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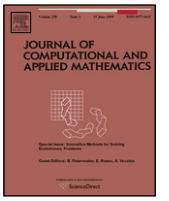

Improving order and efficiency: Composition with a modified Newton's method

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ABSTRACT

In this paper a zero-finding technique for solving nonlinear equations more efficiently than they usually are with traditional iterative methods in which the order of convergence is improved is presented. The key idea in deriving this procedure is to compose a given iterative method with a modified Newton's method that introduces just one evaluation of the function. To carry out this procedure some classical methods with different orders of convergence are used to obtain new methods that can be generalized in Banach spaces.

1. Introduction

Newton's method is one of the best root-finding methods for solving nonlinear equations. Examples of improving the classical formula at the expense of an additional evaluation of the first derivative, an additional evaluation of the function or a change in the point of evaluation are readily found in the literature on the subject (see [1–4]). In these examples, the order of convergence is increased in the neighborhood of a simple root.

A composition of a given iterative method with a modification of Newton's method is presented in this paper. Only one additional evaluation of the function is necessary and the order can be doubled. A result presented in ([3], Theorem 8–1, p. 166) is generalized. Recall that Traub considers a procedure in which the order of convergence is increased by one unit. Namely, let $f(x) = 0$ be a nonlinear equation where $f : D \subset \mathbf{R} \longrightarrow \mathbf{R}$ is sufficiently smooth in a neighborhood I of a simple root α. Traub's theorem states the following:

Theorem 1. *Let $\phi(x)$ be an iterative function of the order of convergence p in a neighborhood I of a simple root α of f. Let*

$$\psi(x) = \phi(x) - \frac{f[\phi(x)]}{f'(x)}.$$

Then $\psi(x)$ is of order $p + 1$.

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2. Main result

In the following we present a technique that consists in an iterative method in two steps, namely,

$$z_n = \phi(x_n), \tag{1}$$

$$x_{n+1} = z_n - \frac{f(z_n)}{f'_q}. \tag{2}$$

In (2) the derivative $f'(z_n)$ has been replaced by f'_q, which is defined as follows:

$$f'_q = q\,\frac{f(z_n)-f(x_n)}{z_n - x_n} + \sum_{k=1}^{q-1} \frac{k-q}{k!} f^{(k)}(x_n)\,(z_n - x_n)^{k-1}. \tag{3}$$

We assume that (1) is of pth order of convergence and we have evaluated the function f and its derivatives up to order equal to $p-1$ at point x. Furthermore, p and q are integers such that $p \geq q \geq 2$. Next, to analyze the order of the two-step iterative method given by (1) and (2), we state the following main result:

Theorem 2. *Let e and E be the errors $e_n = x_n - \alpha$ and $E_n = z_n - \alpha = C_0\,e^p + O(e^{p+1})$ in sequences x_n and z_n respectively. Then the order of the iterative method defined by (1)–(2) is equal to $p+q$. More precisely,*

$$|e_{n+1}| = \begin{cases} |A_{q+1}\,C_0\,e^{p+q}| + O(e^{p+q+1}), & \text{if } p > q, \quad \text{and} \\ \left|\left((-1)^{q+1}\,A_{q+1} + C_0\,A_2\right)\,C_0\,e^{2p}\right| + O(e^{2p+1}), & \text{if } p = q, \end{cases}$$

where $A_k = \frac{f^{(k)}(\alpha)}{k!f'(\alpha)}$, $k \geq 2$.

Proof. Using x instead of x_n and z instead of z_n, and considering Taylor's developments of the functions $f(z)$ and $f'(z)$ in powers of $z-x$, we obtain

$$f(z) = \sum_{i=0}^{q} \frac{f^{(i)}(x)}{i!}(z-x)^i + \frac{f^{(q+1)}(\xi)}{(q+1)!}(z-x)^{q+1}, \tag{4}$$

$$f'(z) = \sum_{j=1}^{q} \frac{f^{(j)}(x)}{(j-1)!}(z-x)^{j-1} + \frac{f^{(q+1)}(\eta)}{q!}(z-x)^{q}, \tag{5}$$

where ξ and η lie between x and z. From (4) we get $f^{(q)}(x)$ that after putting it into (5) yields

