Journal of Computational and Applied Mathematics 231 (2009) 447–472

Contents lists available at ScienceDirect

Journal of Computational and Applied Mathematics

journal homepage: www.elsevier.com/locate/cam

A priori error estimation for the dual mixed finite element method of the elastodynamic problem in a polygonal domain, I

L. Boulaajine [a], M. Farhloul [b,*], L. Paquet [a]

[a] Université de Valenciennes et du Hainaut Cambrésis, MACS, ISTV, F-59313 - Valenciennes Cedex 9, France

[b] Université de Moncton, Département de Mathématiques et de Statistique, Moncton, N.B., E1A 3E9, Canada

ARTICLE INFO

Article history:
Received 25 June 2008
Received in revised form 24 February 2009

MSC:
65M60
65M15
65M50

Keywords:
Sobolev spaces
Elastodynamic
Dual mixed finite element
Newmark scheme
Lagrange multiplier
Hybrid formulation
Error estimation

ABSTRACT

In this paper we analyze a new dual mixed formulation of the elastodynamic system in polygonal domains. In this formulation the symmetry of the strain tensor is relaxed by the rotation of the displacement. For the time discretization of this new dual mixed formulation, we use an explicit scheme. After the analysis of stability of the fully discrete scheme, L^∞ in time, L^2 in space a priori error estimates are derived for the approximation of the displacement, the strain, the pressure and the rotation. Numerical experiments confirm our theoretical predictions.

1. Introduction

The purpose of this paper is the analysis of a finite element method for approximating the linear elastodynamic system using a new dual mixed formulation for the discretization in the spatial variables and an explicit Newmark scheme for the discretization in time. The explicit Newmark scheme is shown to be stable under an appropriate CFL condition. The analysis of an implicit Newmark scheme will be presented in [1].

The analysis of a priori error estimates for the mixed finite element method of a second order hyperbolic system in regular domains using symmetric approximations of the stress was initiated in [2,3] see also [4]. But to our knowledge a similar analysis for the dual mixed formulation of the linear elastodynamic system in nonregular domains, introducing as a new unknown strain tensor, was not yet done. Therefore the goal of this paper is to make this analysis. A priori error estimates are proved for the approximation of the displacement, the strain, the pressure and the rotation, firstly for the semi-discretized solution and then for the completely discretized solution by the explicit Newmark scheme in the time variable.

Over the last two decades there has been considerable interest in the areas of mixed finite element discretizations of the corresponding stationary problem, i.e. the system of linear elasticity; let us quote, for example, [5–10]. The main difficulty

* Corresponding author. Tel.: +1 506 858 7808; fax: +1 506 858 4396.
E-mail addresses: lboulaaj@univ-valenciennes.fr (L. Boulaajine), mohamed.farhloul@umoncton.ca (M. Farhloul), Luc.Paquet@univ-valenciennes.fr (L. Paquet).

0377-0427/$ – see front matter
doi:10.1016/j.cam.2009.03.013

appearing in this problem is finding a way to take into account the symmetry of the strain tensor. In our approach, the symmetry of the strain tensor is relaxed by a Lagrange multiplier, which is nothing else than the rotation.

The outline of this paper is as follows: Section 2 defines some notation, presents the model evolution problem we shall consider and recall two comparison results concerning continuous and discrete Gronwall's inequalities. In Section 3, we define the new dual mixed formulation of the model evolution problem. Section 4 is devoted to some regularity results of the solution of our elastodynamic system in terms of weighted Sobolev spaces. In Section 5, we introduce the semi-discrete mixed formulation and prove the existence and uniqueness of the solution for this formulation and recall some results concerning the inf–sup and coercivity conditions. Then, under some adequate refinement rules of meshes, we establish some error estimates on some interpolation operators and we prove an inverse inequality for the divergence operator. In Section 5.1.1, we derive some error estimates between the exact solution of the mixed problem and the solution of the elliptic projection problem, which will be used in Section 5.1.2 to derive the error estimates between the exact and the semi-discrete solution. Section 6 is concerned with the fully discrete finite element scheme: existence and uniqueness of the solution of the fully discretized problem, stability analysis and a priori error estimates between the exact solution and its fully discrete approximation for the explicit scheme. The proof of the error estimates rest on the introduction of an auxiliary problem: the elliptic projection problem. The numerical experiments of Section 7 confirm our theoretical predictions. In Section 8 we present conclusions.

2. Preliminaries and notations

2.1. The model problem

Let us fix a bounded plane domain Ω with a polygonal boundary. More precisely, we assume that Ω is a simply connected domain and that its boundary Γ is the union of a finite number of linear segments $\bar{\Gamma}_j$, $1 \le j \le n_e$ (Γ_j is assumed to be an open segment). We also fix a partition of $\{1, 2, \dots, n_e\}$ into two subsets $I_{\mathcal{N}}$ and $I_{\mathcal{D}}$. The union Γ_D of the Γ_j, j running over $I_{\mathcal{D}}$, is the part of the boundary Γ, where we assume zero displacement field. The union Γ_N, of the Γ_j, $j \in I_{\mathcal{N}}$, is the part of the boundary Γ where we assume zero traction field.

In this domain Ω, we consider isotropic elastic homogeneous material. Let $u = (u_1, u_2)$ be the displacement field and $f = (f_1, f_2) \in [L^2(\Omega)]^2$ the body force per unit of mass. Thus the displacement field $u = (u_1, u_2)$ satisfies the following equations:

$$\begin{cases} u_{tt} - \operatorname{div} \sigma_s(u) = f & \text{in } [0, T] \times \Omega, \\ u = 0 & \text{on } [0, T] \times \Gamma_D, \\ \sigma_s(u).n = 0 & \text{on } [0, T] \times \Gamma_N, \\ u(0, .) = u_0 & \text{in } \Omega, \\ u_t(0, .) = u_1 & \text{in } \Omega, \end{cases} \tag{2.1}$$

where u_0 and u_1 are the initial conditions on displacements and velocities. n denotes the unit outward normal field along Γ. The stress tensor $\sigma_s(u)$ is defined by

$$\sigma_s(u) := 2\mu\epsilon(u) + \lambda \operatorname{tr} \epsilon(u)\delta. \tag{2.2}$$

The positive constants μ and λ are called the Lamé coefficients. We assume that

$$(\lambda, \mu) \in [\lambda_0, \lambda_1] \times [\mu_1, \mu_2] \tag{2.3}$$

where

$$0 < \mu_1 < \mu_2 \quad \text{and} \quad 0 < \lambda_0 < \lambda_1.$$

As usual, $\epsilon(u)$ denotes the linearized strain tensor (*i.e.*, $\epsilon(u) = \frac{1}{2}(\nabla u + (\nabla u)^T)$) and δ the identity tensor.

For reasons of simplicity in our theoretical analysis,we have chosen homogeneous boundary conditions on both Dirichlet and Neumann boundaries. The extension to nonhomogeneous boundary conditions is done without difficulty. Let us note that numerical tests (see Section 7) are made under the nonhomogeneous surface traction. In what follows, we will use the following notation. For $\tau = (\tau_{ij}) \in [H(\operatorname{div}; \Omega)]^2$, we denote by

$$\operatorname{div}(\tau) = \left(\frac{\partial \tau_{11}}{\partial x_1} + \frac{\partial \tau_{12}}{\partial x_2}, \frac{\partial \tau_{21}}{\partial x_1} + \frac{\partial \tau_{22}}{\partial x_2} \right),$$

$$\operatorname{as}(\tau) = \tau_{21} - \tau_{12}.$$

For $v = (v_1, v_2) \in [H^1(\Omega)]^2$, we recall that

$$\operatorname{rot} v = \frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2}.$$

As usual, we denote by $L^2(.)$ the Lebesgue space of square integrable functions and by $H^s(.), s \geq 0$, the standard Sobolev spaces. The usual norm and semi-norm of $H^s(D)$ are denoted by $\| \cdot \|_{s,D}$ and $| \cdot |_{s,D}$. The inner product in $[L^2(\Omega)]^2$ will be written $(.,.)$. If $\sigma = (\sigma_{ij}), \tau = (\tau_{ij}) \in [L^2(\Omega)]^{2\times 2}$, then we denote by

$$\sigma : \tau = \sum_{i,j} \sigma_{ij}\tau_{ij} \quad \text{and} \quad (\sigma, \tau) = \int_\Omega \sigma : \tau \, dx.$$

We now introduce the Hilbert space

$$[H^1_{\Gamma_D}(\Omega)]^2 := \{v \in [H^1(\Omega)]^2; v|_{\Gamma_D} = 0\}.$$

Finally, in order to avoid excessive use of constants, we use the following notation: $a \lesssim b$ stand for $a \leq c\, b$, with positive constant c independent of a, b, h and Δt.

2.2. Gronwall's inequalities

In this section, we recall two comparison results [11], which will be useful in the stability and convergence analysis of our problem. Let $\phi(.) \geq 0$ be such that $\phi_t(t) \leq \rho\phi(t) + \eta(t)$ for $0 \leq t \leq T$, where $\rho \geq 0$ is some constant and $\eta(.) \geq 0$, $\eta \in L^1([0, T])$. Then

$$\phi(t) \leq e^{\rho T}\left(\phi(0) + \int_0^T \eta(s)\, ds\right), \quad \forall\, t \in [0, T]. \tag{2.4}$$

Let two nonnegative sequences $(k_n)_{n\geq 0}$, $(p_n)_{n\geq 0}$ be given, $g_0 \geq 0$ given also and let us suppose that the sequence $(\phi_n)_{n\geq 0}$ satisfies:

$$\begin{cases} \phi_0 \leq g_0, \\ \phi_n \leq g_0 + \sum_{s=0}^{n-1} p_s + \sum_{s=0}^{n-1} k_s\phi_s, \quad \forall n \geq 1. \end{cases} \tag{2.5}$$

Then

$$\phi_n \leq \left(g_0 + \sum_{s=0}^{n-1} p_s\right) \exp\left(\sum_{s=0}^{n-1} k_s\right), \quad \forall n \geq 1. \tag{2.6}$$

3. The dual mixed formulation

Introducing as new unknowns:

$$\sigma := 2\mu\epsilon(u), \qquad p := -\lambda \text{div}\,(u) \quad \text{and} \quad \omega := \frac{1}{2}\text{rot}\,(u),$$

and the spaces:

$$\Sigma_0 := \{(\tau, q) \in [L^2(\Omega)]^{2\times 2} \times L^2(\Omega);\ \text{div}\,(\tau - q\delta) \in [L^2(\Omega)]^2, (\tau - q\delta).n = 0 \text{ on } \Gamma_N\}, \tag{3.1}$$

$$M := \{(v, \theta) \in [L^2(\Omega)]^2 \times L^2(\Omega)\}, \tag{3.2}$$

we state the dual mixed formulation for our model hyperbolic equation (2.1): find $(\sigma(.), p(.)) \in L^2([0, T]; \Sigma_0)$, $u(.) \in H^2([0, T]; [L^2(\Omega)]^2)$ and $\omega(.) \in L^2([0, T]; L^2(\Omega))$ such that for all $(\tau, q) \in \Sigma_0$, for all $(v, \theta) \in M$ and for a.e. $t \in [0, T]$, we have

$$\begin{cases} \frac{1}{2\mu}(\sigma(t), \tau) + \frac{1}{\lambda}(p(t), q) + (\text{div}\,(\tau - q\delta), u(t)) + (\text{as}\,(\tau), \omega(t)) = 0, \\ (u_{tt}(t), v) - (\text{div}\,(\sigma(t) - p(t)\delta), v) - (\text{as}\,(\sigma(t)), \theta) - (f(t), v) = 0, \\ u(0) = u_0, \qquad u_t(0) = u_1. \end{cases} \tag{3.3}$$

We conclude this section by introducing some notations. We set

$$\underset{\sim}{\sigma} = (\sigma, p), \qquad \underset{\sim}{\tau} = (\tau, q), \qquad \underset{\sim}{u} = (u, \omega), \qquad \underset{\sim}{v} = (v, \theta),$$

$$a(\underset{\sim}{\sigma}, \underset{\sim}{\tau}) := \frac{1}{2\mu}(\sigma, \tau) + \frac{1}{\lambda}(p, q), \quad \forall \underset{\sim}{\sigma},\ \underset{\sim}{\tau} \in \Sigma_0, \tag{3.4}$$

$$b(\underset{\sim}{\tau}, \underset{\sim}{v}) := (\text{div}\,(\tau - q\delta), v) + (\text{as}\,(\tau), \theta), \quad \forall\, \underset{\sim}{\tau} \in \Sigma_0, \forall\, \underset{\sim}{v} \in [L^2(\Omega)]^2 \times L^2(\Omega). \tag{3.5}$$

With these notations, the mixed formulation (3.3) may be rewritten: find $\underset{\sim}{\sigma}(.) = (\sigma(.), p(.)) \in L^2([0,T]; \Sigma_0)$ and $\underset{\sim}{u}(.) = (u(.), \omega(.)) \in H^2([0,T]; [L^2(\Omega)]^2) \times L^2([0,T]; L^2(\Omega))$ such that $u(0) = u_0,\ u_t(0) = u_1$ and for a.e. $t \in [0,T]$:

$$\begin{cases} a(\underset{\sim}{\sigma}(t), \underset{\sim}{\tau}) + b(\underset{\sim}{\tau}, \underset{\sim}{u}(t)) = 0, & \forall \underset{\sim}{\tau} \in \Sigma_0, \\ b(\underset{\sim}{\sigma}(t), \underset{\sim}{v}) + (\mathcal{F}(t), \underset{\sim}{v}) = (u_{tt}(t), v), & \forall \underset{\sim}{v} \in [L^2(\Omega)]^2 \times L^2(\Omega), \end{cases} \tag{3.6}$$

where $(\mathcal{F}(t), \underset{\sim}{v}) := (f(t), v)$.

4. Regularity of the solutions

Let $u \in L^2(0,T; [H^1_{\Gamma_D}(\Omega)]^2)$ be such that $\frac{du}{dt} \in L^2(0,T; [L^2(\Omega)]^2)$, be the solution of (2.1). We consider the Lamé operator defined by

$$L := -\mu\Delta - (\lambda+\mu)\nabla \mathrm{div}.$$

Thus, equivalently u is the weak solution of the problem

$$\begin{cases} u_{tt} + Lu = f & \text{in } [0,T]\times\Omega, \\ u = 0 & \text{on } [0,T]\times\Gamma_D, \\ \sigma_s(u).n = 0 & \text{on } [0,T]\times\Gamma_N, \\ u(0,.) = u_0 & \text{in } \Omega, \\ u_t(0,.) = u_1 & \text{in } \Omega. \end{cases} \tag{4.1}$$

It is well known (see [10] or [12–14]) that the weak solution of the corresponding Lamé system of (4.1) presents vertex singularities. To describe them, we need to introduce the following notations:

Definition 4.1. Let S_j $(1 \le j \le n_e)$ be the vertex of our polygonal domain Ω at the intersection of the sides Γ_j and Γ_{j+1} $(\Gamma_{n_e+1} := \Gamma_1)$. Let us denote by ω_j the measure of the angle at the vertex S_j. By the characteristic equation associated to the vertex S_j, we mean the transcendental equation in the complex variable α:

$$\sin^2(\alpha\omega_j) = \left[\frac{\lambda+\mu}{\lambda+3\mu}\right]^2 \alpha^2 \sin^2\omega_j, \tag{4.2}$$

if S_j is a vertex of Dirichlet type, i.e. $j, j+1 \in I_{\mathcal{D}}$,

$$\sin^2(\alpha\omega_j) = \alpha^2 \sin^2\omega_j, \tag{4.3}$$

if S_j is a vertex of Neumann type, i.e. $j, j+1 \in I_{\mathcal{N}}$,

$$\sin^2(\alpha\omega_j) = \frac{(\lambda+2\mu)^2 - (\lambda+\mu)^2\alpha^2\sin^2\omega_j}{(\lambda+\mu)(\lambda+3\mu)}, \tag{4.4}$$

if S_j is a vertex of mixed type, i.e. $j \in I_{\mathcal{D}}, j+1 \in I_{\mathcal{N}}$ or $j \in I_{\mathcal{N}}, j+1 \in I_{\mathcal{D}}$.

Definition 4.2. For any scalar function $\phi \in C^0(\overline{\Omega})$ such that $\phi(x) > 0$ for every $x \in \bar{\Omega} \setminus \{S_1, S_2, \ldots, S_{n_e}\}$ and any $m, k \in \mathbb{N}$, we define

$$H^{m,k}_\phi(\Omega) = \{v \in H^m(\Omega) \cap H^{m+k}_{loc}(\Omega);\ \phi D^\beta v \in L^2(\Omega),\ \forall \beta \in \mathbb{N}^2 \text{ such that } m < |\beta| \le m+k\}.$$

$H^{m,k}_\phi(\Omega)$ is a Hilbert space equipped with the norm:

$$\|v\|_{m,k;\phi,\Omega} = \left(\|v\|^2_{m,\Omega} + \sum_{m<|\beta|\le m+k} \|\phi D^\beta v\|^2_{0,\Omega}\right)^{1/2}.$$

On this space, we also define the semi-norm:

$$|v|_{m,k;\phi,\Omega} = \left(\sum_{|\beta|=m+k} \|\phi D^\beta v\|^2_{0,\Omega}\right)^{1/2}.$$

We consider also the spaces $L^2(0,T; H^{m,k}_\phi(\Omega))$ endowed with the norm:

$$\|v\|_{L^2(H^{m,k}_\phi)} = \left(\int_0^T \|v\|^2_{m,k;\phi,\Omega}\,\mathrm{d}t\right)^{1/2},$$

and $L^\infty(0,T; H^{m,k}_\phi(\Omega))$ endowed with the norm $\|v\|_{L^\infty(H^{m,k}_\phi)} = \operatorname{ess\,sup}_{0\le t\le T} \|v(t)\|_{m,k;\phi,\Omega}$.

Let us set $\xi = \min_{j=1,\dots,n_e} \xi_j$ where

$$\xi_j = \inf_k \{\mathrm{Re}\,\alpha_{j,k};\ \mathrm{Re}\,\alpha_{j,k} > 0\},$$

where $\alpha_{j,k}$ is solution of the appropriate transcendental equation appearing in Definition 4.1. By ([10], Lemma 2.2), $\xi > \frac{1}{2}$. Let us pick some $\alpha \in]1-\xi, 1/2[$ if $\xi \le 1$, and let us take $\alpha = 0$ if $\xi > 1$.

Now we can give the following regularity result:

Proposition 4.3. *Let us suppose that the appropriate characteristic equation among (4.2)–(4.4) for each vertex of Ω has no root on the vertical line* Re $\alpha = 1$ *in the complex plane. Let $\phi \in C^0(\overline{\Omega})$, as above in Definition 4.2, such that $\phi(x) = r_j(x)^\alpha$ in a neighborhood of the vertex S_j of the polygonal domain Ω for every $j = 1, \dots, n_e$ where $r_j(x) = |x - S_j|$ ($|.|$ means Euclidian norm).*

Let us suppose that:

$$\begin{cases} f \in H^3(0,T;[L^2(\Omega)]^2), \\ u_0,\ u_1,\ f(0) - Lu_0,\ f_t(0) - Lu_1 \in [H^1_{\Gamma_D}(\Omega)]^2, \\ f_{tt}(0) - Lf(0) + L^2 u_0 \in [L^2(\Omega)]^2. \end{cases} \tag{4.5}$$

Then $u \in C(0,T;[H^{1,1}_\phi(\Omega)]^2 \cap [H^1_{\Gamma_D}(\Omega)]^2)$ and $u_{tt} \in L^2(0,T;[H^{1,1}_\phi(\Omega)]^2 \cap [H^1_{\Gamma_D}(\Omega)]^2)$. Consequently $\sigma \in L^\infty(0,T;[H^{0,1}_\phi(\Omega)]^{2\times 2})$, $p \in L^\infty(0,T;H^{0,1}_\phi(\Omega))$ and $\omega \in L^\infty(0,T;H^{0,1}_\phi(\Omega))$. Moreover $\sigma_{tt} \in L^2(0,T;[H^{0,1}_\phi(\Omega)]^{2\times 2})$, $p_{tt} \in L^2(0,T;H^{0,1}_\phi(\Omega))$ and $\omega_{tt} \in L^2(0,T;H^{0,1}_\phi(\Omega))$.

