1-(1-u)\phi(v)} \tag{33}$$

which can be simplified to give the result. □

As quoted in [2,9,10], the sequence $(A_{n,m} - L_{n,m})_n$ converging to γ with an error totally monotonic on the interval $[0, R]$ with $R=\frac{1}{4}$ can be accelerated by the ε-algorithm to obtain an error of order $\mathcal{O}(((2/R-1)-\sqrt{(2/R-1)^2-1})^n) = \mathcal{O}((7-\sqrt{48})^n)$.

Remark. There exists some sequence converging to γ with an error of order $\mathcal{O}(\mathrm{e}^{-8n})$ [1], but they do not provide arithmetic property for γ, as we can do in the last section.

5. Approximation of γ by rational numbers

In the numerical computation of formula

$$L_{n,m} = \sum_{k=0}^{n} \binom{n}{k} \binom{n+k}{k} (-1)^{n+k} \ln(n-m+k+1),$$

the problem is the evaluation of logarithmic functions.

A mean to avoid this drawback is the substitution of $\ln(n-m+k+1)$ by some suitable approximations. We will show now that Padé approximants are good enough to preserve the speed of convergence $\mathcal{O}(4^{-n})$.

Another expression of $L_{n,m}$ is

$$L_{n,m} = \ln(n-m+1) + \sum_{k=0}^{n} \binom{n}{k} \binom{n+k}{k} (-1)^{n+k} \ln\left(1 + \frac{k}{n-m+1}\right). \tag{34}$$

By substituting in $L_{n,m}$, $\ln(n+1+k-m)$ by its Padé approximant $[n/n]$, a rational approximation is obtained as following: for $n-m+1 = 2^p$, $p \in \mathbf{Z}$, let us define

$$\tilde{L}_{n,m} := p[n/n]_{t=1} + \sum_{k=0}^{n} \binom{n}{k} \binom{n+k}{k} (-1)^{n+k} [n/n]_{t=k/(n-m+1)}, \tag{35}$$

where $[n/n]$ is the Padé approximant of $\ln(1+t)$ at $t=0$

$$[n/n]_t = \frac{t \sum_{k=0}^{n} \binom{n}{k} \binom{n+k}{k} \left(\sum_{i=0}^{k-1} \frac{t^{i-k+n}(-1)^i}{i+1}\right)}{\sum_{k=0}^{n} \binom{n}{k} \binom{n+k}{k} t^{n-k}}. \tag{36}$$

If we replace in (8), the quantities $L_{n,m}$ by $\tilde{L}_{n,m}$, we get an approximation of γ by rational numbers:

Theorem 5. *For integers n, m, p such that $n-m+1 = 2^p$*

$$\gamma = A_{n,m} - \tilde{L}_{n,m} + \mathcal{O}(4^{-n}). \tag{37}$$

Proof. The Padé error for the logarithmic function is

$$\ln(1+x) - [n/n]_x = \frac{(-1)^n x^{n+1}}{P_n^*(-1/x)} \int_0^1 \frac{t^n(1-t)^n}{(1+xt)^{n+1}}\,\mathrm{d}t. \tag{38}$$

Let us set

$$L'_{n,m} := \ln(n-m+1) + \sum_{k=0}^{n} \binom{n}{k}\binom{n+k}{k}(-1)^{n+k}[n/n]_{t=k/(n-m+1)}. \tag{39}$$

We have to evaluate the difference $\delta_{n,m} := L_{n,m} - L'_{n,m}$. For sake of simplicity, we set $\zeta_k = k/(n-m+1)$.

$$\delta_{n,m} = \sum_{k=0}^{n} \binom{n}{k}\binom{n+k}{k}(-1)^{n+k}(\ln(1+\zeta_k) - [n/n]_{t=\zeta_k}) \tag{40}$$

$$= \sum_{k=0}^{n} \binom{n}{k}\binom{n+k}{k}(-1)^{k} \frac{\zeta_k^{n+1}}{P_n^*(-\zeta_k^{-1})} \int_0^1 \frac{t^n(1-t)^n}{(1+\zeta_k t)^{n+1}}\,\mathrm{d}t. \tag{41}$$

Since $\zeta_k \in [0, 1]$ and P_n^* has all its roots in $[0, 1]$, $|\zeta_k^n / P_n^*(-\zeta_k^{-1})| \leqslant 1/|P_n^*(-1)|$. On the other hand, the integral

$$\int_0^1 \frac{t^n(1-t)^n}{(1+\zeta_k t)^{n+1}}\,\mathrm{d}t \leqslant 4^{-n} \int_0^1 \frac{1}{(1+\zeta_k t)^{n+1}}\,\mathrm{d}t \leqslant 4^{-n}\frac{1}{n\zeta_k}. \tag{42}$$

