Journal of Computational and Applied Mathematics 235 (2011) 3775–3782

Contents lists available at ScienceDirect

Journal of Computational and Applied Mathematics

journal homepage: www.elsevier.com/locate/cam

Multi-stage solvers optimized for damping and propagation

Rob Haelterman [a,*], Jan Vierendeels [b,1], Dirk Van Heule [a,2], Steve De Ridder [a,3], Helena Bruyninckx [a,4]

[a] Department of Mathematics, Royal Military Academy, Renaissancelaan 30, B-1000 Brussels, Belgium

[b] Department of Flow, Heat and Combustion Mechanics, Ghent University, St-Pietersnieuwstraat 41, B-9000 Gent, Belgium

ARTICLE INFO

Keywords:
Iterative solution
Multi-grid
Multi-stage

ABSTRACT

Explicit multi-stage solvers are routinely used to solve the semi-discretized equations that arise in Computational Fluid Dynamics (CFD) problems. Often they are used in combination with multi-grid methods. In that case, the role of the multi-stage solver is to efficiently reduce the high frequency modes on the current grid and is called a smoother. In the past, when optimizing the coefficients of the scheme, only the damping characteristics of the smoother were taken into account and the interaction with the remainder of the multi-grid cycle was neglected. Recently it had been found that coefficients that result in less damping, but allow for a higher Courant–Friedrichs–Lewy (CFL) number are often superior to schemes that try to optimize damping alone. While this is certainly true for multi-stage schemes used as a stand-alone solver, we investigate in this paper if using higher CFL numbers also yields better results in a multi-grid setting. We compare the results with a previous study we conducted and where a more accurate model of the multi-grid cycle was used to optimize the various parameters of the solver.

We show that the use of the more accurate model results in better coefficients and that in a multi-grid setting propagation is of little importance.

We also look into the gains to be made when we allow the parameters to be different for the pre- and post-smoother and show that even better coefficients can be found in this way.

1. Introduction

Explicit multi-stage schemes are commonly used to solve ordinary differential equations resulting from the space discretization of partial differential equations. When used as a stand-alone solver, such a scheme can either be designed for high order accuracy or to reach steady state quickly when the order of accuracy is less important. As it can easily be shown that low frequency error components are difficult to damp, a multi-stage scheme that improves the propagation of these components generally yields a faster solver; a characteristic of these solvers is the higher CFL number.

When a multi-stage iteration is used in conjunction with multi-grid, the aim is to reduce those low frequency components on a coarser grid; if the number of nodes on the coarser grid is low, a direct solution on that grid can be cost-effective.

* Corresponding author. Tel.: +32 2 7426527.
E-mail addresses: Robby.Haelterman@rma.ac.be (R. Haelterman), Jan.Vierendeels@ugent.be (J. Vierendeels), Dirk.Van.Heule@rma.ac.be (D. Van Heule), Steve.De.Ridder@rma.ac.be (S. De Ridder), Helena.Bruyninckx@rma.ac.be (H. Bruyninckx).
1 Tel.: +32 9 2643586.
2 Tel.: +32 2 7426523.
3 Tel.: +32 2 7426517.
4 Tel.: +32 2 7426343.

0377-0427/$ – see front matter © 2011 Elsevier B.V. All rights reserved.
doi:10.1016/j.cam.2011.01.024

The multi-stage solver is then often designed to eliminate the high frequency components of the error (which cannot be represented on the coarser grid) and is called a smoother. In the past, these coefficients have been optimized by trial and error [1] or by geometric methods [2].

These and other studies [3–5] focused on minimizing the smoothing factor over the high frequency domain. Still, it was realized that designing the multi-stage scheme to be an optimal smoother, as would be desirable for multi-grid to work, would not result in an optimal (i.e. fastest) overall scheme, but that some precedence had to be given to the propagative behavior [6]. Recently the optimization of the damping and propagative efficiencies was expressed within the framework of constrained non-linear optimization [7]. Different objective functions with a variety of constraints were constructed on a heuristical basis and their effect on the convergence rate of the numerical scheme was studied. Multi-stage coefficients that are comparable with values used in the past were found.

