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New representations of Padé, Padé-type, and partial Padé approximants

Claude Brezinski^{a,*}, Michela Redivo-Zaglia^b^a Laboratoire Paul Painlevé, UMR CNRS 8524, UFR de Mathématiques, Université des Sciences et Technologies de Lille, 59655-Villeneuve d'Ascq cedex, France^b Università degli Studi di Padova, Dipartimento di Matematica, Via Trieste 63, 35121-Padova, Italy

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ABSTRACT

Padé approximants are rational functions, with a denominator which does not vanish at zero, and whose series expansion match a given series as far as possible. These approximants are usually written under a rational form. In this paper, we will show how to write them also under two different barycentric forms, and under a partial fraction form, depending on free parameters. According to the choice of these parameters, Padé-type approximants can be obtained under a barycentric or a partial fraction form.

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1. Introduction

This paper describes new mathematical expressions for Padé approximants, and some of their variants. The Padé approximant $[p/q]_f$ of a given formal power series f is a rational function, with a numerator of degree at most p and a denominator of degree at most q which does not vanish at zero, and whose power series expansion in ascending powers of the variable matches the series f up to the term of degree $p + q$ inclusively [1,2]. Thus, a Padé approximant can be understood as a rational Hermite interpolant at zero, and it is usually written under the form of a rational fraction or as the convergent of a certain continued fraction. On the other hand, a rational interpolant can be given under the form of a rational fraction, or as the convergent of a continued fraction, or under a barycentric form.

In this paper, we will show that a Padé approximant can also be written under (at least) two different barycentric rational forms which depend on arbitrary parameters. Such an approximant will be called a *barycentric Padé approximant* (in short BPA). According to the choice of these free parameters, Padé-type approximants are also obtained under this form and we call them *barycentric Padé-type approximant* (in short BPTA). Then, we will show how to write a Padé approximant under a partial fraction form, called a *partial fraction Padé approximant* (in short PFPA). The case of partial Padé approximants [3], where some poles and/or zeros are imposed, could be treated similarly.

2. Rational form

Let f be a formal power series

$$f(t) = c_0 + c_1 t + c_2 t^2 + \dots \quad (1)$$

* Corresponding author. Tel.: +33 0 142465519.

E-mail addresses: Claude.Brezinski@univ-lille1.fr (C. Brezinski), Michela.RedivoZaglia@unipd.it (M. Redivo-Zaglia).<http://dx.doi.org/10.1016/j.cam.2014.07.007>

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We consider the rational function

$$R_{p,q}(t) = \frac{\sum_{i=0}^p a_i t^i}{\sum_{i=0}^q b_i t^i}. \quad (2)$$

If the b_i 's are arbitrarily chosen (with $b_0 \neq 0$), and if the a_i 's are computed by

$$\left. \begin{aligned} a_0 &= c_0 b_0 \\ a_1 &= c_1 b_0 + c_0 b_1 \\ &\vdots \\ a_p &= c_p b_0 + c_{p-1} b_1 + \cdots + c_{p-q} b_q \end{aligned} \right\} \quad (3)$$

with the convention that $c_i = 0$ for $i < 0$, then $R_{p,q}$ is the *Padé-type approximant* of f [4], it is denoted by $(p/q)_f$, and it holds

$$(p/q)_f(t) - f(t) = \mathcal{O}(t^{p+1}).$$

This *accuracy-through-order condition* means that the first $p + 1$ coefficients of the power series expansion of $(p/q)_f$ in ascending powers of the variable t match those of the series f .

Moreover, if the b_i 's are taken as the solution of the system

$$\left. \begin{aligned} 0 &= c_{p+1} b_0 + c_p b_1 + \cdots + c_{p-q+1} b_q \\ &\vdots \\ 0 &= c_{p+q} b_0 + c_{p+q-1} b_1 + \cdots + c_p b_q, \end{aligned} \right\} \quad (4)$$

with $b_0 = 1$ (a rational function is defined up to a multiplying factor), then $R_{p,q}$ is the *Padé approximant* of f [1,2], it is denoted by $[p/q]_f$, and it holds