So,

$$\begin{aligned}
|\delta_{n,m}| &\leqslant \sum_{k=0}^{n} \binom{n}{k}\binom{n+k}{k} \left|\frac{\zeta_k^{n+1}}{P_n^*(-\zeta_k^{-1})}\right| \left|\int_0^1 \frac{t^n(1-t)^n}{(1+\zeta_k t)^{n+1}}\,\mathrm{d}t\right| \\
&\leqslant \sum_{k=0}^{n} \binom{n}{k}\binom{n+k}{k} \frac{\zeta_k}{|P_n^*(-1)|} 4^{-n} \frac{1}{n\zeta_k} \\
&\leqslant \frac{1}{|nP_n^*(-1)|} 4^{-n} \sum_{k=0}^{n} \binom{n}{k}\binom{n+k}{k} = \frac{1}{|nP_n^*(-1)|} 4^{-n} |P_n^*(-1)| = (n4^n)^{-1}.
\end{aligned}$$

The goal is partly reached since the error between $L_{n,m}$ and its approximation is less than $J_{n,m}$. Now, let us consider the approximation of $\ln(n-m+1)$. It is difficult to approximate this number (which tends to infinity) with an error less than 4^{-n}. So, we consider sequences of integers n, such that $n-m+1$ is a power of 2: $n-m+1 = 2^p$. With this hypothesis, $\ln(n-m+1) = p\ln 2$.

In (38), if $x = 1$, $\ln 2 - [n/n]_{x=1} = (-1)^n / P_n^*(-1) \int_0^1 (t^n(1-t)^n/(1+t)^{n+1})\,\mathrm{d}t$. The asymptotic for Legendre polynomials are well known

$$P_n(\alpha) \sim (\alpha + \sqrt{\alpha^2 - 1})^n \quad \text{for } \alpha \in \mathbf{R}\backslash[-1, 1].$$

Thus the shifted Legendre Polynomials satisfy

$$P_n^*(t) \sim ((2t-1) + 2\sqrt{t^2 - t})^n \quad \text{for } t \in \mathbf{R}\backslash[0, 1].$$

The maximum of the fraction $(t(1-t)/(1+t))$ for $t \in [0, 1]$ is obtained for $t = \sqrt{2} - 1$, and its value is $(3 - 2\sqrt{2})$. Thus

$$|\ln 2 - [n/n]_{x=1}| \leqslant \frac{(3-2\sqrt{2})^n}{(3+2\sqrt{2})^n} \ln 2. \tag{43}$$

For $n-m+1 = 2^p$, $\ln(2^p) - p[n/n]_{x=1} \leqslant p(3-2\sqrt{2})^{2n}$ which is a $\mathrm{o}(4^{-n}/n)$. At last, the error $|L_{n,m} - \tilde{L}_{n,m}|$ satisfies

$$|L_{n,m} - \tilde{L}_{n,m}| = \mathcal{O}(4^{-n}/n).$$

Now,

$$\gamma - A_{n,m} + \tilde{L}_{n,m} = J_{n,m} + \tilde{L}_{n,m} - L_{n,m} = \mathcal{O}(4^{-n}) + \mathcal{O}(4^{-n}/n) = \mathcal{O}(J_{n,m}) \tag{44}$$

and the theorem is proved. □

6. Application to the arithmetic property of γ

The irrationality of γ remains an open question. In this section, we prove some sufficient conditions for the proof of the irrationality of Euler's constant. Corollary 6 is similar to the results found in [13,14,8], but it involves only the computation of rational numbers.

On the other hand, Corollary 7 is new because it depends on the decrease of a numerical sequence involving logarithms numbers.