Proof. According to Theorem 30.1 p. 442–443 of [15] we have $u \in H^3(0,T;[H^1_{\Gamma_D}(\Omega)]^2)$ and $u^{(4)} \in L^2(0,T;[L^2(\Omega)]^2)$. In particular $u_{tt} \in L^2(0,T;[H^1_{\Gamma_D}(\Omega)]^2)$ and $(u_{tt})_{tt} \in L^2(0,T;[L^2(\Omega)]^2)$. Knowing that $Lu = f - u_{tt}$, we have $Lu_{tt} = f_{tt} - (u_{tt})_{tt} \in L^2(0,T;[L^2(\Omega)]^2)$. Thus $u_{tt} \in L^2(0,T;[H^1_{\Gamma_D}(\Omega)]^2)$ and $Lu_{tt} \in L^2(0,T;[L^2(\Omega)]^2)$. That is $u_{tt} \in L^2(0,T;D(L))$ where $D(L)$ denotes the domain of the Lamé operator. But $D(L) \hookrightarrow [H^{1,1}_\phi(\Omega)]^2$ by adapting Corollary 2.4 p. 326 of [10]. Thus $u_{tt} \in L^2(0,T;[H^{1,1}_\phi(\Omega)]^2)$ and consequently $\sigma_{tt} \in L^2(0,T;[H^{0,1}_\phi(\Omega)]^{2\times 2})$, $p_{tt} \in L^2(0,T;H^{0,1}_\phi(\Omega))$, $\omega_{tt} \in L^2(0,T;H^{0,1}_\phi(\Omega))$. On the other hand $u_t \in H^2(0,T;[H^1_{\Gamma_D}(\Omega)]^2)$ and $Lu_t = f_t - u_{ttt} \in L^2(0,T;[L^2(\Omega)]^2)$. So that $u_t \in L^2(0,T;D(L))$. And hence,

$$u_t \in L^2(0,T;[H^{1,1}_\phi(\Omega)]^2). \tag{4.6}$$

Similarly, we have $u \in H^2(0,T;[H^1_{\Gamma_D}(\Omega)]^2)$ and $Lu = f - u_{tt} \in L^2(0,T;[L^2(\Omega)]^2)$, so that $u \in L^2(0,T;D(L))$, and hence also $u \in L^2(0,T;[H^{1,1}_\phi(\Omega)]^2)$. From this and (4.6) we get

$$u \in C(0,T;[H^{1,1}_\phi(\Omega)]^2) \subset L^\infty(0,T;[H^{1,1}_\phi(\Omega)]^2).$$

Thus $\sigma \in L^\infty(0,T;[H^{0,1}_\phi(\Omega)]^{2\times 2})$, $p \in L^\infty(0,T;H^{0,1}_\phi(\Omega))$ and $\omega \in L^\infty(0,T;H^{0,1}_\phi(\Omega))$. Moreover $u, u_t \in L^2(0,T;[H^1_{\Gamma_D}(\Omega)]^2)$ implies $u \in C(0,T;[H^1_{\Gamma_D}(\Omega)]^2)$. ■

Proposition 4.4. *Let us suppose that the appropriate characteristic equation among (4.2)–(4.4) for each vertex of Ω has no root on the vertical line* Re $\alpha = 2$ *in the complex plane. Let $\phi \in C^0(\overline{\Omega})$ as in Proposition 4.3 Let us suppose that:*

$$\begin{cases} f \in H^6(0,T;[L^2(\Omega)]^2) \\ f^{(4)} \in L^2(0,T;[H^1(\Omega)]^2) \\ u_0,\ u_1,\ f(0) - Lu_0 \in [H^1_{\Gamma_D}(\Omega)]^2 \\ f^{(1)}(0) - Lu_1 \in [H^1_{\Gamma_D}(\Omega)]^2 \\ f^{(2)}(0) - Lf(0) + L^2 u_0 \in [H^1_{\Gamma_D}(\Omega)]^2 \\ f^{(3)}(0) - Lf^{(1)}(0) + L^2 u_1 \in [H^1_{\Gamma_D}(\Omega)]^2 \\ f^{(4)}(0) - Lf^{(2)}(0) + L^2 f(0) - L^3 u_0 \in [H^1_{\Gamma_D}(\Omega)]^2 \\ f^{(5)}(0) - Lf^{(3)}(0) + L^2 f^{(1)}(0) - L^3 u_1 \in [L^2(\Omega)]^2. \end{cases} \tag{4.7}$$

Then $\sigma_{tttt} \in L^2(0,T;[H^{0,1}_\phi(\Omega)]^{2\times 2})$, $p_{tttt} \in L^2(0,T;H^{0,1}_\phi(\Omega))$, $\omega_{tttt} \in L^2(0,T;H^{0,1}_\phi(\Omega))$ and $u_{tttt} \in L^2(0,T;[H^{1,2}_\phi(\Omega)]^2) \cap C(0,T;[H^1_{\Gamma_D}(\Omega)]^2)$.

Proof. By once more Theorem 30.1 p. 442–443 of [15], it follows that $u \in H^6(0,T;[H^1_{\Gamma_D}(\Omega)]^2)$. By the equation $Lu^{(4)} = f^{(4)} - u^{(6)}$ and the hypothesis $f^{(4)} \in L^2(0,T;[H^1(\Omega)]^2)$, it follows by Corollary 2.4 p. 326 of [10] that $u^{(4)} \in L^2(0,T;[H^{1,2}_\phi(\Omega)]^2)$. This implies the above assertions. ■

5. The semi-discrete mixed formulation

We assume that Ω is discretized by a regular family of triangulations $(\mathcal{T}_h)_{h>0}$ in the sense of [16]. If $T \in \mathcal{T}_h$, then we denote by h_T its diameter. By abuse of notation ([16], remark 17.1 p. 131), h denotes also $\max_{T\in\mathcal{T}_h} h_T$ (the real meaning of h is indicated by the context). We introduce the finite dimensional subspaces $\Sigma_{0,h}$ and $V_h \times W_h$ of Σ_0 and M respectively defined by

$$\Sigma_{0,h} := \{(\tau_h, q_h) \in \Sigma_0;\ \forall T \in \mathcal{T}_h :\ q_{h|T} \in \mathbb{P}_1(T) \text{ and } \tau_{h|T} \in [\mathbb{P}_1(T)]^{2\times 2} \oplus [\mathbb{R}\,\mathrm{Curl}\, b_T]^2\} \tag{5.1}$$

$$V_h \times W_h := \{(v_h, \theta_h) \in M;\ \forall T \in \mathcal{T}_h :\ v_{h|T} \in [\mathbb{P}_0(T)]^2 \text{ and } \theta_{h|T} \in \mathbb{P}_1(T)\}. \tag{5.2}$$

Note that by $\tau_{h|T} \in [\mathbb{P}_1(T)]^{2\times 2} \oplus [\mathbb{R}\,\mathrm{Curl}\, b_T]^2$, we mean that there exist polynomials on T of degree ≤ 1 : $p_{11} \in \mathbb{P}_1(T)$, $p_{12} \in \mathbb{P}_1(T)$, $p_{21} \in \mathbb{P}_1(T)$, $p_{22} \in \mathbb{P}_1(T)$ and two real numbers α_1, α_2 such that

$$\tau_{h|T} = \begin{bmatrix} p_{11} + \alpha_1 \dfrac{\partial b_T}{\partial x_2} & p_{12} - \alpha_1 \dfrac{\partial b_T}{\partial x_1} \\ p_{21} + \alpha_2 \dfrac{\partial b_T}{\partial x_2} & p_{22} - \alpha_2 \dfrac{\partial b_T}{\partial x_1} \end{bmatrix},$$

where b_T denotes the bubble function for the actual triangular element T defined by

$$b_T = 27\lambda_1\lambda_2\lambda_3.$$

$\lambda_1, \lambda_2, \lambda_3$ denote the barycentric coordinates on T. Now we introduce the following semi-discretized problem: Find $(\sigma_h(.), p_h(.)) \in L^2([0, T]; \Sigma_{0,h})$, $u_h(.) \in H^2([0, T]; V_h)$ and $\omega_h(.) \in L^2([0, T]; W_h)$ such that for all $(\tau_h, q_h) \in \Sigma_{0,h}$, for all $(v_h, \theta_h) \in V_h \times W_h$ and for a.e. $t \in [0, T]$, we have:

$$\begin{cases} \dfrac{1}{2\mu}(\sigma_h(t), \tau_h) + \dfrac{1}{\lambda}(p_h(t), q_h) + (\mathrm{div}\,(\tau_h - q_h\delta), u_h(t)) + (\mathrm{as}\,(\tau_h), \omega_h(t)) = 0, \\ (u_{h,tt}(t), v_h) - (\mathrm{div}\,(\sigma_h(t) - p_h(t)\delta), v_h) - (\mathrm{as}\,(\sigma_h(t)), \theta_h) - (f(t), v_h) = 0, \\ u_h(0) = u_{0,h}, \qquad u_{h,t}(0) = u_{1,h}. \end{cases} \tag{5.3}$$

We may think $u_{0,h}$ and $u_{1,h}$ as approximations in V_h of u_0 and u_1 respectively. The initial conditions $u_{0,h}$ and $u_{1,h}$ will be specified later. With the notations (3.4) and (3.5), the semi-discretized problem (5.3) may be rewritten: Find $\underset{\sim h}{\sigma}(.) = (\sigma_h(.), p_h(.)) \in L^2([0, T]; \Sigma_{0,h})$ and $\underset{\sim h}{u}(.) = (u_h(.), \omega_h(.)) \in H^2([0, T]; V_h) \times L^2([0, T]; W_h)$ such that for a.e. $t \in [0, T]$, we have:

$$\begin{cases} a(\underset{\sim h}{\sigma}(t), \underset{\sim h}{\tau}) + b(\underset{\sim h}{\tau}, \underset{\sim h}{u}(t)) = 0, \quad \forall \underset{\sim h}{\tau} = (\tau_h, q_h) \in \Sigma_{0,h}, \\ b(\underset{\sim h}{\sigma}(t), \underset{\sim h}{v}) + (\mathcal{F}(t), \underset{\sim h}{v}) = (u_{h,tt}(t), v_h), \quad \forall \underset{\sim h}{v} = (v_h, \theta_h) \in V_h \times W_h, \\ u_h(0) = u_{0,h}, \qquad u_{h,t}(0) = u_{1,h}. \end{cases} \tag{5.4}$$

The existence and uniqueness of a solution $((\sigma_h(.), p_h(.)), (u_h(.), \omega_h(.)))$ of (5.3) or equivalently to (5.4) are shown in the following lemma:

Lemma 5.1. *A solution $((\sigma_h(.), p_h(.)), (u_h(.), \omega_h(.)))$ of (5.3) or equivalently to (5.4) exists and is unique.*

Proof. The first and the second equation of the evolution problem (5.4) can be rewritten for a.e. $t \in [0, T]$ as

$$\begin{cases} a(\underset{\sim h}{\sigma}(t), \underset{\sim h}{\tau}) + b(\underset{\sim h}{\tau}, \underset{\sim h}{u}(t)) = 0, \quad \forall \underset{\sim h}{\tau} = (\tau_h, q_h) \in \Sigma_{0,h}, \\ b(\underset{\sim h}{\sigma}(t), \underset{\sim h}{v}) = -(f(t) - u_{h,tt}(t), v_h), \quad \forall \underset{\sim h}{v} = (v_h, \theta_h) \in V_h \times W_h. \end{cases} \tag{5.5}$$

We may think the solution $(\underset{\sim h}{\sigma}(t), \underset{\sim h}{u}(t)) \in \Sigma_{0,h} \times (V_h \times W_h)$ of (5.5), for a fixed time, as a solution of the stationary problem: find $(\underset{\sim h}{\sigma}, \underset{\sim h}{u}) \in \Sigma_{0,h} \times (V_h \times W_h)$ solution of

$$\begin{cases} a(\underset{\sim h}{\sigma}, \underset{\sim h}{\tau}) + b(\underset{\sim h}{\tau}, \underset{\sim h}{u}) = 0, \quad \forall \underset{\sim h}{\tau} = (\tau_h, q_h) \in \Sigma_{0,h}, \\ b(\underset{\sim h}{\sigma}, \underset{\sim h}{v}) = (g, v_h), \quad \forall \underset{\sim h}{v} = (v_h, \theta_h) \in V_h \times W_h, \end{cases} \tag{5.6}$$

where $g = -(f(t) - u_{h,tt}(t)) \in [L^2(\Omega)]^2$. We consider the pair of operators (S_h, T_h) defined by

$$(S_h, T_h) : [L^2(\Omega)]^2 \longrightarrow \Sigma_{0,h} \times (V_h \times W_h)$$
$$g \longmapsto (\underset{\sim h}{\sigma}, \underset{\sim h}{u}).$$

The evolution problem (5.5) can be rewritten as

$$\begin{cases} \underset{\sim h}{u}(t) = -T_h\left(P_h^0 f(t) - \dfrac{\mathrm{d}^2 u_h}{\mathrm{d}t^2}(t)\right), \\ \underset{\sim h}{\sigma}(t) = -S_h\left(P_h^0 f(t) - \dfrac{\mathrm{d}^2 u_h}{\mathrm{d}t^2}(t)\right), \end{cases}$$

where P_h^0 is the L^2-orthogonal projection from $[L^2(\Omega)]^2$ onto V_h.
In particular

$$u_h(t) = -T_{h,1}\left(P_h^0 f(t) - \frac{\mathrm{d}^2 u_h}{\mathrm{d}t^2}(t)\right). \tag{5.7}$$

Let us show that the operator $T_{h,1}|V_h : V_h \to V_h$ is invertible. Suppose that $\int_\Omega g.T_{h,1}g \,\mathrm{d}x = 0$. Then from (5.6) we get $a(\underset{\sim h}{\sigma}, \underset{\sim h}{\sigma}) = 0$, i.e.

$$\frac{1}{2\mu}\int_\Omega |\sigma_h|^2 \,\mathrm{d}x + \frac{1}{\lambda}\int_\Omega |p_h|^2 \,\mathrm{d}x = 0.$$

Hence $\sigma_h = 0$ and $p_h = 0$, i.e.

$$\underset{\sim h}{\sigma} = 0. \tag{5.8}$$

By the first equation of (5.6), it now follows that:

$$b(\underset{\sim h}{\tau}, \underset{\sim h}{u}) = 0, \quad \forall \underset{\sim h}{\tau} \in \Sigma_{0,h}.$$

The inf–sup inequality (5.10) yields

$$\underset{\sim h}{u} = (u_h, \omega_h) = 0.$$

Thus in particular, if $\int_\Omega g.T_{h,1}g \,\mathrm{d}x = 0$, then $T_{h,1}g = 0$.

Now, if $g_h \in V_h$ and $\int_\Omega g_h.T_{h,1}g_h \,\mathrm{d}x = 0$, then by (5.8) we have $\underset{\sim h}{\sigma} = 0$ and by the second equation of (5.6), we have $(g_h, v_h) = 0,\ \forall v_h \in V_h$. Thus

$$g_h = 0.$$

Finally, we have proved that $T_{h,1}|V_h : V_h \longrightarrow V_h$ is injective, thus invertible. From (5.7) follows:

$$-P_h^0 f(t) + \frac{\mathrm{d}^2 u_h}{\mathrm{d}t^2}(t) = (T_{h,1}|V_h)^{-1}(u_h(t)). \tag{5.9}$$

Hence

$$\frac{\mathrm{d}^2 u_h}{\mathrm{d}t^2}(t) = P_h^0 f(t) + (T_{h,1}|V_h)^{-1}(u_h(t)).$$

If we consider a basis of the subspace V_h, we obtain an inhomogeneous linear system of differential equations, and if furthermore we fix the initial conditions $u_h(0)$ and $\frac{\mathrm{d}u_h}{\mathrm{d}t}(0)$ in V_h, problem (5.9) has a unique solution. ■

Before discussing some error estimates between the exact solution and its elliptic projection, let us recall some auxiliary results [10]. Adapting the proof of Proposition 4.2 of [10] we obtain:

Proposition 5.2 ([10]). *There exists a strictly positive constant β^*, independent of h, such that*

$$\sup_{\underset{\sim h}{\tau} = (\tau_h, q_h) \in \Sigma_{0,h}} \frac{b(\underset{\sim h}{\tau}, \underset{\sim h}{v})}{\|\underset{\sim h}{\tau}\|_{0,\Omega}} \geq \beta^* \|\underset{\sim h}{v}\|_{0,\Omega}, \quad \forall \underset{\sim h}{v} = (v_h, \theta_h) \in V_h \times W_h. \tag{5.10}$$

Proposition 5.3 ([10]). *The bilinear form $a(.,.)$ defined by (3.4) is coercive uniformly with respect to λ on*

$$K_h := \left\{\underset{\sim h}{\tau} = (\tau_h, q_h) \in \Sigma_{0,h};\ b(\underset{\sim h}{\tau}, \underset{\sim h}{v}) = 0,\ \forall\, \underset{\sim h}{v} = (v_h, \theta_h) \in V_h \times W_h\right\};$$

in other words

$$a(\underset{\sim h}{\tau}, \underset{\sim h}{\tau}) \geq C\|\underset{\sim h}{\tau}\|_{0,\Omega}^2, \quad \forall \underset{\sim h}{\tau} \in K_h, \tag{5.11}$$

with a strictly positive constant C independent of $\lambda > 0$.

Adapting the proof of Proposition 4.4 [10] we obtain:

Proposition 5.4 (*[10]*). *Let $\phi = \phi_\alpha$ be a function as in Proposition 4.3. Then there exists an operator*

$$\Pi_h : \Sigma_0 \cap ([H_\phi^{0,1}(\Omega)]^{2\times 2} \times H_\phi^{0,1}(\Omega)) \longrightarrow \Sigma_{0,h}$$
$$\underset{\sim}{\tau} = (\tau, q) \longmapsto \Pi_h \underset{\sim}{\tau} = (\tau_h, q_h)$$

such that

$$b(\underset{\sim}{\tau} - \Pi_h \underset{\sim}{\tau}, \underset{\sim h}{v}) = 0, \quad \forall \underset{\sim h}{v} = (v_h, \theta_h) \in V_h \times W_h. \tag{5.12}$$

We now recall from [10] three adequate refinement rules of grids imposing constraints on the diameters of the triangles of the triangulations according to their geometrical situation in order to recapture optimal order of convergence of the interpolates.

