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An interior point method for nonlinear optimization with a quasi-tangential subproblem

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ABSTRACT

We present an interior point method for nonlinear programming in this paper. This method follows Byrd and Omojokun's idea of step decomposition, which splits the trial step into a normal step and a tangential step. The method employs a new idea of quasi-tangential subproblem, which is used to generate a tangential step that does not lie strictly on the tangent space of the constraints. Quasi-tangential subproblem is finally formulated into an unconstrained quadratic problem by penalizing the constraints. This method is different and maybe simpler than similar ideas, for example, the relaxed tangential step in trust funnel methods (Gould and Toint, 2010; Curtis, et al., 2017). Also, our method does not need to compute a base of the null space. A line search trust-funnel-like strategy is used to globalize the algorithm. Global convergence theorem is presented and applications to mathematical programs with equilibrium constraints are given.

1. Introduction

In this paper, we describe and analyze an interior point method for nonlinear constrained optimization problem

$$\begin{aligned} &\min f(x) \\ &\text{s.t. } c(x) = 0, \\ &\quad x \geq 0, \end{aligned} \tag{1}$$

where $f : R^n \to R$, $c : R^n \to R^m$ are smooth functions. Clearly, problems with general nonlinear inequality constraints can be equivalently reformulated into this form by using slack variables.

Interior point methods are efficient in treating inequality constraints. They have been intensively studied in the last three decades; see [1–5]. The classical interior point strategy obtains a solution by approximately solving a series of barrier problems of the form

$$\begin{aligned} &\min \varphi^\mu(x) \overset{\text{def}}{=} f(x) - \mu \sum_{i=1}^{n} \ln x^{(i)} \\ &\text{s.t. } c(x) = 0, \end{aligned} \tag{2}$$

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where μ is a barrier parameter which is decreasing and converges to 0. For this solution, some algorithms use Newton (or quasi-Newton) methods [6,7] while some involve SQP (or trust-region) mechanisms. The Newton-like methods get search direction from solving a Newton equation of the perturbed optimality system. Plenty of researches on methods following this algorithmic philosophy have proven their robustness and efficiency [6–9]. Also, some SQP or trust-region based interior point methods have been proposed, many of which have provided promising numerical results [2,4,10–12].

Of all the methods, the step-decomposition approaches, which integrate ideas of interior point methods and Byrd–Omojokun's trust-region idea [13,14], have got plenty of attentions because of the always consistent subproblems and the capacity of infeasibility detection [12]. Our approach follows this framework with a major character that it employs a quasi-tangential subproblem which generates a step not strictly lying on the null space of ∇c_k^T. The key point of the quasi-tangential subproblem is to convert the tangential subproblem into an unconstrained quadratic programming by penalizing the null space constraints. A quasi-tangential step satisfying some necessary conditions is obtained if choosing sufficiently small penalty factor. This strategy also circumvents the cost of computing a base for the null space, which is important in solving tangential subproblem in Byrd–Omojokun-like algorithms.

The idea of quasi-tangential subproblem has some similarity with some methods. Some methods with inexact step computation, in interior points, trust-region or SQP scheme, adopt similar ideas. Curtis et al. [4] used an inexact Newton technique in their interior point method where an inexact tangential step is generated from an inexact Newton equation for the tangential subproblem. Heinkenschloss and Ridzal [15] obtained inexact tangential steps by computing approximate projections of vectors onto the tangent space of the linearized constraints.

A more similar idea is the relaxed tangential step, which was first introduced by Gould and Toint in [16]. In [16], the authors specified conditions that a relaxed tangential step should satisfy. Recently, Curtis, Gould, Robinson and Toint [17] generalized this concept and defined the concepts of relaxed SQP tangential step and very relaxed SQP tangential step in the context of interior point trust funnel algorithm for nonlinear optimization. To achieve the very relaxed (or very relaxed) tangential step, a complex trust-region strategy is used to control the size of normal and tangential steps. The main difference between our method and the relaxed tangential step method is that our method controls the degree of violation of null space constraints directly, while relaxed tangential step controls the size of tangential step by a trust-region.

Another character of our algorithm is that we use a trust-funnel-like strategy to balance the improvements on feasibility and optimality. Trust funnel method was introduced by Gould and Toint in [16] and extended by Curtis et al. [17]. And similar ideas can be found in [18–22] etc. Our ideas mainly differ from these algorithms in the way of computing trial steps and the switch conditions between so called f-iteration and h-iteration. And the mechanism of our method seems simpler than these methods.