$$[p/q]_f(t) - f(t) = \mathcal{O}(t^{p+q+1}).$$

Thus, the first $p + q + 1$ coefficients of the series expansion of $[p/q]_f$ are identical to those of f . Moreover, we have

$$[p/q]_f(t) = \frac{\begin{vmatrix} t^q f_{p-q}(t) & t^{q-1} f_{p-q+1}(t) & \cdots & f_p(t) \\ c_{p-q+1} & c_{p-q+2} & \cdots & c_{p+1} \\ \vdots & \vdots & & \vdots \\ c_p & c_{p+1} & \cdots & c_{p+q} \end{vmatrix}}{\begin{vmatrix} t^q & t^{q-1} & \cdots & 1 \\ c_{p-q+1} & c_{p-q+2} & \cdots & c_{p+1} \\ \vdots & \vdots & & \vdots \\ c_p & c_{p+1} & \cdots & c_{p+q} \end{vmatrix}},$$

with $f_n(t) = c_0 + \cdots + c_n t^n$, the n th partial sum of the series f (f_n is identically zero for $n < 0$).

In the case of a *partial Padé approximant*, a part of its numerator and/or its denominator is arbitrarily chosen, and the remaining part is taken so that its power series expansion matches f as far as possible [3].

3. Barycentric forms (BPA)

In this section, we consider rational functions written under two different barycentric forms.

$$\text{Form 1: } \tilde{R}_{p,q}(t) = \frac{\sum_{i=0}^p \frac{\tilde{a}_i}{\tilde{p}_i - t}}{\sum_{i=0}^q \frac{\tilde{b}_i}{\tilde{z}_i - t}} \quad (5)$$

or

$$\text{Form 2: } \hat{R}_{p,q}(t) = \frac{\sum_{i=0}^p \frac{\hat{a}_i}{1 - \hat{p}_i t}}{\sum_{i=0}^q \frac{\hat{b}_i}{1 - \hat{z}_i t}}, \quad (6)$$

where the \tilde{p}_i 's, the \tilde{z}_i 's, the \hat{p}_i 's, and the \hat{z}_i 's are given points in the complex plane. We assume that all the \tilde{p}_i 's are distinct, and also all the \tilde{z}_i 's, all the \hat{p}_i 's, and all the \hat{z}_i 's.

Obviously, the forms (5) and (6) can be deduced one from each other by setting $\tilde{a}_i = \hat{a}_i / \hat{p}_i$, $\tilde{b}_i = \hat{b}_i / \hat{z}_i$, $\tilde{p}_i = 1 / \hat{p}_i$, and $\tilde{z}_i = 1 / \hat{z}_i$, for all points different from zero. In the sequel, when it is not necessary to distinguish between the two forms and when it is possible to treat them simultaneously, any of them will be simply denoted by $R_{p,q}$, and the parameters by a_i , b_i ,

p_i , and z_i respectively. It must be noticed that, since a rational function is determined apart a common multiplying factor in its numerator and its denominator, there are only $p + q + 1$ independent variables among the a_i 's and the b_i 's, and not $p + q + 2$.

In both cases, we want to determine the coefficients a_i and b_i such that

$$R_{p,q}(t) - f(t) = \mathcal{O}(t^{p+q+1}). \quad (7)$$

Due to this property, and although $R_{p,q}$ is not always identical to the Padé approximant $[p/q]_f$ of the series f as we will see below, such a rational function will be called a *barycentric Padé approximant* and denoted BPA .

Before explaining how to compute the coefficients of such an approximant, let us begin by some important remarks:

1. It is easy to see that, for (5), the \tilde{p}_i 's are poles of $\tilde{R}_{p,q}$ and the \tilde{z}_i 's are zeros of it while, for (6), it is the $1/\tilde{p}_i$'s and the $1/\tilde{z}_i$'s which play these roles. Therefore, if some poles and zeros of f are known, they can be introduced into the construction of the approximant as in the case of partial Padé approximants [3]. Obviously, when some of these poles and zeros (or all of them) are fictitious, they must be chosen outside the domain of the complex plane where a good approximation is wanted. For the form (5), we will assume that $\forall i, \tilde{p}_i \neq 0, \tilde{z}_i \neq 0$. The reason for this condition will be clearly seen in Section 3.1.1. If, in (6), some \tilde{p}_i 's and/or some \tilde{z}_i 's are zero, then the degree of the numerator and/or the degree of the denominator reduces accordingly.
2. After reducing the sum in the numerator of (5) to its common denominator and also the sum in the denominator, $\tilde{R}_{p,q}$ becomes