In [8] we chose to look at a complete 2-grid cycle which closely modeled the interaction between multi-stage solver, defect correction, restriction and prolongation. Multi-stage solvers with an improved performance were found, although their smoothing capacity was sacrificed to some extent.

In this paper we refine the model established in [8] by also looking at the propagation characteristics. Again we are only interested in reaching convergence quickly using a multi-grid scheme, and therefore focus on the multi-stage scheme for which the 2-grid cycle will remove all frequencies as quickly as possible, either by propagation or by damping. We focus on the 1 dimensional advection equation; this limitation is justified, as Hosseini and Alonso [7] have shown that 1D wave equations can serve as a good indication of the performance of real flow solvers.

This paper is organized as follows. In Section 2 we formulate the equation under consideration, the Fourier footprints of the space discretizations and the transmittance function of the multi-stage time-stepping scheme; in Section 3 we model the remainder of the 2-grid cycle and give the transmittance function of its components; in Section 4 we deduce the propagation and damping that results from a transmittance function and formulate the analytical framework for optimization; in Section 5 the results are analyzed in two cases: identical or different parameters for pre- and post-smoother.

Remark. We use **i** as a symbol for the complex unit ($=\sqrt{-1}$) and i as an index.

2. The model: the scalar 1D advection equation

2.1. General formulation

We are interested in the advection equation

$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = 0 \tag{1}$$

to which we add a suitable boundary condition, which in our case are the inlet value of u. Applying a given space discretization to (1) on an equidistant mesh with N nodes results in the semi-discretized nodal equation ($j = 1, \ldots, N$):

$$\frac{\mathrm{d}\tilde{u}_j}{\mathrm{d}t} + \check{r}_j = 0 \tag{2}$$

where the space discretization operator $\check{r}_j$, which is a residual for the steady state solution, is of the form

$$\check{r}_j = \sum_{k=k^-}^{k^+} \beta_{j+k}\tilde{u}_{j+k} \tag{3}$$

where k^- and k^+ depend on the stencil that is used for the space discretization.

If $u_{\mathrm{exact},j}$, is the solution to (2), we can write $u_j = u_{\mathrm{exact},j} + e_j$, where e_j is the (nodal) error of u_j with respect to $u_{\mathrm{exact},j}$.

Eq. (2) can then be transformed into the form

$$\frac{\mathrm{d}\tilde{e}_j}{\mathrm{d}t} + \underbrace{\sum_{k=k^-}^{k^+} \beta_{j+k}\tilde{e}_{j+k}}_{r_j} = 0. \tag{4}$$

We assume that $[\tilde{e}_1 \quad \tilde{e}_2 \quad \cdots \quad \tilde{e}_N]$ can be written as an infinite sum of Fourier modes, of which we now only consider the p-th: $\tilde{e}_{j,p} = \hat{e}_p(t)\mathrm{e}^{\mathbf{i}jp\Delta x}$ ($j = 1, \ldots, N$). A similar expression will be used for r_j. Alternatively, when introducing the phase angle $\theta = p\Delta x$ we get $\tilde{e}_{j,\theta} = \hat{e}_\theta(t)\mathrm{e}^{\mathbf{i}j\theta}$ with $\theta \in [-\pi, \pi]$ ($j = 1, \ldots, N$).

Inserting this in Eq. (4) we get an expression for the amplitude of the error $\hat{e}_\theta$ of the form

$$\frac{\mathrm{d}\hat{e}_\theta}{\mathrm{d}t} = -\hat{r}_\theta = \lambda(\theta)\hat{e}_\theta \tag{5}$$

where we call $\lambda : \mathbb{R} \to \mathbb{C}$ the *Fourier symbol* of the space discretization. Afterward we apply an explicit (single or multi-step) scalar time-integration resulting in

$$\hat{e}_\theta(t + \Delta t) = \mu(\lambda(\theta)) \cdot \hat{e}_\theta(t) \tag{6}$$

with $\mu : \mathbb{C} \to \mathbb{C} : z \mapsto \mu(z)$ being the transmittance function to which we add a stability constraint: $\sup_{\theta\in[-\pi,\pi]} |\mu(\lambda(\theta))| \leq 1$.