Let $(\mathcal{T}_h)_{h>0}$ be a regular family of triangulations on Ω. In the following, we will suppose that $(\mathcal{T}_h)_{h>0}$ satisfies some of the following refinement rules:

R_1: if T is a triangle of $\mathcal{T}_h$ admitting S_j as a vertex, then

$$h_T \lesssim h^{1/(1-\alpha)},$$

(α has been defined just before Proposition 4.3); as usual $h := \max_{T\in\mathcal{T}_h} h_T$;

R_2: if T is a triangle of $\mathcal{T}_h$ admitting no S_j ($j = 1, \ldots, n_e$) as a vertex, then

$$h_T \lesssim h \inf_{x\in T} \phi(x),$$

(ϕ has been defined in Proposition 4.3);

R_3: for all $T \in \mathcal{T}_h$

$$h_T \gtrsim h^\beta,$$

where $\beta \geq 1/(1-\alpha)$.

Remark 5.5. Regular families of meshes satisfying the refinement conditions R_1–R_3 are easily built, see for instance [17].

Corollary 5.6 (*[10]*). *Under the hypotheses* R_1–R_2, *the following error estimate holds for every $q \in H_\phi^{0,1}(\Omega)$,*

$$\|q - P_h^1 q\|_{0,\Omega} \lesssim h|q|_{0,1;\phi,\Omega}, \tag{5.13}$$

where P_h^1 denotes the L^2-orthogonal projection on $\{\theta_h \in L^2(\Omega); \theta_h|_T \in \mathbb{P}_1(T), \forall T \in \mathcal{T}_h\}$.

Corollary 5.7 (*[10]*). *Under the hypotheses* R_1–R_2, *the following error estimate hold for every $\underset{\sim}{\tau} = (\tau, q) \in [H_\phi^{0,1}(\Omega)]^{2\times 2} \times H_\phi^{0,1}(\Omega)$*

$$\| \underset{\sim}{\tau} - \Pi_h \underset{\sim}{\tau} \|_{0,\Omega} \lesssim h(|\tau|_{0,1;\phi,\Omega} + |q|_{0,1;\phi,\Omega}). \tag{5.14}$$

Lemma 5.8. *Under the hypothesis* R_3 *on the regular family of triangulations $(\mathcal{T}_h)_{h>0}$, there exists a constant $C_0 > 0$ independent of h, such that for every vectorfield $v_h \in \{v_h \in H(\operatorname{div}; \Omega); v_h|_T \in [\mathbb{P}_1(T)]^2 \oplus \mathbb{R}\operatorname{Curl} b_T, \forall T \in \mathcal{T}_h\}$:*

$$\|\operatorname{div} v_h\|_{0,\Omega} \leq C_0 h^{-\beta} \|v_h\|_{0,\Omega}. \tag{5.15}$$

Proof. It suffices to apply Definition 4.7 p. 333 of [10] of the Piola transformation, and a simple scaling argument completes the proof. ■

5.1. A priori error estimates

5.1.1. The elliptic projection error estimates

Our next purpose is to derive error estimates for $((\sigma_h(t), p_h(t)), (u_h(t), \omega_h(t)))$. Firstly, we consider the "elliptic projection" of the exact solution. Let us introduce the following discrete elliptic projection problem:

Find $\underset{\sim h}{\widehat{\sigma}}(.) = (\widehat{\sigma}_h(.), \widehat{p}_h(.)) \in L^2([0,T]; \Sigma_{0,h})$, $\underset{\sim h}{\widehat{u}}(.) = (\widehat{u}_h(.), \widehat{\omega}_h(.)) \in L^2([0,T]; V_h \times W_h)$ such that for all $(\tau_h, q_h) \in \Sigma_{0,h}$, for all $(v_h, \theta_h) \in V_h \times W_h$ and for a.e. $t \in [0,T]$, we have:

$$\begin{cases} \dfrac{1}{2\mu}(\widehat{\sigma}_h(t), \tau_h) + \dfrac{1}{\lambda}(\widehat{p}_h(t), q_h) + (\operatorname{div}(\tau_h - q_h\delta), \widehat{u}_h(t)) + (\operatorname{as}(\tau_h), \widehat{\omega}_h(t)) = 0, \\ (u_{tt}(t), v_h) - (\operatorname{div}(\widehat{\sigma}_h(t) - \widehat{p}_h(t)\delta), v_h) - (\operatorname{as}(\widehat{\sigma}_h(t)), \theta_h) - (f(t), v_h) = 0. \end{cases} \tag{5.16}$$

With the notations (3.4) and (3.5), the discrete elliptic projection formulation (5.16) may be rewritten:
find $\underset{\sim h}{\widehat{\sigma}}(.) = (\widehat{\sigma}_h(.), \widehat{p}_h(.)) \in L^2([0,T]; \Sigma_{0,h})$, $\underset{\sim h}{\widehat{u}}(.) = (\widehat{u}_h(.), \widehat{\omega}_h(.)) \in L^2([0,T]; V_h \times W_h)$ such that, for a.e. $t \in [0,T]$, we have

$$\begin{cases} a(\underset{\sim h}{\widehat{\sigma}}(t), \underset{\sim h}{\tau}) + b(\underset{\sim h}{\tau}, \underset{\sim h}{\widehat{u}}(t)) = 0, \quad \forall \underset{\sim h}{\tau} := (\tau_h, q_h) \in \Sigma_{0,h}, \\ b(\underset{\sim h}{\widehat{\sigma}}(t), \underset{\sim h}{v}) + (\mathcal{F}(t), \underset{\sim h}{v}) = (u_{tt}(t), v_h), \quad \forall \underset{\sim h}{v} := (v_h, \theta_h) \in V_h \times W_h. \end{cases} \tag{5.17}$$

We are now in a position to establish optimal error estimates. In the following, we estimate the error between $((\sigma(.), p(.)), (u(.), \omega(.)))$ the exact solution of the mixed problem (3.3) or equivalently (3.6) and $((\widehat{\sigma}_h(.)), \widehat{p}_h(.)), (\widehat{u}_h(.), \widehat{\omega}_h(.))$ the solution of the discrete elliptic projection problem (5.16) or equivalently (5.17).

Proposition 5.9. *Let $(\mathcal{T}_h)_{h>0}$ be a regular family of triangulations on Ω. We suppose that $(\mathcal{T}_h)_{h>0}$ satisfies conditions* R_1 *and* R_2. *Under the hypotheses of Proposition 4.3, the following error estimate holds for a.e. $t \in [0,T]$:*

$$\|\sigma(t) - \widehat{\sigma}_h(t)\|_{0,\Omega} + \|p(t) - \widehat{p}_h(t)\|_{0,\Omega} \lesssim h\left[|u(t)|_{1,1;\phi,\Omega} + |p(t)|_{0,1;\phi,\Omega}\right]. \tag{5.18}$$

Proof. If we subtract (5.17) from (3.6), we get the system in the errors for a.e. $t \in [0,T]$:

$$\begin{cases} a\left(\underset{\sim}{\sigma}(t) - \underset{\sim h}{\widehat{\sigma}}(t), \underset{\sim h}{\tau}\right) + b\left(\underset{\sim h}{\tau}, \underset{\sim}{u}(t) - \underset{\sim h}{\widehat{u}}(t)\right) = 0, \quad \forall \underset{\sim h}{\tau} \in \Sigma_{0,h}, \\ b\left(\underset{\sim}{\sigma}(t) - \underset{\sim h}{\widehat{\sigma}}(t), \underset{\sim h}{v}\right) = 0, \quad \forall \underset{\sim h}{v} \in V_h \times W_h. \end{cases} \tag{5.19}$$

Let $(P_h^0 u(t), P_h^1 \omega(t))$ denote the L^2-orthogonal projection of $(u(t), \omega(t))$ on the space $V_h \times W_h$ and let us set $\Pi_h \underset{\sim}{\sigma}(t) = \Pi_h(\sigma(t), p(t)) = (\sigma_h^*(t), p_h^*(t))$. Eq. (5.12) and the relation (5.19) yield for a.e. $t \in [0,T]$:

$$\begin{cases} a\left(\underset{\sim}{\sigma}(t) - \underset{\sim h}{\widehat{\sigma}}(t), \Pi_h \underset{\sim}{\sigma}(t) - \underset{\sim h}{\widehat{\sigma}}(t)\right) = \left(\operatorname{as}(\widehat{\sigma}_h(t) - \sigma_h^*(t)), \omega(t) - P_h^1\omega(t)\right), \\ b\left(\Pi_h \underset{\sim}{\sigma}(t) - \underset{\sim h}{\widehat{\sigma}}(t), \underset{\sim h}{v}\right) = 0, \quad \forall \underset{\sim h}{v} \in V_h \times W_h. \end{cases} \tag{5.20}$$

The first equation of (5.20) implies

$$\begin{aligned} a\left(\Pi_h \underset{\sim}{\sigma}(t) - \underset{\sim h}{\widehat{\sigma}}(t), \Pi_h \underset{\sim}{\sigma}(t) - \underset{\sim h}{\widehat{\sigma}}(t)\right) = {} & \frac{1}{2\mu}\left(\sigma_h^*(t) - \sigma(t), \sigma_h^*(t) - \widehat{\sigma}_h(t)\right) + \frac{1}{\lambda}\left(p_h^*(t) - p(t), p_h^*(t) - \widehat{p}_h(t)\right) \\ & + \left(\operatorname{as}(\widehat{\sigma}_h(t) - \sigma_h^*(t)), \omega(t) - P_h^1\omega(t)\right). \end{aligned} \tag{5.21}$$

Thus due to Proposition 5.3, we have

$$\|\Pi_h \underset{\sim}{\sigma}(t) - \underset{\sim h}{\widehat{\sigma}}(t)\|_{0,\Omega} \lesssim \left[\|\Pi_h \underset{\sim}{\sigma}(t) - \underset{\sim}{\sigma}(t)\|_{0,\Omega} + \|\omega(t) - P_h^1\omega(t)\|_{0,\Omega}\right]. \tag{5.22}$$

Using (5.13) and Corollary 5.7, we get

$$\begin{aligned} \|\Pi_h \underset{\sim}{\sigma}(t) - \underset{\sim h}{\widehat{\sigma}}(t)\|_{0,\Omega} &\lesssim h\left[|\sigma(t)|_{0,1;\phi,\Omega} + |p(t)|_{0,1;\phi,\Omega} + |\omega(t)|_{0,1;\phi,\Omega}\right] \\ &\lesssim h\left[|u(t)|_{1,1;\phi,\Omega} + |p(t)|_{0,1;\phi,\Omega}\right]. \end{aligned} \tag{5.23}$$

Finally, (5.18) follows from (5.23), Corollary 5.7 and the triangle inequality. ■

Proposition 5.10. *Under the hypotheses of Proposition 5.9, the following error estimates hold for a.e. $t \in [0,T]$:*

$$\|\sigma_{tt}(t) - \widehat{\sigma}_{h,tt}(t)\|_{0,\Omega} + \|p_{tt}(t) - \widehat{p}_{h,tt}(t)\|_{0,\Omega} \lesssim h\left[|u_{tt}(t)|_{1,1;\phi,\Omega} + |p_{tt}(t)|_{0,1;\phi,\Omega}\right], \tag{5.24}$$

$$\|\omega_{tt}(t) - \widehat{\omega}_{h,tt}(t)\|_{0,\Omega} + \|P_h^0 u_{tt}(t) - \widehat{u}_{h,tt}(t)\|_{0,\Omega} \lesssim h\left[|u_{tt}(t)|_{1,1;\phi,\Omega} + |p_{tt}(t)|_{0,1;\phi,\Omega}\right], \tag{5.25}$$

$$\|u_{tt}(t) - \widehat{u}_{h,tt}(t)\|_{0,\Omega} \lesssim h\left[|u_{tt}(t)|_{1,1;\phi,\Omega} + |u_{tt}(t)|_{1,\Omega} + |p_{tt}(t)|_{0,1;\phi,\Omega}\right]. \tag{5.26}$$

Proof. Let us consider the second derivative with respect to time of system (5.19):

$$\begin{cases} a\left(\underset{\sim}{\sigma}_{tt}(t)-\underset{\sim}{\widehat{\sigma}}_{h,tt}(t),\underset{\sim}{\tau}_h\right)+b\left(\underset{\sim}{\tau}_h,\underset{\sim}{u}_{tt}(t)-\underset{\sim}{\widehat{u}}_{h,tt}(t)\right)=0, & \forall\,\underset{\sim}{\tau}_h\in\Sigma_{0,h},\\ b\left(\underset{\sim}{\sigma}_{tt}(t)-\underset{\sim}{\widehat{\sigma}}_{h,tt}(t),\underset{\sim}{v}_h\right)=0, \quad \forall\,\underset{\sim}{v}_h\in V_h\times W_h. \end{cases} \tag{5.27}$$

Firstly, let us observe that with the same techniques as in Proposition 5.9, we get from (5.27), the following estimate:

$$\|\underset{\sim}{\sigma}_{tt}(t)-\underset{\sim}{\widehat{\sigma}}_{h,tt}(t)\|_{0,\Omega}\lesssim h\left[|u_{tt}(t)|_{1,1;\phi,\Omega}+|p_{tt}(t)|_{0,1;\phi,\Omega}\right], \tag{5.28}$$

which proves (5.24).

To prove (5.25), we shall use the uniform inf–sup condition (5.10). Firstly, it follows from the first equation of (3.5) and (5.27) that

$$b\left(\underset{\sim}{\tau}_h,\left(P_h^0u_{tt}(t),P_h^1\omega_{tt}(t)\right)-\underset{\sim}{\widehat{u}}_{h,tt}(t)\right)=-a\left(\underset{\sim}{\sigma}_{tt}(t)-\underset{\sim}{\widehat{\sigma}}_{h,tt}(t),\underset{\sim}{\tau}_h\right)+\left(\mathrm{as}\,(\tau_h),P_h^1\omega_{tt}(t)-\omega_{tt}(t)\right),\quad \forall\underset{\sim}{\tau}_h\in\Sigma_{0,h}.$$

Thus by the uniform inf–sup condition (5.10), we have

$$\|P_h^0u_{tt}(t)-\widehat{u}_{h,tt}(t)\|_{0,\Omega}+\|P_h^1\omega_{tt}(t)-\widehat{\omega}_{h,tt}(t)\|_{0,\Omega}\lesssim\left[\|\underset{\sim}{\sigma}_{tt}(t)-\underset{\sim}{\widehat{\sigma}}_{h,tt}(t)\|_{0,\Omega}+\|P_h^1\omega_{tt}(t)-\omega_{tt}(t)\|_{0,\Omega}\right]. \tag{5.29}$$

Finally, (5.25) and (5.26) follow from (5.28), (5.29), (5.13), (1.47) p. 27 of [18] (or (45) p. 624 of [19]) and the triangle inequality. ■

Remark 5.11. Under the hypotheses of Proposition 4.4, if we consider fourth order derivatives with respect to t instead of second ones of the system of errors (5.19), and using similar techniques as above, we obtain the following estimate for a.e. $t\in[0,T]$:

$$\|u_{tttt}(t)-\widehat{u}_{h,tttt}(t)\|_{0,\Omega}\lesssim h\left[|u_{tttt}(t)|_{1,1;\phi,\Omega}+|u_{tttt}(t)|_{1,\Omega}+|p_{tttt}(t)|_{0,1;\phi,\Omega}\right]. \tag{5.30}$$

If instead, we consider third order derivatives with respect to t of the system of errors (5.19), we obtain the following estimate for a.e. $t\in[0,T]$:

$$\|u_{ttt}(t)-\widehat{u}_{h,ttt}(t)\|_{0,\Omega}\lesssim h\left[|u_{ttt}(t)|_{1,1;\phi,\Omega}+|u_{ttt}(t)|_{1,\Omega}+|p_{ttt}(t)|_{0,1;\phi,\Omega}\right]. \tag{5.31}$$

5.1.2. Error estimates for the evolution problem

Before giving optimal error estimates for our mixed method, we choose the initial conditions $u_{0,h}$ and $u_{1,h}$ in the semi-discretized problem (5.3) or equivalently (5.4), as the elliptic projections of the initial conditions u_0 and u_1 respectively. We can now derive the following error estimates:

Theorem 5.12. *Under the hypotheses of Proposition 5.9, the following error estimates hold:*

$$\|\sigma-\sigma_h\|_{L^\infty(L^2)}+\|p-p_h\|_{L^\infty(L^2)}\lesssim h\left[||u_{tt}||_{L^2(H_\phi^{1,1})}+|p_{tt}|_{L^2(H_\phi^{0,1})}+|u|_{L^\infty(H_\phi^{1,1})}+|p|_{L^\infty(H_\phi^{0,1})}\right], \tag{5.32}$$

$$\|\omega-\omega_h\|_{L^\infty(L^2)}+\|P_h^0u-u_h\|_{L^\infty(L^2)}\lesssim h\left[||u_{tt}||_{L^2(H_\phi^{1,1})}+|p_{tt}|_{L^2(H_\phi^{0,1})}+|u|_{L^\infty(H_\phi^{1,1})}+|p|_{L^\infty(H_\phi^{0,1})}\right], \tag{5.33}$$

$$\|u-u_h\|_{L^\infty(L^2)}\lesssim h\left[||u_{tt}||_{L^2(H_\phi^{1,1})}+|p_{tt}|_{L^2(H_\phi^{0,1})}+||u||_{L^\infty(H_\phi^{1,1})}+|p|_{L^\infty(H_\phi^{0,1})}\right]. \tag{5.34}$$

Proof. Let $((\widehat{\sigma}_h(t),\widehat{p}_h(t)),(\widehat{\omega}_h(t),\widehat{u}_h(t)))$ be the elliptic projection of $((\sigma(t),p(t)),(\omega(t),u(t)))$ and set

$$\varepsilon_h(t)=\sigma_h(t)-\widehat{\sigma}_h(t),\qquad \chi_h(t)=u_h(t)-\widehat{u}_h(t),\qquad \psi_h(t)=\omega_h(t)-\widehat{\omega}_h(t)\quad\text{and}\quad r_h(t)=p_h(t)-\widehat{p}_h(t).$$

We may then write the error system in the form

$$\begin{cases}\dfrac{1}{2\mu}(\varepsilon_h(t),\tau_h)+\dfrac{1}{\lambda}(r_h(t),q_h)+(\mathrm{div}\,(\tau_h-q_h\delta),\chi_h(t))+(\mathrm{as}\,(\tau_h),\psi_h(t))=0,\\ (\mathrm{div}\,(\varepsilon_h(t)-r_h(t)\delta),v_h)+(\mathrm{as}\,(\varepsilon_h(t)),\theta_h)=(u_{h,tt}(t)-u_{tt}(t),v_h),\end{cases} \tag{5.35}$$

for all $\underset{\sim}{\tau}_h=(\tau_h,q_h)\in\Sigma_{0,h}$ and all $\underset{\sim}{v}_h=(v_h,\theta_h)\in V_h\times W_h$.

Choosing the initial conditions for the semi-discrete problem implies that $\chi_h(0) = 0$ and $\chi_{h,t}(0) = 0$. Afterwards, from the first equation of (5.35) at time $t = 0$ with $(\tau_h, q_h) = (\varepsilon_h(0), r_h(0))$ and the fact that $(\mathrm{as}\,(\varepsilon_h(0)), \theta_h) = 0, \forall \theta_h \in W_h$, follows that $\varepsilon_h(0) = 0$ and $r_h(0) = 0$.