$$\tilde{R}_{p,q}(t) = \frac{\tilde{N}_p(t) \prod_{i=0}^q (\tilde{z}_i - t)}{\tilde{D}_q(t) \prod_{i=0}^p (\tilde{p}_i - t)},$$

where \tilde{N}_p is a polynomial of degree p and \tilde{D}_q a polynomial of degree q .

If $\forall i, \tilde{p}_i \neq \tilde{z}_i$, then $\tilde{R}_{p,q}$ has a numerator and a denominator both of degree $p + q + 1$ at most. Thus, the order of approximation of $\tilde{R}_{p,q}$ seems to be one less than the order of approximation of the Padé-type approximant with the same degrees [4]. However, the ratio $\tilde{N}_p(t)/\tilde{D}_q(t)$ is determined apart a common multiplying factor in \tilde{N}_p and \tilde{D}_q and, thus, $\tilde{R}_{p,q}$ is, in fact, the partial Padé approximant of f [3] with the \tilde{p}_i 's and the \tilde{z}_i 's as poles and zeros respectively.

Obviously, some (or all) of the \tilde{p}_i 's and the \tilde{z}_i 's can be poles and zeros of f if they are known. Otherwise, they are fictitious as explained above, but they can be selected so that $\tilde{R}_{p,q}$ possesses other interesting properties such as, for example, the preservation of as many moments of f as possible. However, they cannot be chosen after the \tilde{a}_i 's and the \tilde{b}_i 's have been computed since, as we will see below, these coefficients depend on them.

If some of the \tilde{p}_i 's coincide with some of the \tilde{z}_i 's, then a cancellation occurs and it lowers the degrees accordingly. If, when $p < q$, $\tilde{p}_i = \tilde{z}_i$ for $i = 0, \dots, p$, the product in the denominator disappears and the product in the numerator reduces to $\prod_{i=p+1}^q (\tilde{z}_i - t)$. Thus $\tilde{z}_{p+1}, \dots, \tilde{z}_q$ are zeros of $\tilde{R}_{p,q}$. When $q < p$ and $\tilde{p}_i = \tilde{z}_i$ for $i = 0, \dots, q$, it is the product in the numerator which disappears and the product in the denominator reduces to $\prod_{i=q+1}^p (\tilde{p}_i - t)$. Thus $\tilde{p}_{q+1}, \dots, \tilde{p}_p$ are poles of $\tilde{R}_{p,q}$.

If $q = 0$, $\tilde{R}_{p,0}$ becomes the Padé-type approximant $(p/p + 1)_f$ of the series f .

Similar remarks hold for (6).

3. For both forms, $\forall i, p_i = z_i$ and $p = q$, then $R_{p,p}$ has a numerator and a denominator both of degree p at most. Thus, thanks to the condition (7), $R_{p,p}$ is the usual Padé approximant $[p/p]_f$ of f , and it is such that $R_{p,p}(t) - f(t) = \mathcal{O}(t^{2p+1})$. Due to its uniqueness, this approximant is, in theory, independent of the choice of the p_i 's. However, in practice, the choice of the p_i 's can influence the stability of the approximant, an important issue yet to be studied.

Barycentric Padé approximants (which, in this case, are true Padé approximants) with arbitrary degrees in the numerator and in the denominator can be constructed as follows. Let us write f as

$$f(t) = c_0 + \dots + c_{n-1}t^{n-1} + t^n f^n(t) \quad \text{with } f^n(t) = c_n + c_{n+1}t + \dots$$

