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Spectral analysis of the finite Hankel transform and circular prolate spheroidal wave functions

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ABSTRACT

In this paper, we develop two practical methods for the computation of the eigenvalues as well as the eigenfunctions of the finite Hankel transform operator. These different eigenfunctions are called circular prolate spheroidal wave functions (CPSWFs). This work is motivated by the potential applications of the CPSWFs as well as the development of practical methods for computing their values. Also, in this work, we should prove that the CPSWFs form an orthonormal basis of the space of Hankel band-limited functions, an orthogonal basis of $L^2([0, 1])$ and an orthonormal system of $L^2([0, +\infty[)$. Our computation of the CPSWFs and their associated eigenvalues is done by the use of two different methods. The first method is based on a suitable matrix representation of the finite Hankel transform operator. The second method is based on the use of an efficient quadrature method based on a special family of orthogonal polynomials. Also, we give two Maple programs that implement the previous two methods. Finally, we present some numerical results that illustrate the results of this work.

1. Introduction

In a famous series of papers, [1–4], Slepian, Landau and Pollack have given a complete and detailed study of the subject of prolate spheroidal wave functions. Starting from a non-classical version of the uncertainty principle, they have shown that among the set of band-limited functions with bandwidth $c > 0$, the most concentrated functions on the interval $[-1, 1]$, are the solutions of the following eigenvalue problem

$$F_c(\psi_{n,c})(x) = \int_{-1}^{1} \frac{\sin c(x-y)}{\pi(x-y)} \psi_{n,c}(y)\mathrm{d}y = \lambda_n(c)\psi_{n,c}(x), \quad x \in \mathbf{R}. \tag{1}$$

Since the operator F_c is written as $F_c = Q_c^* Q_c$, where Q_c is the finite Fourier transform operator given by

$$Q_c(f)(x) = \int_{-1}^{1} \mathrm{e}^{\mathrm{i}cxy} f(y)\mathrm{d}y, \quad x \in \mathbf{R}, \tag{2}$$

then the PSWFs $(\psi_{n,c})_{n\geq 0}$ are also the eigenfunctions of Q_c. For more details, the reader is referred to [4]. Moreover, in [2], the authors have proved that these PSWFs have the desirable properties to form an orthogonal basis of $L^2([-1, 1])$, an

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orthonormal system of $L^2(\mathbf{R})$ and more importantly, an orthonormal basis of B_c, the Paley–Wiener space of band-limited functions given by

$$B_c = \left\{ f \in L^2(\mathbf{R}),\ \mathrm{Supp}^t \widehat{f} \subset [-c, c] \right\},$$

where $\mathrm{Supp}^t \widehat{f}$ denotes the support of $\widehat{f}(x)$. Here, $\widehat{f}(x)$ is the Fourier transform of $f(x)$. Although, the PSWFs have very desirable properties, they have been and still regarded as tedious functions that are difficult to compute and with no standard representation in terms of elementary functions. Recently, a special interest is devoted to the development of new methods for the computation of the PSWFs, (see for example [5–8]) for the 1D case and [9,3] for the bi-dimensional case. The interest for the PSWFs is partly due to their promising applications in different fields such as pure and applied mathematics, physics and digital signal processing, (see for example [10–14]).

On the other hand, the CPSWFs have found very interesting applications in different area. For example, in Optics, they are used in the reconstruction of optical objects with super-resolution from diffraction-limited images, [15]. They are also used in the study of confocal laser modes or wave aberrations, (see for example [16]). In MRI (magnetic resonance imaging) the CPSWFs have been used in the frame work of 2D PSWFs to get an appropriate tradeoff between spatial and temporal resolution and increase data acquisition speed for dynamic magnetic resonance imaging, [17]. In Astrophysics, the CPSWFs are used in the study and the better understanding of stellar coronagraphy, [16].

The promising potential applications of the CPSWFs as well as the development of practical methods for computing their values are the main motivations behind this work. In this paper, we are interested in the study of the following eigenvalue problem

$$\widetilde{H}_{c,N}(\phi)(x) = \int_0^1 y J_N(cxy)\, \phi(y) \mathrm{d}y = \beta \phi(x), \quad x \geq 0. \tag{3}$$

Here c, N are two real numbers satisfying $c > 0$, $N > -1$ and $J_N(\cdot)$ is the Bessel function of the first type and order N, given by

$$J_N(y) = \left(\frac{y}{2}\right)^N \sum_{j=0}^{\infty} (-1)^j \frac{(y)^{2j}}{4^j j!\, \Gamma(N + j + 1)}, \quad y \in \mathbf{R}. \tag{4}$$

The operator $\widetilde{H}_{c,N}$ is called the finite Hankel transform operator. It is well suited for solving several initial boundary problems that model many problems from physics and engineering, see for example [18].

As already mentioned in [3], the spectrum of $\widetilde{H}_{c,N}$ is closely related to the spectrum of the second version of the finite Hankel transform operator given via the following eigenvalue problem,

$$H_{c,N}(\psi)(x) = \int_0^1 \sqrt{cxy} J_N(cxy)\, \psi(y) \mathrm{d}y = \gamma \psi(x), \quad x \geq 0. \tag{5}$$

The reader can easily check that the operator $H_{c,N}$ is self-adjoint and compact. Note that the eigenvalues and the eigenfunctions of the operators $H_{c,N}$ and $\widetilde{H}_{c,N}$ are related to each other by the following relations

$$\forall\, n \geq 0, \quad \gamma_{n,N}(c) = \sqrt{c}\beta_{n,N}(c), \qquad \psi^N_{n,c}(x) = \sqrt{x}\phi^N_{n,c}(x), \quad x \geq 0. \tag{6}$$

We should mention that the eigenfunctions given by (3) are used as the radial parts of the multi-dimensional PSWFs on the unit sphere, (see [3]). Moreover, we show that the CPSWFs, the eigenfunctions of $H_{c,N}$, constitute an orthogonal basis of $L^2([0, 1])$, an orthonormal basis of the space of Hankel band-limited functions.

In this work, we are interested in developing two methods for computing the spectrum and the eigenfunctions of the compact operators $H_{c,N}$ and $\widetilde{H}_{c,N}$. Unlike other existing methods, our proposed methods do not use the differential operator that commutes with the finite Hankel transform integral operator. Moreover, they are easy to use and provide highly accurate numerical approximations. Also, since our methods do not depend on any differential operator, they have the nice property to be easily adapted to the computation of the spectrum and the eigenfunctions of more general compact integral operators. We should mention that some integral operator based computational methods of the classical prolate spheroidal functions have been already given in [7,8]. Our first method for computing the CPSWFs is based on an appropriate matrix representation A of the operator $H_{c,N}$. This representation uses a special orthonormal basis of $L^2([0, 1])$ with basis functions given by

$$T_{k,N}(x) = \sqrt{2(2k + N + 1)}\, x^{N+1/2} P_k^{(N,0)}(1 - 2x^2), \quad k \geq 0. \tag{7}$$

Here, $N > -1$ is a fixed real number and $P_k^{(N,0)}(\cdot)$ denotes the Jacobi polynomial of degree k. We should mention that these basis functions have been already used in [3] to tridiagonalize the differential operator that commutes with the integral operator $H_{c,N}$. If $(\gamma_{n,N}(c))_{n\geq 0}$ denotes the infinite set of the eigenvalues of $H_{c,N}$ arranged in the decreasing order

$$|\gamma_{0,N}(c)| > |\gamma_{1,N}(c)| > \cdots > |\gamma_{n,N}(c)| > \cdots,$$

then, we prove that for any positive integer M, the first M eigenvalues of $H_{c,N}$ are approximated accurately by the first M eigenvalues of an appropriate submatrix of A. Moreover we show how to compute the eigenfunction $\psi_{n,c}^N(x)$ associated with the eigenvalue $\gamma_{n,N}(c)$. Note that unlike Slepian's method given in [3], our first method is not restricted to the positive integer values of N and it is valid for any real number $N > -1$.

Our second method for the computation of the spectrum of $H_{c,N}$ is based on an efficient quadrature method applied to the integral operator $\widetilde{H}_{c,N}$. This method is based on the use of a special set of orthonormal polynomials $(Q_n(x))_{n\geq 0}$, over $[0, 1]$ and with respect to the measure $d\alpha(x) = x\mathrm{d}x$. Based on some desirable properties of the orthogonal polynomials, we provide a rapid method for computing the different quadrature nodes and weights. This second method has the advantage of working well for small as well as relatively large values of the parameter c. However and unlike the first method, this second method provides approximations of the values of $\psi_{n,c}^N$ at the quadrature nodes only. To overcome this problem, we provide the reader with an efficient interpolation method for the approximation of $\psi_{n,c}^N(x)$ along the interval $[0, 1]$.

This paper is organized as follows. In Section 2, we give some properties of the CPSWFs and prove the exponential decay of their expansion coefficients with respect to the orthonormal basis given by (7). In Section 3, we describe the matrix representation technique for solving the eigenvalue problem (5). In Section 4, we develop a quadrature method that approximates with high accuracy the eigenvalues and the eigenfunctions of the operator $\widetilde{H}_{c,N}$ given by (3). In Section 5, we give two maple programs that implement the methods described in the previous two sections. Finally, in Section 6, we provide the reader with some numerical results that illustrate the results of this work.

2. Circular prolate spheroidal wave functions

In this section, we prove some desirable properties of the CPSWFs $(\psi_{n,c}^N)_{n\geq 0}$, the set of the eigenfunctions of $H_{c,N}$. We let HB_c denotes the space of Hankel band-limited functions given by

$$HB_c = \left\{ f \in L^2([0, +\infty[), \mathrm{Supp}^t\left(\mathcal{H}_N(f)\right) \subseteq [0, c] \right\}.$$

Here $\mathcal{H}_N$ is the Hankel transform operator defined by

$$\mathcal{H}_N(f)(x) = \int_0^{+\infty} \sqrt{y} J_N(xy) f(y)\mathrm{d}y, \quad f \in L^2([0, +\infty[).$$

Next, we assume that the eigenfunctions $\psi_{n,c}^N(x)$ are normalized by the following rule

$$\int_0^1 \left(\psi_{n,c}^N(x)\right)^2 \mathrm{d}x = c\gamma_{n,N}(c)^2. \tag{8}$$

The following proposition gives us some important properties of the CPSWFs.