$$\begin{aligned} f'(z) &= \sum_{j=1}^{q-1} \frac{f^{(j)}(x)}{(j-1)!}(z-x)^{j-1} + \frac{f^{(q+1)}(\eta)}{q!}(z-x)^{q} \\ &\quad + \frac{q}{z-x}\left[f(z) - \sum_{i=0}^{q-1} \frac{f^{(i)}(x)}{i!}(z-x)^i - \frac{f^{(q+1)}(\xi)}{(q+1)!}(z-x)^{q+1}\right] \\ &= f'_q + T_q, \end{aligned} \tag{6}$$

where f'_q is given in (3) and

$$T_q = \left(\frac{f^{(q+1)}(\eta)}{q!} - \frac{q f^{(q+1)}(\xi)}{(q+1)!}\right)(z-x)^q. \tag{7}$$

Taking into account (7) and developing T_q in powers of $z-x$, we get

$$T_q = \frac{f^{(q+1)}(x)}{(q+1)!}\,(z-x)^q + O\left((z-x)^{q+1}\right). \tag{8}$$

From

$$f(x) = f'(\alpha)\left(e + \sum_{k=2}^{q+1} A_k\,e^k + O(e^{q+2})\right),$$

we have $f^{(q+1)}(x) = f'(\alpha)\left[(q+1)!\,A_{q+1} + O(e)\right]$.

Similarly, from $f(z) = f'(\alpha)\left(E + A_2\,E^2 + O(E^3)\right)$, we get

$$f'(z) = f'(\alpha)\left[1 + 2\,A_2\,E + O(E^2)\right]. \tag{9}$$

Table 1
Efficiencies.

	$p=2$	$p=3$	$p=4$
Method $\phi(x)$	$2^{1/2}\approx 1.414$	$3^{1/3}\approx 1.442$	$4^{1/4}\approx 1.414$
$q=2$	$4^{1/3}\approx 1.587$	$5^{1/4}\approx 1.495$	$6^{1/5}\approx 1.431$
$q=3$	–	$6^{1/4}\approx 1.565$	$7^{1/5}\approx 1.476$
$q=4$	–	–	$8^{1/5}\approx 1.516$

Due to the fact that $z-x=E-e$, if we develop (8) in Taylor's series at point α, we obtain

$$\begin{aligned} T_q &= f'(\alpha)\,A_{q+1}\,(E-e)^q + O(e^{q+1}) \\ &= f'(\alpha)\,(-1)^q\,A_{q+1}\,e^q + O(e^{q+1}). \end{aligned} \tag{10}$$

It follows from (9) and (10) that (6) becomes

$$\begin{aligned} f_q' &= f'(z) - T_q \\ &= f'(\alpha)\left[1 + 2\,A_2\,C_0\,e^p + O(e^{p+1}) + (-1)^{q+1}A_{q+1}\,e^q + O(e^{q+1})\right]. \end{aligned}$$

Now we distinguish two cases:

$$f_q' = \begin{cases} f'(\alpha)\left[1 + (-1)^{q+1}A_{q+1}\,e^q + O(e^{q+1})\right], & \text{if } p > q, \\ f'(\alpha)\left[1 + \left((-1)^{q+1}A_{q+1} + 2\,A_2\,C_0\right)\,e^q + O(e^{q+1})\right], & \text{if } p = q. \end{cases}$$

Subtracting α from the two sides of (2) and assuming that $p>q$, then from the previous expression of f_q', we get

$$\begin{aligned} e_{n+1} &= E - \frac{E + O(E^2)}{1 + (-1)^{q+1}A_{q+1}\,e^q + O(e^{q+1})} \\ &= (-1)^{q+1}A_{q+1}\,e^q\,E + O(e^{q+1}E). \end{aligned}$$

If $p=q$, then

$$\begin{aligned} e_{n+1} &= E - \frac{E + A_2\,E^2 + O(E^3)}{1 + \left[(-1)^{q+1}A_{q+1} + 2\,A_2\,C_0\right]\,e^q + O(e^{q+1})} \\ &= \left[(-1)^{q+1}A_{q+1} + 2\,A_2\,C_0\right]\,e^q\,E - A_2\,E^2 + O(e^{q+1}E). \end{aligned}$$

Replacing E by $E = C_0\,e^p + O(e^{p+1})$ the proof is complete. □

Eq. (3) was used in [5] for $q=3$. Other works related to family (3) can be found in [6,7].