The balance of this paper is organized as follows. In the next section, we describe the design of the algorithm in detail. In Section 3, we show that the proposed algorithm is well-defined while the global convergence is shown in Section 4. In Section 5, we report preliminary numerical results. Finally, some further remark is given in Section 6.

Notations: We use $\|\cdot\|$ to denote the Euclidean norm $\|\cdot\|_2$. Subscript k refers to iteration indices and superscript (i) is the ith component of a vector.

2. Algorithm description

We motivate the main algorithm in this section. We first introduce the main framework of primal–dual interior point methods, then deduce the method with quasi-tangential subproblem from some ideas of trust-region methods.

2.1. The primal–dual barrier method

The Karush–Kuhn–Tucker (KKT) conditions of the barrier problem (2) cause the following nonlinear system

$$\begin{pmatrix} \nabla f(x) + \nabla c(x)\lambda - z \\ -\mu X^{-1}e + z \\ c(x) \end{pmatrix} = 0 \tag{3}$$

where $\lambda \in R^m$ and $0 \le z \in R^n$ are Lagrangian multipliers and $X = \mathrm{diag}(x_1, x_2, \ldots, x_n)$. Multiplying the second row of (3) by X, we obtain the system

$$\begin{pmatrix} \nabla f(x) + \nabla c(x)\lambda - z \\ Xz - \mu e \\ c(x) \end{pmatrix} = 0. \tag{4}$$

This may be viewed as a perturbed KKT system for the original problem (1). The optimality error for the barrier problem is defined, based on (4) as [9]

$$E_\mu(x, \lambda, z); = \left\{ \frac{\|\nabla f(x) + \nabla c(x)\lambda - z\|}{s_d}, \frac{\|Xz - \mu e\|}{s_c}, \|c(x)\| \right\}$$

with scaling parameters $s_d, s_c \ge 1$ defined as

$$s_d = \max\left\{ s_{\max}, \frac{\|\lambda\|_1 + \|z\|_1}{m+n} \right\} / s_{\max}, \quad s_c = \max\left\{ s_{\max}, \frac{\|z\|_1}{n} \right\} / s_{\max},$$

where $s_{\max} > 1$. Correspondingly, we use $E_0(x, \lambda, z)$ to measure the optimality error for the original problem (1). A typical algorithmic framework of barrier methods for (1) is as follows.

ALGORITHM 1: OUT LOOP

Step 0 Choose an initial value for the barrier parameter $\mu_0 > 0$, and select the parameter $\kappa_\epsilon > 0$, and the stop tolerance ϵ. Choose the starting point x_0, λ_0, z_0. Set $j := 0$.

Step 1 If $E_0(x_j, \lambda_j, z_j) \le \epsilon$, stop.

Step 2 Apply an SQP method, starting from x_j, to find an approximate solution x_{j+1} for the barrier problem, with Lagrange multipliers λ_{j+1}, z_{j+1} satisfying $E_{\mu_j}(x_{j+1}, \lambda_{j+1}, z_{j+1}) \le \kappa_\epsilon \mu_j$.

Step 3 Choose $\mu_{j+1} \in (0, \mu_j)$, set $j := j + 1$ and go to Step 1.

Remark. To achieve fast local convergence algorithm, the barrier parameter μ needs to be updated carefully [1,6,23]. We will follow the approach suggested by Byrd, Liu and Nocedal [23] and will specify the details in Section 5.

The primary work of Algorithm 1 lies clearly in Step 2, where an approximate solution of (2) is found. Primal–dual interior point methods apply Newton's method to the perturbed KKT system (4) and modify step-size so that the inequality $(x, z) \ge 0$ is satisfied strictly. A primal–dual linear system is given as

$$\begin{pmatrix} H & \nabla c(x) & -I \\ Z & 0 & X \\ \nabla c(x)^T & 0 & 0 \end{pmatrix} \begin{pmatrix} d_x \\ d_\lambda \\ d_z \end{pmatrix} = - \begin{pmatrix} \nabla f(x) + \nabla c(x)\lambda - z \\ Xz - \mu e \\ c(x) \end{pmatrix},$$

where H is the Hessian of the Lagrangian function and $Z = \mathrm{diag}(z_1, \ldots, z_n)$. Eliminating d_z by

$$d_z = -z + \mu X^{-1}e - X^{-1}Zd_x, \tag{5}$$

and defining $\lambda^+ = \lambda + d_\lambda$, we have the iteration

$$\begin{pmatrix} H + X^{-1}Z & \nabla c(x) \\ \nabla c(x)^T & 0 \end{pmatrix} \begin{pmatrix} d_x \\ \lambda^+ \end{pmatrix} = - \begin{pmatrix} \nabla \varphi^\mu(x) \\ c(x) \end{pmatrix}.$$