Proposition 1. *Under the above notation and hypothesis, the eigenfunctions $\psi_{n,c}^N(x)$ satisfy the following properties:*

(P_1) *For any integer $n \geq 0$, $\psi_{n,c}^N$ is Hankel band-limited function with bandwidth c.*

(P_2) *The set $\mathcal{B} = \{\psi_{n,c}^N, , n \geq 0\}$ is an orthogonal basis of $L^2([0, 1])$, an orthonormal system of $L^2([0, +\infty[)$ and an orthonormal basis of HB_c.*

Proof. We prove property (P_1) by showing that

$$\forall\, \xi \geq 0, \quad \mathcal{H}_N(\psi_{n,c}^N)(\xi) = \frac{1}{c\gamma_{n,N}(c)} \psi_{n,c}^N\left(\frac{\xi}{c}\right) \chi_{[0,c]}(\xi). \tag{9}$$

It is well known, (see [18]), that the operator $\mathcal{H}_N$ is its proper inverse, that is $\mathcal{H}_N^{-1} = \mathcal{H}_N$. Hence, if $f \in L^2([0, +\infty[)$, then

$$f(x) = \int_0^{+\infty} \int_0^{+\infty} \sqrt{xy} J_N(xy) \sqrt{ty} J_N(ty) f(t)\mathrm{d}t\mathrm{d}y, \quad \text{a.e. } x \in [0, +\infty[. \tag{10}$$

Moreover, from (5), one has

$$\psi_{n,c}^N(x) = \frac{1}{\gamma_{n,N}(c)} \int_0^{+\infty} \sqrt{cxy} J_N(cxy) \psi_{n,c}^N(y) \chi_{[0,1]}(y)\mathrm{d}y. \tag{11}$$

Hence,

$$\begin{aligned} \mathcal{H}_N(\psi_{n,c}^N(\xi)) &= \frac{1}{\gamma_{n,N}(c)} \int_0^{+\infty} \int_0^{+\infty} \sqrt{cxy} J_N(cxy) \psi_{n,c}^N(y) \chi_{[0,1]}(y)\mathrm{d}y \sqrt{x\,\xi} J_N(x\xi)\mathrm{d}x, \quad Y = cy, \\ &= \frac{1}{c\gamma_{n,N}(c)} \int_0^{+\infty} \int_0^{+\infty} \sqrt{xY} J_N(xY) \psi_{n,c}^N(Y/c) \chi_{[0,c]}(Y)\mathrm{d}y \sqrt{x\,\xi} J_N(x\xi)\mathrm{d}x. \end{aligned} \tag{12}$$

By using the equality (12) together with (10), one concludes that

$$\mathcal{H}_N(\psi^N_{n,c}(\xi)) = \frac{1}{c\gamma_{n,N}(c)}\psi^N_{n,c}\left(\frac{\xi}{c}\right)\chi_{[0,c]}(\xi), \quad \text{a.e. } \xi \geq 0. \tag{13}$$

Moreover, from (5), it is easy to see that $\psi^N_{n,c}$ is continuous over $[0, +\infty[$. Consequently, the equality (13) holds for any $\xi \geq 0$.

The proof of (P_2) is easily done as follows. Since $\mathcal{B}$ is the set of the different eigenfunctions of the self-adjoint Hilbert Schmidt operator $H_{c,N}$, over $L^2([0, 1])$, then $\mathcal{B}$ is an orthogonal basis of $L^2([0, 1]) \cap \left[\mathrm{Ker}(H_{c,N})\right]^{\perp}$. Moreover, it is easy to check that the operator $H_{c,N}$ is one to one on $L^2([0, 1])$. Hence, $\mathcal{B}$ is an orthogonal basis of $L^2([0, 1])$. Also, by using (8) and (9) and Parseval formula for the Hankel transform, (see [18]), one gets

$$\begin{aligned}
\int_0^{+\infty} \psi^N_{n,c}(t)\psi^N_{m,c}(t)\mathrm{d}t &= \int_0^{+\infty} \mathcal{H}_N(\psi^N_{n,c})(t)\mathcal{H}_N(\psi^N_{m,c})(t)\mathrm{d}t \\
&= \frac{1}{c^2\gamma_{n,N}(c)\gamma_{m,N}(c)}\int_0^c \psi^N_{n,c}\left(\frac{t}{c}\right)\psi^N_{m,c}\left(\frac{t}{c}\right)\mathrm{d}t, \quad t = c\,u \\
&= \frac{1}{c\gamma_{n,N}(c)\gamma_{m,N}(c)}\int_0^1 \psi^N_{n,c}(u)\psi^N_{m,c}(u)\mathrm{d}u \\
&= \frac{1}{c\gamma_{n,N}(c)\gamma_{m,N}(c)}c\gamma_{n,N}(c)^2\delta_{nm} = \delta_{nm}.
\end{aligned}$$

Finally, we prove that $\mathcal{B}$ is an orthonormal basis of HB_c. Let $f \in HB_c$, then for any $x \geq 0$, we have

$$f(x) = \mathcal{H}_N^{-1}\left(\mathcal{H}_N(f)\right)(x) = \int_0^{+\infty} \sqrt{xy}J_N(xy)\mathcal{H}_N(f)(y)\mathrm{d}y \tag{14}$$

$$= \int_0^c \sqrt{xy}J_N(xy)\mathcal{H}_N(f)(y)\mathrm{d}y, \quad y = cu$$

$$= c\int_0^1 \sqrt{cxu}J_N(cxu)\mathcal{H}_N(f)(cu)\mathrm{d}u. \tag{15}$$

As for the classical Fourier band-limited functions, it can be easily checked that a Hankel band-limited function is an entire function over $[0, +\infty[$. Moreover, since $\mathcal{B}$ is an orthogonal basis of $L^2([0, 1])$, then by using (9), it is clear that there exists a sequence of real numbers $(\alpha_n)_n \in l^2(\mathbf{N})$ such that

$$\forall u \in [0, 1], \quad \mathcal{H}_N(f)(cu) = \sum_{n\geq 0}\alpha_n\psi^N_{n,c}(u). \tag{16}$$

Moreover, Mercer's theorem (see [19]), applied to $H_{c,N}H^*_{c,N}$ implies that $\left(\gamma_{n,N}(c)\right)_n \in l^2(\mathbf{N})$. Hence, the sequence $\left(\alpha_n\gamma_{n,N}(c)\right)_n \in l^1(\mathbf{N})$. Hence, by combining the equalities (15) and (16), one concludes that

$$f(x) = c\int_0^1 \sqrt{cxu}J_N(cxu)\left(\sum_{n\geq 0}\alpha_n\gamma_{n,N}(c)\frac{\psi^N_{n,c}(u)}{\gamma_{n,N}(c)}\right)\mathrm{d}u, \quad x \geq 0. \tag{17}$$

Since the function $u \to \sqrt{cxu}J_N(cxu)\frac{\psi^N_{n,c}(u)}{\gamma_{n,N}(c)} \in C([0, 1])$ and since $(\alpha_n\gamma_{n,N}(c))_n \in l^1(\mathbf{N})$, then by using (5) and (17), one gets

$$f(x) = \sum_{n\geq 0}\alpha_n\gamma_{n,N}(c)\int_0^1 \sqrt{cxy}J_N(cxu)\frac{\psi^N_{n,c}(u)}{\gamma_{n,N}(c)}\mathrm{d}u = \sum_{n\geq 0}\alpha_n\gamma_{n,N}(c)\psi^N_{n,c}(x). \quad \square$$

To proceed further, we need the following notation.

Notation: In what follows, we let $[x]$ denotes the integer part of the real x. Also, we let $\Psi^N_{n,c}(x)$ denotes the re-normalized eigenfunction of (5) given by

$$\Psi^N_{n,c}(x) = \frac{1}{\sqrt{c}\,\gamma_{n,N}(c)}\psi^N_{n,c}(x), \quad x \geq 0.$$

It is important to mention that as in the classical case, the eigenvalues $\gamma_{n,N}(c)$ associated with the CPSWFs decay exponentially. This is given by the following lemma. Note that the proof of this lemma is based on some techniques very similar to those used in [3] in the classical case.

Lemma 1. *For any integer n satisfying* $n \geq \left[\frac{e\,c-N+1}{2}\right]$, *we have*

$$|\gamma_{n,N}(c)| < \sqrt{e/(2\pi)}\left(\frac{\Gamma(n+1)\Gamma(n+N+1)}{\Gamma(2n+N+1)}\mathrm{e}^{\int_0^c p(t,n)\,\mathrm{d}t}\right)\mathrm{e}^{-(2n+N+1)\log 2}. \tag{18}$$

Here $p(t,n)$ *is the function defined on* $[0,c]$, *by*

$$p(0,n)=0, \qquad p(t,n)=\frac{(\Psi_{n,t}^N(1))^2-1}{2t}-\frac{2n+N+1}{t}, \quad t\in]0,c]. \tag{19}$$

Proof. In [3], the author has proved that

$$\lim_{c\to 0}\frac{\gamma_{n,N}(c)}{c^{2n+N+1/2}}=\frac{(-1)^n n!\Gamma(n+N+1)}{2^{2n+N+1}\Gamma(2n+N+1)\Gamma(2n+N+2)}. \tag{20}$$

Here, $\Gamma(\cdot)$ denotes the gamma function defined by $\Gamma(x)=\int_0^{+\infty}t^{x-1}\mathrm{e}^{-t}\,\mathrm{d}t, x>-1$. Also, note that the re-normalized eigenfunction $\Psi_{n,c}^N(x)$ satisfies the following condition $\int_0^1\Psi_{n,c}^N(x)^2\mathrm{d}x=1$. Moreover, in [3], it has been shown that

$$\frac{\partial\gamma_{n,N}(c)}{\partial c}=\frac{\gamma_{n,N}(c)}{2c}\left((\psi_{n,c}^N(1))^2-1\right). \tag{21}$$

Equality (21) is equivalent to

$$\frac{\partial\log\gamma_{n,N}(c)}{\partial c}=\frac{1}{2c}\left((\psi_{n,c}^N(1))^2-1\right). \tag{22}$$

Consider a real $0<c_0<c$, then by integrating (22) over $[c_0,c]$, one obtains

$$\log\gamma_{n,N}(c)=\log\gamma_{n,N}(c_0)+\int_{c_0}^c\frac{2n+N+1/2}{t}\mathrm{d}t+\int_{c_0}^c p(t,n)\mathrm{d}t. \tag{23}$$

The following result given in [3],

$$\Psi_{n,t}^N(1)=(-1)^n\sqrt{2(2n+N+1)}\left(1+\frac{N^2t^2}{4(2n+N)^2(2n+N+2)^2}\right)+O(t^4),$$

shows that the function $p(t,n)$ is continuous over $[0,c]$. By exponentiating the members of (23), one gets

$$\gamma_{n,N}(c)=\gamma_{n,N}(c_0)\left(\frac{c}{c_0}\right)^{2n+N+1/2}\mathrm{e}^{\int_{c_0}^c p(t,n)\mathrm{d}t}. \tag{24}$$

By letting c_0 go to 0 in the equality (24) together with (20), one obtains the following equality

$$\gamma_{n,N}(c)=\frac{(-1)^n n!\,\Gamma(n+N+1)c^{2n+N+1/2}}{2^{2n+N+1}\Gamma(2n+N+1)\Gamma(2n+N+2)}\mathrm{e}^{\int_0^c p(t,n)\mathrm{d}t}. \tag{25}$$

Moreover, by using the following consequence of Stirling's formula, (see [20])

$$\forall s>0, \quad \Gamma(s+1)\geq\sqrt{2\pi}s^{s+1/2}\mathrm{e}^{-s}, \tag{26}$$

we conclude that for all $n\geq\left[\frac{e\,c-N+1}{2}\right]$

$$\begin{aligned}\frac{c^{2n+N+1/2}}{\Gamma(2n+N+2)} &\leq \frac{\mathrm{e}^{2n+N+1}c^{2n+N+1/2}}{\sqrt{2\pi}(2n+N+1)^{2n+N+3/2}}\leq\left(\frac{e\,c}{2n+N+1}\right)^{2n+N+1/2}\frac{\sqrt{e/(2\pi)}}{2n+N+1}\\ &\leq\frac{\sqrt{e/(2\pi)}}{2n+N+1}.\end{aligned}$$