The classical definition of efficiency index defined in [8–10,3] as $EI = m^{1/r}$, where m is the local order of convergence of the method and r is the number of evaluations of the functions per step has been used. By applying the improvement in the order obtained in Theorem 2 the efficiency is increased considerably. In the case in which the first step in the iterative method is of pth order and there are p evaluations of the functions per iteration, the efficiency index is $EI = p^{1/p}$. By increasing the value of $q \geq 2$ we can obtain $EI = (p+q)^{1/p+1}$ in the modified iterative methods. In Table 1 we give several values for efficiency.

3. New methods

In this section, we explore some methods that give the best efficiency indexes for $2 \leq p \leq 4$. From these methods, it is clear that $q=p$. We also give the expression of the asymptotic error constant. It has also been computed symbolically in a different manner to the way in which it was calculated in Theorem 2.

- For the case $p=2$. We choose Newton's method as the $z=\phi(x)$ method. If we write

$$z_n = \psi_2^2(x_n) = x_n - u(x_n) \quad \text{and} \quad f_2' = 2\,[x_n, z_n] - f'(x_n),$$

where $u(x_n) = \frac{f(x_n)}{f'(x_n)}$ and $[x_n, z_n] = \frac{f(z_n)-f(x_n)}{z_n - x_n}$, we have

$$x_{n+1} = z_n - \frac{f(z_n)}{f_2'}.$$

More explicitly,

$$x_{n+1} = \psi_2^4(x_n) = x_n - u(x_n) - \frac{f(x_n - u(x_n))}{2\,[x_n, z_n] - f'(x_n)}.$$

Note that the expression of error in Newton's method is $E = A_2\, e_n^2 + O_3$. The order goes from 2 to 4 in the method described here and the error difference equation is

$$e_{n+1} = A_2 \left(A_2^2 - A_3\right)\, e_n^4 + O_5,$$

which agrees with the result of Theorem 2 for this particular case.

• For the case $p = 3$. We use Chebyshev's method [2,11] as the $z = \phi(x)$ method. If we write

$$z_n = \psi_3^3(x_n) = x_n - \left(1 + \frac{1}{2}L(x_n)\right)\, u(x_n),$$

and $f_3' = 3\,[x_n, z_n] - 2f'(x_n) - \frac{1}{2}f^{(2)}(x_n)\,(z_n - x_n)$, where $L(x_n) = \frac{f^{(2)}(x_n)}{f'(x_n)}\, u(x_n)$, we have $x_{n+1} = \psi_3^6(x_n) = z_n - \frac{f(z_n)}{f_3'}$. Note that the error in Chebyshev's method is $E = (2\,A_2 - A_3)\, e_n^3 + O_4$. The improved method presented here is of 6th order and the error difference equation is

$$e_{n+1} = (2\,A_2 - A_3)\, \left(A_4 - A_2A_3 + 2\,A_2^3\right)\, e_n^6 + O_7,$$

as shown in Theorem 2.

• For the case $p = 4$. The method $z = \phi(x)$ considered is Schröeder's method [12]. Writing

$$z_n = \psi_4^4(x_n) = x_n - \left(1 + \frac{1}{2}\, L(x_n)\ - \frac{1}{6}M(x_n)\, u(x_n)^2\right) u(x_n),$$

and

$$f_4' = 4\,[x_n, z_n] - 3f'(x_n) - f^{(2)}(x_n)\,(z_n - x_n) - \frac{1}{6}f^{(3)}(x_n)\,(z_n - x_n)^2,$$

where

$$M(x_n) = \frac{f^{(3)}(x_n)}{f'(x_n)} - 3\left(\frac{f^{(2)}(x_n)}{f'(x_n)}\right)^2, \quad \text{then we have } x_{n+1} = \psi_4^8(x_n) = z_n - \frac{f\,(z_n)}{f_4'}.$$

The error in Schröeder's method is $E = \left(5\,A_2^3 - 5\,A_2\,A_3 + A_4\right)\, e_n^4 + O_5$. The improved method presented here is of 8th order and the error equation is

$$e_{n+1} = \left(5\,A_2^3 - 5\,A_2\,A_3 + A_4\right)\, \left(5\,A_2^4 - 5\,A_2^2\,A_3 + A_2\,A_4 - A_5\right)\, e_n^8 + O_9,$$

as given in Theorem 2.