2.2. Time discretization

Integrating the ODE of Eq. (2) with a single-stage explicit time-stepping scheme results in the following discretization

$$\tilde{u}_j(t + \Delta t) = \tilde{u}_j(t) - \check{r}_j \Delta t \tag{7}$$

or applied to resp. Eqs. (4) and (5)

$$\tilde{e}_j(t + \Delta t) = \tilde{e}_j(t) - \Delta t r_j \tag{8}$$

$$\hat{e}_\theta(t + \Delta t) = \underbrace{(1 + \Delta t \lambda(\theta))}_{\mu(\lambda(\theta))} \hat{e}_\theta(t). \tag{9}$$

An m-stage (Runge–Kutta) time-stepping scheme is given by (after dropping the subscripts):

$$\begin{aligned}
&U^{(o)} = \tilde{u}(t)\\
&U^{(1)} = U^{(o)} - \alpha_1 \Delta t \check{r}^{(o)}\\
&\vdots\\
&U^{(l)} = U^{(o)} - \alpha_l \Delta t \check{r}^{(l-1)}\\
&\vdots\\
&U^{(m)} = U^{(o)} - \alpha_m \Delta t \check{r}^{(m-1)}\\
&\tilde{u}(t + \Delta t) = U^{(m)}.
\end{aligned} \tag{10}$$

As a consequence of Eq. (10) we can write

$$\hat{e}_\theta(t + \Delta t) = \underbrace{\left(1 + \sum_{l=1}^{m} \left(\prod_{i=m-l+1}^{m} \alpha_i \right) (\Delta t \lambda(\theta))^l \right)}_{\mu(\lambda(\theta))} \hat{e}_\theta(t). \tag{11}$$

As the multi-stage parameters α_i are always encountered in conjunction with Δt we can regroup them into a new parameter $\tilde{\alpha}_i = \alpha_i \Delta t$ without this restriction. We chose not to do so to clearly emphasize the effect of the time-step (and hence the CFL number) in the parameters.

2.3. Space discretization

If we discretize (1) (with $a = 1$ for simplicity) with a first order upwind discretization (U1) we find the following space discretization for the interior nodes

$$\check{r}_j = \frac{\tilde{u}_j - \tilde{u}_{j-1}}{\Delta x} \tag{12}$$

$$\lambda(\theta) = \frac{\mathrm{e}^{-\mathbf{i}\theta} - 1}{\Delta x}. \tag{13}$$

In the case of a second order upwind discretization (U2) we find

$$\check{r}_j = \frac{3\tilde{u}_j - 4\tilde{u}_{j-1} + \tilde{u}_{j-2}}{2\Delta x} \tag{14}$$

$$\lambda(\theta) = -\frac{3 - 4\mathrm{e}^{-\mathbf{i}\theta} + \mathrm{e}^{-2\mathbf{i}\theta}}{2\Delta x}. \tag{15}$$

In the case of a third order upwind-biased discretization (K3) we find

$$\check{r}_j = \frac{2\tilde{u}_{j+1} + 3\tilde{u}_j - 6\tilde{u}_{j-1} + \tilde{u}_{j-2}}{6\Delta x} \tag{16}$$

$$\lambda(\theta) = -\frac{3 + 2\mathrm{e}^{\mathbf{i}\theta} - 6\mathrm{e}^{-\mathbf{i}\theta} + \mathrm{e}^{-2\mathbf{i}\theta}}{6\Delta x}. \tag{17}$$

3. The effect of the multi-grid cycle

If we want to optimize the coefficients of the multi-stage time-stepping scheme used as a smoother in a multi-grid setting, we can add the effect of the defect correction of the multi-grid.