Let $R_{p,p}(t)$ be the barycentric Padé approximant of the series f^n constructed as above with $\forall i, p_i = z_i$. Thus, it is the unique Padé approximant $[p/p]_{f^n}$ of f^n , and the approximant

$$R_{n+p,p}(t) = c_0 + \dots + c_{n-1}t^{n-1} + t^n R_{p,p}(t)$$

satisfies

$$R_{n+p,p}(t) - f(t) = \mathcal{O}(t^{n+2p+1}).$$

Thus it is identical to the Padé approximant $[n + p/p]_f$ independently of the choice of the p_i 's. Similarly, write f as

$$f(t) = t^{-n} f^{-n}(t) \quad \text{with } f^{-n}(t) = 0 + 0t + \dots + 0t^{n-1} + c_0t^n + c_1t^{n+1} + \dots$$

The approximant

$$R_{p,p+n}(t) = t^{-n} R_{p+n,p+n}(t),$$

where $R_{p+n,p+n}$ is the barycentric Padé approximant of the series f^{-n} , satisfies

$$R_{p,n+p}(t) - f(t) = \mathcal{O}(t^{n+2p+1}).$$

Thus it is identical to the Padé approximant $[p/n + p]_f$ independently of the choice of the p_i 's.

4. Let us remind that if, in (5), $p = q$, $\forall i$, $\tilde{p}_i = \tilde{z}_i$ and $\tilde{a}_i = w_i f(\tilde{p}_i)$, $\tilde{b}_i = w_i \neq 0$, then $\tilde{R}_{p,p}(\tilde{p}_i) = f(\tilde{p}_i)$ independently of the choice of the w_i [5]. Thus, the w_i 's can be chosen so that, in addition, $\tilde{R}_{p,p}$ matches the series f as far as possible as proposed in [6].

3.1. Computation of the coefficients

Since the accuracy-through-order condition (7) contains $p + q + 1$ relations while $R_{p,q}$ has $p + q + 2$ coefficients to be determined, an additional condition needs to be imposed. It is the so-called *normalization condition*. Because $R_{p,q}$ approximates f around zero, its denominator should not vanish at this point. Thus, since a rational function is determined apart a common multiplying factor in its numerator and its denominator, it is convenient to choose the normalization condition for (5) and (6) as

$$\sum_{i=0}^q \frac{\tilde{b}_i}{\tilde{z}_i} = 1 \quad \text{for (5), and} \quad \sum_{i=0}^q \hat{b}_i = 1 \quad \text{for (6).} \quad (8)$$

3.1.1. Form 1

For the form (5), having chosen the \tilde{p}_i 's and the \tilde{z}_i 's, the accuracy-through-order condition (7) can be written

$$\sum_{i=0}^p \frac{\tilde{a}_i / \tilde{p}_i}{1 - t / \tilde{p}_i} = (c_0 + c_1 t + c_2 t^2 + \dots) \sum_{i=0}^q \frac{\tilde{b}_i / \tilde{z}_i}{1 - t / \tilde{z}_i}.$$

But, we have $1/(1 - t/\tilde{p}_i) = 1 + t/\tilde{p}_i + t^2/\tilde{p}_i^2 + \dots$, and a similar expansion for $1/(1 - t/\tilde{z}_i)$. Thus, after replacement, the preceding relation becomes

$$\sum_{i=0}^p \frac{\tilde{a}_i}{\tilde{p}_i} \left(1 + \frac{t}{\tilde{p}_i} + \frac{t^2}{\tilde{p}_i^2} + \dots \right) = (c_0 + c_1 t + c_2 t^2 + \dots) \sum_{i=0}^q \frac{\tilde{b}_i}{\tilde{z}_i} \left(1 + \frac{t}{\tilde{z}_i} + \frac{t^2}{\tilde{z}_i^2} + \dots \right).$$

Identifying the coefficients of the identical powers of t in both sides, and taking into account the normalization condition lead to the system of equations for computing the coefficients \tilde{a}_i and b_i

$$\left. \begin{aligned} \sum_{i=0}^q \frac{\tilde{b}_i}{\tilde{z}_i} &= 1 \\ \sum_{i=0}^p \frac{\tilde{a}_i}{\tilde{p}_i^{k+1}} - \sum_{i=0}^q \tilde{b}_i \sum_{j=0}^{k-1} \frac{c_j}{\tilde{z}_i^{k-j+1}} &= c_k, \quad k = 0, \dots, p+q. \end{aligned} \right\} \quad (9)$$

Obviously, the sum on j is empty for $k = 0$.