We then differentiate the first equation of (5.35) with respect to time to obtain

$$\frac{1}{2\mu}(\varepsilon_{h,t}(t), \tau_h) + \frac{1}{\lambda}(r_{h,t}(t), q_h) + (\mathrm{div}\,(\tau_h - q_h\delta), \chi_{h,t}(t)) + (\mathrm{as}\,(\tau_h), \psi_{h,t}(t)) = 0. \tag{5.36}$$

Now taking $(\tau_h, q_h) = (\varepsilon_h(t), r_h(t))$ in this last equality, we obtain

$$\frac{1}{2\mu}(\varepsilon_{h,t}(t), \varepsilon_h(t)) + \frac{1}{\lambda}(r_{h,t}(t), r_h(t)) + (\mathrm{div}\,(\varepsilon_h(t) - r_h(t)\delta), \chi_{h,t}(t)) + (\mathrm{as}\,(\varepsilon_h(t)), \psi_{h,t}(t)) = 0. \tag{5.37}$$

The second equation of (5.35) with $(v_h, \theta_h) = (\chi_{h,t}(t), \psi_{h,t}(t))$ gives

$$(\mathrm{div}\,(\varepsilon_h(t) - r_h(t)\delta), \chi_{h,t}(t)) + (\mathrm{as}\,(\varepsilon_h(t)), \psi_{h,t}(t)) = (u_{h,tt}(t) - u_{tt}(t), \chi_{h,t}(t)). \tag{5.38}$$

Subtracting (5.38) from (5.37) gives

$$\begin{aligned}\frac{1}{2\mu}\frac{\mathrm{d}}{\mathrm{d}t}\|\varepsilon_h(t)\|^2_{0,\Omega} + \frac{1}{\lambda}\frac{\mathrm{d}}{\mathrm{d}t}\|r_h(t)\|^2_{0,\Omega} &= 2(u_{tt}(t) - u_{h,tt}(t), \chi_{h,t}(t)) \\ &= 2(u_{tt}(t) - \widehat{u}_{h,tt}(t), \chi_{h,t}(t)) - \frac{\mathrm{d}}{\mathrm{d}t}\|\chi_{h,t}(t)\|^2_{0,\Omega}.\end{aligned} \tag{5.39}$$

Using the Cauchy–Schwarz inequality to bound the right-hand side of (5.39) we obtain

$$\begin{aligned}\frac{1}{2\mu}\frac{\mathrm{d}}{\mathrm{d}t}\|\varepsilon_h(t)\|^2_{0,\Omega} + \frac{1}{\lambda}\frac{\mathrm{d}}{\mathrm{d}t}\|r_h(t)\|^2_{0,\Omega} + \frac{\mathrm{d}}{\mathrm{d}t}\|\chi_{h,t}(t)\|^2_{0,\Omega} &= 2(u_{tt}(t) - \widehat{u}_{h,tt}(t), \chi_{h,t}(t)) \\ &\le \|u_{tt}(t) - \widehat{u}_{h,tt}(t)\|^2_{0,\Omega} + \|\chi_{h,t}(t)\|^2_{0,\Omega}.\end{aligned} \tag{5.40}$$

Now applying Gronwall's inequality (2.4) to (5.40) we get

$$\frac{1}{2\mu}\|\varepsilon_h(t)\|^2_{0,\Omega} + \frac{1}{\lambda}\|r_h(t)\|^2_{0,\Omega} + \|\chi_{h,t}(t)\|^2_{0,\Omega} \le \mathrm{e}^T \int_0^T \|u_{tt}(s) - \widehat{u}_{h,tt}(s)\|^2_{0,\Omega}\,\mathrm{d}s.$$

Thanks to (5.26) one can write

$$\begin{aligned}&\frac{1}{2\mu}\|\varepsilon_h(t)\|^2_{0,\Omega} + \frac{1}{\lambda}\|r_h(t)\|^2_{0,\Omega} + \|\chi_{h,t}(t)\|^2_{0,\Omega} \\ &\quad\lesssim h^2\left[\int_0^T |u_{tt}(s)|^2_{1,1;\phi,\Omega}\,\mathrm{d}s + \int_0^T |u_{tt}(s)|^2_{1,\Omega}\,\mathrm{d}s + \int_0^T |p_{tt}(s)|^2_{0,1;\phi,\Omega}\,\mathrm{d}s\right].\end{aligned} \tag{5.41}$$

Taking the square root of (5.41) and using assumption (2.3) on λ and μ give us

$$\|\varepsilon_h(t)\|_{0,\Omega} + \|r_h(t)\|_{0,\Omega} \lesssim h\left[|u_{tt}|_{L^2(H^{1,1}_\phi)} + |u_{tt}|_{L^2(H^1)} + |p_{tt}|_{L^2(H^{0,1}_\phi)}\right]. \tag{5.42}$$

Therefore, (5.42), (5.18) and the triangle inequality, we get

$$\begin{aligned}&\|\sigma(t) - \sigma_h(t)\|_{0,\Omega} + \|p(t) - p_h(t)\|_{0,\Omega} \\ &\quad\lesssim h\left[|u_{tt}|_{L^2(H^{1,1}_\phi)} + |u_{tt}|_{L^2(H^1)} + |p_{tt}|_{L^2(H^{0,1}_\phi)} + |u(t)|_{1,1;\phi,\Omega} + |p(t)|_{0,1;\phi,\Omega}\right].\end{aligned} \tag{5.43}$$

Taking the supremum over all $t \in [0, T]$ in this last inequality we get (5.32).

Proceeding similarly as in the proof of Proposition 5.10, we get:

$$\begin{aligned}&\|\omega(t) - \omega_h(t)\|_{0,\Omega} + \|P^0_h u(t) - u_h(t)\|_{0,\Omega} \\ &\quad\lesssim \left[\|\sigma(t) - \sigma_h(t)\|_{0,\Omega} + \|p(t) - p_h(t)\|_{0,\Omega} + \|P^1_h\omega(t) - \omega(t)\|_{0,\Omega}\right].\end{aligned} \tag{5.44}$$

This last inequality combined with (5.43), (5.13) and the triangle inequality we get

$$\begin{aligned}&\|\omega(t) - \omega_h(t)\|_{0,\Omega} + \|P^0_h u(t) - u_h(t)\|_{0,\Omega} \\ &\quad\lesssim h\left[|u_{tt}|_{L^2(H^{1,1}_\phi)} + |u_{tt}|_{L^2(H^1)} + |p_{tt}|_{L^2(H^{0,1}_\phi)} + |u(t)|_{1,1;\phi,\Omega} + |p(t)|_{0,1;\phi,\Omega}\right].\end{aligned} \tag{5.45}$$

Taking the supremum over all $t \in [0, T]$ in this last inequality we get (5.33). Using furthermore the bound on the error of the P^0_h projection (1.47) p.27 of [18] (or (45) p. 624 of [19]) and the triangle inequality, we obtain (5.34). ■

6. The fully discrete mixed finite element scheme

6.1. Notation

Let $\Delta t := \frac{T}{N} > 0$ denote the time step size and define $t_i = i\Delta t$ $(i = 0, 1, \ldots, N)$, $t_N = T$ and $t_0 = 0$. For any function ϕ of time, let ϕ^n denote $\phi(t_n)$. We denote by $t^{n+\frac{1}{2}} := \frac{t^n+t^{n+1}}{2}$, $\phi^{n+\frac{1}{2}} := \frac{\phi^n+\phi^{n+1}}{2}$, and we define the following discrete temporal derivatives:

$$\Delta_t\phi^n := \frac{\phi^{n+1}-\phi^{n-1}}{2\Delta t}, \qquad \Delta_t\phi^{n+\frac{1}{2}} := \frac{\phi^{n+1}-\phi^n}{\Delta t}, \qquad \Delta_t^2\phi^n := \frac{\phi^{n+1}-2\phi^n+\phi^{n-1}}{(\Delta t)^2}.$$

We can easily see that we have

$$\Delta_t^2\phi^n := \frac{\Delta_t\phi^{n+\frac{1}{2}}-\Delta_t\phi^{n-\frac{1}{2}}}{\Delta t} \quad \text{and} \quad \Delta_t\phi^n := \frac{\Delta_t\phi^{n+\frac{1}{2}}+\Delta_t\phi^{n-\frac{1}{2}}}{2}. \tag{6.1}$$

6.2. The explicit Newmark scheme

The explicit-in-time discrete mixed formulation is as follows: Find $(\sigma_h^{n+1}, p_h^{n+1}) \in \Sigma_{0,h}$, and $(u_h^{n+1}, \omega_h^{n+1}) \in V_h \times W_h$ such that

$$u_h^0 = \hat{u}_h(0), \tag{6.2}$$

$$u_h^{-1} = \hat{u}_h(-\Delta t) \simeq \hat{u}_h(0) - \Delta t\, \hat{u}_{h,t}(0) + \frac{\Delta t^2}{2}\hat{u}_{h,tt}(0), \tag{6.3}$$

and

$$\begin{cases} \frac{1}{2\mu}(\sigma_h^{n+1}, \tau_h) + \frac{1}{\lambda}(p_h^{n+1}, q_h) + (\operatorname{div}(\tau_h - q_h\delta), u_h^{n+1}) + (\operatorname{as}(\tau_h), \omega_h^{n+1}) = 0, \quad \forall n \geq -1, \\ (\operatorname{as}(\sigma_h^{n+1}), \theta_h) = 0, \quad \forall n \geq -1, \\ (\Delta_t^2 u_h^n, v_h) - (\operatorname{div}(\sigma_h^n - p_h^n\delta), v_h) - (f^n, v_h) = 0, \quad \forall n \geq 0, \end{cases} \tag{6.4}$$

$\forall(\tau_h, q_h) \in \Sigma_{0,h}, \forall(v_h, \theta_h) \in V_h \times W_h$.

The existence and uniqueness of a solution to problem (6.4) is provided by the following lemma:

Lemma 6.1. *A solution $\left((\sigma_h^{n+1}, p_h^{n+1}), (u_h^{n+1}, \omega_h^{n+1})\right)$ of (6.4) exists and is unique.*

Proof. Let us consider $n \geq -1$. With every $\left((\sigma_h^{n+1}, p_h^{n+1}), \omega_h^{n+1}\right) \in \Sigma_{0,h} \times W_h$, we associate the element of its dual $\Sigma'_{0,h} \times W'_h$:

$$\begin{pmatrix} (\tau_h, q_h) \longmapsto \frac{1}{2\mu}(\sigma_h^{n+1}, \tau_h) + \frac{1}{\lambda}(p_h^{n+1}, q_h) + (\operatorname{as}(\tau_h), \omega_h^{n+1}) \\ \theta_h \longmapsto (\operatorname{as}(\sigma_h^{n+1}), \theta_h) \end{pmatrix}.$$

Let us call this mapping T_h^n; it is a linear mapping from $\Sigma_{0,h} \times W_h$ into its dual. We have to prove that T_h^n is bijective. But the arrival and departure spaces have the same dimension. Thus, by a well-known theorem of linear algebra it suffices to prove that T_h^n is injective. Thus, let $\left((\sigma_h^{n+1}, p_h^{n+1}), \omega_h^{n+1}\right) \in \Sigma_{0,h} \times W_h$ be such that:

$$\frac{1}{2\mu}(\sigma_h^{n+1}, \tau_h) + \frac{1}{\lambda}(p_h^{n+1}, q_h) + (\operatorname{as}(\tau_h), \omega_h^{n+1}) = 0, \quad \forall(\tau_h, q_h) \in \Sigma_{0,h}, \tag{6.5}$$

$$(\operatorname{as}(\sigma_h^{n+1}), \theta_h) = 0, \quad \forall\theta_h \in W_h. \tag{6.6}$$

From (6.6), it follows that $(\operatorname{as}(\sigma_h^{n+1}), \omega_h^{n+1}) = 0$ and then by taking $(\tau_h, q_h) = (\sigma_h^{n+1}, p_h^{n+1})$ in (6.5), we get

$$\frac{1}{2\mu}\|\sigma_h^{n+1}\|_{0,\Omega}^2 + \frac{1}{\lambda}\|p_h^{n+1}\|_{0,\Omega}^2 = 0,$$

which implies that

$$\sigma_h^{n+1} = 0, \qquad p_h^{n+1} = 0.$$

Thus (6.5) reduces to:

$$(\operatorname{as}(\tau_h), \omega_h^{n+1}) = 0.$$

By inf–sup inequality (5.10), with $(v_h, \theta_h) = (0, \omega_h^{n+1})$, we get $\omega_h^{n+1} = 0$.

Thus, we have proved that the mapping T_h^n is an isomorphism from $\Sigma_{0,h} \times W_h$ into its dual. ■

Let us now explain how to construct the solution to system (6.4). Firstly, let us recall that u_h^0 and u_h^{-1} are given by the initial conditions. Starting with u_h^0, we deduce from the first two equations of system (6.4) with $n = -1$ by the invertibility of T_h^{-1}: $\sigma_h^0, p_h^0, \omega_h^0$. From the third equation of system (6.4) with $n = 0$ and u_h^0, u_h^{-1}, we are now able to deduce u_h^1. Returning to the first two equations of system (6.4) with $n = 0$, we deduce by the invertibility of T_h^0: $\sigma_h^1, p_h^1, \omega_h^1$. Using the third equation of system (6.4) with $n = 1$, we deduce u_h^2 and so on.

Note, that it follows by uniqueness from the first two equations of system (6.4) with $n = -1$ using $u_h^0 = \hat{u}_h(0)$ that

$$\sigma_h^0 = \hat{\sigma}_h(0), p_h^0 = \hat{p}_h(0) \quad \text{and} \quad \omega_h^0 = \hat{\omega}_h(0). \tag{6.7}$$

6.3. The stability of the fully discrete explicit scheme

Before the statement of the result concerning the stability of the fully discrete scheme, we begin by the proof of the following lemma:

Lemma 6.2. *Under the hypothesis* R_3, *we have*

$$\begin{aligned} &\|\Delta_t u_h^{N+\frac12}\|_{0,\Omega}^2 + \frac{1}{2\mu}\|\sigma_h^{N+\frac12}\|_{0,\Omega}^2 - \frac{\Delta t^2}{8\mu}\|\Delta_t\sigma_h^{N+\frac12}\|_{0,\Omega}^2 + \frac{1}{\lambda}\|p_h^{N+\frac12}\|_{0,\Omega}^2 - \frac{\Delta t^2}{4\lambda}\|\Delta_t p_h^{N+\frac12}\|_{0,\Omega}^2 \\ &\quad = 2\Delta t\sum_{n=1}^{N}(f^n, \Delta_t u_h^n) + \|\Delta_t u_h^{\frac12}\|_{0,\Omega}^2 + \frac{1}{2\mu}\|\sigma_h^{\frac12}\|_{0,\Omega}^2 - \frac{(\Delta t)^2}{8\mu}\|\Delta_t\sigma_h^{\frac12}\|_{0,\Omega}^2 + \frac{1}{\lambda}\|p_h^{\frac12}\|_{0,\Omega}^2 - \frac{(\Delta t)^2}{4\lambda}\|\Delta_t p_h^{\frac12}\|_{0,\Omega}^2. \end{aligned} \tag{6.8}$$

Proof. Subtracting from the first equation of (6.4) at time $(n+1)\Delta t$, the same equation at time $(n-1)\Delta t$, we obtain for all $(\tau_h, q_h) \in \Sigma_{0,h}$:

$$\frac{1}{2\mu}(\sigma_h^{n+1} - \sigma_h^{n-1}, \tau_h) + \frac{1}{\lambda}(p_h^{n+1} - p_h^{n-1}, q_h) + (\mathrm{div}\,(\tau_h - q_h\delta), u_h^{n+1} - u_h^{n-1}) + (\mathrm{as}\,(\tau_h), \omega_h^{n+1} - \omega_h^{n-1}) = 0. \tag{6.9}$$

Taking $(\tau_h, q_h) = \frac{1}{2\Delta t}(\sigma_h^n, p_h^n)$ in (6.9) and using the fact that $(\mathrm{as}\,(\sigma_h^n), \theta_h) = 0,\ \forall\theta_h \in W_h$, we get

$$\frac{1}{2\mu}(\Delta_t\sigma_h^n, \sigma_h^n) + \frac{1}{\lambda}(\Delta_t p_h^n, p_h^n) + (\mathrm{div}\,(\sigma_h^n - p_h^n\delta), \Delta_t u_h^n) = 0. \tag{6.10}$$

The third equation of (6.4) with $v_h = \Delta_t u_h^n$, becomes

$$(\Delta_t^2 u_h^n, \Delta_t u_h^n) - (\mathrm{div}\,(\sigma_h^n - p_h^n\delta), \Delta_t u_h^n) = (f^n, \Delta_t u_h^n). \tag{6.11}$$

Adding (6.11) and (6.10) yields

$$(\Delta_t^2 u_h^n, \Delta_t u_h^n) + \frac{1}{2\mu}(\Delta_t\sigma_h^n, \sigma_h^n) + \frac{1}{\lambda}(\Delta_t p_h^n, p_h^n) = (f^n, \Delta_t u_h^n). \tag{6.12}$$

It follows from (6.1):

$$\frac{1}{2\Delta t}\left(\Delta_t u_h^{n+\frac12} - \Delta_t u_h^{n-\frac12}, \Delta_t u_h^{n+\frac12} + \Delta_t u_h^{n-\frac12}\right) + \frac{1}{2\mu}(\Delta_t\sigma_h^n, \sigma_h^n) + \frac{1}{\lambda}(\Delta_t p_h^n, p_h^n) = (f^n, \Delta_t u_h^n). \tag{6.13}$$

Thus

$$\frac{1}{2\Delta t}\left(\|\Delta_t u_h^{n+\frac12}\|_{0,\Omega}^2 - \|\Delta_t u_h^{n-\frac12}\|_{0,\Omega}^2\right) + \frac{1}{2\mu}(\Delta_t\sigma_h^n, \sigma_h^n) + \frac{1}{\lambda}(\Delta_t p_h^n, p_h^n) = (f^n, \Delta_t u_h^n). \tag{6.14}$$

Now we are going to transform the last two terms on the left-hand side of this last equation. We have

$$\begin{aligned} \Delta_t\sigma_h^n &= \frac{\sigma_h^{n+1} - \sigma_h^{n-1}}{2\Delta t} = \frac{\sigma_h^{n+1} + \sigma_h^n - \sigma_h^n - \sigma_h^{n-1}}{2\Delta t} = \frac{\sigma_h^{n+\frac12} - \sigma_h^{n-\frac12}}{\Delta t} \\ \sigma_h^n &= \frac{\sigma_h^{n+1} + \sigma_h^n}{4} + \frac{\sigma_h^{n-1} + \sigma_h^n}{4} - \frac{(\Delta t)^2}{4}\left(\frac{\sigma_h^{n+1} - 2\sigma_h^n + \sigma_h^{n-1}}{(\Delta t)^2}\right). \end{aligned}$$