1. Pre-smoothing with an m-stage time-stepping scheme.
2. Restriction by full weighting.
3. Defect correction on the coarse grid with a coarsening factor of 2. We assume that the defect equation on the coarse grid is solved exactly. While we implicitly assume this is done by a direct solver, it could be done with an accurate iterative scheme.
4. Prolongation by linear interpolation.
5. Post-smoothing with an m-stage time-stepping scheme.

The effect of the smoother has been looked at in the previous paragraphs and will henceforth be characterized by the transmittance functions μ_{pre}, μ_{post}.

A model for the combined transmittance of restriction, defect correction and prolongation was established in [8], which we call μ_{DC}:

$$\mu_{DC}(\lambda(\theta)) = \left(1 - \left(\cos\frac{\theta}{2}\right)^4 \frac{2\lambda(\theta)}{\lambda(2\theta)}\right) \quad \text{for } \theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \tag{18}$$

$$= 1 \quad \text{for } \theta \in [-\pi, \pi] \setminus \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]. \tag{19}$$

The combined 2-grid cycle can then be expressed as:

$$\hat{e}_\theta(t + \Delta t) = \mu_{\text{post}}(\lambda(\theta))\mu_{DC}(\lambda(\theta))\mu_{\text{pre}}(\lambda(\theta))\hat{e}_\theta(t). \tag{20}$$

Remark. Although the above model represents a 2-grid cycle, it could also be used for any type of multi-grid cycle (V, W or F), as long as the defect correction with respect to the coarsest grid is solved exactly.

4. Quantifying damping and propagation

When we look at the way an iterative scheme eliminates error modes, we see that there are two mechanisms at play: damping and propagation. We will establish an analytical expression that allows us to quantify both, based on the transmittance function. If we only look at the transmittance of the smoother we will be interested in the high frequency modes, i.e. those with $\theta \in \left[-\pi, -\frac{\pi}{2}\right] \bigcup \left[\frac{\pi}{2}, \pi\right]$. If a complete 2-grid cycle is considered, the objective is to eliminate all modes as quickly as possible.

In previous studies [9,5] the main focus lay on the damping properties of the smoother, meaning that it was tried to keep the amplification factor $|\mu(\lambda(\theta))|$ of the smoother as small as possible for the high frequencies (possibly with a maximal number of zeros). Still, it was realized that some attention had to be paid to the CFL number. In [7,10] the authors went a step further and proposed a number of minimizers combining CFL number (to quantify propagation) and amplification factor (to quantify damping). The main conclusion was that the contribution of propagation to convergence is far superior to that of damping, provided that one does not find itself at the limits of stability. Implicitly these studies assumed that the same scheme was used for the pre- and post-smoother while the quantitative effect of the defect correction was ignored.

In [8] we showed that, in order to obtain optimal performance, the smoother had to be tuned to be as complementary as possible to the defect correction.

The transmittance function (either of the smoother, or the complete 2-grid cycle) can be split as

$$\mu(\lambda(\theta)) = \sigma(\theta)\mathrm{e}^{-\mathbf{i}\omega(\theta)} \tag{21}$$

where $\sigma, \omega : \mathbb{R} \to \mathbb{C}$.

From the modulus we get the stability condition

$$\sup_{\theta \in [-\pi, \pi]} |\sigma(\theta)| \leq 1. \tag{22}$$

From Eq. (6) we obtain;

$$\begin{aligned} &\hat{e}_\theta(t + \Delta t) = \sigma(\theta)\mathrm{e}^{-\mathbf{i}\omega(\theta)}\hat{e}_\theta(t) \\ &\Rightarrow \tilde{e}_{j,\theta}(t + \Delta t) = \sigma(\theta)\hat{e}_\theta(t)\mathrm{e}^{\mathbf{i}p\left(j\Delta x - \frac{\omega(\theta)}{\theta}\Delta x\right)} \\ &\Rightarrow \tilde{e}_{j,\theta}(t + \Delta t) = \sigma(\theta)\hat{e}_\theta(t)\mathrm{e}^{\mathbf{i}(j\theta - \omega(\theta))}. \end{aligned} \tag{23}$$

As such, the amplitude of the p-th mode is modulated by $\sigma(\theta)$, while the argument is modified by $\omega(\theta)$; that is, it is shifted by a distance $\frac{\omega(\theta)}{\theta}\Delta x$ (to the right if $\frac{\omega(\theta)}{\theta}\Delta x > 0$).