Hence,

$$|\gamma_{n,N}(c)|\leq\mathrm{e}^{-(2n+N+1)\log 2}\frac{\sqrt{e/(2\pi)}\Gamma(n+1)\Gamma(n+N+1)}{\Gamma(2n+N+1)}\mathrm{e}^{\int_0^c p(t,n)\,\mathrm{d}t}. \quad \square$$

Remark 1. Numerical evidences show that for any integer $n \geq 0$, for all $t \geq 0$, $p(t, n) \leq 0$. Hence, the above decay bound can be replaced with the following better bound.

$$|\gamma_{n,N}(c)| \leq \frac{1}{2^{2n+N+1}} \frac{\sqrt{e/(2\pi)}\Gamma(n+1)\Gamma(n+N+1)}{\Gamma(2n+N+1)}. \tag{27}$$

Our computations of the CPSWFs are heavily based on the use of the Jacobi polynomials. Note that for any two real numbers $\alpha, \beta > -1$, the Jacobi polynomial $P_n^{(\alpha,\beta)}(x)$ of degree n is given by the following Rodrigues formula

$$P_n^{(\alpha,\beta)}(x) = \frac{(-1)^n}{2^n n!}(1-x)^{-\alpha}(1+x)^{-\beta}\frac{\mathrm{d}^n}{\mathrm{d}x^n}\left[(1-x)^{\alpha+n}(1+x)^{\beta+n}\right], \quad n \geq 0. \tag{28}$$

Moreover, if

$$a_n = \frac{2^{(\alpha+\beta+1)}\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{n!(\alpha+\beta+2n+1)\Gamma(\alpha+\beta+n+1)}, \quad n \geq 0,$$

then the set $\{\frac{1}{\sqrt{a_n}}P_n^{(\alpha,\beta)}(x), n \in \mathbf{N}\}$ is an orthonormal basis of $(L^2[-1, 1], \mathrm{d}\mu)$, where $\mathrm{d}\mu(x) = (1-x)^\alpha(1+x)^\beta \mathrm{d}x$. For more details, the reader may consult Reference [21].

In [3], the author has considered an integer $N \geq 0$ and has used the following set of functions

$$T_{k,N}(x) = \sqrt{2(2k+N+1)}x^{N+\frac{1}{2}}P_k^{(N,0)}(1-2x^2), \quad k \geq 0, \tag{29}$$

to get a symmetric tridiagonal matrix representation of a differential operator having the same eigenfunctions as the operator $H_{c,N}$. Moreover, since $\mathcal{B} = \{T_{k,N}(x),\ k \in \mathbf{N}\}$ is an orthonormal basis of $(L^2[0, 1], \mathrm{d}x)$, then for any real $N > -1$ and any integer $n \geq 0$, $\psi_{n,c}^N(x)$ has the following series expansion with respect to $\mathcal{B}$,

$$\psi_{n,c}^N(x) = \sum_{k\geq 0} d_{k,n}^N T_{k,N}(x), \quad x \in [0, 1]. \tag{30}$$

In what follows, we shall prove that basis $\mathcal{B}$ is also well adapted for the matrix representation of the operator $H_{c,N}$. This is partly due to the existence of exact and explicit simple formulae for the different moments of the basis functions $T_{k,N}(x)$. This is the subject of the following proposition.

Proposition 2. *For any integers $j,\ k \geq 0$ and any real number $N > -1$, let $M_{j,k}$ denotes the moment of $T_{k,N}$ given by*

$$M_{j,k} = \int_0^1 x^{N+\frac{1}{2}+2j}T_{k,N}(x)\mathrm{d}x. \tag{31}$$

Then, we have

$$M_{j,k} = \begin{cases} (-1)^k\sqrt{k+\dfrac{N+1}{2}}\dfrac{j!}{(j-k)!}\dfrac{\Gamma(N+j+1)}{\Gamma(N+j+k+2)} & \text{if } j \geq k \geq 0 \\ 0 & \text{if } 0 \leq j < k. \end{cases} \tag{32}$$

Proof. Since

$$M_{j,k} = c_{N,k}\int_0^1 x^{N+\frac{1}{2}+2j}x^{N+\frac{1}{2}}P_k^{N,0}(1-2x^2)\mathrm{d}x, \quad c_{N,k} = \sqrt{2(2k+N+1)},$$

then by the change of variable $u = 1-2x^2$ applied to the above integral and by using the Rodrigues formula, one obtains

$$\begin{aligned} M_{j,k} &= \frac{c_{N,k}}{4.2^{N+j}}\int_{-1}^1 (1-u)^{N+j}\frac{(-1)^k}{2^k k!}(1-u)^{-N}\frac{\mathrm{d}^k}{\mathrm{d}x^k}\left((1-u)^{N+k}(1+u)^k\right)\mathrm{d}u \\ &= \frac{c_{N,k}(-1)^k}{2^{N+j+2+k}k!}\int_{-1}^1 (1-u)^j\frac{\mathrm{d}^k}{\mathrm{d}x^k}\left((1-u)^{N+k}(1+u)^k\right)\mathrm{d}u. \end{aligned} \tag{33}$$

Note that, for any integer $1 \leq l \leq k$, if $f_l(u) = \frac{\mathrm{d}^{k-l}}{\mathrm{d}x^{k-l}}\left((1-u)^{N+k}(1+u)^k\right)$, then $f_l(1) = f_l(-1) = 0$. Hence, by applying a successive k integrations by parts to (33), one gets

$$\begin{aligned} M_{j,k} &= \frac{c_{N,k}(-1)^k}{2^{N+j+2+k}k!}(j)_k\int_{-1}^1 (1-u)^{N+j}(1+u)^k\mathrm{d}u, \qquad v = \frac{1+u}{2} \\ &= \frac{c_{N,k}(-1)^k 2^{N+j+k}}{2^{N+j+2+k}k!}\int_0^1 v^{N+j}(1-v)^k\mathrm{d}v \\ &= \frac{c_{N,k}(-1)^k}{4k!}B(N+j+1, k+1). \end{aligned}$$

Here $(a)_l$ denotes the shifted factorial defined by

$$\forall l > 0, \quad (a)_l = a(a+1)\cdots(a+l-1), \quad (a)_0 = 1.$$

It is well known, (see [21]), that for $\mathrm{Re}(x) > 0$ and $\mathrm{Re}(y) > 0$, we have

$$B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}\mathrm{d}t = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}. \tag{34}$$

Hence, by using equality (34), one concludes that, for all $j \geq k \geq 0$, we have

$$M_{j,k} = \frac{c_{N,k}(-1)^k(j)_k}{4k!}\frac{\Gamma(N+j+1)\Gamma(k+1)}{\Gamma(N+j+k+2)} = (-1)^k\sqrt{k+\frac{N+1}{2}}\frac{j!}{(j-k)!}\frac{\Gamma(N+j+1)}{\Gamma(N+j+k+2)}. \tag{35}$$

Next, if $0 \leq j \leq k$ and since $\deg P_l^{N,0}(1-2x^2) = 2l$, where $\deg P_l^{N,0}(1-2x^2)$ is the degree of the polynomial $P_l^{N,0}(1-2x^2)$, then we have $x^{2j} = \sum_{l=0}^{j}\alpha_l P_l^{N,0}(1-2x^2)$ or equivalently $x^{N+\frac{1}{2}+2j} = \sum_{l=0}^{j}\alpha_l T_{l,N}(x)$. Hence $M_{j,k} = \sum_{l=0}^{j}\alpha_l\int_0^1 T_{l,N}(x)T_{k,N}(x)\mathrm{d}x$. By using the orthonormality of $\left\{T_{k,N}(x)\right\}_{k\geq 0}$ over $[0,1]$, we finally obtain $\langle T_{l,N}, T_{k,N}\rangle = 0$, where $\langle .,.\rangle$ denotes the usual inner product of $L^2([0,1])$. Consequently, for all $0 \leq l \leq j < k$, it follows, $M_{j,k} = 0$. □

We should mention that in [3], the author has shown that if $N \geq 0$ is an integer, then for $x > 1$ the eigenfunction $\psi_{n,c}^N(x)$ is given as follows

$$\psi_{n,c}^N(x) = \frac{1}{\gamma_{n,N}(c)}\sum_{k\geq 0} d_{k,n}^N\sqrt{2(2k+N+1)}\frac{J_{N+2k+1}(cx)}{\sqrt{cx}}, \quad N \in \mathbf{N}, x \geq 1. \tag{36}$$

The proof in [3] is restricted to the integer values of N and is based on finding a complicated second order differential equation with a solution given by $F_{N,n}(x) = \int_0^1\sqrt{xy}J_N(xy)T_{N,n}(y)\mathrm{d}y$. The following proposition generalizes (36) to the case of any real number $N > -1$. The proof of this proposition is based on Proposition 2 and it is much more easier than Slepian's proof, (see [3]).

Proposition 3. *Under the above notations, for any real number $N > -1$ and for all $x \geq 1$, we have*

$$\psi_{n,c}^N(x) = \frac{1}{\gamma_{n,N}(c)}\sum_{k\geq 0} d_{k,n}^N\sqrt{2(2k+N+1)}\frac{J_{N+2k+1}(cx)}{\sqrt{cx}}. \tag{37}$$

Proof. Since $\psi_{n,c}^N(x)$ is an eigenfunction of $H_{c,N}$ associated with the eigenvalue $\gamma_{n,N}(c)$, then by using (30), one gets $\forall x \geq 1$,

$$\begin{aligned}
\psi_{n,c}^N(x) &= \frac{1}{\gamma_{n,N}(c)}\int_0^1\sqrt{cxy}J_N(cxy)\psi_{n,c}^N(y)\mathrm{d}y \\
&= \frac{1}{\gamma_{n,N}(c)}\int_0^1\sqrt{cxy}J_N(cxy)\sum_{k\geq 0} d_{k,n}^N T_{k,N}(y) \\
&= \frac{1}{\gamma_{n,N}(c)}\sum_{k\geq 0} d_{k,n}^N\sum_{j=0}^{+\infty}\int_0^1\sqrt{cxy}\left(\frac{cxy}{2}\right)^N\frac{(-1)^j(cxy)^{2j}}{4^j j!\Gamma(N+j+1)}T_{k,N}(y)\mathrm{d}y.
\end{aligned} \tag{38}$$