It is easy to see that the step generated by this system coincides with the solution of the following primal–dual QP subproblem

$$\begin{aligned} &\min \nabla \varphi^\mu(x)^T d + \frac{1}{2} d^T \widetilde{W} d \\ &\text{s.t. } c(x) + \nabla c(x)^T d = 0, \end{aligned} \tag{6}$$

where $\widetilde{W} = H + X^{-1}Z$. Step computation of our algorithm is based on this model.

2.2. Normal subproblem and quasi-tangential subproblem

A trust-region constraint

$$\|d\| \le \Delta \tag{7}$$

is always introduced in (6) so as to obtain global convergence and to allow for the case where $\widetilde{W}$ is not positive definite on the null space of $\nabla c(x)^T$. It is well known [24] that (6) with (7) can be inconsistent when the trust-region Δ is so small that even the shortest step d that satisfies the constraints in (6) is excluded by the trust-region. One of the common used strategies to make the constraints consistent is the step decomposition method of Byrd [13] and Omojokum [14] in which the total step of the algorithm is split into normal and tangential steps. The normal step v is a move toward the satisfaction of the constraints, and is defined as the solution of the normal subproblem

$$\begin{aligned} &\min \frac{1}{2} \|c(x) + \nabla c(x)^T v\|^2 \\ &\text{s.t. } \|v\| \le \xi \Delta, \end{aligned} \tag{8}$$

with $\xi \in (0, 1)$. The tangential step t aims to reduce $\varphi^{\mu}(x)$ on the null space of $\nabla c(x)^T$, and is generated by solving the tangential subproblem

$$\min_t \, (\nabla \varphi^{\mu}(x) + \widetilde{W} v)^T t + \frac{1}{2} t^T \widetilde{W} t$$
$$\text{s.t. } \nabla c(x)^T t = 0,$$
$$\|v + t\| \le \Delta.$$

Note that (8) is an trust-region model for the nonlinear least square problem

$$\min \ \frac{1}{2} h(x)^2,$$

where

$$h(x) = \|c(x)\|.$$

As a classical but still popular method for this problem, Levenberg–Marquardt methods [25–27] compute a search direction by solving the following linear system

$$(\nabla c_k \nabla c_k^T + \eta_k I) v = -\nabla c_k c_k, \tag{9}$$

where η_k is a positive parameter. Sometimes it is considered to be the progenitor of the trust-region approach for general unconstrained optimization [28]. The following lemma from Nocedal and Wright [28] shows connection between the solutions of (8) and (9).

Lemma 2.1. *The vector v_k^{LM} is a solution of the trust-region subproblem (8) if and only if v_k^{LM} is feasible and there is a scalar $\eta > 0$ such that*

$$(\nabla c_k \nabla c_k^T + \eta I) v_k^{LM} = -\nabla c_k c_k,$$
$$\eta(\Delta - \|v_k^{LM}\|) = 0.$$

Levenberg–Marquardt method is globalized by a line search strategy, which is less costly in computation than trust-region method. Furthermore, researchers have shown that if $\eta_k = \|c_k\|^{\delta}$, $\delta \in [1, 2]$, fast local convergence can be achieved without nonsingularity assumption, see Yamashita and Fukushima [29], Fan and Yuan [30], Zhang [31], Kimiaei [32] etc. These facts motivate us to compute the normal step by the following hybrid method:

$$v_k = \begin{cases} \arg\min \ \|c_k + \nabla c_k^T v\|^2 + \|c_k\|^{\delta} \|v\|^2, & \text{if } \nabla c_k \text{ is rank deficient,} \\ \arg\min \|c_k + \nabla c_k^T v\|^2, & \text{otherwise,} \end{cases} \tag{10}$$

where $\delta \in (1, 2)$ is a fixed constant.