Using these last two equalities, we can write

$$\begin{aligned}
\frac{1}{2\mu}(\Delta_t\sigma_h^n,\sigma_h^n) &= \frac{1}{2\mu}\left(\frac{\sigma_h^{n+\frac12}-\sigma_h^{n-\frac12}}{\Delta t},\frac{\sigma_h^{n+1}+\sigma_h^n}{4}+\frac{\sigma_h^{n-1}+\sigma_h^n}{4}-\frac{(\Delta t)^2}{4}\left(\frac{\sigma_h^{n+1}-2\sigma_h^n+\sigma_h^{n-1}}{(\Delta t)^2}\right)\right)\\
&= \frac{1}{2\mu}\left(\frac{\sigma_h^{n+\frac12}-\sigma_h^{n-\frac12}}{\Delta t},\frac{\sigma_h^{n+1}+\sigma_h^n}{4}+\frac{\sigma_h^{n-1}+\sigma_h^n}{4}-\frac{(\Delta t)^2}{4}\Delta_t^2\sigma_h^n\right)\\
&= \frac{1}{2\mu}\left(\frac{\sigma_h^{n+\frac12}-\sigma_h^{n-\frac12}}{\Delta t},\frac{\sigma_h^{n+1}+\sigma_h^n}{4}+\frac{\sigma_h^{n-1}+\sigma_h^n}{4}-\frac{(\Delta t)^2}{4}\frac{\Delta_t\sigma_h^{n+\frac12}-\Delta_t\sigma_h^{n-\frac12}}{\Delta t}\right)\\
&= \frac{1}{2\mu}\frac{1}{2\Delta t}\left(\sigma_h^{n+\frac12}-\sigma_h^{n-\frac12},\sigma_h^{n+\frac12}+\sigma_h^{n-\frac12}\right)-\frac{1}{2\mu}\frac{(\Delta t)^2}{4}\left(\Delta_t\sigma_h^n,\frac{\Delta_t\sigma_h^{n+\frac12}-\Delta_t\sigma_h^{n-\frac12}}{\Delta t}\right)\\
&= \frac{1}{4\mu\Delta t}\left(\|\sigma_h^{n+\frac12}\|_{0,\Omega}^2-\|\sigma_h^{n-\frac12}\|_{0,\Omega}^2\right)-\frac{1}{2\mu}\frac{(\Delta t)^2}{4}\left(\frac{\Delta_t\sigma_h^{n+\frac12}+\Delta_t\sigma_h^{n-\frac12}}{2},\frac{\Delta_t\sigma_h^{n+\frac12}-\Delta_t\sigma_h^{n-\frac12}}{\Delta t}\right)\\
&\quad \text{from (6.1)}\\
&= \frac{1}{4\mu\Delta t}\left(\|\sigma_h^{n+\frac12}\|_{0,\Omega}^2-\|\sigma_h^{n-\frac12}\|_{0,\Omega}^2\right)-\frac{\Delta t}{16\mu}\left(\|\Delta_t\sigma_h^{n+\frac12}\|_{0,\Omega}^2-\|\Delta_t\sigma_h^{n-\frac12}\|_{0,\Omega}^2\right).
\end{aligned}$$

In the same way, we get:

$$\frac{1}{\lambda}(\Delta_t p_h^n,p_h^n)=\frac{1}{2\lambda\Delta t}\left(\|p_h^{n+\frac12}\|_{0,\Omega}^2-\|p_h^{n-\frac12}\|_{0,\Omega}^2\right)-\frac{\Delta t}{8\lambda}\left(\|\Delta_t p_h^{n+\frac12}\|_{0,\Omega}^2-\|\Delta_t p_h^{n-\frac12}\|_{0,\Omega}^2\right).$$

Using the previous two identities, Eq. (6.14) can be rewritten:

$$\begin{aligned}
&\frac{1}{2\Delta t}\left(\|\Delta_t u_h^{n+\frac12}\|_{0,\Omega}^2-\|\Delta_t u_h^{n-\frac12}\|_{0,\Omega}^2\right)+\frac{1}{4\mu\Delta t}\left(\|\sigma_h^{n+\frac12}\|_{0,\Omega}^2-\|\sigma_h^{n-\frac12}\|_{0,\Omega}^2\right)\\
&-\frac{\Delta t}{16\mu}\left(\|\Delta_t\sigma_h^{n+\frac12}\|_{0,\Omega}^2-\|\Delta_t\sigma_h^{n-\frac12}\|_{0,\Omega}^2\right)+\frac{1}{2\lambda\Delta t}\left(\|p_h^{n+\frac12}\|_{0,\Omega}^2-\|p_h^{n-\frac12}\|_{0,\Omega}^2\right)\\
&-\frac{\Delta t}{8\lambda}\left(\|\Delta_t p_h^{n+\frac12}\|_{0,\Omega}^2-\|\Delta_t p_h^{n-\frac12}\|_{0,\Omega}^2\right)=(f^n,\Delta_t u_h^n).
\end{aligned}\tag{6.15}$$

Summing Eqs. (6.15) from $n=1,\ldots,N$, we obtain (6.8). ■

Theorem 6.3. *Under the hypothesis* R_3, *the explicit scheme defined by* (6.2)–(6.4) *is stable if the following CFL condition is satisfied:* $\Delta t<\min\{\frac{1}{C_0\sqrt{2\mu+\lambda}}h^\beta,1\}$, *i.e.,*

$$\begin{aligned}
&\alpha_0\|\Delta_t u_h^{N+\frac12}\|_{0,\Omega}^2+\frac{1}{2\mu}\|\sigma_h^{N+\frac12}\|_{0,\Omega}^2+\frac{1}{\lambda}\|p_h^{N+\frac12}\|_{0,\Omega}^2\\
&\quad\le\left[\|\Delta_t u_h^{\frac12}\|_{0,\Omega}^2+\frac{1}{\lambda}\|p_h^{\frac12}\|_{0,\Omega}^2+\frac{1}{2\mu}\|\sigma_h^{\frac12}\|_{0,\Omega}^2+T\left(\max_{t\in[0,T]}\|f(t)\|_{0,\Omega}\right)^2\right]\times\exp\left(T\alpha_0^{-1}\right),
\end{aligned}\tag{6.16}$$

$$\beta^*\left(\|u_h^{N+\frac12}\|_{0,\Omega}+\|\omega_h^{N+\frac12}\|_{0,\Omega}\right)\le\frac{1}{2\mu}\|\sigma_h^{N+\frac12}\|_{0,\Omega}+\frac{1}{\lambda}\|p_h^{N+\frac12}\|_{0,\Omega},\tag{6.17}$$

where

$$\alpha_0=\frac12-\frac{C_0^2}{2}(2\mu+\lambda)\frac{\Delta t^2}{h^{2\beta}}\quad(\beta \text{ is defined in } \mathrm{R}_3)$$

and β^* *is the constant of the inf–sup condition defined in* (5.10)*.*

Proof. Subtracting from the first equation of (6.4) at time $(N+1)\Delta t$, the same equation at time $N\Delta t$, we obtain for all $(\tau_h,q_h)\in\Sigma_{0,h}$:

$$\frac{1}{2\mu}(\sigma_h^{N+1}-\sigma_h^N,\tau_h)+\frac{1}{\lambda}(p_h^{N+1}-p_h^N,q_h)+(\operatorname{div}(\tau_h-q_h\delta),u_h^{N+1}-u_h^N)+(\operatorname{as}(\tau_h),\omega_h^{N+1}-\omega_h^N)=0.$$

Thus

$$\frac{1}{2\mu}\Delta t(\Delta_t\sigma_h^{N+\frac{1}{2}},\tau_h)+\frac{1}{\lambda}\Delta t(\Delta_t p_h^{N+\frac{1}{2}},q_h)+\Delta t(\operatorname{div}(\tau_h-q_h\delta),\Delta_t u_h^{N+\frac{1}{2}})+\Delta t(\operatorname{as}(\tau_h),\Delta_t\omega_h^{N+\frac{1}{2}})=0.$$

Choosing $(\tau_h,q_h)=(\Delta_t\sigma_h^{N+\frac{1}{2}},\Delta_t p_h^{N+\frac{1}{2}})$ and, applying Cauchy–Schwarz inequality and inequality (5.15), we obtain:

$$\begin{aligned}
&\frac{1}{2\mu}(\Delta_t\sigma_h^{N+\frac{1}{2}},\Delta_t\sigma_h^{N+\frac{1}{2}})+\frac{1}{\lambda}(\Delta_t p_h^{N+\frac{1}{2}},\Delta_t p_h^{N+\frac{1}{2}})=-\left(\operatorname{div}(\Delta_t\sigma_h^{N+\frac{1}{2}}-\Delta_t p_h^{N+\frac{1}{2}}\delta),\Delta_t u_h^{N+\frac{1}{2}}\right)\\
&\quad\le\|\operatorname{div}(-\Delta_t\sigma_h^{N+\frac{1}{2}}+\Delta_t p_h^{N+\frac{1}{2}}\delta)\|_{0,\Omega}\|\Delta_t u_h^{N+\frac{1}{2}}\|_{0,\Omega}\\
&\quad\le C_0\,h^{-\beta}\|-\Delta_t\sigma_h^{N+\frac{1}{2}}+\Delta_t p_h^{N+\frac{1}{2}}\delta\|_{0,\Omega}\|\Delta_t u_h^{N+\frac{1}{2}}\|_{0,\Omega}\\
&\quad\le C_0\,h^{-\beta}\left(\|\Delta_t\sigma_h^{N+\frac{1}{2}}\|_{0,\Omega}+\|\Delta_t p_h^{N+\frac{1}{2}}\|_{0,\Omega}\right)\|\Delta_t u_h^{N+\frac{1}{2}}\|_{0,\Omega}\\
&\quad\le C_0\sqrt{2}\,h^{-\beta}\sqrt{\lambda+2\mu}\left(\frac{1}{2\mu}\|\Delta_t\sigma_h^{N+\frac{1}{2}}\|_{0,\Omega}^2+\frac{1}{\lambda}\|\Delta_t p_h^{N+\frac{1}{2}}\|_{0,\Omega}^2\right)^{\frac{1}{2}}\|\Delta_t u_h^{N+\frac{1}{2}}\|_{0,\Omega}.
\end{aligned}$$

Thus

$$\sqrt{\frac{1}{2\mu}\|\Delta_t\sigma_h^{N+\frac{1}{2}}\|_{0,\Omega}^2+\frac{1}{\lambda}\|\Delta_t p_h^{N+\frac{1}{2}}\|_{0,\Omega}^2}\le\sqrt{2}C_0h^{-\beta}\sqrt{\lambda+2\mu}\|\Delta_t u_h^{N+\frac{1}{2}}\|_{0,\Omega}.$$

From (6.8) we get

$$\begin{aligned}
&\left(1-\frac{1}{2}C_0^2(2\mu+\lambda)\frac{\Delta t^2}{h^{2\beta}}\right)\|\Delta_t u_h^{N+\frac{1}{2}}\|_{0,\Omega}^2+\frac{1}{2\mu}\|\sigma_h^{N+\frac{1}{2}}\|_{0,\Omega}^2+\frac{1}{\lambda}\|p_h^{N+\frac{1}{2}}\|_{0,\Omega}^2\\
&\quad\le 2\Delta t\sum_{n=1}^{N}(f^n,\Delta_t u_h^n)+\|\Delta_t u_h^{\frac{1}{2}}\|_{0,\Omega}^2+\frac{1}{2\mu}\|\sigma_h^{\frac{1}{2}}\|_{0,\Omega}^2+\frac{1}{\lambda}\|p_h^{\frac{1}{2}}\|_{0,\Omega}^2-\frac{(\Delta t)^2}{8\mu}\|\Delta_t\sigma_h^{\frac{1}{2}}\|_{0,\Omega}^2-\frac{(\Delta t)^2}{4\lambda}\|\Delta_t p_h^{\frac{1}{2}}\|_{0,\Omega}^2\\
&\quad\le\Delta t\sum_{n=1}^{N}\|f^n\|_{0,\Omega}^2+\Delta t\sum_{n=1}^{N}\|\Delta_t u_h^n\|_{0,\Omega}^2+\|\Delta_t u_h^{\frac{1}{2}}\|_{0,\Omega}^2+\frac{1}{2\mu}\|\sigma_h^{\frac{1}{2}}\|_{0,\Omega}^2+\frac{1}{\lambda}\|p_h^{\frac{1}{2}}\|_{0,\Omega}^2\\
&\qquad-\frac{(\Delta t)^2}{8\mu}\|\Delta_t\sigma_h^{\frac{1}{2}}\|_{0,\Omega}^2-\frac{(\Delta t)^2}{4\lambda}\|\Delta_t p_h^{\frac{1}{2}}\|_{0,\Omega}^2.
\end{aligned}\tag{6.18}$$

Moreover

$$\begin{aligned}
\sum_{n=1}^{N}\|\Delta_t u_h^n\|_{0,\Omega}^2&=\sum_{n=1}^{N}\left\|\frac{\Delta_t u_h^{n+\frac{1}{2}}+\Delta_t u_h^{n-\frac{1}{2}}}{2}\right\|_{0,\Omega}^2\\
&\le\frac{1}{2}\left(\sum_{n=1}^{N}\|\Delta_t u_h^{n+\frac{1}{2}}\|_{0,\Omega}^2+\sum_{n=0}^{N-1}\|\Delta_t u_h^{n+\frac{1}{2}}\|_{0,\Omega}^2\right)\\
&=\frac{1}{2}\left(2\sum_{n=1}^{N-1}\|\Delta_t u_h^{n+\frac{1}{2}}\|_{0,\Omega}^2+\|\Delta_t u_h^{N+\frac{1}{2}}\|_{0,\Omega}^2+\|\Delta_t u_h^{\frac{1}{2}}\|_{0,\Omega}^2\right)\\
&\le\sum_{n=0}^{N-1}\|\Delta_t u_h^{n+\frac{1}{2}}\|_{0,\Omega}^2+\frac{1}{2}\|\Delta_t u_h^{N+\frac{1}{2}}\|_{0,\Omega}^2.
\end{aligned}\tag{6.19}$$

Thus (6.18) becomes

$$\begin{aligned}
&\left(\frac{1}{2}-\frac{1}{2}C_0^2(2\mu+\lambda)\frac{\Delta t^2}{h^{2\beta}}\right)\|\Delta_t u_h^{N+\frac{1}{2}}\|_{0,\Omega}^2+\frac{1}{2\mu}\|\sigma_h^{N+\frac{1}{2}}\|_{0,\Omega}^2+\frac{1}{\lambda}\|p_h^{N+\frac{1}{2}}\|_{0,\Omega}^2\\
&\quad\le\Delta t\sum_{n=1}^{N}\|f^n\|_{0,\Omega}^2+\Delta t\sum_{n=0}^{N-1}\|\Delta_t u_h^{n+\frac{1}{2}}\|_{0,\Omega}^2+\|\Delta_t u_h^{\frac{1}{2}}\|_{0,\Omega}^2+\frac{1}{2\mu}\|\sigma_h^{\frac{1}{2}}\|_{0,\Omega}^2+\frac{1}{\lambda}\|p_h^{\frac{1}{2}}\|_{0,\Omega}^2\\
&\qquad-\frac{(\Delta t)^2}{8\mu}\|\Delta_t\sigma_h^{\frac{1}{2}}\|_{0,\Omega}^2-\frac{(\Delta t)^2}{4\lambda}\|\Delta_t p_h^{\frac{1}{2}}\|_{0,\Omega}^2,
\end{aligned}\tag{6.20}$$

due to our hypothesis that the time step Δt is < 1. Let us note that by

$$\Delta t < \frac{1}{C_0\sqrt{2\mu+\lambda}}h^{\beta}, \tag{6.21}$$

that $\alpha_0 := \frac{1}{2} - \frac{1}{2}C_0^2(2\mu+\lambda)\frac{\Delta t^2}{h^{2\beta}} > 0$.
Inequality (6.20) implies a fortiori:

$$\begin{aligned}&\left(\frac{1}{2} - \frac{1}{2}C_0^2(2\mu+\lambda)\frac{\Delta t^2}{h^{2\beta}}\right)\|\Delta_t u_h^{N+\frac{1}{2}}\|_{0,\Omega}^2 + \frac{1}{2\mu}\|\sigma_h^{N+\frac{1}{2}}\|_{0,\Omega}^2 + \frac{1}{\lambda}\|p_h^{N+\frac{1}{2}}\|_{0,\Omega}^2\\ &\quad\le \Delta t\sum_{n=1}^{N}\|f^n\|_{0,\Omega}^2 + \Delta t\sum_{n=0}^{N-1}\|\Delta_t u_h^{n+\frac{1}{2}}\|_{0,\Omega}^2 + \|\Delta_t u_h^{\frac{1}{2}}\|_{0,\Omega}^2 + \frac{1}{2\mu}\|\sigma_h^{\frac{1}{2}}\|_{0,\Omega}^2 + \frac{1}{\lambda}\|p_h^{\frac{1}{2}}\|_{0,\Omega}^2.\end{aligned} \tag{6.22}$$

Discrete Gronwall's inequality (2.6) to (6.22) yields

$$\begin{aligned}&\alpha_0\|\Delta_t u_h^{N+\frac{1}{2}}\|_{0,\Omega}^2 + \frac{1}{2\mu}\|\sigma_h^{N+\frac{1}{2}}\|_{0,\Omega}^2 + \frac{1}{\lambda}\|p_h^{N+\frac{1}{2}}\|_{0,\Omega}^2\\ &\quad\le \left[\|\Delta_t u_h^{\frac{1}{2}}\|_{0,\Omega}^2 + \frac{1}{\lambda}\|p_h^{\frac{1}{2}}\|_{0,\Omega}^2 + \frac{1}{2\mu}\|\sigma_h^{\frac{1}{2}}\|_{0,\Omega}^2 + \Delta t\sum_{n=1}^{N}\|f^n\|_{0,\Omega}^2\right]\times\exp\left(\Delta t\sum_{n=0}^{N-1}\alpha_0^{-1}\right)\\ &\quad\le \left[\|\Delta_t u_h^{\frac{1}{2}}\|_{0,\Omega}^2 + \frac{1}{\lambda}\|p_h^{\frac{1}{2}}\|_{0,\Omega}^2 + \frac{1}{2\mu}\|\sigma_h^{\frac{1}{2}}\|_{0,\Omega}^2 + T\left(\max_{t\in[0,T]}\|f(t)\|_{0,\Omega}\right)^2\right]\times\exp\left(T\alpha_0^{-1}\right),\end{aligned}$$

which is inequality (6.16).

Inequality (6.17) follows from the equation

$$\frac{1}{2\mu}(\sigma_h^{\left(N+\frac{1}{2}\right)},\tau_h) + \frac{1}{\lambda}(p_h^{\left(N+\frac{1}{2}\right)},q_h) + (\mathrm{div}\,(\tau_h - q_h\delta), u_h^{\left(N+\frac{1}{2}\right)}) + (\mathrm{as}\,(\tau_h), \omega_h^{\left(N+\frac{1}{2}\right)}) = 0, \tag{6.23}$$

by the inf–sup condition (5.10).

Inequalities (6.16) and (6.17) imply that the quantities $\|u_h^{N+\frac{1}{2}}\|_{0,\Omega}$, $\|\sigma_h^{N+\frac{1}{2}}\|_{0,\Omega}$, $\|p_h^{N+\frac{1}{2}}\|_{0,\Omega}$, $\|\omega_h^{N+\frac{1}{2}}\|_{0,\Omega}$ are bounded independently of N, therefore proving the stability of the explicit scheme defined by (6.2)–(6.4) under the CFL condition: $\Delta t < \min(\frac{1}{C_0\sqrt{2\mu+\lambda}}h^{\beta}, 1)$. ■

6.4. *A priori error estimates for the fully discrete explicit scheme*

We shall prove the optimal error estimates between the solution of the fully discrete explicit-in-time mixed finite element problem and the solution of the continuous problem. To this end, we start by the proofs of the following Lemmas:

Lemma 6.4. *Let $(\mathcal{T}_h)_{h>0}$ be a regular family of triangulations on Ω. We suppose that $(\mathcal{T}_h)_{h>0}$ satisfies the refinement rules* R1–R3. *Let $((\widehat{\sigma}_h(t_n),\widehat{p}_h(t_n)), (\widehat{\omega}_h(t_n),\widehat{u}_h(t_n)))$ be the elliptic projection of $((\sigma(t_n),p(t_n)), (\omega(t_n),u(t_n)))$ and set*

$$\varepsilon_h^n = \sigma_h^n - \widehat{\sigma}_h(t_n), \qquad \chi_h^n = u_h^n - \widehat{u}_h(t_n), \qquad \psi_h^n = \omega_h^n - \widehat{\omega}_h(t_n) \quad \text{and} \quad r_h^n = p_h^n - \widehat{p}_h(t_n).$$

Under the hypotheses of Proposition 4.4 and $\Delta t < \zeta\frac{1}{C_0\sqrt{2\mu+\lambda}}h^{\beta}$ with $0<\zeta<1$, we have

$$\|\varepsilon_h^{\bullet}\|_{L^\infty(L^2)} + \|r_h^{\bullet}\|_{L^\infty(L^2)} \lesssim \Delta t\sum_{j=0}^{N-1}\|u_{tt}(t_j) - \Delta_t^2\widehat{u}_h(t_j)\|_{0,\Omega}, \tag{6.24}$$

where $\varepsilon_h^{\bullet}$ denotes the mapping $t_n \mapsto \varepsilon_h^n$ and $r_h^{\bullet}$ the mapping $t_n\mapsto r_h^n$.