Note that the permutation of the integral and the summation signs are made possible thanks to the fast decay of the sequence $(d_{k,n}^N)_k$ (see Theorem 1) and the uniform convergence over $[0,1]$ of the series $S_x(y) = \sum_{j=0}^{+\infty}\frac{(-1)^j(cxy)^{2j}}{4^j j!\Gamma(N+j+1)}T_{k,N}(y)$, (see the proof of Theorem 2). Hence, by using the equality (38) together with Proposition 2, one concludes that

$$\begin{aligned}
\psi_{n,c}^N(x) &= \frac{1}{\gamma_{n,N}(c)}\sum_{k\geq 0} d_{k,n}^N\sum_{j=0}^{+\infty}\frac{(-1)^j(cx)^{2j+N+\frac{1}{2}}}{2^N 4^j j!\Gamma(N+j+1)}\int_0^1 y^{N+2j+\frac{1}{2}}T_{k,N}(y)\mathrm{d}y \\
&= \frac{1}{\gamma_{n,N}(c)}\sum_{k\geq 0} d_{k,n}^N\sum_{j=0}^{+\infty}\frac{(-1)^j(cx)^{2j+N+\frac{1}{2}}}{2^N 4^j j!\Gamma(N+j+1)}M_{j,k} \\
&= \frac{1}{\gamma_{n,N}(c)}\sum_{k\geq 0} d_{k,n}^N\sum_{j\geq k}^{+\infty}\frac{(-1)^j(cx)^{2j+N+\frac{1}{2}}}{2^N 4^j j!\Gamma(N+j+1)}(-1)^k\sqrt{k+\frac{N+1}{2}}\frac{j!\Gamma(N+j+1)}{(j-k)!\Gamma(N+j+k+2)} \\
&= \frac{1}{\gamma_{n,N}(c)\sqrt{cx}}\sum_{k\geq 0} d_{k,n}^N\sqrt{k+\frac{N+1}{2}}\sum_{J\geq 0}^{+\infty}\frac{(-1)^{J+k}(cx)^{2J+2k+N+1}}{2^N 4^{J+k}}(-1)^k\frac{1}{J!\Gamma(N+J+2k+2)}
\end{aligned}$$

$$= \frac{1}{\gamma_{n,N}(c)} \sum_{k\geq 0} d_{k,n}^N \sqrt{2(2k+N+1)} \frac{J_{N+2k+1}(cx)}{\sqrt{cx}}. \quad \square$$

The following theorem shows that coefficients $d_{k,n}^N$ in (30) decay exponentially.

Theorem 1. *For any $k \geq k_N = \left[\frac{ec-N+1}{2}\right]$, we have*

$$|d_{k,n}^N| \leq c_N \frac{e^{\frac{c^2}{4(N+2k+2)}}}{\sqrt{\pi}|\gamma_{n,N}(c)|} \left(\frac{1}{2}\right)^{N+k+1}, \tag{39}$$

where c_N is a positive constant only depending on N.

Proof. We note that for any real number $N > -1$, there exists $M_N > 0$, such that $\sup_{x\geq 0} x(J_N(x))^2 \leq M_N$. The proof of this result will be included in the proof of Theorem 4 of Section 3. Let $c_N = \sqrt{M_N}$, then the previous inequality combined with (5) and (8) give us for all $x \geq 0$

$$\begin{aligned} |\psi_{n,c}^N(x)| &\leq \frac{1}{|\gamma_{n,N}(c)|} \left(\int_0^1 cxyJ_N^2(cxy)\mathrm{d}y\right)^{\frac{1}{2}} \|\psi_{n,c}^N\|_{2,[0,1]} \\ &\leq \frac{\sqrt{M_N}}{|\gamma_{n,N}(c)|} \|\psi_{n,c}^N\|_{2,[0,1]} = \frac{c_N}{|\gamma_{n,N}(c)|}\sqrt{c}|\gamma_{n,N}(c)| = c_N\sqrt{c}. \end{aligned}$$

Hence, we have

$$\int_0^1 \frac{|\psi_{n,c}^N(y)|}{\sqrt{cy}}\mathrm{d}y \leq 2c_N.$$

To show the decay of the sequence $\left(d_{k,n}^N\right)_{k\in\mathbf{N}}$ we use the following equality

$$\int_0^1 J_N(cxy)\sqrt{cxy}T_{k,N}(y)\mathrm{d}y = \sqrt{2(2k+N+1)}\frac{J_{N+2k+1}(cx)}{\sqrt{cx}}, \tag{40}$$

which can be seen from the proof of Proposition 3. Note that $\psi_{n,c}^N$ satisfies

$$\forall x \geq 0, \quad \int_0^1 \sqrt{cxy}J_N(cxy)\psi_{n,c}^N(y)\mathrm{d}y = \gamma_{n,N}(c)\psi_{n,c}^N(x). \tag{41}$$

By multiplying (41) by $T_{k,N}(x)$ and integrating it from 0 to 1, one gets,

$$\int_0^1 T_{k,N}(x)\left(\int_0^1 \sqrt{cxy}J_N(cxy)\psi_{n,c}^N(y)\mathrm{d}y\right)\mathrm{d}x = \gamma_{n,N}(c)\int_0^1 T_{k,N}(x)\psi_{n,c}^N(x)\mathrm{d}x.$$

By using Fubini's theorem and equality (40), we obtain

$$\sqrt{2(2k+N+1)}\int_0^1 \psi_{n,c}^N(y)\frac{J_{N+2k+1}(y)}{\sqrt{cy}}\mathrm{d}y = \gamma_{n,N}(c)d_{k,n}^N,$$

or equivalently,

$$d_{k,n}^N = \frac{\sqrt{2(2k+N+1)}}{\gamma_{n,N}(c)}\int_0^1 \psi_{n,c}^N(y)\frac{J_{N+2k+1}(cy)}{\sqrt{cy}}\mathrm{d}y. \tag{42}$$

Using equality (42) together with the series expansion of Bessel's function, we conclude that

$$\begin{aligned} d_{k,n}^N &= \frac{\sqrt{2(2k+N+1)}}{\gamma_{n,N}(c)}\int_0^1 \psi_{n,c}^N(y)\frac{J_{N+2k+1}(cy)}{\sqrt{cy}}\mathrm{d}y \\ &= \frac{\sqrt{2(2k+N+1)}}{\gamma_{n,N}(c)}\int_0^1 \psi_{n,c}^N(x)\left(\frac{1}{2}cx\right)^{N+2k+1}\frac{1}{\sqrt{cx}}\sum_{j=0}^{+\infty}\frac{\left(\frac{-1}{4}c^2x^2\right)^j}{j!\Gamma(N+2k+j+2)}\mathrm{d}x \\ &= \frac{\sqrt{2(2k+N+1)}}{\gamma_{n,N}(c)}\sum_{j=0}^{+\infty}\frac{1}{j!\Gamma(N+2k+j+2)}\int_0^1 \psi_{n,c}^N(x)\left(\frac{1}{2}cx\right)^{N+2k+1}\left(\frac{-1}{4}c^2x^2\right)^j\frac{1}{\sqrt{cx}}\mathrm{d}x \end{aligned}$$

$$= \frac{\sqrt{2(2k+N+1)}}{\gamma_{n,N}(c)} \sum_{j=0}^{+\infty} \frac{\left(\frac{c}{2}\right)^{N+2k+1}}{j!\underbrace{(N+2k+1+j)\cdots(N+2k+2)}_{P(j)}\Gamma(N+2k+2)} C(N,n,j,k),$$

where

$$C(N,n,j,k) = \int_0^1 \psi_{n,c}^N(x) x^{N+2k+1} \left(\frac{-1}{4}c^2x^2\right)^j \frac{1}{\sqrt{cx}}\mathrm{d}x.$$

Note that

$$|C(N,n,j,k)| \le \left(\int_0^1 |\psi_{n,c}^N(x)| \frac{1}{\sqrt{cx}}\mathrm{d}x\right)\left(\frac{c^2}{4}\right)^j \le 2c_N \left(\frac{c^2}{4}\right)^j.$$

Moreover, since $P(j) \ge (N+2k+2)^j$, then we have

$$\begin{aligned} |d_{k,n}^N| &\le 2c_N \frac{\sqrt{2(2k+N+1)}}{|\gamma_{n,N}(c)|} \frac{\left(\frac{c}{2}\right)^{N+2k+1}}{\Gamma(N+2k+2)} \sum_{j=0}^{+\infty} \frac{\left(\frac{c^2}{4}\right)^j}{j!(N+2k+2)^j} \\ &\le 2c_N \frac{\sqrt{2(2k+N+1)}}{|\gamma_{n,N}(c)|} \frac{\left(\frac{c}{2}\right)^{N+2k+1}}{\Gamma(N+2k+2)} \mathrm{e}^{\frac{c^2}{4(N+2k+2)}}. \end{aligned} \tag{43}$$

Using the inequality (26), we obtain for all $k \ge k_N = \left[\frac{ec-N+1}{2}\right]$

$$\begin{aligned} \frac{\left(\frac{c}{2}\right)^{N+2k+1}}{\Gamma(N+2k+2)} &\le \frac{1}{\sqrt{2\pi}} \frac{\mathrm{e}^{N+2k+1}\left(\frac{c}{2}\right)^{N+2k+1}}{(N+2k+1)^{N+2k+\frac{3}{2}}} \\ &= \frac{1}{\sqrt{2\pi}} \left(\frac{ec}{2(N+2k+1)}\right)^{N+2k+1} \frac{1}{2\sqrt{N+2k+1}} \\ &\le \frac{1}{\sqrt{2\pi}} \left(\frac{1}{2}\right)^{N+2k+1} \frac{1}{2\sqrt{N+2k+1}}. \end{aligned} \tag{44}$$

Combining (43) and (44), one obtains the desired inequality

$$\forall\, k \ge \left[\frac{ec-N+1}{2}\right], \quad |d_{k,n}^N| \le c_N \frac{\mathrm{e}^{\frac{c^2}{4(N+2k+2)}}}{\sqrt{\pi}|\gamma_{n,N}(c)|} \left(\frac{1}{2}\right)^{N+k+1}. \quad \square \tag{45}$$

3. Matrix representation of finite Hankel transform operator and spectral analysis

In this paragraph, we study a matrix representation of the operator $H_{c,N}$, given by (5). This strategy was also followed in [6]. This representation will provide us with an efficient method for computing accurate approximations of the different eigenvalues of this operator. This is given by the following theorem.