In tangential subproblem, the tangential constraint $\nabla c(x)^T t = 0$ is used to prevent t from jeopardizing the infeasibility reduction that normal step v_k just obtains. We note that this effect can be achieved by only requiring

$$\|\nabla c_k^T t\| \le \xi_k \tag{11}$$

for an appropriate positive scalar ξ_k. Similar observation was made by Gould and Toint in [16]. On the other hand, it is quite likely that the concession on linearized feasibility that (11) makes can lead to a considerable improvement in objective value over a reasonable step. However, replacing $\nabla c_k^T t = 0$ by (11) will increase the difficulty of solving tangential subproblem. But this difficulty can be easily circumvented by adding the item $\frac{1}{2\nu_k} t^T \nabla c_k \nabla c_k^T t$ to the quadratic objective function where $\nu_k > 0$ acts as a penalty factor. Hereby, we get the quasi-tangential subproblem

$$\min_t \ (\nabla \varphi_k^{\mu} + \widetilde{W}_k v_k)^T t + \frac{1}{2} t^T \left(\widetilde{W}_k + \frac{1}{\nu_k} \nabla c_k \nabla c_k^T + \zeta_k I \right) t. \tag{12}$$

where $\zeta_k \ge 0$ is a regularization parameter to make the matrix $(\widetilde{W}_k + \frac{1}{\nu_k} \nabla c_k \nabla c_k^T + \zeta_k I)$ sufficiently positive definite.

2.3. Trust-funnel-like approach for the barrier problem

We use a trust-funnel-like [16,21] line search method for the approximate solution of the barrier problem (2) with a given barrier parameter $\mu = \mu_j$. Given the iterate x_k and the corresponding Lagrange multiplier λ_k, z_k, a trust-funnel-like method for (2) pursuits the solution iteratively in a progressively stricter trust funnel defined by

$$h(x) \le h_k^{\max},$$

where and $h_k^{\max}$ is a non-increasing limit on infeasibility.

Now we specify the choices of ν_k. First, the parameter ν_k should be a positive scalar that ensures the positive definiteness of the symmetric matrix $\widetilde{W}_k + \frac{1}{\nu_k} \nabla c_k \nabla c_k^T$. If this positive definiteness is not satisfied, one can regularized it by term $\zeta_k I$

such that

$$d^T(\widetilde{W}_k + \frac{1}{\nu_k}\nabla c_k \nabla c_k^T + \zeta_k I)d \geq b_1\|d\|^2 \tag{13}$$

for any $d \in R^n$ with $b_1 > 0$. Second, ν_k controls how inexactly that the quasi-tangential step t_k, which solves (12), lies on the null space of ∇c_k^T. If

$$-(\nabla \varphi_k^\mu)^T(v_k + t_k) \geq \sigma_1 h_k^{\sigma_2}, \tag{14}$$

where σ_1, σ_2 are positive constants, then t_k should satisfy

$$\|\nabla c_k^T t_k\| \leq \kappa_1(h_k^{\max} - \|c_k + \nabla c_k^T v_k\|), \tag{15}$$

where $\kappa_1 \in (0, 1)$. Otherwise, t_k is required to satisfy

$$\|\nabla c_k^T t_k\| \leq \kappa_2(h_k - \|c_k + \nabla c_k^T v_k\|), \tag{16}$$

where $\kappa_2 \in (0, 1)$. As we will show later, such a ν_k is guaranteed unless an infeasible stationary point, which is a stationary point of the problem

$$\min_{x \geq 0} \frac{1}{2}h(x)^2 \tag{17}$$

is found. This situation always implies that the problem is locally infeasible. Then, the algorithm stops and reports infeasible stationarity.

We describe algorithm for updating ν_k in Algorithm 2.

ALOGRITHM 2: UPDATING ν_k

Step 0 Set $\nu_k := \max\{\min\{\nu_{k-1}, h_k\}, \underline{\nu}\}$ and $\zeta_k = 0$, where $\underline{\nu}$ is a preset positive parameter.

Step 1 If $\widetilde{W}_k + \frac{1}{\nu_k}\nabla c_k \nabla c_k^T + \zeta_k I$ is not positive definite, then choose $\zeta_k > 0$ such that (13) holds and set $\underline{\nu} := \underline{\nu}/2$.