Thus, the coefficients of the series expansion of $\tilde{R}_{p,q}(t) = \tilde{d}_0 + \tilde{d}_1 t + \tilde{d}_2 t^2 + \dots$ are given by

$$\begin{aligned} \tilde{d}_k &= c_k, \quad k = 0, \dots, p+q, \\ \tilde{d}_k &= \sum_{i=0}^p \frac{\tilde{a}_i}{\tilde{p}_i^{k+1}} - \sum_{i=0}^q \tilde{b}_i \sum_{j=0}^{k-1} \frac{\tilde{d}_j}{\tilde{z}_i^{k-j+1}}, \quad k = p+q+1, p+q+2, \dots \end{aligned}$$

3.1.2. Form 2

For the form (6), the condition (7) gives

$$\sum_{i=0}^p \hat{a}_i (1 + \hat{p}_i t + \hat{p}_i^2 t^2 + \dots) = (c_0 + c_1 t + c_2 t^2 + \dots) \sum_{i=0}^q \hat{b}_i (1 + \hat{z}_i t + \hat{z}_i^2 t^2 + \dots).$$

Identifying the coefficients of the identical powers of t in both sides, and taking into account the normalization condition lead to the system of equations for determining the coefficients \hat{a}_i and \hat{b}_i (the sum on j is empty for $k = 0$)

$$\left. \begin{aligned} \sum_{i=0}^q \hat{b}_i &= 1 \\ \sum_{i=0}^p \hat{a}_i \hat{p}_i^k - \sum_{i=0}^q \hat{b}_i \sum_{j=0}^{k-1} c_j \hat{z}_i^{k-j} &= c_k, \quad k = 0, \dots, p+q. \end{aligned} \right\} \quad (10)$$

The coefficients of the series expansion of $\hat{R}_{p,q}(t) = \hat{d}_0 + \hat{d}_1 t + \hat{d}_2 t^2 + \dots$ are given by

$$\begin{aligned} \hat{d}_k &= c_k, \quad k = 0, \dots, p+q, \\ \hat{d}_k &= \sum_{i=0}^p \hat{a}_i \hat{p}_i^k - \sum_{i=0}^q \hat{b}_i \sum_{j=0}^{k-1} \hat{d}_j \hat{z}_i^{k-j}, \quad k = p+q+1, p+q+2, \dots \end{aligned}$$

3.2. Barycentric Padé-type approximants (BPTA)

Consider again the rational functions (5) and (6) and assume now that the coefficients b_i in their respective denominators are arbitrarily chosen. Then, the coefficients of their numerators can be computed by solving the system (11)

$$\sum_{i=0}^p \frac{\tilde{a}_i}{\tilde{p}_i^{k+1}} = \sum_{i=0}^q \tilde{b}_i \sum_{j=0}^{k-1} \frac{c_j}{\tilde{z}_i^{k-j+1}}, \quad k = 0, \dots, p \quad (11)$$

for (5), or the system (12) for (6)

$$\sum_{i=0}^p \hat{a}_i \hat{p}_i^k = \sum_{i=0}^q \hat{b}_i \sum_{j=0}^{k-1} c_j \hat{z}_i^{k-j}, \quad k = 0, \dots, p. \quad (12)$$

In both cases, the rational function $R_{p,q}$ which is obtained satisfies

$$R_{p,q}(t) - f(t) = \mathcal{O}(t^{p+1}),$$

and, thanks to this property, it is called a *barycentric Padé-type approximant* (see [4]) and denoted BPTA.

Similarly, a part of the numerator and/or a part of the denominator can be fixed thus leading to a *barycentric partial Padé-type approximant* in the style of [3].

The b_i 's could be chosen so that $R_{p,q}$ satisfies some additional properties as explained above.