Proof. From (6.4) and (5.16) follows that the quantities ε_h^n, χ_h^n, ψ_h^n and r_h^n are linked by the system of equations for all $(\tau_h,q_h)\in\Sigma_{0,h}$, for all $\theta_h\in W_h$ and for all $v_h\in V_h$:

$$\begin{cases}\frac{1}{2\mu}(\varepsilon_h^{n+1},\tau_h) + \frac{1}{\lambda}(r_h^{n+1},q_h) + (\mathrm{div}\,(\tau_h - q_h\delta),\chi_h^{n+1}) + (\mathrm{as}\,(\tau_h),\psi_h^{n+1}) = 0,\\ (\mathrm{as}\,(\varepsilon_h^{n+1}),\theta_h) = 0,\\ (\mathrm{div}\,(\varepsilon_h^n - r_h^n\delta), v_h) = (-u_{tt}(t_n) + \Delta_t^2\widehat{u}_h(t_n) + \Delta_t^2\chi_h^n, v_h).\end{cases} \tag{6.25}$$

Note that $\varepsilon_h^0 = 0$, $\psi_h^0 = 0$ and $r_h^0 = 0$ from (6.7). Furthermore $\chi_h^0 = 0$ and $\chi_h^{-1} = 0$ from (6.2) and (6.3) respectively and thus $\Delta_t\chi_h^{-\frac{1}{2}} = 0$.

If we apply the difference operator Δ_t to the first equation of (6.25), we obtain:

$$\frac{1}{2\mu}\left(\Delta_t\varepsilon_h^{n+\frac{1}{2}},\tau_h\right)+\frac{1}{\lambda}\left(\Delta_t r_h^{n+\frac{1}{2}},q_h\right)+\left(\operatorname{div}(\tau_h-q_h\delta),\Delta_t\chi_h^{n+\frac{1}{2}}\right)+\left(\operatorname{as}(\tau_h),\Delta_t\psi_h^{n+\frac{1}{2}}\right)=0. \tag{6.26}$$

Now taking $(\tau_h,q_h)=\left(\varepsilon_h^{n+\frac{1}{2}},r_h^{n+\frac{1}{2}}\right)$ in this last equality and using the second equation of (6.25), we obtain

$$\frac{1}{2\mu}\left(\Delta_t\varepsilon_h^{n+\frac{1}{2}},\varepsilon_h^{n+\frac{1}{2}}\right)+\frac{1}{\lambda}\left(\Delta_t r_h^{n+\frac{1}{2}},r_h^{n+\frac{1}{2}}\right)+\left(\operatorname{div}\left(\varepsilon_h^{n+\frac{1}{2}}-r_h^{n+\frac{1}{2}}\delta\right),\Delta_t\chi_h^{n+\frac{1}{2}}\right)=0. \tag{6.27}$$

The last equation of (6.25) with $v_h=\frac{1}{2}(\Delta_t\chi_h^{n+\frac{1}{2}}+\Delta_t\chi_h^{n-\frac{1}{2}})$ gives

$$\left(\operatorname{div}(\varepsilon_h^n-r_h^n\delta),\frac{1}{2}\left(\Delta_t\chi_h^{n+\frac{1}{2}}+\Delta_t\chi_h^{n-\frac{1}{2}}\right)\right)=\left(-u_{tt}(t_n)+\Delta_t^2\widehat{u}_h(t_n)+\Delta_t^2\chi_h^n,\frac{1}{2}\left(\Delta_t\chi_h^{n+\frac{1}{2}}+\Delta_t\chi_h^{n-\frac{1}{2}}\right)\right). \tag{6.28}$$

Subtracting (6.28) from (6.27), we get

$$\begin{aligned}&\frac{1}{2\mu}\left(\Delta_t\varepsilon_h^{n+\frac{1}{2}},\varepsilon_h^{n+\frac{1}{2}}\right)+\frac{1}{\lambda}\left(\Delta_t r_h^{n+\frac{1}{2}},r_h^{n+\frac{1}{2}}\right)+\frac{1}{2}\left(\operatorname{div}(\varepsilon_h^{n+1}-r_h^{n+1}\delta),\Delta_t\chi_h^{n+\frac{1}{2}}\right)\\&\quad-\frac{1}{2}\left(\operatorname{div}(\varepsilon_h^n-r_h^n\delta),\Delta_t\chi_h^{n-\frac{1}{2}}\right)=\left(u_{tt}(t_n)-\Delta_t^2\widehat{u}_h(t_n)-\Delta_t^2\chi_h^n,\frac{1}{2}\left(\Delta_t\chi_h^{n+\frac{1}{2}}+\Delta_t\chi_h^{n-\frac{1}{2}}\right)\right).\end{aligned}$$

Thus

$$\begin{aligned}&\frac{1}{2\mu}\left(\Delta_t\varepsilon_h^{n+\frac{1}{2}},\varepsilon_h^{n+\frac{1}{2}}\right)+\frac{1}{\lambda}\left(\Delta_t r_h^{n+\frac{1}{2}},r_h^{n+\frac{1}{2}}\right)+\frac{1}{2}\left(\operatorname{div}(\varepsilon_h^{n+1}-r_h^{n+1}\delta),\Delta_t\chi_h^{n+\frac{1}{2}}\right)\\&\quad-\frac{1}{2}\left(\operatorname{div}(\varepsilon_h^n-r_h^n\delta),\Delta_t\chi_h^{n-\frac{1}{2}}\right)+\left(\Delta_t^2\chi_h^n,\frac{1}{2}\left(\Delta_t\chi_h^{n+\frac{1}{2}}+\Delta_t\chi_h^{n-\frac{1}{2}}\right)\right)\\&\quad=\left(u_{tt}(t_n)-\Delta_t^2\widehat{u}_h(t_n),\frac{1}{2}\left(\Delta_t\chi_h^{n+\frac{1}{2}}+\Delta_t\chi_h^{n-\frac{1}{2}}\right)\right).\end{aligned} \tag{6.29}$$

We expand (6.29) to get

$$\begin{aligned}&\frac{1}{2\mu}\left(\frac{\varepsilon_h^{n+1}-\varepsilon_h^n}{\Delta t},\frac{\varepsilon_h^{n+1}+\varepsilon_h^n}{2}\right)+\frac{1}{\lambda}\left(\frac{r_h^{n+1}-r_h^n}{\Delta t},\frac{r_h^{n+1}+r_h^n}{2}\right)+\frac{1}{2}\left(\operatorname{div}(\varepsilon_h^{n+1}-r_h^{n+1}\delta),\Delta_t\chi_h^{n+\frac{1}{2}}\right)\\&\quad-\frac{1}{2}\left(\operatorname{div}(\varepsilon_h^n-r_h^n\delta),\Delta_t\chi_h^{n-\frac{1}{2}}\right)+\left(\frac{\Delta_t\chi_h^{n+\frac{1}{2}}-\Delta_t\chi_h^{n-\frac{1}{2}}}{\Delta t},\frac{\Delta_t\chi_h^{n+\frac{1}{2}}+\Delta_t\chi_h^{n-\frac{1}{2}}}{2}\right)\\&\quad=\left(u_{tt}(t_n)-\Delta_t^2\widehat{u}_h(t_n),\frac{1}{2}(\Delta_t\chi_h^{n+\frac{1}{2}}+\Delta_t\chi_h^{n-\frac{1}{2}})\right).\end{aligned}$$

So

$$\begin{aligned}&\frac{1}{2\Delta t}\left(\frac{1}{2\mu}(\|\varepsilon_h^{n+1}\|_{0,\Omega}^2-\|\varepsilon_h^n\|_{0,\Omega}^2)\right)+\frac{1}{2\Delta t}\left(\frac{1}{\lambda}(\|r_h^{n+1}\|_{0,\Omega}^2-\|r_h^n\|_{0,\Omega}^2)\right)+\frac{1}{2}\left(\operatorname{div}(\varepsilon_h^{n+1}-r_h^{n+1}\delta),\Delta_t\chi_h^{n+\frac{1}{2}}\right)\\&\quad-\frac{1}{2}\left(\operatorname{div}(\varepsilon_h^n-r_h^n\delta),\Delta_t\chi_h^{n-\frac{1}{2}}\right)+\frac{1}{2\Delta t}\left(\|\Delta_t\chi_h^{n+\frac{1}{2}}\|_{0,\Omega}^2-\|\Delta_t\chi_h^{n-\frac{1}{2}}\|_{0,\Omega}^2\right)\\&\quad=\left(u_{tt}(t_n)-\Delta_t^2\widehat{u}_h(t_n),\frac{1}{2}(\Delta_t\chi_h^{n+\frac{1}{2}}+\Delta_t\chi_h^{n-\frac{1}{2}})\right).\end{aligned} \tag{6.30}$$

Multiplying (6.30) by $2\Delta t$, replacing n by j and summing these equations from $j=0$ to $n-1$ yield

$$\begin{aligned}&\frac{1}{2\mu}(\|\varepsilon_h^n\|_{0,\Omega}^2-\|\varepsilon_h^0\|_{0,\Omega}^2)+\frac{1}{\lambda}(\|r_h^n\|_{0,\Omega}^2-\|r_h^0\|_{0,\Omega}^2)+\Delta t\left(\operatorname{div}(\varepsilon_h^n-r_h^n\delta),\Delta_t\chi_h^{n-\frac{1}{2}}\right)\\&\quad-\Delta t\left(\operatorname{div}(\varepsilon_h^0-r_h^0\delta),\Delta_t\chi_h^{-\frac{1}{2}}\right)+\|\Delta_t\chi_h^{n-\frac{1}{2}}\|_{0,\Omega}^2-\|\Delta_t\chi_h^{-\frac{1}{2}}\|_{0,\Omega}^2\\&\quad=\Delta t\sum_{j=0}^{n-1}\left(u_{tt}(t_j)-\Delta_t^2\widehat{u}_h(t_j),\Delta_t\chi_h^{j+\frac{1}{2}}+\Delta_t\chi_h^{j-\frac{1}{2}}\right).\end{aligned} \tag{6.31}$$

Recall that $\varepsilon_h^0 = 0$, $r_h^0 = 0$ and $\Delta_t \chi_h^{\frac{-1}{2}} = 0$. Thus (6.31) becomes

$$\frac{1}{2\mu}\|\varepsilon_h^n\|_{0,\Omega}^2 + \frac{1}{\lambda}\|r_h^n\|_{0,\Omega}^2 - \Delta t\left(\operatorname{div}(-\varepsilon_h^n + r_h^n\delta),\, \Delta_t \chi_h^{n-\frac{1}{2}}\right) + \|\Delta_t \chi_h^{n-\frac{1}{2}}\|_{0,\Omega}^2$$
$$= \Delta t \sum_{j=0}^{n-1}\left(u_{tt}(t_j) - \Delta_t^2 \widehat{u}_h(t_j),\, \Delta_t \chi_h^{j+\frac{1}{2}} + \Delta_t \chi_h^{j-\frac{1}{2}}\right). \tag{6.32}$$

By the Cauchy–Schwarz inequality and the inverse inequality (5.15), and as $\Delta t < \zeta \frac{h^\beta}{C_0\sqrt{2\mu+\lambda}}$ by hypothesis, we obtain

$$\begin{aligned}
\left|\Delta t\left(\operatorname{div}(-\varepsilon_h^n + r_h^n\delta),\, \Delta_t \chi_h^{n-\frac{1}{2}}\right)\right| &\le \Delta t\|\operatorname{div}(-\varepsilon_h^n + r_h^n\delta)\|_{0,\Omega}\|\Delta_t \chi_h^{n-\frac{1}{2}}\|_{0,\Omega}\\
&\le \Delta t C_0 h^{-\beta}\| - \varepsilon_h^n + r_h^n\delta\|_{0,\Omega}\|\Delta_t \chi_h^{n-\frac{1}{2}}\|_{0,\Omega}\\
&\le \sqrt{2}\Delta t C_0 h^{-\beta}(\|\varepsilon_h^n\|_{0,\Omega} + \|r_h^n\|_{0,\Omega})\|\Delta_t \chi_h^{n-\frac{1}{2}}\|_{0,\Omega}\\
&\le 2\Delta t C_0 h^{-\beta}\sqrt{2\mu+\lambda}\left(\frac{1}{2\mu}\|\varepsilon_h^n\|_{0,\Omega}^2 + \frac{1}{\lambda}\|r_h^n\|_{0,\Omega}^2\right)^{\frac{1}{2}}\|\Delta_t \chi_h^{n-\frac{1}{2}}\|_{0,\Omega}\\
&\le \zeta\left(\frac{1}{2\mu}\|\varepsilon_h^n\|_{0,\Omega}^2 + \frac{1}{\lambda}\|r_h^n\|_{0,\Omega}^2 + \|\Delta_t \chi_h^{n-\frac{1}{2}}\|_{0,\Omega}^2\right).
\end{aligned} \tag{6.33}$$

As $0 < \zeta < 1$, it follows from (6.32) using the preceding inequality:

$$\begin{aligned}
\frac{1}{2\mu}\|\varepsilon_h^n\|_{0,\Omega}^2 + \frac{1}{\lambda}\|r_h^n\|_{0,\Omega}^2 + \|\Delta_t \chi_h^{n-\frac{1}{2}}\|_{0,\Omega}^2 &\le C\Delta t\sum_{j=0}^{n-1}\left|\left(u_{tt}(t_j) - \Delta_t^2\widehat{u}_h(t_j),\, \Delta_t \chi_h^{j+\frac{1}{2}} + \Delta_t \chi_h^{j-\frac{1}{2}}\right)\right|\\
&\le C\Delta t\sum_{j=0}^{n-1}\|u_{tt}(t_j) - \Delta_t^2\widehat{u}_h(t_j)\|_{0,\Omega}\|\Delta_t \chi_h^{j+\frac{1}{2}} + \Delta_t \chi_h^{j-\frac{1}{2}}\|_{0,\Omega}\\
&\le 2C\Delta t\sum_{j=0}^{n-1}\|u_{tt}(t_j) - \Delta_t^2\widehat{u}_h(t_j)\|_{0,\Omega}\|\Delta_t^{\bullet+\frac{1}{2}}\chi_h\|_{L^\infty(L^2)},
\end{aligned}$$

since $\|\Delta_t \chi^{j+\frac{1}{2}}\|_{0,\Omega} \le \|\Delta_t^{\bullet+\frac{1}{2}}\chi_h\|_{L^\infty(L^2)} := \sup_{0\le j\le N-1}\|\Delta_t \chi_h^{j+\frac{1}{2}}\|_{0,\Omega}$. Then

$$\begin{aligned}
\frac{1}{2\mu}\|\varepsilon_h^n\|_{0,\Omega}^2 + \frac{1}{\lambda}\|r_h^n\|_{0,\Omega}^2 + \|\Delta_t \chi_h^{n-\frac{1}{2}}\|_{0,\Omega}^2 &\le 2C\Delta t\|\Delta_t^{\bullet+\frac{1}{2}}\chi_h\|_{L^\infty(L^2)}\sum_{j=0}^{N-1}\|u_{tt}(t_j) - \Delta_t^2\widehat{u}_h(t_j)\|_{0,\Omega}\\
&\le \frac{1}{3}\|\Delta_t^{\bullet+\frac{1}{2}}\chi_h\|_{L^\infty(L^2)}^2 + 3C^2(\Delta t)^2\left(\sum_{j=0}^{N-1}\|u_{tt}(t_j) - \Delta_t^2\widehat{u}_h(t_j)\|_{0,\Omega}\right)^2.
\end{aligned} \tag{6.34}$$

Taking the supremum on n on the left-hand side of (6.34) we get

$$\frac{1}{2\mu}\|\varepsilon_h^\bullet\|_{L^\infty(L^2)}^2 + \frac{1}{\lambda}\|r_h^\bullet\|_{L^\infty(L^2)}^2 + \|\Delta_t^{\bullet+\frac{1}{2}}\chi_h\|_{L^\infty(L^2)}^2 \le \|\Delta_t^{\bullet+\frac{1}{2}}\chi_h\|_{L^\infty(L^2)}^2 + 9C^2(\Delta t)^2\left(\sum_{j=0}^{N-1}\|u_{tt}(t_j) - \Delta_t^2\widehat{u}_h(t_j)\|_{0,\Omega}\right)^2.$$

This last inequality and assumption on μ and λ yield (6.24). ■

Lemma 6.5. *Under the hypotheses of Lemma 6.4, we have*

$$\begin{aligned}
\|\varepsilon_h^\bullet\|_{L^\infty(L^2)} + \|r_h^\bullet\|_{L^\infty(L^2)} \lesssim h\Big[&|u_{tt}|_{L^\infty(H_\phi^{1,1})} + |u_{tt}|_{L^\infty(H^1)} + |p_{tt}|_{L^\infty(H_\phi^{0,1})} + |u_{ttt}|_{L^\infty(H_\phi^{1,1})} + |u_{ttt}|_{L^\infty(H^1)} + |p_{ttt}|_{L^\infty(H_\phi^{0,1})}\Big]\\
&+ (\Delta t)^2\|u_{tttt}\|_{L^\infty(L^2)}.
\end{aligned} \tag{6.35}$$

Proof. To obtain (6.35), we have to bound the right-hand side of the inequality (6.24). We write

$$u_{tt}(t_j) - \Delta_t^2 \widehat{u}_h(t_j) = u_{tt}(t_j) - \Delta_t^2 u(t_j) + \Delta_t^2 u(t_j) - \Delta_t^2 \widehat{u}_h(t_j).$$

If we denote by R_h the elliptic projection operator, we can write

$$\Delta_t^2 u(t_j) - \Delta_t^2 \widehat{u}_h(t_j) = (I - R_h)\Delta_t^2 u(t_j) = (I - R_h)\frac{u(t_{j+1}) - 2u(t_j) + u(t_{j-1})}{(\Delta t)^2}.$$

By Taylor expansion with integral remainder up to the second order term, we have

$$u(t_{j-1}) = u(t_j) - \Delta t\, u_t(t_j) + \frac{(\Delta t)^2}{2} u_{tt}(t_j) + \frac{1}{2}\int_{t_j}^{t_{j-1}} (t_{j-1} - s)^2 u_{ttt}(s)\, \mathrm{d}s,$$

and

$$u(t_{j+1}) = u(t_j) + \Delta t\, u_t(t_j) + \frac{(\Delta t)^2}{2} u_{tt}(t_j) + \frac{1}{2}\int_{t_j}^{t_{j+1}} (t_{j+1} - s)^2 u_{ttt}(s)\, \mathrm{d}s.$$