Step 2 Compute t_k.

Step 3 If t_k satisfies (14), then stop if (15) is also satisfied. Otherwise, stop if (16) is satisfied.

Step 4 Set $\nu_k := \nu_k/2$, go to Step 2.

Now we have got the search direction $d_k = v_k + t_k$. The multiplier vector corresponding to the next iterate is estimated by

$$\lambda_{k+1} = \frac{1}{\nu_k}\nabla c_k^T t_k. \tag{18}$$

The line search along d_k is performed by first determining $\alpha_k^{\max}$ which is the maximal α satisfying the fraction-to-the-boundary rule

$$x_k^{(i)} + \alpha d_k^{(i)} \geq (1-\tau)x_k^{(i)},\ i = 1, 2, \ldots, n \tag{19}$$

where $\tau \in (0, 1)$ is a parameter close to 1 with respect to the iteration j of the out loop.

If (14) holds, then we call the kth iteration an f-iteration and x_k an $f-$iterate because it is quite reasonable to expect considerable reduction on objective function in this case. Hence, we require the step-size $\alpha \in (0, \alpha_k^{\max}]$ to satisfy

$$\varphi_k^\mu - \varphi^\mu(x_k + \alpha d_k) \geq -\rho\alpha(\nabla\varphi_k^\mu)^T d_k. \tag{20}$$

The requirement for the feasibility on the new iterate is relatively rough. We require

$$h(x_k + \alpha d_k) \leq h_k^{\max}. \tag{21}$$

In the case where (14) fails, which indicates that the infeasibility is significant while sufficient reduction of objective function is not ensured, we call the kth iteration an h-iteration and x_k an h-iterate. We search $\alpha \in (0, \alpha_k^{\max}]$ satisfying

$$h(x_k + \alpha d_k) \leq (1-\rho)h_k + \rho\|c_k + \alpha\nabla c_k^T d_k\| \tag{22}$$

in this case.

From (5), we obtain the estimate of the new dual variables

$$z_{k+1} = \mu X_k^{-1} e - X_k^{-1} Z_k d_k. \tag{23}$$

For the convergence proof, we require the "primal–dual barrier Hessian" $X_k^{-1}Z_k$ do not deviate arbitrarily much from the "primal Hessian" μX_k^{-2}. To do this, we reset [9]

$$z_{k+1}^{(i)} := \max\left\{ \min\left\{ z_{k+1}^{(i)}, \frac{\kappa_\sigma \mu}{x_{k+1}^{(i)}} \right\}, \frac{\mu}{\kappa_\sigma x_{k+1}^{(i)}} \right\}, \quad i = 1, 2, \ldots, n \tag{24}$$

for some fixed $\kappa_\sigma > 1$ after each step. Such safeguards not only benefit the convergence analysis but also work satisfactorily in practice [9,33,34].

After obtaining a new iterate, the limit on feasibility of the new iterate is set as [16]

$$h_{k+1}^{\max} = \begin{cases} h_k^{\max} & \text{if } x_k \text{ is an } f\text{-iterate}, \\ \max\{\kappa_h h_k^{\max}, \bar{\kappa}_h h_k + (1-\bar{\kappa}_h) h_{k+1}\}, & \text{if } x_k \text{ is an } h\text{-iterate}. \end{cases} \tag{25}$$

Now, we are ready to summarize all the details of this line search trust-funnel-like approach for the barrier problem. Suppose that the current outer loop iteration is j and that the parameter μ, τ are available, and that the last iteration finished with the primal–dual vector (x_j, λ_j, z_j), where $(x_j, z_j) > 0$. The detailed description is given in Algorithm 3.

Algorithm 3: Inner Loop

Step 0 Choose $\delta \in (1, 2]$, $\sigma_1, \sigma_2 > 0$, $\kappa_1, \kappa_2 \in (0, 1)$, $\rho \in (0, 1)$, $\kappa_\sigma > 1$, $\underline{\nu} > 0$ and $\nu_0 \geq \underline{\nu}$. Initialize the primal–dual entry as $(x_0, \lambda_0, z_0) = (x_j, \lambda_j, z_j)$. Let

$$h_0^{\max} = \max\{h_0, \min(10, E_\mu(x_0, \lambda_0, z_0))\}.$$

Set $\mu = \mu_j$ and $k := 0$.