Theorem 2. *Consider a real number $N > -1$ and let $A = [a_{kk'}(c)]_{k,k'\ge 0}$ be the infinite matrix given by*

$$a_{kk'}(c) = \sum_{j \ge \max(k,k')} \frac{(-1)^j c^{N+2j+\frac{1}{2}}}{2^N j!\, 4^j \Gamma(N+j+1)} M_{j,k} M_{j,k'}. \tag{46}$$

Here, the $M_{j,k}$ are as given in (35). The spectrum of $H_{c,N}$ coincides with the spectrum of A. Moreover the coefficients of A have the following decay estimate

$$|a_{kk'}(c)| \le \frac{c^{N+2K}\mathrm{e}^{\frac{c^2}{4}}}{2^{N+2K+1}K!\Gamma(N+K+1)\Gamma(N+2K+1)}; \qquad K = \max(k,k'); \quad k,k' \ge 0.$$

Moreover if $|k-k'| \ge \frac{c^2}{4}$, then

$$|a_{kk'(c)}| \le \frac{c^{N+2K}}{2^{N+2K+1}K!\Gamma(N+K+1)\Gamma(N+2K+1)}; \qquad K = \max(k,k'); \quad k,k' \ge 0.$$

Proof. Since $H_{c,N}$ has a symmetric kernel K_N, given by $K_N(x,y)=\sqrt{cxy}J_N(cxy)$ and since $K_N(x,y)\in L^2([0,1]^2)$, then $H_{c,N}$ is a Hilbert–Schmidt self-adjoint operator. Consider the orthonormal basis $\mathcal{B}$ of $L^2([0,1])$ given by $\mathcal{B}=\{T_{k,N}(x),\ k\geq 0\}$, then

$$\begin{aligned}H_{c,N}T_{k,N}(x) &= \int_0^1 \sqrt{cxy}J_N(cxy)T_{k,N}(y)\mathrm{d}y\\ &= \int_0^1 \sum_{j=0}^{+\infty}\frac{(cxy)^{N+2j+\frac{1}{2}}(-1)^j}{2^N j!4^j\Gamma(N+j+1)}T_{k,N}(y)\mathrm{d}y,\quad x\in[0,1].\end{aligned}$$

It is well known, (see [21]) that

$$\sup_{y\in[-1,1]}\left|P_k^{(N,0)}(y)\right| = \begin{cases}\dfrac{\Gamma(N+k)}{k!\Gamma(N+1)}, & \text{if } N\geq -\dfrac{1}{2},\\ \sim 1/\sqrt{k} & \text{if } -\dfrac{1}{2}>N>-1.\end{cases} \tag{47}$$

Hence, it is clear that

$$\forall x\in[0,1],\quad S_x(y)=\sum_{j\geq 0}\frac{(cxy)^{N+2j+\frac{1}{2}}(-1)^j}{2^N j!4^j\Gamma(N+j+1)}T_{k,N}(y)\mathrm{d}y$$

converges uniformly in $[0,1]$. Consequently, we have

$$\begin{aligned}H_{c,N}(T_{k,N})(x) &= c^{N+\frac{1}{2}}\sum_{j\geq 0}\frac{c^{2j}x^{N+2j+\frac{1}{2}}(-1)^j}{2^N j!4^j\Gamma(N+j+1)}\int_0^1 y^{N+2j+\frac{1}{2}}T_{k,N}(y)\mathrm{d}y\\ &= c^{N+\frac{1}{2}}\sum_{j\geq 0}\frac{c^{2j}(-1)^j}{2^N j!4^j\Gamma(N+j+1)}M_{j,k}x^{N+2j+\frac{1}{2}}.\end{aligned}$$

Since for all $j<k$, we have $M_{j,k}=0$. Then, for all $k,k'\geq 0$, it follows that

$$a_{kk'}(c)=\langle H_{c,N}T_{k,N},T_{k',N}\rangle = c^{N+\frac{1}{2}}\sum_{j\geq 0}\frac{c^{2j}(-1)^j}{2^N j!4^j\Gamma(N+j+1)}M_{j,k}M_{j,k'}.$$

Hence, if $A=[a_{kk'}(c)]_{k,k'\geq 0}$, then A is the matrix representation of $H_{c,N}$ with respect to the orthonormal basis $\mathcal{B}$. Consequently, $H_{c,N}$ and A have the same spectrum.

To prove the decay estimate of the coefficients $a_{kk'}(c)$, we first note that for all $j\geq k\geq 0$, we have

$$|M_{j,k}|\leq\left(\int_0^1 x^{2N+1+4j}\mathrm{d}x\right)^{\frac{1}{2}}\left(\int_0^1 (T_{k,N}(x))^2\mathrm{d}x\right)^{\frac{1}{2}}=\sqrt{\frac{1}{2(N+2j+1)}}.$$

Moreover, from (32), one has

$$M_{j,k}M_{j,k'}=(-1)^{k+k'}|M_{j,k}||M_{j,k'}|.$$

Since for all $j<k$ we have $M_{j,k}=0$. Then for all $k,k'\geq 0$, it follows that

$$(-1)^{k+k'}a_{kk'}(c)=c^{N+\frac{1}{2}}\sum_{j\geq\max(k,k')}\frac{c^{2j}(-1)^j}{2^N j!4^j\Gamma(N+j+1)}|M_{j,k}M_{j,k'}|.$$

If $K=\max(k,k')$, then

$$|a_{kk'}(c)|\leq\frac{c^{N+\frac{1}{2}}}{2^N\Gamma(N+K+1)}\frac{1}{2(N+2K+1)}\sum_{j\geq K}\frac{\left(\frac{c^2}{4}\right)^j}{j!}.$$

Since $\sum_{j\geq K}\frac{\left(\frac{c^2}{4}\right)^j}{j!}\leq\frac{\left(\frac{c^2}{4}\right)^K}{K!}\mathrm{e}^{\frac{c^2}{4}}$, then

$$|a_{kk'}(c)|\leq\frac{c^{N+2K+\frac{1}{2}}}{2^{N+1}(N+2K+1)4^K K!}\mathrm{e}^{\frac{c^2}{4}}=\epsilon_{N,K}(c)\mathrm{e}^{\frac{c^2}{4}}.$$

By using (35), one gets

$$(-1)^{k+k'}a_{kk'}(c) = c^{N+\frac{1}{2}}\sum_{j\geq K}\frac{c^{2j}(-1)^j}{2^N j!4^j\Gamma(N+j+1)}|M_{j,k}M_{j,k'}| = c^{N+\frac{1}{2}}\sum_{j\geq K}(-1)^j\alpha_j^{k,k'}(c),$$

with

$$\frac{\alpha_{j+1}^{k,k'}(c)}{\alpha_j^{k,k'}(c)} = \frac{c^2(j+1)(N+j+1)}{4(j+1-k)(j+1-k')(N+j+k+2)(N+j+k'+2)}.$$

This shows that if $|k-k'|\geq\frac{c^2}{4}$, then $\forall j\geq K=\max(k,k')$, we have

$$0<\frac{\alpha_{j+1}^{k,k'}(c)}{\alpha_j^{k,k'}(c)}<\frac{c^2}{4}\frac{1}{|k-k'|}\frac{j(j+1)}{(j+k+1)(j+k'+1)}<1.$$

Hence $(-1)^{k+k'}a_{kk'}(c)$ is written as the remainder of a convergent alternating series with decreasing coefficients. Consequently

$$|a_{kk'}(c)|\leq\frac{c^{N+\frac{1}{2}}}{2^N}\frac{c^{2K}}{4^K K!\Gamma(N+K+1)}|M_{j,k}M_{j,k'}|\leq\frac{c^{N+\frac{1}{2}+2K}}{2^N4^K K!\Gamma(N+K+1)}\frac{1}{2(N+2K+1)}=\epsilon_{N,K}(c).\quad\square$$

Next, if $\mathrm{eig}(H_{c,N})=\{\gamma_{n,N}(c),n\geq 0\}$, where $\mathrm{eig}(H_{c,N})$ is the set of the eigenvalues of $H_{c,N}$ and $\gamma_{n,N}(c)$ are arranged in the decreasing order of their magnitudes, $|\gamma_{0,N}|\geq|\gamma_{1,N}|\geq\cdots|\gamma_{n,N}(c)|\geq\cdots\geq 0$, then by Theorem 2, we know that $\mathrm{eig}(H_{c,N})=\lambda(A)$. In practice one can only consider a finite order submatrix $A_L(c)=[a_{kk'}(c)]_{0\leq k,k'\leq L}$ of A, and consider the eigenvalues of $A_L(c)$ as approximate eigenvalues of the first K eigenvalues of $H_{c,N}$ for some $0<K<L$. The following theorem shows that for any positive integer K, one can get highly accurate approximation of the first K eigenvalues of $H_{c,N}$ by considering the first K eigenvalues of an appropriate submatrix of A. The proof of this theorem is based on the well known Weyl perturbation theorem, (see [22]).

Theorem 3 (*Weyl's Perturbation Theorem*). *Let A and B be hermitian matrices (finite or infinite order). Then*

$$\max_j|\lambda_j(A)-\lambda_j(B)|\leq\|A-B\|.$$

Here $\|A\|=\sup_{\|x\|=1}|\langle x,Ax\rangle|$ and $\lambda_j(A)$, $\lambda_j(B)j\geq 0$ are the eigenvalues of A and B, arranged in the decreasing order of their absolute values.

Theorem 4. *For any $\epsilon>0$ there exists a positive integer L_ϵ such that $\forall K\geq L_\epsilon$, we have*

$$|\mu(j)-\gamma_{j,N}(c)|<\epsilon,\quad\forall\,0\leq j\leq K,\tag{48}$$

where the $\mu(j)$ and $\gamma_{j,N}(c)$ are the decreasing sequences of the eigenvalues of $A_K(c)$ and $H_{c,N}$ respectively.

Proof. The proof of the theorem is divided into two parts.

First part. We prove that for any real numbers $N>-1$, $c>0$, there exists a constant M_N such that $\|H_{c,N}\|^2\leq M_N$. It is clear that under the above conditions, the kernel $K_N(x,y)=\sqrt{cxy}J_N(cxy)\in L^2([0,1]^2)$ and consequently $H_{c,N}$ is a Hilbert Schmidt operator. Also, it is well known that if $\|H_{c,N}\|_{\mathcal{H}}=\left(\int_0^1\int_0^1(K_N(x,y))^2\mathrm{d}x\mathrm{d}y\right)^{1/2}$ is the Hilbert Schmidt norm of $H_{c,N}$, then $\|H_{c,N}\|\leq\|H_{c,N}\|_{\mathcal{H}}$. On the other hand, if $Y_N(x)$ denotes the Bessel function of the second kind and of order N, where $N\geq 0$ is a positive integer, then it is shown in [23], that the function $\eta(.)$ defined by $\eta(x)=x(J_N^2(x)+Y_N^2(x))$, $x>0$ is decreasing. Moreover $Y_N(x)$ has an infinite number of zeros on $]0,+\infty[$. Let z_0 denote the first zeros of $Y_N(x)$, then for all $x\geq z_0$, we have

$$xJ_N^2(x)\leq z_0[J_N^2(z_0)+Y_N^2(z_0)]=z_0J_N^2(z_0)=M_{0,N}.\tag{49}$$

Moreover, since $x\to xJ_N^2(x)$ is continuous on $[0,z_0]$, then

$$\forall x\in[0,z_0],\quad xJ_N^2(x)\leq M_{1,N}.\tag{50}$$

If $M_N=\max(M_{0,N},M_{1,N})$, then by combining (49) and (50) one concludes that

$$x\,(J_N(x))^2\leq M_N,\quad\forall\,x\geq 0.\tag{51}$$

Hence, for any integer $N \geq 0$, we have

$$\|H_{c,N}\|^2_{\mathcal{H}} = \int_0^1 \int_0^1 cxy(J_N(cxy))^2 \mathrm{d}x\mathrm{d}y \leq \int_0^1 \int_0^1 M_N \mathrm{d}x\mathrm{d}y = M_N.$$

Next, for an arbitrary real number $N > -1$, we use (4) to get the following inequality,

$$\forall x, y \in [0, 1], \quad |J_N(cxy)| \leq \left(\frac{cxy}{2}\right)^N \frac{\mathrm{e}^{c^2/4}}{\Gamma(N+1)}.$$

Consequently, we obtain again,

$$\int_0^1 \int_0^1 cxy(J_N(cxy))^2 \mathrm{d}x\mathrm{d}y \leq \frac{c^{1+2N}\mathrm{e}^{c^2/2}}{(\Gamma(N+1))^2 4^N} \int_0^1 \int_0^1 x^{1+2N} y^{1+2N} \mathrm{d}x\mathrm{d}y = M'_N.$$

Second part. In this part, we prove (48). Let $A = [a_{kk'}(c)]_{k,k\geq 0}$ and $A_L = [a_{kk'}(c)]_{0\leq k,k\leq L}$, then $A - A_L$ is a matrix representation of a Hilbert–Schmidt operator $\epsilon_{c,L}$ with Hilbert–Schmidt norm given by $\|\epsilon_{c,L}\|^2_{\mathcal{H}} = \sum_{k,k'>L}(a_{kk'}(c))^2$, but from part I, we showed that $\|\epsilon_{c,L}\|_{\mathcal{H}} \to 0$ as $L \to +\infty$. Hence, $\forall \epsilon$, $\exists L_\epsilon \in \mathbf{N}$ such that $\|\epsilon_{c,L}\|_{\mathcal{H}} < \epsilon$. By Weyl's perturbation theorem, we have

$$\max_{0\leq j\leq L_\epsilon} |\lambda_j(A) - \lambda_j(A_L)| \leq \|A - A_L\| \leq \|A - A_L\|_{\mathcal{H}} = \|\epsilon_{c,L}\|_{\mathcal{H}} < \epsilon.$$

This concludes the proof. □

Remark 2. For small values of the parameter c and for reasonable small values of the tolerance ϵ, the value of L_ϵ given by the above theorem is not large. As an example, for $c = 15$, $\epsilon = 10^{-30}$, we have found that $L_\epsilon = 40$.