4. Numerical experiments and comparison

We have tested the preceding methods with seven functions using the Maple computer algebra system. We have computed the root of each function for the initial approximation x_0. The iterative method was stopped when $|f(x_n)| < \eta$; tolerance η was chosen to be equal to 0.5×10^{-3000}.

The functions tested are the same as those presented in [13]. Table 2 shows the functions; the initial approximation, which is the same for all the methods; and the root with seven significant digits.

Table 3 shows, for each method and function, the number of iterations needed to compute the root to the level of precision described. The notation works as follows: Newton's iterative method and the modified method are written as ψ_2^2 $(p = 2)$ and ψ_2^4 $(p = 2, q = 2)$. Chebyshev's methods are represented by ψ_3^3 $(p = 3)$, ψ_3^5 $(p = 3, q = 2)$ and ψ_3^6 $(p = 3, q = 3)$. For Schröeder's method and the modified method we have ψ_4^4 $(p = 4)$, ψ_4^6 $(p = 4, q = 2)$, ψ_4^7 $(p = 4, q = 3)$ and ψ_4^8 $(p = 4, q = 4)$. Namely, the notation used is ψ_p^{p+q}.

In Table 3, we emphasize the low cost of the iteration functions ψ_2^4 and ψ_3^6, which show faster computation time than the other methods considered. In general the results are excellent: the order is maximized and the total number of function evaluations is lowest for iterative methods ψ_2^4 and ψ_3^6. Choosing one of them would depend on the cost of evaluating the first and second derivatives of the function with respect to the cost of evaluating the function.

Finally, we conclude that the new iterative methods ψ_2^4 and ψ_3^6 presented in this paper can compete with other efficient equation solvers, such as Newton's, Chebyshev's and Schröeder's methods (ψ_2^2, ψ_3^3 and ψ_4^4 respectively).

Table 2
Test functions, their roots and their initial points.

Function	α	x_0
$f_1(x) = x^3 - 3x^2 + x - 2$	2.893289	2.5
$f_2(x) = x^3 + \cos x - 2$	1.172578	1.5
$f_3(x) = 2\sin x + 1 - x$	2.380061	2.5
$f_4(x) = (x+1)\,\mathrm{e}^{-x} - 1$	0.557146	1.0
$f_5(x) = \mathrm{e}^{x^2+7x-30} - 1$	3.0	2.94
$f_6(x) = \mathrm{e}^{-x} + \cos(x)$	1.746140	1.5
$f_7(x) = x - 3\ln x$	1.857184	2.0

Table 3
Iteration number and total number of function evaluations (TNFE).

	ψ_2^2	ψ_2^4	ψ_3^3	ψ_3^5	ψ_3^6	ψ_4^4	ψ_4^6	ψ_4^7	ψ_4^8
$f_1(x)$	13	7	9	6	6	7	5	5	5
$f_2(x)$	13	7	8	6	5	7	5	5	5
$f_3(x)$	11	6	8	5	5	6	5	4	4
$f_4(x)$	13	7	8	6	5	7	5	5	5
$f_5(x)$	14	7	9	6	6	7	5	5	5
$f_6(x)$	11	6	8	5	5	6	5	4	4
$f_7(x)$	12	6	8	5	5	6	5	5	4
Iter	87	46	58	39	37	46	35	33	32
TNFE	174	**138**	174	156	**148**	184	175	165	160