Step 1 If $E_\mu(x_k, \lambda_k, z_k) \leq \kappa_\epsilon \mu$, return.

Step 2 Compute the normal step v_k by solving (10). If $v_k = 0$ and $h_k > 0$, stop.

Step 3 Use Algorithm 2 to update ν_k and compute the tangential step t_k.

Step 4 Let $d_k = v_k + t_k$. Determine λ_{k+1} by (18). Compute z_{k+1} by (23) and reset it by (24).

Step 5 Set $\alpha = \alpha_k^{\max}$ with $\alpha_k^{\max}$ defined by (19).

Step 6 If (14) holds, go to Step 7. Otherwise, go to Step 8.

Step 7 *f-iteration*

Step 7.1 Set $x_k(\alpha) = x_k + \alpha d_k$.
Step 7.2 If $x_k(\alpha)$ satisfies (20) and (21), go to Step 9.
Step 7.3 Let $\alpha = \alpha/2$ and go to Step 7.1.

Step 8 *h-iteration*

Step 8.1 Set $x_k(\alpha) = x_k + \alpha d_k$.
Step 8.2 If $x_k(\alpha)$ satisfies (22), go to Step 9.
Step 8.3 Let $\alpha = \alpha/2$ and go to Step 8.1.

Step 9 Set $\alpha_k = \alpha$, $x_{k+1} = x_k + \alpha_k d_k$. Compute $h_k^{\max}$ defined by (25). Set $k := k + 1$, go to Step 1.

Remark. Algorithm 3 has chances to stop at Step 2. In this case, the point x_k is an stationary point of (17), which always implies local infeasibility of the problem.

3. Well-definedness of the algorithms

Note that inner loops are used in the above algorithms. In Algorithm 2, we try to find a proper regularization parameter ν_k by repeated reducing it, and in Algorithm 3, we keep having the step-size α_k, so as to find an acceptable step. We will show that, under certain assumptions, both the inner loops will terminate finitely, i.e., the algorithms are well-defined.

We begin by recalling a result concerning a direct consequence of the definition of $h_k^{\max}$.

Lemma 3.1 ([16]). *The sequence $\{h_k^{\max}\}$ is non-increasing and the inequality*

$$0 \le h_l \le h_k^{\max}$$

holds for all $l \ge k$.

Next we show that the parameter ν_k will admit the requirements if it becomes small enough.

Lemma 3.2. *Suppose that ∇f_k and ∇c_k are Lipschitz continuous on a bounded open convex set Ω containing all the iterates generated by Algorithm 3, that $\{H_k\}$ is bounded, and that the Algorithm 3 does not terminate at x_k. Then either a step t_k satisfying (14) and (15), or a step satisfying (16) but does not satisfy (14) will be found if ν_k becomes small enough.*

Proof. First, we give an estimate of the scale of $\|t_k\|$. By the first order necessary condition of the quasi-tangential subproblem (12), step t_k satisfies

$$\nabla\varphi_k^\mu + \widetilde{W}_k v_k + \left(\widetilde{W}_k + \frac{1}{\nu_k}\nabla c_k \nabla c_k^T + \zeta_k I\right) t_k = 0. \tag{26}$$

Taking the inner products of this equation with the vector t_k and rearranging the resulted equality, we have

$$t_k^T\left(\widetilde{W}_k + \frac{1}{\nu_k}\nabla c_k \nabla c_k^T + \zeta_k I\right) t_k = -(\nabla\varphi_k^\mu + \widetilde{W}_k v_k)^T t_k. \tag{27}$$

By Cauchy–Schwarz inequality and (13), we have that

$$b_1\|t_k\|^2 \le \|(\nabla\varphi_k^\mu + \widetilde{W}_k v_k)^T t_k\|\|t_k\|.$$

If $t_k \ne 0$, then it follows that

$$\|t_k\| \le \frac{1}{b_1}\|\nabla\varphi_k^\mu + \widetilde{W}_k v_k\|. \tag{28}$$

If $t_k = 0$, then (28) is trivially true.