A mode can disappear from the system by

- propagation, determined by $\omega(\theta)$
- damping, determined by $\sigma(\theta)$.

We define by L the length of the domain and by ε the required reduction of the amplitude of the error with respect to the initial value. A mode has disappeared if, after n steps, either

$$\left|\sum_{i=1}^{n} \frac{\omega(\theta)}{\theta}\Delta x\right| \geq L \quad \text{or} \quad \left|\prod_{i=1}^{n} \sigma(\theta)\right| < \varepsilon. \tag{24}$$

If we use the same iterative scheme (which can mean the whole multi-grid cycle) throughout the whole solution process, then all modes will have disappeared when

$$\max_{\theta\in\mathcal{T}} \left\{\min\left(n\left|\frac{\omega(\theta)}{\theta}\Delta x\right| - L, |\sigma(\theta)|^n - \varepsilon\right)\right\} = 0. \tag{25}$$

Conversely, the number of iterations needed to damp out the different modes, n_{damp}, is given by

$$n_{\text{damp}}(\theta) = \frac{\log\varepsilon}{\log|\sigma(\theta)|} \tag{26}$$

and the number of steps needed to convect those modes out of the system, n_{conv}, is given by

$$n_{\text{conv}}(\theta) = \left|\frac{\theta L}{\omega(\theta)\Delta x}\right|. \tag{27}$$

The best scheme is thus the one that minimizes the following function

$$\mathcal{I} = \max_{\theta\in\mathcal{T}}\{\min(n_{\text{damp}}(\theta), n_{\text{conv}}(\theta))\} \tag{28}$$

with the constraint (22).

If we only want to optimize the coefficients of the smoother, we limit ourselves to high frequency modes $\left(\mathcal{T} = [-\pi,\pi]\setminus\left[-\frac{\pi}{2},\frac{\pi}{2}\right]\right)$ and take the transmittance of the multi-stage time-stepping. If we are interested in coefficients that optimize the whole cycle, we take the corresponding transmittance and look at the whole range of θ ($\mathcal{T} = [-\pi,\pi]$) when minimizing $\mathcal{I}$.

Remarks.

1. We like to point out that the optimal choice of parameters will now depend on L and ε, both of which are problem-, grid- and user-dependent.
2. Although we have not stated this explicitly, the formula above does not take into account any reflection at the end of the domain.
3. The constraint in (22) can be relaxed, as long as the instability is not that strong that the iteration blows up before the instable modes have been propagated out of the mesh.

5. Results

We now take a closer look at the advection equation (1), discretized with the U1, U2 and K3 schemes and look into the coefficients of the multi-stage solver that give the fastest convergence speed. For the Dirichlet boundary condition the method of ghost cells was used for the higher order discretization schemes.

We compare the optimal coefficients that were found in [8,5] and those obtained by optimizing $\mathcal{I}$ in Eq. (28). For all sets of coefficients we give $\sigma_{\max} = \sqrt[2m]{\sup_{\theta\in[-\pi,\pi]}|\sigma(\theta)|}$, which is a measure for the damping performance of the 2-grid cycle, averaged over the number of stages of the Runge–Kutta smoother.

We take $a = 1$ and $\Delta x = 1$ for simplicity, as the aim of this study is to validate the new approach to find the optimal coefficients and to verify to which extent the use of Fourier analysis can yield good coefficients for the higher order discretization schemes.

5.1. Previous studies

The optimal coefficients from [5] for the U1, U2 and K3 discretization are found in resp. Tables 1–3. Here the objective was to have as much zeros for the transmittance of the smoother in the high frequency range as possible. The quantitative effect of the defect correction was ignored and the same coefficients were used for the pre- and post-smoothing.

The optimal coefficients [8] found in resp. Tables 4–6. Here the objective function was $\sigma_{\max}$, and the pre- and post-smoother were identical.