Note that the computation of accurate approximations of the different eigenvalues and eigenfunctions of $H_{c,N}$ is done as follows. By using the matrix representation $A(c) = [a_{kk'}(c)]_{k,k'\geq 0}$ of $H_{c,N}$ it is clear that for an integer $n \geq 0$, the sequence $\left(d^N_{k,n}\right)_{k\geq 0}$ is nothing but the eigenvector corresponding to $\gamma_{n,N}(c)$. Moreover by Theorem 4, one can get highly accurate approximation $\mu(n)$ of $\gamma_{n,N}(c)$ by using a submatrix of A of order $L > 0$. The eigenvector $\left(\tilde{d}^N_{k,n}\right)_{0\leq k\leq L}$ is taken as a good approximation in the L^2-norm of $\left(d^N_{k,n}\right)_{k\geq 0}$. Consequently

$$\tilde{\psi}^N_{n,c}(x) = \sum_{k=0}^{L} \tilde{d}^N_{k,n} T_{k,N}(x), \quad x \in [0, 1] \tag{52}$$

is the approximation of the exact eigenfunction $\psi^N_{n,c}(x)$ on the interval $[0, 1]$. The eigenvector $\left(\tilde{d}^N_{k,n}\right)_{0\leq k\leq L}$ is normalized so that $\sum_{k=0}^{L} |d^N_{k,n}|^2 = 1$. An approximation of the analytic continuation of the normalized $\psi^N_{n,c}$ is given by the following formula

$$\tilde{\psi}^N_{n,c}(x) = \frac{1}{\mu_n(c)} \sum_{k=0}^{L} \tilde{d}^N_{k,n} \sqrt{2(2k+N+1)} \frac{J_{N+2k+1}(cx)}{\sqrt{cx}}, \quad x > 1. \tag{53}$$

4. Approximate spectrum of $\widetilde{H}_{c,N}$ by a quadrature method

In this paragraph, we develop a quadrature method for approximating the spectrum and the eigenfunctions of the operator $\widetilde{H}_{c,N}$ given by (3). Then by using (6), one obtains approximations of the CPSWFs and their associated eigenvalues. We should mention that for the classical finite Fourier transform, some efficient generalized Gaussian type quadrature methods on the unit circle have been successfully used to approximate the 1D PSWFs and their eigenvalues, (see [5]). Recently, in [9], the author has provided a quadrature method for the approximation of the 2D PSWFs on the unit circle and it is essentially based on an appropriate quadrature method for solving the eigenvalue problem (5). In this paragraph, we propose a different method for solving (5) through a quadrature method adapted to (3). This method is heavily based on a special set of orthonormal polynomials. It can be considered as a generalization of a similar method we have developed in [24]. To proceed further, we need the following mathematical preliminaries.

4.1. Mathematical preliminaries

We first construct a set of orthonormal polynomials over $[0, 1]$ and with respect to the measure $\mathrm{d}\alpha(x) = x\mathrm{d}x$. This set of polynomials is then used for the construction of a Gaussian quadrature for problem (3). A candidate of a set of polynomials $\{Q_n(x),\ n \geq 0\}$ satisfying the previous conditions is given by the following Rodrigues Formula

$$Q_n(x) = c_n \frac{1}{x} \frac{\mathrm{d}^n}{\mathrm{d}x^{2n}} \left[x^{n+1}(1-x)^n\right], \quad n \geq 0. \tag{54}$$

Here c_n is a normalization constant to be fixed in what follows. From the equality (54), it is clear that for any integer $n \geq 0$, Q_n is a polynomial of degree n. Moreover, one can easily check that for any two integers $0 \leq m \leq n$, n integrations by parts give us

$$\int_0^1 Q_n(x)Q_m(x)\,x\mathrm{d}x = c_n c_m(-1)^m \int_0^1 x^{n+1}(1-x)^n \frac{\mathrm{d}^n}{\mathrm{d}x^{2n}}\left[\frac{1}{x}\frac{\mathrm{d}^m}{\mathrm{d}x^m}(x^{m+1}(1-x)^m)\right]\mathrm{d}x$$
$$= \begin{cases} 0 & \text{if } m < n, \\ (c_n)^2 \dfrac{n!(2n+1)!}{(n+1)!}B(n+2,n+1) = (c_n)^2 \dfrac{(n!)^2}{2n+2} & \text{if } m = n. \end{cases}$$

Hence, the desired set of orthonormal polynomials $Q_n(x)$, $n \geq 0$, is given by

$$Q_n(x) = \frac{\sqrt{2n+2}}{n!}\frac{1}{x}\frac{\mathrm{d}^n}{\mathrm{d}x^n}\left[x^{n+1}(1-x)^n\right], \quad n \geq 0. \tag{55}$$

It is well known, (see for example [21]), that if $\{P_n(x),\ n \geq 0\}$ is a set of orthogonal polynomials over an interval $[a, b]$, then this family satisfies a three-term recursion formula of the following type

$$P_{n+1}(x) = (A_n x + B_n)P_n(x) - C_n P_{n-1}(x), \quad n \geq 0. \tag{56}$$

Moreover, if the highest coefficient of P_n is $k_n > 0$, then

$$A_n = \frac{k_{n+1}}{k_n}, \qquad C_n = \frac{A_n}{A_{n-1}}, \quad \forall\, n \geq 1. \tag{57}$$

Note that the sequence $(B_n)_n$ given in (56) can be computed by equating the coefficient of x^n from both sides of this equality. In the special case of the orthonormal polynomials $Q_n(x)$ given by (55), straightforward computations show that for any integer $n \geq 0$, we have $k_n = \frac{\sqrt{2n+2}}{n!}\frac{(2n+1)!}{(n+1)!}$. Hence, sequence $Q_n(x)$ satisfy the recursion formula (56) with the sequences $(A_n)_n$, $(B_n)_n$ and $(C_n)_n$ given explicitly as follows

$$A_n = \frac{2(2n+3)}{\sqrt{(n+1)(n+2)}}, \qquad C_n = \frac{2n+3}{2n+1}\sqrt{\frac{n}{n+2}}, \qquad B_n = -4\frac{(n+1)^{3/2}}{(2n+1)\sqrt{n+2}}, \quad n \geq 1, \tag{58}$$

with the starting terms $Q_0(x) = \sqrt{2},\ Q_1(x) = 4 - 6x$.

Also, note that from the theory of Classical Orthogonal Polynomials, (see [21]), one concludes that for all $n \geq 0$, the introduced polynomials $Q_n(x)$ has n different zeros inside. Moreover, these n different zeros are simply given as the eigenvalues of a tridiagonal symmetric matrix D of order n, given by

$$D = \left[d_{i,j}\right]_{1\leq i,j\leq n}, \qquad d_{i,i} = -\frac{B_{i-1}}{A_{i-1}}, \qquad d_{i,i+1} = d_{i+1,i} = \frac{-1}{A_{i-1}}, \qquad d_{i,j} = 0 \quad \text{if } j \neq i-1, i, i+1, \tag{59}$$

where the A_i and B_i are given by (58).

Remark 3. Note that formula (59) provides us with an excellent and very fast method for the computation of the different zeros of $Q_n(x)$, even if the degree n is large.

4.2. A quadrature method and approximate spectrum of $H_{c,N}$

In this paragraph, we use the previous results to construct a quadrature method for the eigenvalue problem (3). A Gaussian quadrature method of order $2n$, associated with an orthogonal $P_n(x)$, defined on $[a, b]$, and orthogonal with respect to a measure $d\alpha(x)$, is given by [21]),

$$\int_a^b f(x)\mathrm{d}\alpha(x) \approx \sum_{k=1}^n \omega_k f(x_k), \tag{60}$$

where $f \in C([a, b])$, the different nodes $(x_k)_{1\leq k\leq n}$ are the different zeros of $P_n(x)$. It is interesting to note that the different quadrature weights $(\omega_k)_{1\leq k\leq n}$ are simply given by the following practical formula

$$\omega_k = -\frac{k_{n+1}}{k_n}\frac{1}{P_{n+1}(x_k)P_n'(x_k)}, \quad 1 \leq k \leq n. \tag{61}$$

Here, k_n is the highest coefficient of $P_n(x)$. It is well known, (see [21]), that if $f \in C^{2n}([a, b], \mathbf{R})$, then we have the following desirable formula for estimating the error of the quadrature formula (60),

$$\int_a^b f(x)\mathrm{d}\alpha(x) = \sum_{k=1}^n \omega_k f(x_k) + \frac{1}{k_n^2}\frac{f^{(2n)}(\eta)}{(2n)!}, \quad a \leq \eta \leq b. \tag{62}$$

In the special case of our set of orthonormal polynomials $Q_n(x),\ n \geq 0$, the Gaussian quadrature (60) becomes

$$\int_0^1 f(x)x\mathrm{d}x \approx \sum_{k=1}^{n} \omega_k f(x_k), \qquad \omega_k = -\frac{2(2n+3)}{\sqrt{n^2+3n+2}} \frac{1}{Q_{n+1}(x_k)Q_n'(x_k)}, \quad 1 \leq k \leq n, \tag{63}$$

where the nodes x_k are the eigenvalues of the matrix D given by (59). In the special case where $\mathrm{d}\alpha(x) = x\mathrm{d}x$ and $f(x) = J_N(cxy)$, $y \in [0, 1]$ is fixed, Eq. (62) becomes

$$\int_0^1 J_N(cxy)\, x\mathrm{d}x = \sum_{k=1}^{n} \omega_k J_N(cx_k y) + \frac{(n!(n+1)!)^2}{2(n+1)(2n)!((2n+1)!)^2} \frac{\mathrm{d}^{2n}}{\mathrm{d}x^{2n}}\left[J_N(cxy)\right](\eta), \quad 0 \leq \eta \leq 1. \tag{64}$$

The following theorem borrowed from [24] provides a discretization formula for eigenproblem (3) as well as an interpolation formula for the approximate CPSWFs.