Summing these two equations and dividing by $(\Delta t)^2$, we obtain:

$$\begin{aligned}\Delta_t^2 u(t_j) &= u_{tt}(t_j) + \frac{1}{2(\Delta t)^2}\left(\int_{t_j}^{t_{j-1}} (t_{j-1} - s)^2 u_{ttt}(s)\, \mathrm{d}s + \int_{t_j}^{t_{j+1}} (t_{j+1} - s)^2 u_{ttt}(s)\, \mathrm{d}s\right)\\ &= u_{tt}(t_j) + \frac{1}{2(\Delta t)^2}\left(-\int_{-\Delta t}^{0} (\Delta t + t)^2 u_{ttt}(t + t_j)\, \mathrm{d}t + \int_{0}^{\Delta t} (\Delta t - t)^2 u_{ttt}(t + t_j)\, \mathrm{d}t\right). \end{aligned} \tag{6.36}$$

Applying the operator $I - R_h$ to both sides, we obtain:

$$\begin{aligned}(I - R_h)\Delta_t^2 u(t_j) = (I - R_h)u_{tt}(t_j) + \frac{1}{2(\Delta t)^2}\Bigg(&-\int_{-\Delta t}^{0} (\Delta t + t)^2 (I - R_h)u_{ttt}(t + t_j)\, \mathrm{d}t\\ &+ \int_{0}^{\Delta t} (\Delta t - t)^2 (I - R_h)u_{ttt}(t + t_j)\, \mathrm{d}t\Bigg).\end{aligned}$$

Taking norms, we obtain:

$$\begin{aligned}\|(I - R_h)\Delta_t^2 u(t_j)\|_{0,\Omega} &\le \|(I - R_h)u_{tt}(t_j)\|_{0,\Omega} + \frac{1}{2}\frac{\sqrt{\Delta t}}{\sqrt{5}}\left[\left(\int_{-\Delta t}^{0} \|(I - R_h)u_{ttt}(t + t_j)\|_{0,\Omega}^2\, \mathrm{d}t\right)^{\frac{1}{2}}\right.\\ &\qquad \left. + \left(\int_{0}^{\Delta t} \|(I - R_h)u_{ttt}(t + t_j)\|_{0,\Omega}^2\, \mathrm{d}t\right)^{\frac{1}{2}}\right]\\ &\le \|(I - R_h)u_{tt}(t_j)\|_{0,\Omega} + \frac{\sqrt{2}}{2}\frac{\sqrt{\Delta t}}{\sqrt{5}}\left(\int_{-\Delta t}^{\Delta t} \|(I - R_h)u_{ttt}(t + t_j)\|_{0,\Omega}^2\, \mathrm{d}t\right)^{\frac{1}{2}}\\ &= \|(I - R_h)u_{tt}(t_j)\|_{0,\Omega} + \frac{\sqrt{\Delta t}}{\sqrt{10}}\left(\int_{-\Delta t + t_j}^{\Delta t + t_j} \|(I - R_h)u_{ttt}(s)\|_{0,\Omega}^2\, \mathrm{d}s\right)^{\frac{1}{2}}.\end{aligned}$$

And hence, we have found:

$$\|(I - R_h)\Delta_t^2 u(t_j)\|_{0,\Omega} \le \|(I - R_h)u_{tt}(t_j)\|_{0,\Omega} + \frac{\sqrt{\Delta t}}{\sqrt{10}}\left(\int_{-\Delta t + t_j}^{\Delta t + t_j} \|(I - R_h)u_{ttt}(s)\|_{0,\Omega}^2\, \mathrm{d}s\right)^{\frac{1}{2}}.$$

From the continuous-in-time error estimate (5.26), we obtain

$$\|(I - R_h)u_{tt}(t_j)\|_{0,\Omega} \lesssim h\left[|u_{tt}|_{L^\infty(H_\phi^{1,1})} + |u_{tt}|_{L^\infty(H^1)} + |p_{tt}|_{L^\infty(H_\phi^{0,1})}\right],$$

and by a similar argument as the one used in Remark 5.11:

$$\Delta t \int_{-\Delta t+t_j}^{\Delta t+t_j} \|(I-R_h)u_{ttt}(s)\|_{0,\Omega}^2\, \mathrm{d}s \lesssim \Delta t\, h^2 \left[\int_{-\Delta t+t_j}^{\Delta t+t_j} \left(|u_{ttt}(s)|_{1,1;\phi,\Omega}^2 + |u_{ttt}(s)|_{1,\Omega}^2 + |p_{ttt}(s)|_{0,1;\phi,\Omega}^2\right) \mathrm{d}s\right]$$

$$\lesssim (\Delta t)^2 h^2 \left[|u_{ttt}|^2_{L^\infty(H_\phi^{1,1})} + |u_{ttt}|^2_{L^\infty(H^1)} + |p_{ttt}|^2_{L^\infty(H_\phi^{0,1})}\right].$$

By the stability condition (6.21) we can write

$$\sqrt{\Delta t}\left(\int_{-\Delta t+t_j}^{\Delta t+t_j} \|(I-R_h)u_{ttt}(s)\|_{0,\Omega}^2\, \mathrm{d}s\right)^{\frac{1}{2}} \lesssim h^{\beta+1}\left[|u_{ttt}|_{L^\infty(H_\phi^{1,1})} + |u_{ttt}|_{L^\infty(H^1)} + |p_{ttt}|_{L^\infty(H_\phi^{0,1})}\right]$$

$$\lesssim h\left[|u_{ttt}|_{L^\infty(H_\phi^{1,1})} + |u_{ttt}|_{L^\infty(H^1)} + |p_{ttt}|_{L^\infty(H_\phi^{0,1})}\right].$$

Summing these last inequalities from $j=0$ to $j=N-1$, we get

$$\Delta t \sum_{j=0}^{N-1} \|(I-R_h)\Delta_t^2 u(t_j)\|_{0,\Omega}$$

$$\lesssim h\left[|u_{tt}|_{L^\infty(H_\phi^{1,1})} + |u_{tt}|_{L^\infty(H^1)} + |p_{tt}|_{L^\infty(H_\phi^{0,1})} + |u_{ttt}|_{L^\infty(H_\phi^{1,1})} + |u_{ttt}|_{L^\infty(H^1)} + |p_{ttt}|_{L^\infty(H_\phi^{0,1})}\right] \times \sum_{j=0}^{N-1}\Delta t$$

$$\lesssim h\left[|u_{tt}|_{L^\infty(H_\phi^{1,1})} + |u_{tt}|_{L^\infty(H^1)} + |p_{tt}|_{L^\infty(H_\phi^{0,1})} + |u_{ttt}|_{L^\infty(H_\phi^{1,1})} + |u_{ttt}|_{L^\infty(H^1)} + |p_{ttt}|_{L^\infty(H_\phi^{0,1})}\right]. \tag{6.37}$$

Using once again Taylor expansion with integral remainder, but up to the third order term this time, we get:

$$u(t_{j-1}) = u(t_j) - \Delta t\, u_t(t_j) + \frac{(\Delta t)^2}{2}u_{tt}(t_j) - \frac{(\Delta t)^3}{6}u_{ttt}(t_j) + \frac{1}{6}\int_{t_j}^{t_{j-1}} (t_{j-1}-s)^3 u_{tttt}(s)\,\mathrm{d}s,$$

and

$$u(t_{j+1}) = u(t_j) + \Delta t\, u_t(t_j) + \frac{(\Delta t)^2}{2}u_{tt}(t_j) + \frac{(\Delta t)^3}{6}u_{ttt}(t_j) + \frac{1}{6}\int_{t_j}^{t_{j+1}} (t_{j+1}-s)^3 u_{tttt}(s)\,\mathrm{d}s.$$

Thus

$$|\Delta_t^2 u(t_j) - u_{tt}(t_j)| = \left|\frac{1}{6(\Delta t)^2}\left(\int_{-\Delta t}^{0} (\Delta t + t)^3 u_{tttt}(t+t_j)\,\mathrm{d}t + \int_0^{\Delta t} (\Delta t - t)^3 u_{tttt}(t+t_j)\,\mathrm{d}t\right)\right|$$

$$\leq \frac{1}{6}\frac{(\Delta t)^{\frac{3}{2}}}{\sqrt{7}}\left[\left(\int_{-\Delta t}^{0}\left(u_{tttt}(t+t_j)\right)^2\,\mathrm{d}t\right)^{\frac{1}{2}} + \left(\int_0^{\Delta t}\left(u_{tttt}(t+t_j)\right)^2\,\mathrm{d}t\right)^{\frac{1}{2}}\right]$$

$$\lesssim \Delta t^{\frac{3}{2}}\left(\int_{-\Delta t}^{\Delta t}\left(u_{tttt}(t+t_j)\right)^2\,\mathrm{d}t\right)^{\frac{1}{2}}.$$

Hence

$$\|\Delta_t^2 u(t_j) - u_{tt}(t_j)\|_{0,\Omega} \lesssim (\Delta t)^{\frac{3}{2}}\left(\int_{-\Delta t+t_j}^{\Delta t+t_j} \|u_{tttt}(s)\|_{0,\Omega}^2\,\mathrm{d}s\right)^{\frac{1}{2}}. \tag{6.38}$$

So that

$$\Delta t \sum_{j=0}^{N-1} \|\Delta_t^2 u(t_j) - u_{tt}(t_j)\|_{0,\Omega} \lesssim (\Delta t)^2 \|u_{tttt}\|_{L^\infty(L^2)} \sum_{j=0}^{N-1}\Delta t$$

$$\lesssim (\Delta t)^2 \|u_{tttt}\|_{L^\infty(L^2)}. \tag{6.39}$$

By (6.37) and (6.39), (6.24) becomes (6.35). ■

We are now in a position to establish optimal error estimates between the solution of the fully discrete explicit-in-time mixed finite element problem and the solution of the continuous problem.

Theorem 6.6. *Let $(\mathcal{T}_h)_{h>0}$ be a regular family of triangulations on Ω. We suppose that $(\mathcal{T}_h)_{h>0}$ satisfies the refinement rules* R1–R3. *Under the hypotheses of Proposition 4.4 and $\Delta t < \zeta \frac{1}{C_0\sqrt{2\mu+\lambda}} h^\beta$ with $0 < \zeta < 1$, the following error estimates hold*

$$\|\sigma - \sigma_h^\bullet\|_{L^\infty(L^2)} + \|p - p_h^\bullet\|_{L^\infty(L^2)} \lesssim h\left[|u|_{L^\infty(H^{1,1}_\phi)} + |u_{tt}|_{L^\infty(H^{1,1}_\phi)} + |u_{tt}|_{L^\infty(H^1)} + |u_{ttt}|_{L^\infty(H^{1,1}_\phi)} + |u_{ttt}|_{L^\infty(H^1)}\right] + (\Delta t)^2\|u_{tttt}\|_{L^\infty(L^2)}, \tag{6.40}$$

$$\|\omega - \omega_h^\bullet\|_{L^\infty(L^2)} + \|P_h^0 u - u_h^\bullet\|_{L^\infty(L^2)} \lesssim h\left[|u|_{L^\infty(H^{1,1}_\phi)} + |u_{tt}|_{L^\infty(H^{1,1}_\phi)} + |u_{tt}|_{L^\infty(H^1)} + |u_{ttt}|_{L^\infty(H^{1,1}_\phi)} + |u_{ttt}|_{L^\infty(H^1)}\right] + (\Delta t)^2\|u_{tttt}\|_{L^\infty(L^2)}, \tag{6.41}$$

$$\|u - u_h^\bullet\|_{L^\infty(L^2)} \lesssim h\left[|u|_{L^\infty(H^{1,1}_\phi)} + |u_{tt}|_{L^\infty(H^{1,1}_\phi)} + |u_{tt}|_{L^\infty(H^1)} + |u_{ttt}|_{L^\infty(H^{1,1}_\phi)} + |u_{ttt}|_{L^\infty(H^1)} + |u|_{L^\infty(H^1)}\right] + (\Delta t)^2\|u_{tttt}\|_{L^\infty(L^2)}, \tag{6.42}$$

where $\sigma_h^\bullet$ denotes the mapping $t_n \mapsto \sigma_h^n$, $p_h^\bullet$ the mapping $t_n \mapsto p_h^n$, $\omega_h^\bullet$ the mapping $t_n \mapsto \omega_h^n$ and $u_h^\bullet$ the mapping $t_n \mapsto u_h^n$.

Proof. By (6.35), the triangle inequality and taking the supremum on t in (5.18), we then find

$$\|\sigma - \sigma_h^\bullet\|_{L^\infty(L^2)} + \|p - p_h^\bullet\|_{L^\infty(L^2)} \lesssim h\Big[|u|_{L^\infty(H^{1,1}_\phi)} + |u_{tt}|_{L^\infty(H^{1,1}_\phi)} + |u_{tt}|_{L^\infty(H^1)} + |p|_{L^\infty(H^{0,1}_\phi)} + |p_{tt}|_{L^\infty(H^{0,1}_\phi)} + |u_{ttt}|_{L^\infty(H^{1,1}_\phi)} + |u_{ttt}|_{L^\infty(H^1)} + |p_{ttt}|_{L^\infty(H^{0,1}_\phi)}\Big] + (\Delta t)^2\|u_{tttt}\|_{L^\infty(L^2)}.$$

By the inf–sup inequality (5.10) and the triangle inequality, we get

$$\|\omega(t_n) - \omega_h^n\|_{0,\Omega} + \|P_h^0 u(t_n) - u_h^n\|_{0,\Omega} \lesssim \left[\|\sigma(t_n) - \sigma_h^n\|_{0,\Omega} + \|p(t_n) - p_h^n\|_{0,\Omega} + \|P_h^1\omega(t_n) - \omega(t_n)\|_{0,\Omega}\right].$$

Taking the supremum on n, we get

$$\|\omega - \omega_h^\bullet\|_{L^\infty(L^2)} + \|P_h^0 u - u_h^\bullet\|_{L^\infty(L^2)} \lesssim \left[\|\sigma - \sigma_h^\bullet\|_{L^\infty(L^2)} + \|p - p_h^\bullet\|_{L^\infty(L^2)} + \|P_h^1\omega - \omega\|_{L^\infty(L^2)}\right].$$

Using the P_h^1-projection error inequality (5.13), we hence find

$$\|\omega - \omega_h^\bullet\|_{L^\infty(L^2)} + \|P_h^0 u - u_h^\bullet\|_{L^\infty(L^2)} \lesssim h\Big[|u|_{L^\infty(H^{1,1}_\phi)} + |u_{tt}|_{L^\infty(H^{1,1}_\phi)} + |u_{tt}|_{L^\infty(H^1)} + |p|_{L^\infty(H^{0,1}_\phi)} + |p_{tt}|_{L^\infty(H^{0,1}_\phi)} + |u_{ttt}|_{L^\infty(H^{1,1}_\phi)} + |u_{ttt}|_{L^\infty(H^1)} + |p_{ttt}|_{L^\infty(H^{0,1}_\phi)}\Big] + (\Delta t)^2\|u_{tttt}\|_{L^\infty(L^2)}.$$

The triangle inequality and the P_h^0-projection error inequality (1.47) p.27 of [18] (or (45) p. 624 of [19]), give us

$$\|u - u_h^\bullet\|_{L^\infty(L^2)} \lesssim h\Big[|u|_{L^\infty(H^{1,1}_\phi)} + |u_{tt}|_{L^\infty(H^{1,1}_\phi)} + |u_{tt}|_{L^\infty(H^1)} + |p|_{L^\infty(H^{0,1}_\phi)} + |p_{tt}|_{L^\infty(H^{0,1}_\phi)} + |u_{ttt}|_{L^\infty(H^{1,1}_\phi)} + |u_{ttt}|_{L^\infty(H^1)} + |p_{ttt}|_{L^\infty(H^{0,1}_\phi)} + |u|_{L^\infty(H^1)}\Big] + (\Delta t)^2\|u_{tttt}\|_{L^\infty(L^2)}. \quad \blacksquare$$

7. Implementation and numerical results

In this section we will confirm our theoretical analysis by numerical tests. We begin by introducing the so-called "Hybrid formulations" [7,20,21] for solving the explicit scheme (6.2)–(6.4). The numerical results are presented on an L-shaped domain. Given $f : [0, T] \times \Omega \longrightarrow \mathbb{R}^2$, a surface force density $g : [0, T] \times \Gamma_N \longrightarrow \mathbb{R}^2$ and the initial conditions u_0 and u_1, the displacement field u satisfies the following equations:

$$\begin{cases} u_{tt} - \operatorname{div}\sigma_s(u) = f & \text{in } [0, T] \times \Omega, \\ u = 0 & \text{on } [0, T] \times \Gamma_D, \\ \sigma_s(u).n = g & \text{on } [0, T] \times \Gamma_N, \\ u(0, .) = u_0 & \text{in } \Omega, \\ u_t(0, .) = u_1 & \text{in } \Omega. \end{cases} \tag{7.1}$$

7.1. Explicit-in-time scheme

Let us mention that our explicit-in-time scheme (6.2)–(6.4) is fast and easy to implement. In order to fit the complex geometry of the boundary, the mesh which must be moreover refined accordingly to the rules R_1– R_3, will usually contain elements of very small size, which implies, because of the CFL stability condition, the use of a very small time step. Thus the explicit scheme is essentially appropriate when we are interested only in the behavior of the solution for a short time after the initial time.