Taking the inner products of (26) with the vector $\nu_k t_k$, moving terms not involving $\nabla c_k^T t_k$ to the right hand side and using (28) and Cauchy–Schwarz inequality, we obtain

$$\|\nabla c_k^T t_k\|^2 \le \frac{\nu_k}{b_1}\|\nabla\varphi_k^\mu + \widetilde{W}_k v_k\|^2\left(1 + \frac{1}{b_1}\|\widetilde{W}_k + \zeta_k I\|\right). \tag{29}$$

Note that if $v_k = 0$ and the algorithm does not terminate, then by Step 2 of Algorithm 3, we have that $h_k = 0$. Since Algorithm 3 does not stop at x_k, the step t_k cannot be 0. In fact, if $t_k = 0$, then by (26), we have $\nabla\varphi_k^\mu = 0$. Then we have

$$c_k = 0 \text{ and } \nabla\varphi_k^\mu + \nabla c_k^T 0 = 0,$$

which implies that x_k is a first order stationary point of the barrier problem. The fact that $t_k \ne 0$ and (26) imply that

$$-(\nabla\varphi_k^\mu)^T t_k = t_k^T\left(\widetilde{W}_k + \frac{1}{\nu_k}\nabla c_k \nabla c_k^T + \zeta_k I\right) t_k \ge b_1\|t_k\|^2 > 0,$$

which yields (14). Then, from (29), condition (15) is satisfied if

$$0 < \nu_k \le \frac{b_1\kappa_1^2(h_k^{\max})^2}{\|\nabla\varphi_k^\mu\|^2\left(1 + \frac{1}{b_1}\|\widetilde{W}_k + \zeta_k I\|\right)}.$$

If $v_k \ne 0$, then from (29), both (15) and (16) are satisfied if

$$0 < \nu_k \le \frac{b_1\min\{\kappa_1^2(h_k^{\max} - \|c_k + \nabla c_k^T v_k\|)^2, \kappa_2^2(h_k - \|c_k + \nabla c_k^T v_k\|)^2\}}{\|\nabla\varphi_k^\mu + \widetilde{W}_k v_k\|^2\left(1 + \frac{1}{b_1}\|\widetilde{W}_k + \zeta_k I\|\right)}.$$

Thus, the claim is true in this case. $\square$

Next, we state the finite termination of the line search.

Lemma 3.3. *Suppose that* $h_k > 0$.

(a) *If* ∇c_k *is of full rank, then*

$$\|c_k\| - \|c_k + \nabla c_k^T v_k\| = \|c_k\|.$$

(b) *If* ∇c_k *is rank deficient, then*

$$\|c_k\| - \|c_k + \nabla c_k^T v_k\| \geq \frac{\|c_k\|^{\delta-1}}{2}\|v_k\|.$$

Proof. (a) If ∇c_k is of full column rank, then by definition

$$v_k = -\nabla c_k(\nabla c_k^T \nabla c_k)^{-1} c_k,$$

which implies $c_k + \nabla c_k^T v_k = 0$. Then we have

$$\|c_k\| - \|c_k + \nabla c_k^T v_k\| = \|c_k\|.$$

(b) By the definition of v_k for (10), we have

$$\|c_k + \nabla c_k^T v_k\|^2 \leq \|c_k + \nabla c_k^T v_k\|^2 + \|c_k\|^\delta \|v_k\|^2 \leq \|c_k\|^2,$$

which yields

$$\begin{aligned}\|c_k\| - \|c_k + \nabla c_k^T v_k\| \geq & \frac{\|c_k\|^\delta \|v_k\|^2}{\|c_k\| + \|c_k + \nabla c_k^T v_k\|} \\ \geq & \frac{\|c_k\|^\delta \|v_k\|^2}{2\|c_k\|} = \frac{\|c_k\|^{\delta-1}}{2}\|v_k\|. \quad \square\end{aligned}$$

Lemma 3.4. *Denote by* L_{dc} *the Lipschitz constant for* ∇c. *Then*

$$\|c(x_k + \alpha d_k)\| \leq (1-\alpha)\|c_k\| + \alpha\|c_k + \nabla c_k^T d_k\| + \frac{1}{2}\alpha^2 L_{dc}\|d_k\|^2.$$