4. Partial fraction form (PFPA)

Let us consider now the rational function

$$R_{k,k+1}(t) = \sum_{i=0}^k \frac{a_i}{1 - p_i t}. \quad (13)$$

It has a denominator of degree $k+1$ and a numerator of degree k . We want to compute the a_i 's and the p_i 's such that this rational function be identical to the Padé approximant $[k/k+1]_f$ of the series f . Such an approximant will be called a *partial fraction Padé approximant* and denoted PFPA. It can be obtained by a slight variation of a method due to the French mathematician and hydraulics engineer Gaspard Clair François Marie Riche, Baron de Prony (Chamelet, 22 July 1755–Asnières-sur-Seine, 29 July 1839) for interpolation by a sum of exponential functions [7]. This method is used in signal analysis and recovery (see, for example, [8,9]). Applied to our case, this variant is as follows (see, for example, [10, pp. 141–142]).

We want to have

$$\sum_{i=0}^k \frac{a_i}{1 - p_i t} = \sum_{i=0}^k a_i (1 + p_i t + p_i^2 t^2 + \dots) = c_0 + c_1 t + c_2 t^2 + \dots,$$

which leads, by identification of the powers of t on both sides, to

$$\sum_{i=0}^k a_i p_i^j = c_j, \quad j = 0, \dots, 2k+1. \quad (14)$$

The denominator of the rational function (13) is

$$Q(t) = (1 - p_0 t) \cdots (1 - p_k t) = b_0 + b_1 t + \cdots + b_{k+1} t^{k+1},$$

where $b_0 = 1$. Let us first compute its coefficients. Multiply the first equation in (14) (that is the equation for $j = 0$) by b_0 , the second one (that is corresponding to $j = 1$) by b_1 , and so on up to the $(k + 2)$ th equation (that is the equation for $j = k + 1$) by b_{k+1} , and sum them up. Begin again the same process starting from the second equation in (14) (that is for $j = 1$) which is multiplied by b_0 , multiply the third equation by b_1 , and so on up to the $(k + 3)$ th equation (that is the equation for $j = k + 2$) by b_{k+1} , and sum them up. Continue the process until all equations in (14) have been used. We finally obtain

$$\sum_{j=0}^{k+1} b_j c_{j+n} = \sum_{j=0}^{k+1} b_j \sum_{i=0}^k a_i p_i^{j+n}, \quad n = 0, \dots, k,$$

which can be written as

$$\sum_{j=0}^{k+1} b_j c_{j+n} = \sum_{i=0}^k a_i p_i^n \sum_{j=0}^{k+1} b_j p_i^j = \sum_{i=0}^k a_i p_i^n Q(p_i) = 0, \quad n = 0, \dots, k.$$

Since $b_0 = 1$, the other coefficients b_1, \dots, b_{k+1} of the polynomial Q are solution of the linear system

$$\sum_{j=1}^{k+1} b_j c_{j+n} = -c_n, \quad n = 0, \dots, k. \quad (15)$$

It must be noticed that, for obtaining the system giving the coefficients of the denominator of the Padé approximant, the numbering of the b_i 's has to be reversed in this system. After solving (15), the zeros p_0, \dots, p_k of Q can be computed (for example, by the QR algorithm as the eigenvalues of the companion matrix of the coefficients of the polynomial Q) and, finally, a_0, \dots, a_k are obtained by solving the linear system consisting in the first $k + 1$ equations in (14). Notice that this system is singular if the p_i 's are not distinct and, in this case, $[k/k + 1]_f$ cannot be written under the form (13) but, possibly, under a partial fraction form involving powers in its denominator. This case arises if the denominator of $[k/k + 1]_f$ has multiple zeros. It is easy to see that the system (15) is identical to the system (4) when $p = k$ and $q = k + 1$, and after reversing the numbering of its coefficients. There exist several procedures for improving the numerical stability of Prony's method [11], and it is even possible to avoid the computation of the coefficients of Q [12].

Let c be the linear functional on the vector space of polynomials defined by

$$c(x^i) = c_i, \quad i = 0, 1, \dots$$

Then, the system (15) can be written

$$c(x^n Q(x)) = 0, \quad n = 0, \dots, k.$$

Thus, Q is the polynomial of degree $k + 1$ belonging to the family of *formal orthogonal polynomials* with respect to c . Such polynomials, introduced by Wynn [13], play an important role in the algebraic and in the analytic theory of Padé approximation [4,2,14]. Via formal orthogonal polynomials, Padé approximants are also related to formal Gaussian quadrature procedures [15], as first showed by Zinn-Justin in the case of Stieltjes series [16]. An interesting reference on the connection between these topics is [17].