5. Generalization to several variables

In this section, we are concerned with the technique presented in (1)–(3) for Banach spaces. The technique is used to solve the nonlinear operator equation $F(x) = 0$, where F is defined in an open convex subset D of Banach space X with values in Banach space Y. Let us use $\mathcal{L}(X, Y)$ to denote the space of bounded linear operators from X to Y. An operator $[x, z] \in \mathcal{L}(X, Y)$ can be said to be a first-order divided difference for operator F at points x and z $(x \neq z)$ (see [14–20]) if the following equality holds:

$$[x, z]\,(z - x) = F(z) - F(x). \tag{11}$$

Using Eq. (11), iterative methods (1)–(3) in Banach spaces when $q = 2$, for example, are described by the following algorithms:

$$\begin{aligned} z_n &= \Phi(x_n), \\ M_2 &= 2\,[x_n, z_n] - F'(x_n), \\ x_{n+1} &= z_n - M_2^{-1} F(z_n). \end{aligned} \tag{12}$$

Note that our scheme only involves first-order divided differences, which makes it interesting from a numerical point of view. Moreover, it enables us to solve the linear system $M_2\, y_n = F(z_n)$ rather than computing $y_n = M_2^{-1} F(z_n)$.

5.1. Numerical example

Finally, in this section we give an example in which we apply the method given in (12). We have tested the following methods: Newton's method (Ψ_2^2), Chebyshev's method (Ψ_3^3), that can be found in [21], and an accelerated variant of Newton's method (Ψ_2^4), which is defined in (12), where we take Newton's method to be the $\Phi(x)$ method.

When the number of repeated digits is η the calculation is stopped:

$$\|x_{n+1} - x_n\|_\infty < \epsilon = 0.5 \times 10^{-\eta},$$

where $\eta = 350$ and $\eta = 1400$. The order of convergence can be approximated using the computational order of convergence (COC)

$$\kappa = \frac{\log\left(\|x_{n+1} - \alpha\|_\infty / \|x_n - \alpha\|_\infty\right)}{\log\left(\|x_n - \alpha\|_\infty / \|x_{n-1} - \alpha\|_\infty\right)}, \tag{13}$$

which is a generalization of the order of convergence defined for a single variable in [4].

Example. The following statement appears in Ralston [10]. It proposes to calculate the complex root of the equation $\mathrm{e}^z = z$ with the smallest imaginary part. There are several ways to face this problem. Here, we solve it by considering the system of nonlinear equations

$$\begin{cases} \mathrm{e}^{x_1}\,\cos x_2 - x_1 = 0, \\ \mathrm{e}^{x_1}\,\sin x_2 - x_2 = 0, \end{cases}$$

Table 4
Number of iterations and COC for each method and example, for $\eta = 350\,(1400)$.

Method	# Iterations	κ
Ψ_2^2	9 (11)	2.00
Ψ_3^3	7 (10)	3.00
Ψ_2^4	7 (8)	4.00

that is obtained from the original equation equating its real and imaginary parts

$$e^{x_1+ix_2} = x_1 + i\,x_2.$$

We apply the methods referred to in this numerical example by taking $(0.2,\ 1.1)^T$ as a starting point.

Table 4 shows the number of iterations needed to reach ϵ to the level of precision described with $\eta = 350\,(1400)$, for the three methods Ψ_2^2 (the classic Newton's method), Ψ_3^3 (Chebyshev's method) and Ψ_2^4 (the modified Newton's method defined in (12)). Furthermore, the COC for this method, κ, is also calculated. Note that the third method, Ψ_2^4, is numerically considered to be of 4th order and works better in terms of both order and efficiency because in each iteration it is only necessary to compute two evaluations of the function, $F(x)$ and $F(z)$, the evaluation of the derivative, $F'(x)$, and the solution of two linear systems in order to calculate the inverses of $F'(x)$ and the linear operator M defined in (12).

6. Concluding remarks

In this paper we introduced a technique for accelerating the order of convergence of a given iterative process with an additional evaluation of the function. Furthermore, we analyzed the new schemes obtained from three particular cases: Newton's, Chebyshev's and Schröeder's methods. Efficiency was improved in all cases and consideration was given to the numerical implementation of the methods obtained.

The procedure was generalized to several variables. Furthermore, we have applied these methods to examples for which we calculated the number of steps needed to reach a given level of precision and the computational order of convergence given in [4]. The family of iterative methods that is introduced in this paper uses multi-precision and adaptive floating-point arithmetic with low computing time. It is worth noting that the improvement in efficiency afforded by these methods could be especially significant for methods in which the evaluation of the derivatives is more expensive than the evaluation of the function.

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