Table 1
Optimal m-stage coefficients for smoothing. Advection equation U1 according to [5].

	$m=1$	$m=2$	$m=3$	$m=4$	$m=5$	$m=6$
α_1	1	0.3333	0.1481	0.0833	0.0533	0.0370
α_2		1	0.4000	0.2069	0.1263	0.0851
α_3			1	0.4265	0.2375	0.1521
α_4				1	0.4414	0.2562
α_5					1	0.4512
α_6						1
Δt	1/2	1	1.5	2	5/2	3
$\sigma_{\max}$	0.7056	0.7159	0.7703	0.8071	0.8329	0.8521

Table 2
Optimal m-stage coefficients for smoothing. Advection equation U2 according to [5].

	$m=2$	$m=3$	$m=4$	$m=5$	$m=6$
α_1	0.4242	0.1918	0.1084	0.0695	0.0482
α_2	1	0.4929	0.2602	0.1602	0.1085
α_3		1	0.5052	0.2898	0.1885
α_4			1	0.5060	0.3050
α_5				1	0.5063
α_6					1
Δt	0.4693	0.6936	0.9214	1.1508	1.3805
$\sigma_{\max}$	0.8655	0.8839	0.9000	0.9127	0.9222

Table 3
Optimal m-stage coefficients for smoothing. Advection equation K3 according to [5].

	$m=2$	$m=3$	$m=4$	$m=5$	$m=6$
α_1	0.6621	0.2884	0.1666	0.1067	0.0742
α_2	1	0.5010	0.3027	0.1979	0.1393
α_3		1	0.5275	0.3232	0.2198
α_4			1	0.5201	0.3302
α_5				1	0.5181
α_6					1
Δt	0.8276	1.3254	1.7320	2.1668	2.5975
$\sigma_{\max}$	0.8491	0.8426	0.8531	0.8668	0.8788

5.2. This study

5.2.1. Identical pre- and post-smoother

In a first phase we search for the parameters that minimize $\mathcal{I}$ in Eq. (28) when the pre- and post-smoother are identical. The purpose is to investigate to which degree propagation has to be taken into account when looking for the optimal coefficients of the smoother. A relatively short domain length of $L = 100$ and high damping requirement of $\varepsilon = 10^{-10}$ was chosen in order to promote the effect of elimination by propagation.

We modeled the algorithm in Matlab 7.0. Because $\mathcal{I}$ was highly non-linear and non-derivable, two different search methods were used. The first consisted in taking an interval in which each of the coefficients would probably lie (based on previous results) and create a large number of coefficient sets at random in this interval. These were used as starting points for Matlab's *fminsearch* function (which uses the Nelder–Mead simplex method). Of all the local minima found in this way, the best was retained. A second search pattern used a genetic algorithm that gave an optimal set of coefficients which were then further refined with *fminsearch*.

The resulting coefficients were almost identical to those found in [8], with only the very low frequency modes sometimes disappearing first by propagation. When averaging over the number of stages in the smoother this confirmed the findings of [8] that a lower number of stages was computationally the most efficient, in contrast to the findings of most previous studies. Experiments on real solvers closely followed those of the model.

As we found that the values were not different enough from those in Tables 4–6 and the gains negligible, we have chosen not to copy them here. It shows however that in a multi-grid setting the propagation effect is insignificant, even for short domain lengths. We point out, that in the current framework the defect correction was solved exactly, by a non-specified solver. If on the coarsest grid, which typically has a limited number of nodes, an iterative solver were to be used, then – just as for stand-alone solvers – it will be beneficial to add some weight to the propagative behavior of that solver.

5.2.2. Different pre- and post-smoother

In a second phase we allowed the coefficients for the pre- and post-smoothing to be different and optimized for $\mathcal{I}$. Except for the first iteration, the classical multi-grid cycle will result in a sequence on the finest grid where the m-stage

Table 4
Optimal m-stage coefficients for a smoother when looking at the complete 2-grid cycle but with identical pre- and post-smoothing. Advection equation using U1.