Theorem 5. *Let N be a fixed positive integer and let ϵ be an arbitrary real number satisfying $0 < \epsilon < 1$. Let $K_\epsilon = \max\left([2(ec+N+2)], \left[\frac{\log(5/(8\pi\epsilon))}{\log 2} + N + 2\right]\right)$, then under the above notation, we have*

$$\sup_{x\in[0,1]} \left| \phi_{n,c}^N(x) - \frac{1}{\beta_{n,N}(c)} \sum_{j=1}^{K_\epsilon} \omega_j J_N(cxy_j)\phi_{n,c}^N(y_j) \right| < \frac{c\,\epsilon}{|\beta_{n,N}(c)|}. \tag{65}$$

Here, $(y_j)_{1\leq j\leq n}$ denote the different zeros of the orthogonal polynomial $Q_n(x)$ and $\phi_{n,c}^N(\cdot)$, $\beta_{n,N}(c)$ are given analogously to (3).

Remark 4. Although the sequence $(|\beta_{n,N}(c)|)_n$ has a fast decay to zero, the approximation formula given by the Theorem 5 remains practical even for relatively large values of n. This is mainly due to the log-term in the expression of K_ϵ.

Remark 5. As in the classical finite Fourier case, (see [5]), the accuracy of the interpolation formula (65) depends essentially on the accuracy of the approximation of the integral $I_N(y) = \int_0^1 xJ_N(cxy)\mathrm{d}x$ by a quadrature method. If a $2n$th order quadrature method based on an orthogonal Legendre polynomial over $[0, 1]$, is used to approximate $I_N(y)$, then one gets

$$\int_0^1 xJ_N(cxy)\mathrm{d}x = \sum_{k=1}^{n} \omega_k x_k J_N(cx_k y) + \frac{(n!)^4}{(2n+1)((2n)!)^3} \frac{\mathrm{d}^{2n}}{\mathrm{d}x^{2n}}\left[xJ_N(cxy)\right](\eta), \quad 0 \leq \eta \leq 1. \tag{66}$$

By comparing (64) and (66), one concludes that our proposed quadrature method is slightly more accurate than the classical Gauss quadrature method. Moreover, the error analysis of our proposed quadrature method is easier to handle than in the case of Gauss quadrature. For more details, the reader is referred to [24].

As a result of Theorem 5, we obtain the following discretization scheme for the eigenvalue problem (3),

$$\sum_{j=1}^{K_\epsilon} \omega_j J_N(cx_i y_j)\widetilde{\phi}_{n,c}^N(y_j) = \widetilde{\beta}_{n,N}(c)\widetilde{\phi}_{n,c}^N(x_i), \quad 1 \leq i, j \leq K_\epsilon, \tag{67}$$

where the x_i, y_j and the ω_i denote the different nodes and weights of our proposed quadrature method. If A_K denotes the square matrix of order K, defined by

$$A = \left[\omega_j J_N(cx_i y_j)\right]_{1\leq i,j\leq K}, \tag{68}$$

then the set of the eigenvalues of A_K defines approximate values of a finite subset of the eigenvalues of the operator $\widetilde{H}_{c,N}$, introduced by (3). Moreover, for any integer $0 \leq n \leq K$, the eigenvector $\widetilde{U}_n$ corresponding to the approximate eigenvalue $\widetilde{\beta}_{n,N}(c)$ is given by $\widetilde{U}_n = \left[\widetilde{\phi}_{n,c}^N(x_i)\right]_{1\leq i\leq K}$. Finally, to provide approximate values $\Phi_{n,c}^N(x)$ of $\widetilde{\phi}_{n,c}^N(x)$ along the interval $[0, 1]$, we use the following interpolation formula based on [24].

$$\Phi_{n,c}^N(x) = \frac{1}{\beta_{n,N}(c)} \sum_{j=1}^{K} \omega_j J_N(cxy_j)\widetilde{\phi}_{n,c}^N(y_j), \quad 0 \leq x \leq 1. \tag{69}$$

Finally, approximate values of the CPSWFs along the interval $[0, 1]$, are given by the following formula,

$$\widetilde{\psi}_{n,c}^N(x) = \sqrt{x}\Phi_{n,c}^N(x), \quad 0 \leq x \leq 1. \tag{70}$$

Remark 6. As predicted by Theorem 5, interpolation formula (69) is highly accurate. As an example, for $c = 10$, $N = 2$, $n = 3$ and $K = 35$, we have numerically found that the relative approximation error over $[0, 1]$ of the previous formula is given by

$$\sup_{x\in[0,1]}\left|\frac{\Phi^N_{n,c}(x)-\widetilde{\phi}^N_{n,c}(x)}{\widetilde{\phi}^N_{n,c}(x)}\right| = \left|\frac{\Phi^N_{n,c}(x_0)-\widetilde{\phi}^N_{n,c}(x_0)}{\widetilde{\phi}^N_{n,c}(x_0)}\right| = 6.00481\text{e}{-13}.$$

Here, $x_0 = 0.92216$ and the references values on $[0, 1]$ of the exact eigenfunction $\widetilde{\phi}^N_{n,c}(\cdot)$ are obtained by using a higher order quadrature formula with $K = 70$ quadrature nodes. Moreover, we have found that the $L^2([0, 1], x\mathrm{d}x)-$ approximation error for this example is given by

$$\left\|\Phi^N_{n,c}(x)-\widetilde{\phi}^N_{n,c}(x)\right\|_2 = 3.66235\text{e}{-18}.$$

5. Computation of the spectrum of the finite Hankel transform operators: Maple programs

In this paragraph, using the results of Sections 3 and 4, we construct two Maple programs for computing the CPSWFs and their associated eigenvalues. For a given bandwidth $c > 0$, and a real number $N > -1$, the first program computes the Kth order submatrix A_K of A, the matrix representation of $H_{c,N}$. Then, the program computes accurate approximate values of the first eigenvalues of $H_{c,N}$. Also, the program uses formulae (52) and (53) and computes approximate values of the normalized CPSWFs along the domain $[0, \infty[$.

Program 1: Computation of the CPSWFs and their eigenvalues by method 1.

```
> # Initialization
>  with(linalg):Digits:=40:N:=1:c:=20:
   if floor(c) <= 40 then K:=max(floor(2*c),20+floor(c)):else
   K:= max(floor(c)+20,floor(6/5*c)): fi:
>  # To use a different value of K,
   # enable the following command and insert your choice of K.
>  #K:= :
> # Compute the different moments M_{lk}
> for l from 0 to K do M[l,0]:=evalf(sqrt((N+1)/2)/(N+l+1)):od:
  l:='l':for k from 1 to K do for l from 0 to K do M[l,k]:=0:od:od:
  l:='l':k:='k':for k from 1 to K do for l from k to K do
  M[l,k]:=(-1)^k*evalf(sqrt((2*k+N+1)/2)*l!/(l-k)!*GAMMA(N+l+1)/GAMMA(N+l+k+2)):od;od:
> # Compute the eigenvalues and the eigenvectors of H_{c,N}.
> b:=array(1..K,1..K):k:='k':j:='j':for k from 0 to K-1 do for l from 0 to K-1 do
  b[k+1,l+1]:=evalf(sum((-1/4)^j/(2^N*j!*GAMMA(N+j+1))*M[j,k]*M[j,l]*c^(N+2*j+1/2),
  j=max(l,k)..K)):od:od:
> F:=eigenvectors(b):L:=[seq(abs(F[i][1]),i=1..K)]:L:=sort(L):
> #  Reorder the eigenvalues of H_{c,N}
> for m from 1 to K do R[m]:=0:od:for i from 1 to K do for j from 1 to K do
  if((abs(L[i])-abs(F[j][1]))=0) then R[i]:=j:else R[j]:=R[j]+0:fi:od:od:
> # To print the ordered eigenvalues, enable the following command.
> # i:='i':for i from 1 to K do print(i-1,abs(F[R[K+1-i]][1]));od;
> # Computation of the basis T_{N,k}
> Y:='Y':PP[0]:= 1:PP[1]:= 1.0/2*((N+2)*Y+N):for k from 1 to K do
  PP[k+1]:= expand(1/(2*(k+1)*(k+N+1)*(2*k+N))*((2*k+N+1)*(N^{2}+Y*(N+2*k+2)*(N+2*k))
  *PP[k]-2*k*(k+N)*(2*k+N+2)*PP[k-1])):od:x:='x':Y:=1.0-2.0*x^{2}:
  for k from 0 to K do T[k]:= expand(sqrt(2.0*(2*k+N+1))*x^(N+1/2)*PP[k]):od:
> # construct the different normalized  circular PSWFs
> x:='x':  psi:=proc(n,x) a:=evalf(sqrt(sum((F[R[K+1-n]][3,1][kk])^{2},kk=1..K))):
  if(x=1) then evalf(1.0/a*sum(F[R[K+1-n]][3,1][kk]*T[kk-1],kk=1..K));
  else (-Heaviside(x-1)+Heaviside(x))*evalf(1.0/a*sum(F[R[K+1-n]][3,1][kk]*T[kk-1],
  kk=1..K))+Heaviside(x-1)* evalf(1/F[R[K+1-n]][1]*sum(1/a*F[R[K+1-n]][3,1][kk]*
  sqrt(2*(2*kk+N-1))*BesselJ(N+2*kk-1,c*x)/sqrt(c*x),kk=1..K)):fi:end:
> # To plot the graph of psi_{n,c}(x) over [0,b],
  # enable the following command with your choice of n and b.
> # x:='x':n:=1:b:=2:plot(psi(n+1,x),x=0..b);
```

The second program computes approximate values of the CPSWFs and their eigenvalues by using the quadrature method of Section 4. More precisely, for the given values of $c > 0$, $N > -1$ and the positive integer K, the program computes the different K quadrature nodes and weights. Then, it computes approximate values of the first eigenvalues $\beta_{n,N}(c)$. Finally the program uses formula (70) and computes approximate values of the $\widetilde{\phi}^N_{n,c}(x)$ along the interval [0, 1].