7.1.1. Hybrid formulation

Firstly, we introduce the enlarged space $\widetilde{\Sigma}_h$ (with respect to $\Sigma_{h,0}$) by suppressing the requirement for its elements to have continuous normal component at the interfaces of the triangulation $\mathcal{T}_h$:

$$\widetilde{\Sigma}_h := \{(\tau_h, q_h) \in [L^2(\Omega)]^{2\times 2} \times L^2(\Omega);\ \forall T \in \mathcal{T}_h : q_{h|T} \in \mathbb{P}_1(T) \text{ and } \tau_{h|T} \in [\mathbb{P}_1(T)]^{2\times 2} \oplus [\mathbb{R}\,\mathrm{Curl}\, b_T]^2\}.$$

We introduce the following hybrid formulation: Find $(\widetilde{\sigma}_h^{n+1}, \widetilde{p}_h^{n+1}, \lambda_h^{n+1}) \in \widetilde{\Sigma}_h \times \Lambda_h$ and $(\widetilde{u}_h^{n+1}, \widetilde{\omega}_h^{n+1}) \in M_h$ such that

$$\widetilde{u}_h^0 = \hat{u}_h(0), \qquad \widetilde{u}_h^{-1} = \hat{u}_h(-\Delta t) \simeq \hat{u}_h(0) - \Delta t\, \hat{u}_{h,t}(0) + \frac{\Delta t^2}{2}\hat{u}_{h,tt}(0),$$

$$\begin{cases} \dfrac{1}{2\mu}(\widetilde{\sigma}_h^{n+1}, \tau_h) + \dfrac{1}{\lambda}(\widetilde{p}_h^{n+1}, q_h) + \displaystyle\sum_{T\in\mathcal{T}_h}\int_T \mathrm{div}\,(\tau_h - q_h\delta).\widetilde{u}_h^{n+1}\,\mathrm{d}x + (\mathrm{as}\,(\tau_h), \widetilde{\omega}_h^{n+1}) \\ \quad - \displaystyle\sum_{T\in\mathcal{T}_h}\int_{\partial T} \lambda_h^{n+1}(\tau_h - q_h\delta).n_T\,\mathrm{d}s = 0, \quad \forall\,(\tau_h, q_h) \in \widetilde{\Sigma}_h,\ \forall n \ge -1, \\ (\mathrm{as}\,(\widetilde{\sigma}_h^{n+1}), \theta_h) = 0, \quad \forall\,\theta_h \in W_h, \forall n \ge -1, \\ (\Delta_t^2 \widetilde{u}_h^n, v_h) - \displaystyle\sum_{T\in\mathcal{T}_h}\int_T \mathrm{div}\,(\widetilde{\sigma}_h^n - \widetilde{p}_h^n\delta).v_h\,\mathrm{d}x - (f^n, v_h) = 0, \quad \forall\, v_h \in V_h, \forall n \ge 0, \\ \displaystyle\sum_{T\in\mathcal{T}_h}\int_{\partial T} \mu_h(\widetilde{\sigma}_h^{n+1} - \widetilde{p}_h^{n+1}\delta).n_T\,\mathrm{d}s = \sum_{T\in\mathcal{T}_h}\int_{\partial T\cap\Gamma_N} \mu_h.g^{n+1}\,\mathrm{d}s, \quad \forall\,\mu_h \in \Lambda_h, \forall n \ge -1, \end{cases} \tag{7.2}$$

where

$$\Lambda_h := \{\mu_h \in [L^2(\mathcal{E}_h)]^2;\ \mu_{h|e} \in [\mathbb{P}_1(e)]^2,\ \forall e \in \mathcal{E}_h \text{ and } \mu_{h|e} = 0,\ \forall e \subset \Gamma_D\}.$$

$\mathcal{E}_h$ denotes the set of all edges in $\mathcal{T}_h$ and $V_h \times W_h$ is defined in (5.2). It is easy to prove that the hybrid formulation (7.2) is equivalent to the explicit-in-time discrete mixed formulation (6.4), i.e. that $\widetilde{\sigma}_h^n = \sigma_h^n$, $\widetilde{p}_h^n = p_h^n$, $\widetilde{u}_h^n = u_h^n$ and $\widetilde{\omega}_h^n = \omega_h^n$. Taking as test functions $v_h 1_T$ in the third equation of the system (7.2), we get explicitly u_h^{n+1}

$$u_{h|T}^{n+1} = \Delta t^2\left[\mathrm{div}\,(\sigma_{h|T}^n - p_{h|T}^n\delta) + P_{h|T}^0 f^n\right] + 2u_{h|T}^n - u_{h|T}^{n-1}, \quad \forall n \ge 0. \tag{7.3}$$

Still noting $\sigma_h^{n+1},\ p_h^{n+1},\ u_h^{n+1},\ \omega_h^{n+1}$ and λ_h^{n+1} the vectors of the degrees of freedom of these same unknowns, the algebraic equations generated by (7.2) have the following form

$$\begin{cases} A\sigma_h^{n+1} - E^T\lambda_h^{n+1} + H^T\omega_h^{n+1} = -F_1^{n+1}, \\ Pp_h^{n+1} + G^T\lambda_h^{n+1} = F_2^{n+1}, \\ H\sigma_h^{n+1} = 0, \\ E\sigma_h^{n+1} - Gp_h^{n+1} = F_3^{n+1}, \end{cases} \tag{7.4}$$

where $A,\ E,\ H,\ P,\ G$ are the corresponding matrices of the bilinear forms of the different terms in the system (7.2), and $F_1^{n+1},\ F_2^{n+1}$ are second member vectors at the $(n+1)$th step obtained by replacing the variable u_h^{n+1} in the first equation of the system (7.2) by its value (7.3) obtained from the third equation and putting these terms on the right-hand side. Finally F_3^{n+1} corresponds to the traction on the Neumann boundary Γ_N at the $(n+1)$th step. In system (7.4), we start by eliminating σ_h^{n+1} and p_h^{n+1} and after we eliminate ω_h^{n+1}. These eliminations are made element by element. After this procedure, we end with the following system:

$$R\lambda_h^{n+1} = \mathcal{F}^{n+1}, \tag{7.5}$$

where

$$R = EA^{-1}E^T - EA^{-1}H^T(HA^{-1}H^T)^{-1}HA^{-1}E^T + GP^{-1}G^T,$$

and

$$\mathcal{F}^{n+1} = EA^{-1}F_1^{n+1} + GP^{-1}F_2^{n+1} - EA^{-1}H^T(HA^{-1}H^T)^{-1}HA^{-1}F_1^{n+1} + F_3^{n+1}.$$

The matrix R is symmetric and positive definite.

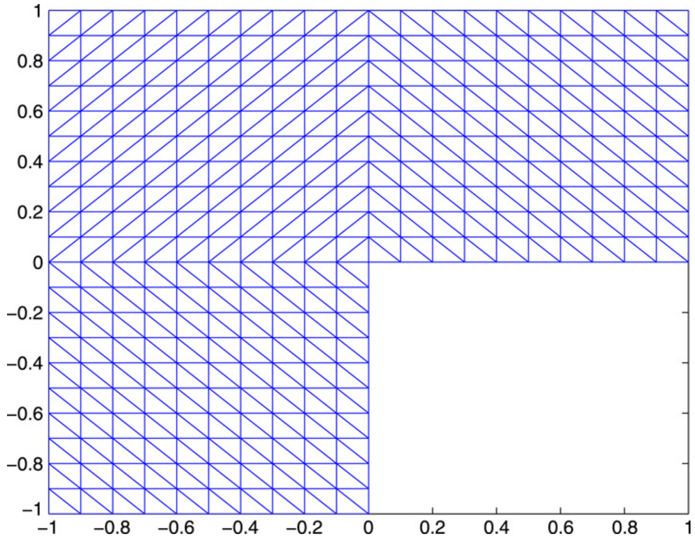

Fig. 1. Uniform meshes.

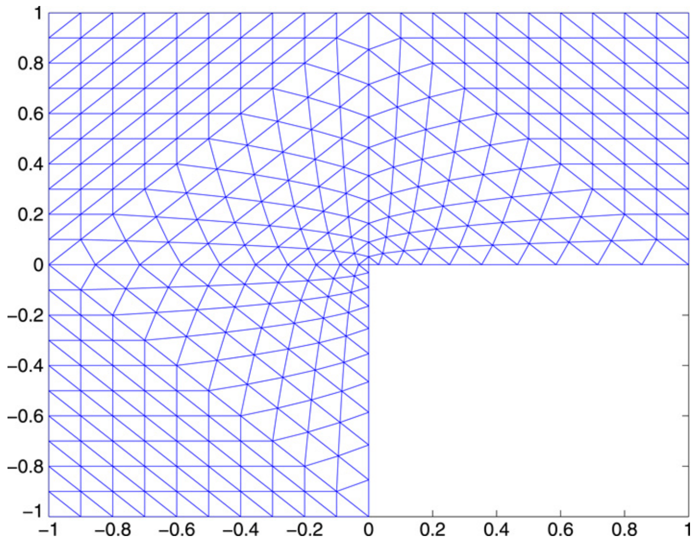

Fig. 2. Refined meshes.

Table 1
Convergence results when using uniform meshes at $T = 1$ s.

h	Pressure errors	Strain errors	Displacement errors	Rotational errors
2.828427e−001	7.379843e−002	3.733002e−001	6.127325e−002	1.263216e−001
1.414214e−001	5.071551e−002	2.513737e−001	3.054023e−002	8.408249e−002
9.428090e−002	4.070448e−002	2.002361e−001	2.027724e−002	6.671641e−002
7.071068e−002	3.480387e−002	1.706050e−001	1.518150e−002	5.674623e−002
5.656854e−002	3.081811e−002	1.507419e−001	1.212897e−002	5.008866e−002

7.1.2. Numerical test with the explicit Newmark scheme

We now present some numerical results on a test problem in the L-shaped domain

$$\Omega =]-1, 1[^2 \setminus ([0, 1[\times]-1, 0])$$

whose exact solution is the tensor product of the function $t \mapsto e^{-t}$ with a singularity of the stationary Navier equation ([22], Section 4.2 p. 52) arising at the reentrant corner of the L-shaped domain. The numerical tests are performed with $T = 1$ (second). Using polar coordinates (r, θ), $0 \leq \theta \leq \frac{3\pi}{2}$, which are centered at the reentrant corner, we consider as analytical

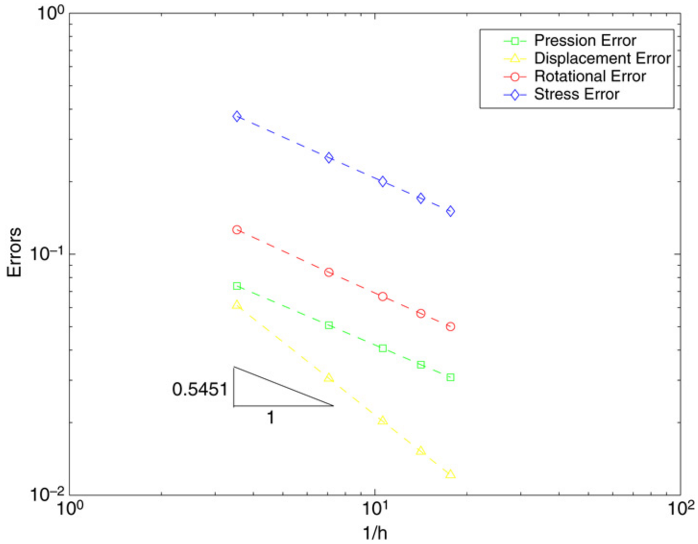

Fig. 3. Errors as a function of $1/h$ for uniform meshes.

solution:

$$u(r,\theta,t)=\mathrm{e}^{-t}r^{\alpha_1}\overrightarrow{\phi}_{\alpha_1}(\theta)$$

where

$$\overrightarrow{\phi}_{\alpha_1}(\theta)_1=C_1(\rho+\tau)\{\cos((\alpha_1-2)\theta)-\cos(\alpha_1\theta)\}+C_2((\rho+\tau)\sin((\alpha_1-2)\theta)+(\rho-3\tau)\sin(\alpha_1\theta)),$$
$$\overrightarrow{\phi}_{\alpha_1}(\theta)_2=C_1(-(\rho+\tau)\sin((\alpha_1-2)\theta)+(3\rho-\tau)\sin(\alpha_1\theta))+C_2(\rho+\tau)\{\cos((\alpha_1-2)\theta)-\cos(\alpha_1\theta)\}.$$

The parameters are

$$C_1=(\rho+\tau)\sin((\alpha_1-2)\omega)-(3\tau-\rho)\sin(\alpha_1\omega),$$
$$C_2=(\rho+\tau)\{\cos(\alpha_1\omega)-\cos((\alpha_1-2)\omega)\},$$
$$\rho=\frac{\lambda+\mu}{\mu}(\alpha_1-1)-2,\qquad \tau=\frac{\lambda+\mu}{\mu}(\alpha_1+1)+2,$$

where $\alpha_1=0.548643149483$ is the smallest strictly positive solution of the transcendental equation (4.2) for $\omega=\frac{3\pi}{2}$, $\lambda=1000,\ \mu=20$. In conformity, which what we have said just after Definition 4.2, we have to choose $\alpha\in]1-\alpha_1,\frac{1}{2}[$. We fix $\Delta t=10^{-5}$ a very small time step because of the CFL stability condition. Consequently $N=\frac{T}{\Delta t}=10^5$. All numerical results will be presented at the final time $T=1$ ($N=10^5$). The initial conditions u_h^0 and u_h^{-1} are chosen as the elliptic projection of $u(0)$ and $u(-\Delta t)$, i.e. as follows $u_h^0=\hat{u}_h(0)$, $u_h^{-1}=\hat{u}_h(-\Delta t)$. We use two kinds of meshes. The first one (uniform) is obtained by dividing each of the intervals $[0,1]$ and $[-1,0]$ into n subintervals of length $\frac{1}{n}$, and then each square of sidelength $\frac{1}{n}$ is divided into two triangles (see Fig. 1 where we have chosen $n=10$). The second kind of meshes (refined meshes) are obtained from the first ones by refinement near the reentrant corner $(0,0)$ according to Raugel's procedure [17] in order to satisfy the refinement rules $\mathrm{R}_1,\ \mathrm{R}_2,\ \mathrm{R}_3$ stated just after Proposition 5.4. Namely, Ω is divided into six big triangles; on the three ones which do not contain $(0,0)$, a uniform mesh is used; on the other hand each big triangle admitting $(0,0)$ as a vertex is divided into strips according to the ratios $(\frac{i}{n})^\beta,\ 1\le i\le n$, where $\beta\ge\frac{1}{(1-\alpha)}$ along the sides which end up at $(0,0)$ and finally each of these strips divided uniformly (see Fig. 2 where we have chosen $n=10$ and $\beta=1.8$). We then represent the variations of the errors $\|\sigma_h^N-\sigma(T)\|_{0,\Omega}$, $\|p_h^N-p(T)\|_{0,\Omega}$, $\|u_h^N-u(T)\|_{0,\Omega}$ and $\|\omega_h^N-\omega(T)\|_{0,\Omega}$, with respect to the mesh size h, in Figs. 3 and 4. A double logarithmic scale was used such that the slopes of the curves yield the order of convergence $O(h)$ for refined meshes (see Fig. 4) according to the theoretical results, and $O(h^{\frac{2}{3}})$ for uniform meshes (see Fig. 3) due to the singular behavior of the solution. In Tables 1 and 2, we summarize the results on the errors for uniform meshes and refined meshes respectively.

8. Conclusion

We have constructed and analyzed a finite element method for approximating the elastodynamic system using the dual mixed formulation for spatial discretization and an explicit Newmark scheme in the time variable. In our analysis, we take

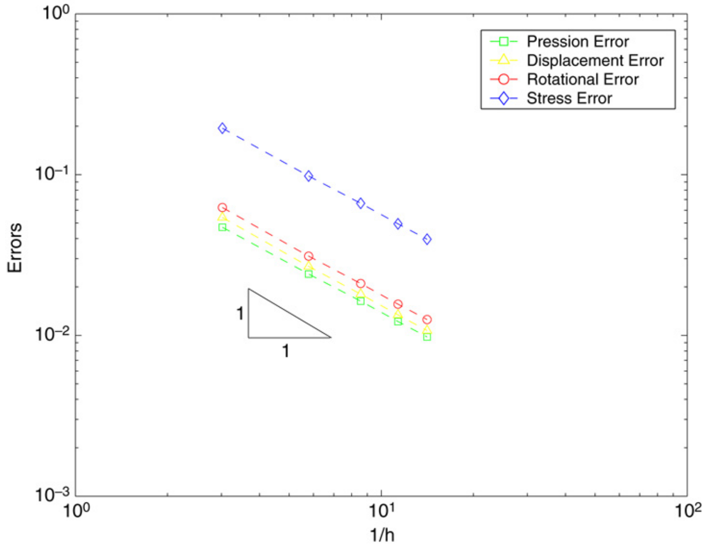

Fig. 4. Errors as a function of $1/h$ for refined meshes.

Table 2
Convergence results when using refined meshes at $T = 1$ s.

h	Pressure errors	Strain errors	Displacement errors	Rotational errors
3.307907e−001	4.706629e−002	1.943443e−001	5.413231e−002	6.239506e−002
1.727505e−001	2.408332e−002	9.796571e−002	2.686243e−002	3.111419e−002
1.167855e−001	1.634564e−002	6.628362e−002	1.806031e−002	2.100029e−002
8.819391e−002	1.219502e−002	4.939540e−002	1.340491e−002	1.563465e−002
7.084489e−002	9.788028e−003	3.962408e−002	1.072136e−002	1.253589e−002

into account the singularities of the solution due to the geometric singularities of the boundary. Optimal order L^∞-in-time and L^2-in-space a priori error estimates are derived and a quadratic convergence rate in time for the fully discretized scheme has been established for the explicit Newmark numerical scheme. As mentioned in the introduction, we will present, in [1], the analysis of the method proposed in this paper by using an implicit Newmark scheme for the time discretization.

References

[1] L. Boulaajine, M. Farhloul, L. Paquet, A priori error estimation for the dual mixed finite element method of the elastodynamic problem in a polygonal domain, II (in preparation).
[2] E. Bécache, P. Joly, C. Tsogka, A new family of mixed finite elements for the linear elastodynamic problem, SIAM J. Numer. Anal. 39 (2002) 2109–2132.
[3] Ch.G. Makridakis, On mixed finite element methods for linear elastodynamics, Numer. Math. 61 (1992) 235–260.
[4] E.W. Jenkins, B. Rivière, M.F. Wheeler, A priori error estimates for mixed finite element approximations of the acoustic wave equation, SIAM J. Numer. Anal. 40 (2002) 1698–1715.
[5] D.N. Arnold, G. Awanou, Rectangular mixed finite elements for elasticity, Math. Models Methods Appl. Sci. 15 (2005) 1417–1429.
[6] D.N. Arnold, R.S. Falk, R. Winther, Mixed finite element methods for linear elasticity with weakly imposed symmetry, Math. of Comput. 76 (2007) 1699–1723.
[7] F. Brezzi, M. Fortin, Mixed and Hybrid Finite Element Methods, Springer, New York, 1991.
[8] M. Farhloul, M. Fortin, Dual hybrid methods for the elasticity and the Stokes problems: A unified approach, Numer. Math. 76 (1997) 419–440.
[9] M. Farhloul, M. Fortin, A mixed nonconforming finite element for the elasticity and the Stokes problems, Math. Models Methods Appl. Sci. 9 (1999) 1179–1199.
[10] M. Farhloul, L. Paquet, Refined mixed finite element method for the elasticity problem in a polygonal domain, Numer. Methods Partial Differ. Equ. 18 (2002) 323–339.
[11] V. Thomée, Galerkin Finite Element Methods for Parabolic Problems, Springer, New York, 1997.
[12] P. Grisvard, Singularités en Elasticité, Arch. Ration. Mech. Anal. 107 (1989) 157–180.
[13] P. Grisvard, Singularities in Boundary Value Problems, in: RMA, 22, Masson, Paris, 1992, Springer, New York.
[14] M. Dauge, Elliptic boundary value problems in corner domains, Smoothness and asymptotics of solutions, in: Lecture Notes in Mathematics, vol. 1341, Springer, 1988.
[15] J. Wloka, Partial Differential Equations, English edition, Cambridge University Press, 1992.
[16] P.G. Ciarlet, Basic Error Estimates for Elliptic Problems, in: P.G. Ciarlet, J.L. Lions (Eds.), Handbook of Numerical Analysis, Vol. II. Finite Element Methods (Part 1), North-Holland, 1991, pp. 17–351.
[17] G. Raugel, Résolution numérique par une méthode d'élements finis du problème de Dirichlet pour le Laplacien dans un polygône, C. R. Acad. Sci. Paris Sér. I 286 (1978) 791–794.
[18] H. El Sossa, Quelques méthodes d'éléments finis mixtes raffinées basées sur l'utilisation des champs de Raviart-Thomas, Thèse de l'Université de Valenciennes, France, 2001.
[19] H. El Sossa, L. Paquet, Refined mixed finite element method for the Poisson problem in a polygonal domain with a reentrant corner, Adv. Math. Sci. Appl. 12 (2002) 607–643.

[20] B.F. Fraeijs de Veubeke, Displacement and equilibrium models in finite element method, in: O.C. Zienkiewicz, G.S. Holister (Eds.), Stress Analysis, Wiley, New York, 1965, pp. 145–197 (Chapter 9).
[21] A. Quarteroni, A. Valli, Numerical Approximation of Partial Differential Equations, Springer, New York, 1994.
[22] P. Grisvard, Boundary value problems in plane polygons instructions for use, in: EDF Bulletin de la Direction des Etudes et Recherches, Série C Mathématiques Informatique, vol. 1, 1986, pp. 21–59 (in French).