Obviously, if the p_i 's are all distincts, then $a_i = -p_i P(1/p_i)/Q'(1/p_i)$ where P is the numerator of $[k/k + 1]_f$.

If the p_i 's are arbitrary distinct points and if the a_i 's are solutions of the first $k + 1$ equations in (14), then $R_{k,k+1}$ is only a Padé-type approximant of f . In this case, the p_i 's could be chosen so that $R_{k,k+1}$ satisfies some additional properties.

Partial fraction Padé approximants can also be written as $\sum_{i=0}^k a_i/(p_i - t)$, and treated similarly. Analogous forms can be derived for Partial Padé approximants [3].

4.1. Numerical examples

Let us now give some numerical examples showing the interest of the barycentric forms. Both forms give similar results as expected.

4.1.1. Example 1

We consider the following function, and its series expansion

$$f(t) = \frac{\tan(\omega t)}{\omega t} = 1 + \frac{1}{3} \omega^2 t^2 + \frac{2}{15} \omega^4 t^4 + \frac{17}{315} \omega^6 t^6 + \frac{62}{2835} \omega^8 t^8 + \dots$$

This function has poles at odd multiples of $\pi/(2\omega)$, and zeros at odd multiples of π/ω , except at 0.

With $\omega = 4$, $p = q = 4$ and taking for the \tilde{p}_i 's $\pm\pi/(2\omega)$, $\pm 3\pi/(2\omega)$, $5\pi/(2\omega)$ and for the \tilde{z}_i 's $\pm\pi/\omega$, $\pm 3\pi/\omega$, $5\pi/\omega$, which are the five first poles and zeros of f respectively, we obtain the results of Fig. 1.

Adding a uniformly distributed random perturbation between $[-0.0001, +0.0001]$ to the c_i 's leads to the results of Fig. 2. For $t \in [-1.5, +1.5]$, the error of the true Padé approximant is in the interval $[3.5528 \times 10^{-5}, 2.8663 \times 10^4]$, and for the barycentric Padé approximant computed either by the system (9) (form 1) or by the system (10) (form 2) it belongs to $[4.8921 \times 10^{-8}, 1.8136 \times 10^3]$.

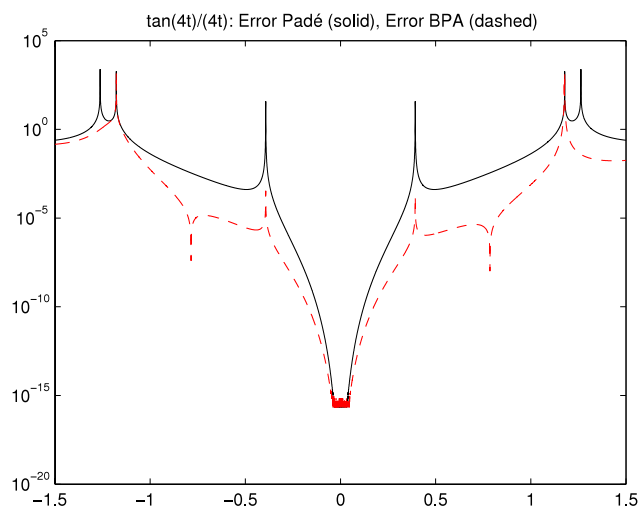


Fig. 1. Error of Padé (solid) and barycentric Padé (dashed) approximants for $\tan(4t)/(4t)$.

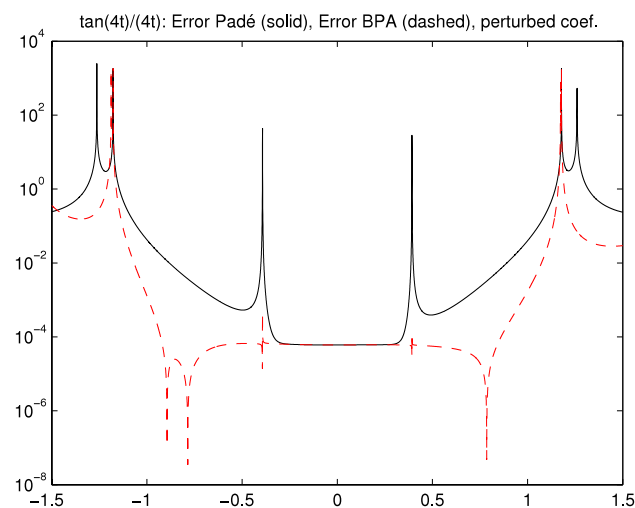


Fig. 2. Error of Padé (solid) and barycentric Padé (dashed) approximants for $\tan(4t)/(4t)$ with perturbed coefficients.