Corollary 6. *Let us denote by* $\{x\}$ *the fractional part of the real number* x:

$$\{x\} := x - \lfloor x \rfloor$$

and $d_n := LCM(1, \ldots, n)$ (*lower common divisor*). *If for some integer* m, $\{d_{2^p+m-1}(-1)^m \tilde{L}_{2^p+m-1,m}\}$ *does not converge to* 0 *when* p *tends to infinity, then* γ *is irrational.*

Proof. Suppose that γ is rational, then there exists a pair of integers A, B such that $\gamma = A/B$. Then, for n greater than some integer N, $d_n\gamma$ is an integer. For integers n, m, p such that $n - m + 1 = 2^p$, the relation

$$\gamma = A_{n,m} - \tilde{L}_{n,m} + \mathcal{O}(J_{n,m}),$$

leads to

$$d_n\gamma = d_n A_{n,m} - d_n \tilde{L}_{n,m} + d_n \mathcal{O}(J_{n,m}).$$

$A_{n,m} = 2H_n$, so $d_n A_{n,m}$ is an integer, and thus the fractional part of $(-1)^m d_n \tilde{L}_{n,m}$ is equal to the fractional part of the positive sequence $(-1)^m d_n \mathcal{O}(J_{n,m})$ which converges to zero since $\lim_n d_n^{1/n} = e$ [11].

So, if for some integer m, the fractional part $(\{d_{2^p+m-1}(-1)^m \tilde{L}_{2^p+m-1,m}\})_p$ does not converge to 0, then γ is irrational. □

Another sufficient condition comes from the property of the error term in the asymptotic formula (20) and from the upper and lower bound of the LCM$(1, \ldots, n)$:

Corollary 7. *Let* $\mathcal{P}$ be *the following property*: *A sequence* $(x_n)_n$ *satisfies* $\mathcal{P}$ *if*

$$\forall N \in \mathbf{N}, \quad \exists n \geqslant N, \quad x_n - x_{n+1} < 0.$$

If for some integer m, *the sequence* $\{d_{2^p}(-1)^m L_{2^p,m}\}$ *satisfies* $\mathcal{P}$ *then* γ *is irrational.*

Proof. For the proof, we exploit the property of totally monotonic sequences (TMS).

A sequence u_n is called TMS if there exists a non negative measure $\mathrm{d}\mu$ with infinitely many points of increase such that

$$\forall n \in \mathbf{N}, \quad u_n = \int_0^\infty x^n \, \mathrm{d}\mu(x).$$

If the support of the measure $\mathrm{d}\mu$ is the interval $[0, 1/R]$, then $\forall n$, $u_{n+1}/u_n \leqslant R$ and $\lim_n u_{n+1}/u_n = R$. If $R = 1$, it is equivalent to

$$\forall n \in \mathbf{N}, \quad \forall k \in \mathbf{N}, \quad (-1)^k \Delta^k(u_n) > 0,$$

where $\Delta^0(u_n) := u_n$ and $\Delta^{k+1} u_n = \Delta^k u_{n+1} - \Delta^k u_n$. (See [16, p. 108].)

The previous properties can be applied to the sequence $J_{n,m}$ for which we prove some convergence properties. Since $\{d_n(-1)^m J_{n,m}\} = d_n(-1)^m J_{n,m} = \{d_n(-1)^m L_{n,m}\}$, if they are not satisfied by $\{d_n(-1)^m L_{n,m}\}$ then γ is irrational.

First we will prove that $J_{n,m}$ satisfies $d_{2n}(-1)^m J_{2n,m} < d_n(-1)^m J_{n,m}$: the numbers d_n and $J_{n,m}$ satisfy $2^n \leqslant d_n < \mathrm{e}^{1.039n}$ (see [15, pp. 12–13] for the lower bound and [11] for the upper one) $J_{n+1,m}/J_{n,m} < \frac{1}{4}$ (property of totally monotonic sequence [16, p. 135]).

$$\frac{d_n J_{n,m}}{d_{2n} J_{2n,m}} > \frac{2^n}{\mathrm{e}^{1.039\times 2n}} 4^n > 1.0014.$$

Thus, for all integer m, $(d_{2^p}(-1)^m J_{2^p,m})_{p\in\mathbf{N}}$ is a positive decreasing sequence, converging to 0. So, if $(\{d_{2^p}(-1)^m L_{2^p,m}\})_p$ is nondecreasing for p greater than any integer, then γ is irrational. $\square$

Consequence: If for some m, $\{d_{2^p}(-1)^m L_{2^p,m}\}$ satisfies $\mathscr{P}$, then γ is irrational.

Suppose that for some m, and some p $\{d_{2^p}(-1)^m L_{2^p,m}\} < \{d_{2^{p+1}}(-1)^m L_{2^{p+1},m}\}$, then it implies that if γ is rational then its denominator is greater than 2^p.

Numerical computation show that it is true for $m = 1,\ p = 15$. So, if γ is rational, then its denominator is greater than $2^{15} = 32768$.

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