	$m=1$	$m=2$	$m=3$	$m=4$	$m=5$	$m=6$
α_1	1	0.3745	0.1824	0.1084	0.0717	0.0514
α_2		1	0.4907	0.2713	0.1733	0.1209
α_3			1	0.5465	0.3275	0.2199
α_4				1	0.5868	0.3704
α_5					1	0.6210
α_6						1
Δt	0.5000	0.9985	1.4985	1.9999	2.4983	2.9504
$\sigma_{\max}$	0.7056	0.7046	0.7475	0.7790	0.8011	0.8201

Table 5
Optimal m-stage coefficients for a smoother when looking at the complete 2-grid cycle but with identical pre- and post-smoothing. Advection equation using U2.

	$m=2$	$m=3$	$m=4$	$m=5$	$m=6$
α_1	0.5333	0.2451	0.1427	0.0934	0.0659
α_2	1	0.6606	0.3528	0.2217	0.1521
α_3		1	0.7072	0.4101	0.2703
α_4			1	0.7267	0.4436
α_5				1	0.7342
α_6					1
Δt	0.3879	0.5890	0.7900	0.9906	1.1925
$\sigma_{\max}$	0.8557	0.8636	0.8721	0.8797	0.8863

Table 6
Optimal m-stage coefficients for a smoother when looking at the complete 2-grid cycle but with identical pre- and post-smoothing. Advection equation using K3.

	$m=2$	$m=3$	$m=4$	$m=5$	$m=6$
α_1	0.7268	0.3006	0.1753	0.1276	0.0944
α_2	1	0.6874	0.37514	0.2538	0.1895
α_3		1	0.6811	0.4152	0.2988
α_4			1	0.6768	0.4458
α_5				1	0.6877
α_6					1
Δt	0.6499	1.0537	1.3984	1.7256	1.9769
$\sigma_{\max}$	0.8241	0.7861	0.7894	0.8018	0.8137

post-smoother is followed by the m-stage pre-smoother, which at that point effectively double as a $2m$-stage smoother. If we look closely at the transmittance of an optimal $2m$-stage smoother in Tables 4–6 and compared it with the square of the transmittance of the optimal m-stage smoother from the same tables, we see that they are not identical. The reason is that the transmittance function of a smoother of order $2m$ cannot always be split as the product of two real transmittance functions of order m. An opportunity therefore exists to improve the performance by choosing different coefficients for pre- and post-smoothing.

The optimal performance for U1, U2 and K3 was obtained for $m = 2$.

The optimal coefficients for the U1 scheme are $(\alpha_1, \alpha_2) = (0.2805, 1.0000)$, $\Delta t = 1.0000$ for pre-smoothing and $(\alpha_1, \alpha_2) = (0.6839, 1.0000)$, $\Delta t = 1.0000$ for post-smoothing.

For U2 this is $(\alpha_1, \alpha_2) = (1.4984, 1.000)$, $\Delta t = 0.3634$ for pre-smoothing and $(\alpha_1, \alpha_2) = (0.2976, 1.000)$, $\Delta t = 0.4551$ for post-smoothing.

For K3 this is $(\alpha_1, \alpha_2) = (2.3486, 1.0000)$, $\Delta t = 0.4710$ for pre-smoothing and $(\alpha_1, \alpha_2) = (0.3203, 1.0000)$, $\Delta t = 0.9763$ for post-smoothing.

We need to point out that the sequence of pre- and post-smoothers can be inverted without any change, except for the first iteration; for that reason preference for the pre-smoother should be given to the scheme with the best damping properties for the high frequency modes in order to limit aliasing.

Also note that in this study pre- or post-smoother taken separately will not always result in a stable scheme, but the combination of both will.

The gain for U1, U2 and K3 with this approach with respect to the old values was resp. 14%, 28% and 42%. Again, only a very limited number of error modes disappeared by propagation first.

The faster convergence when using different pre- and post-smoothing is illustrated for the K3 discretization in Fig. 1.