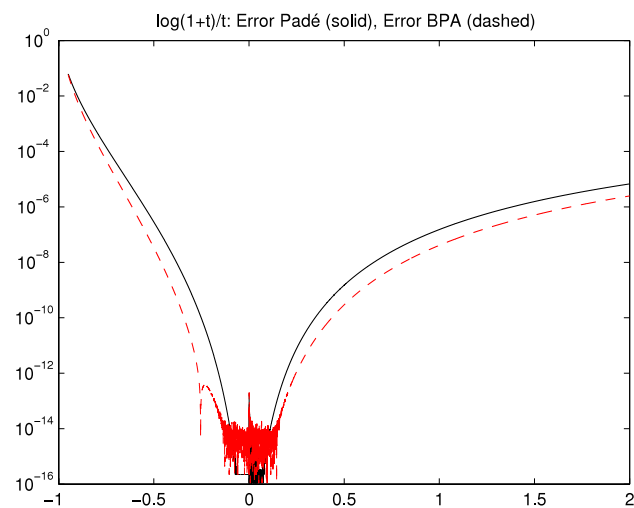


Fig. 3. Error of Padé (solid) and barycentric Padé (dashed) approximants for $\log(1+t)/t$, $p_i \in [-10, -1]$, $z_i \in [-10, -2]$.

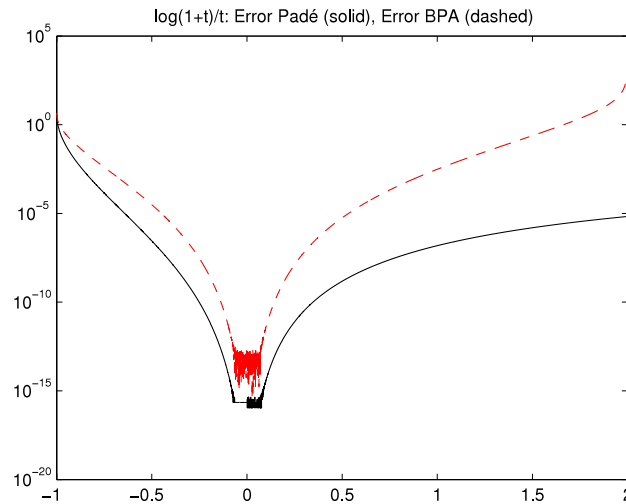


Fig. 4. Error of Padé (solid) and barycentric Padé (dashed) approximants for $\log(1+t)/t$, $p_i \in [2, 5]$, $z_i \in [-3, -2]$.

4.1.2. Example 2

We consider the series

$$f(t) = \frac{\log(1+t)}{t} = 1 - \frac{t}{2} + \frac{t^2}{3} - \frac{t^3}{4} + \dots$$

which converges in the unit disk and on the unit circle except at the point -1 since there is a cut from -1 to $-\infty$. For $p = q = 4$, the p_i 's equidistant in $[-10, -1]$ and the z_i 's equidistant in $[-10, -2]$, we obtain the results of Fig. 3. The numerical results highly depend on these choices as can be seen by comparing Fig. 3, corresponding to $p_i \in [-10, -1]$, $z_i \in [-10, -2]$, and Fig. 4, for which $p_i \in [2, 5]$, $z_i \in [-3, -2]$.

5. Conclusion

This paper is an addition to the vast literature on Padé approximation, and is only an introduction to these barycentric and partial fraction forms in order to show how to compute their coefficients. Their main features are discussed and some numerical experiments show the interest of these new representations. However, the main problem which remains to be studied is the influence of the choice of the free parameters involved in their construction, a choice related to the important issues of their robustness [18] and their stability [19].

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