Program 2: Computation of the CPSWFs and their eigenvalues by method 2.

```
# Initialization
> restart;with(linalg):Digits:=50:c:=30.0:N:=1:K:=50:x:='x':
> # Compute the orthogonal polynomials Q_n(x) by Rodrigues Formula.
> for n from K to K+1 do a:=evalf((n!)/(sqrt(2.0*n+2.0))):
  Q[n]:=expand((-1)^n/(a)*1.0/x*diff((x^(n+1)*(1-x)^n,x$n))):
  k[n]:=evalf((sqrt(2*n+2)*(2*n+1)!)/(n!*(n+1)!)):od:dQ[K]:=diff(Q[K],x):
> # Compute the K nodes as the eigenvalues of the tridiagonal matrix D1
> n:=K:A[0]:=sqrt(2.0)*3.0:B[0]:=-evalf(2.0*sqrt(2.0)):for i from 1 to K do
  A[i]:=evalf(2*(2*i+3)/(sqrt((i+1)*(i+2)))):
  B[i]:=evalf(-4*(i+1)^(3/2))/((2*i+1)*sqrt(i+2)):od:
  D1:=array(1..K,1..K):for i from 1 to K do for j from 1 to K do D1[i,j]:=0.0:od:od:
  i:='i':for i from 1 to K-1 do D1[i,i]:=-evalf(B[i-1]/A[i-1]):
  D1[i,i+1]:=evalf(-1/A[i-1]):  D1[i+1,i]:=D1[i,i+1]:od:D1[K,K]:=-evalf(B[K-1]/A[K-1]):
  X:=eigenvals(D1):
> # Compute the different nodes by the use of formula (63).
> for l from 1 to K do x:=X[l]:W[l]:=evalf(-k[K+1]/k[K]*1.0/(Q[K+1]*dQ[K])):od:x:='x':
> # Check the accuracy of the  approximate quadrature weights
  #and change the precision if necessary.
> a:=evalf(sum(W[ii],ii=1..K)):if (abs(a-0.5)>=10^(-7)) then
  print(‘A precision problem: restart the program with a higher value of Digits‘); fi:
> # Compute the eigenvalues of A
> Digits:=40:A2:=array(1..K,1..K):for i from 1 to K do for j from 1 to K do
  A2[i,j]:=W[j]*BesselJ(N,c*X[i]*X[j]):od:od:F:=eigenvectors(A2):
  beta:=sort([seq(abs(F[i][1]),i=1..K)]):
> # Reorder the eigenvalues Gamma_{n,N}=|sqrt(c)*beta_{n,N}|
> for m from 1 to K do R[m]:=0:od:for i from 1 to K do for j from 1 to K do
  if((abs(beta[i])-abs(F[j][1]))=0) then R[i]:=j:else  R[j]:=R[j]+0:fi:od:od:
  i:='i':for i from 1 to K do Gamma[i]:=evalf(sqrt(c)*beta[K+1-i]):od:
> # To print the ordered eigenvalues,   enable the following command.
> #i:='i':for i from 1 to K do print(i,Gamma[i]);od:
> # Construct the Circular PSWFs according to formulae (69) and (70).
> psi:=proc(n,x) sqrt(x)*1/F[R[K+1-n]][1]*
  sum(W[kk]*BesselJ(N,c*x*X[kk])*F[R[K+1-n]][3,1][kk],kk=1..K):end:
> # To plot the graph of psi_{n,c}(x), enable  the following command.
> # n:=1: plot(psi(n+1,x),x=0..1);
```

6. Numerical results

To illustrate the results of Section 3, we have considered different values of the bandwidth c and the parameter N. Also, we have used the method of Section 3 to construct for each value of c, the corresponding square matrix $\widetilde{A}_K$ of order $K+1 = 31$. The integer $\widetilde{K}$ used to truncate the series representing the different coefficients $a_{kk'}(c)$ is also set to 30. Table 1 shows the eigenvalues $|\gamma_{n,N}(c)|$, for the values of $0 \le n \le 35$, $N = 1$. Moreover, we have used (52) and (53) with a maximum truncation order $L = 20$, and obtained accurate approximations to the normalized CPSWFs $\Psi^N_{n,c}(x)$ along the interval $[0, +\infty[$. Table 2 lists the approximate values $\widetilde{\Psi}^1_{0,c}(x)$ of $\Psi^1_{0,c}(x)$, for different values of x. The different approximation errors in absolute value $|\Psi^1_{0,c}(x) - \widetilde{\Psi}^1_{0,c}(x)|$ are given by Table 3. Note that reference values of $\Psi^1_{0,c}(x)$ are obtained by the method of Section 3 with large enough values of the parameter K. The plots of the CPSWFs $\widetilde{\Psi}^1_{n,c}(x)$, for $c = 5, 10$ and $n = 0, 1$ are given by Figs. 1 and 2, respectively. Moreover, in Fig. 3, we have given the graphs of $\widetilde{\Psi}^1_{0,c}(x)$, for $c = 15, 20, 25$.

In [3], the author has used a differential operator based method for the computation of the eigenvalues $\lambda_{n,N}(c) = c(\gamma_{n,N}(c))^2$. Tabulated numerical approximations of the $\lambda_{n,N}(c)$ with different values of $0.1 \le c \le 17$, $0 \le n \le 3$ and $0 \le N \le 2$ are given in [3]. By comparing these tabulated results with those obtained by our Maple programs, we have found that both programs provide highly accurate approximate values of the $\lambda_{n,N}(c)$. Moreover, the results given in [3] agree with our results to 7 digits after the decimal point.

For large values of the bandwidth c, we have computed the spectrum of $H_{c,N}$ by the use of the quadrature method of Section 4. Table 4 shows the obtained numerical values of $(-1)^n\gamma_{n,N}(c)$ obtained by this method with $K = 120$, $N = 1$

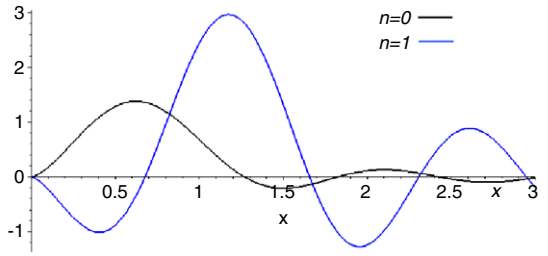

Fig. 1. Graph of $\widetilde{\Psi}^1_{n,c}(x)$, for $c = 5$ and $n = 0, 1$.

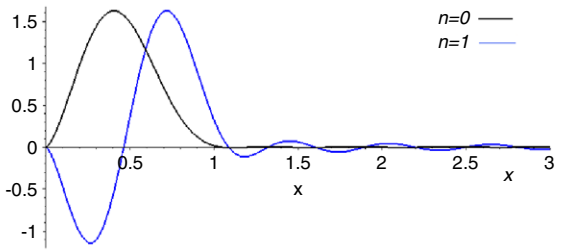

Fig. 2. Graph of $\widetilde{\Psi}^1_{n,c}(x)$, for $c = 10$ and $n = 0, 1$.

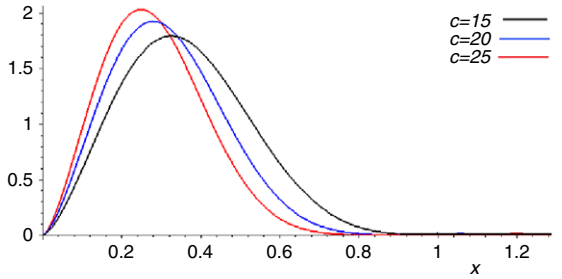

Fig. 3. Graphs of $\widetilde{\Psi}^1_{0,c}(x)$, for various values of c.

Table 1
Values of $(-1)^n\gamma_{n,N}(c)$, $N = 1$ obtained by the matrix representation technique.

n	$c = 5$	$c = 10$	$c = 15$	$c = 20$	$c = 25$
0	4.328318175e−01	3.162250925e−01	2.581988895e−01	2.236067978e−01	2.000000000e−01
5	1.196803059e−07	3.284074138e−04	2.733547500e−03	1.954103152e−01	1.999229711e−01
10	6.405453090e−19	1.883448918e−12	1.113343361e−08	4.955905388e−06	4.968332728e−04
15	3.064597888e−32	9.281147765e−23	3.234887597e−17	2.713156779e−13	2.891621901e−10
20	6.113522480e−47	1.898881328e−34	3.839225644e−27	5.806750350e−22	5.953367985e−18
25	1.102382564e−62	3.508271891e−47	4.099371147e−38	1.107359178e−31	1.070060302e−26
30	2.871151025e−79	9.359081975e−61	6.312637325e−50	3.036036535e−42	2.747172638e−36
35	1.480100480e−96	4.941159911e−75	1.922863730e−62	1.644460526e−53	1.389720689e−46

Table 2
Values of $\widetilde{\Psi}^1_{0,c}(x)$ for different values of x.

c	$x = 0.5$	$x = 1.0$	$x = 10$
5	1.284982709234547	6.786508504640555e−01	1.073274758507817e−02
10	1.491587327754714	1.732760660998238e−02	2.573063153232900e−05
15	1.179651185830145	2.277910990652706e−04	2.542232373640525e−06
20	0.815969572910883	2.428300661805786e−06	3.327010256089710e−08
25	0.526090079255421	2.322645561508451e−08	2.878140653809407e−10

and different values of n and c. We should mention that all the above numerical results are obtained by the use of the Maple programs of the previous section. Also, for small values of the parameter c ($c \leq 100$), the method of Section 3 is more appropriate to use. It provides highly accurate approximations to the spectrum of the finite Hankel transform

Table 3
The approximation error $|\Psi^1_{0,c}(x) - \widetilde{\Psi}^1_{0,c}(x)|$.

c	$x = 0.5$	$x = 1$	$x = 10$
5	1.1442838286e−13	6.2982408797e−13	1.8550801301e−14
10	1.5787484492e−14	1.4403313533e−13	3.7308645328e−15
15	2.7529448221e−13	1.7946177540e−12	2.4168978400e−14
20	1.9856873138e−13	4.0101035991e−12	1.1262346357e−14
25	3.8239830938e−13	4.0120555290e−12	8.5623318288e−15

Table 4
Values of $(-1)^n \gamma_{n,N}(c)$, $N = 1$.

n	$c = 100$	n	$c = 200$
0	1.00000000000e−01	0	7.07106781187e−02
20	1.00000000000e−01	50	7.07106781187e−02
30	9.45634141276e−02	60	7.06774639718e−02
40	4.43455780887e−09	70	8.85617955333e−07
50	4.80218801960e−20	80	1.30912062099e−15
60	6.41679645801e−33	90	6.11717011336e−26

operator. Moreover, it has the advantage of providing very good approximations to the values of the CPSWFs along the interval $[0, +\infty[$ without the use of any interpolation formula. Nonetheless, for larger values of c, method 1 becomes slow and no longer practical. This is due to the increase in the computation load of the truncated matrix representation of $H_{c,N}$. Method 2 of Section 4 is better adapted for moderate large values of the parameter c. This second method has the advantage of handling larger values of the bandwidth c. As other known quadrature based methods for computing the PSWFs or the CPSWFs and unlike our first method, our second method has to use an interpolation formula in order to approximate the values of the CPSWFs on $[0, 1]$.

7. Conclusion

We have given some extended theoretical results concerning the spectral properties of the finite Hankel transform operator $H_{c,N}$. Also, we have developed two numerical methods for the accurate approximation of the eigenvalues and the eigenfunctions of the latter. The first method is restricted to small values of the parameter c. It has the advantage of providing accurate approximations of the CPSWFs, the eigenfunctions of $H_{c,N}$ along the domain $[0, +\infty[$. The second method is based on a special quadrature formula. It is well adapted for moderate large values of c and provide accurate approximations of the CPSWFs along the interval $[0, 1]$. Also, we have provided the reader with two Maple programs. Program 1 and program 2 implement method 1 and method 2, respectively. Note that the computation of the K quadrature nodes and weights required by method 2, has to be performed by the use of a high enough precision arithmetic. This is the reason why this second method cannot be applied for very large values of the parameter c. Nonetheless, in [24], one of us has developed a composite Gaussian based quadrature method for high frequency CPSWFs. This last method is well adapted for the accurate computation of the spectrum of $H_{c,N}$, where the parameter c may vary from few hundreds to several thousands.

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