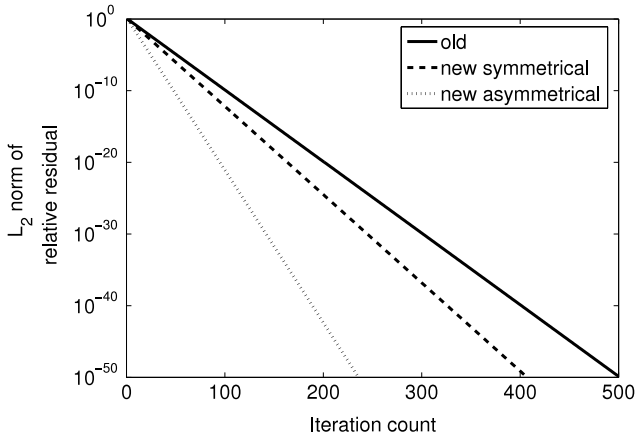

Fig. 1. Convergence history of the L2-norm of the relative residual for the solution of the advection equation (1) with K3 discretization and $m = 2$. "Old" = coefficients from [5]; "new symmetrical" = coefficients for this study with identical pre- and post-smoother; "new asymmetrical" = optimal coefficients with different pre- and post-smoother.

6. Conclusions

We have included the quantitative effect of the propagation and damping of error modes in a multi-grid setting that uses an iterative multi-stage smoother. While the convergence speed of single-grid solvers benefits from higher CFL numbers, we have found no such effect for multi-grid solvers, although some previous studies did propose higher CFL numbers in order to speed up convergence.

We have been able to improve convergence speed however by modeling the various components of the multi-grid cycle and taking their interplay into account. The results obtained in this way showed that a low number of stages was computationally more efficient.

Even more gains were obtained when in the same optimization environment we allowed for different multi-stage parameters during pre- and post-smoothing. The best improvement was obtained for the K3 scheme, in the order of 40%.

We have thus shown that the optimal solver cannot be found unless we look at the combined effect of every module of a multi-grid cycle and quantify its transmittance and that the convergence of multi-grid solvers is almost uniquely determined by damping.

References

[1] A. Jameson, Numerical solution of the Euler equations for compressible inviscid fluids, in: F. Angrand, A. Dervieux, J.A. Désidéri, R. Glowinski (Eds.), Numerical Methods for the Euler Equations of Fluid Dynamics, SIAM, 1985, pp. 199–245.
[2] C.H. Tai, Acceleration techniques for explicit Euler codes, Ph.D. Thesis, University of Michigan, 1990.
[3] B. Gustafsson, P. Lötstedt, Analysis of multigrid methods applied to hyperbolic equations, in: Proc. 3rd Copper Mountain Conf. on Multigrid Methods, 1987.
[4] A. Jameson, Solution of the Euler equations for two-dimensional transonic flow by a multigrid method, Appl. Math. Comput. 13 (1983) 327–355.
[5] B. Van Leer, C.H. Tai, K.G. Powell, Design of optimally smoothing multi-stage schemes for the Euler equations, AIAA Paper 89-1923, 1989.
[6] E. Dick, K. Riemslagh, Multi-staging of Jacobi relaxation in multigrid methods for steady Euler equations II, J. Comput. Appl. Math. 59 (3) (1995) 339–348.
[7] K. Hosseini, J.J. Alonso, Optimization of multistage coefficients for explicit multigrid flow solvers, in: 16th AIAA Computational Fluid Dynamics Conference, AIAA, Orlando, FL, 2003, pp. 2003–3705.
[8] R. Haelterman, J. Vierendeels, D. Van Heule, A generalization of the Runge–Kutta iteration, J. Comput. Appl. Math. (2008) doi:10.1016/j.cam.2008.04.021.
[9] L.A. Catalano, H. Deconinck, Two-dimensional optimization of smoothing properties of multi-stage schemes applied to hyperbolic equations, Technical Note 173, Von Karman Institute for Fluid Dynamics, 1990.
[10] K. Hosseini, Practical implementation of robust preconditioners for optimized multistage flow solvers, Ph.D. Thesis, Stanford University, 2005.