Proof. By convexity of $\|\cdot\|$ and Lipschitz continuity, we have

$$\begin{aligned}\|c(x_k + \alpha d_k)\| = & \left\| c_k + \alpha \int_0^1 \nabla c(x_k + \alpha\tau d_k)^T d_k d\tau \right\| \\ = & \left\| c_k + \alpha \int_0^1 (\nabla c(x_k + \alpha\tau d_k)^T d_k - \nabla c_k^T d_k) d\tau + \alpha \nabla c_k^T d_k \right\| \\ \leq & \|c_k + \alpha \nabla c_k^T d_k\| + \left\| \alpha \int_0^1 (\nabla c(x_k + \alpha\tau d_k)^T d_k - \nabla c_k^T d_k) d\tau \right\| \\ \leq & (1-\alpha)\|c_k\| + \alpha\|c_k + \nabla c_k^T d_k\| + \alpha \int_0^1 \|\nabla c(x_k + \alpha\tau d) - \nabla c_k\| \|d_k\| d\tau \\ \leq & (1-\alpha)\|c_k\| + \alpha\|c_k + \nabla c_k^T d_k\| + \frac{1}{2}\alpha^2 L_{dc}\|d_k\|^2. \quad \square\end{aligned}$$

Lemma 3.5. *Suppose that* ∇f_k *and* ∇c_k *are Lipschitz continuous on a bounded open convex set* Ω *containing all the iterates generated by Algorithm 3, that* $\{H_k\}$ *is bounded, and that the Algorithm 3 does not terminate at* x_k. *Suppose also that* t_k *satisfies* (14) *and* (15). *Then there exists a positive scalar* α_k^f *such that for any* $\alpha \in (0, \alpha_k^f]$, *both* (20) *and* (21) *are satisfied.*

Proof. First, we show that $-(\nabla \varphi_k^\mu)^T d_k > 0$. If $h_k > 0$, then by (14)

$$-\nabla(\varphi_k^\mu)^T d_k = -\nabla(\varphi_k^\mu)^T(s_k + t_k) \geq \sigma_1 h_k^{\sigma_2} > 0.$$

If $h_k = 0$, then by the proof of Lemma 3.2, we also have $-\nabla(\varphi_k^\mu)^T d_k > 0$.

Using the Lipschitz continuity of $\nabla f(x)$, we have, with a Lipschitz constant L_{df}, that for any step-size $\alpha \in (0, \alpha_k^{\max}]$

$$\begin{aligned}&|f(x_k) - f(x_k + \alpha d_k) - (-\alpha \nabla f_k^T d_k)| \\ &\leq \alpha \sup_{x_k^\tau \in [x_k, x_k + \alpha d_k]} \|\nabla f(x_k^\tau) - \nabla f_k\| \|d_k\| \\ &\leq \alpha^2 L_{df}\|d_k\|^2.\end{aligned}$$

Similarly, for any $i = 1, \dots, n$,

$$\begin{aligned}
&\left| -\mu \sum_{i=1}^{n} \ln x_k^{(i)} + \mu \sum_{i=1}^{n} \ln(x_k^{(i)} + \alpha d_k^{(i)}) - \mu\alpha \sum_{i=1}^{n} \frac{d_k^{(i)}}{x_k^{(i)}} \right| \\
&\le \mu \sum_{i=1}^{n} \left| -\ln x_k^{(i)} + \sum_{i=1}^{n} \ln(x_k^{(i)} + \alpha d_k^{(i)}) - \alpha \frac{d_k^{(i)}}{x_k^{(i)}} \right| \\
&\le \frac{\mu\alpha^2}{1-\tau} \sum_{i=1}^{n} \left(\frac{d_k^{(i)}}{x_k^{(i)}} \right)^2 \le \frac{\mu\alpha^2}{(1-\tau)\min_i (x_k^{(i)})^2} \|d_k\|^2.
\end{aligned}$$

Using the above two inequalities, we have

$$|\varphi_k^\mu - \varphi^\mu(x_k + \alpha d_k) - (-\alpha(\nabla\varphi_k^\mu)^T d_k)| \le \alpha^2 \left(L_{df} + \frac{\mu}{(1-\tau)\min_i (x_k^{(i)})^2} \right) \|d_k\|^2.$$

This inequality implies

$$\frac{|\varphi_k^\mu - \varphi^\mu(x_k + \alpha d_k) - (-\alpha(\nabla\varphi_k^\mu)^T d_k)|}{-\alpha(\nabla\varphi_k^\mu)^T d_k} \le \alpha \frac{\left(L_{df} + \frac{\mu}{(1-\tau)\min_i (x_k^{(i)})^2} \right) \|d_k\|^2}{-(\nabla\varphi_k^\mu)^T d_k}. \tag{30}$$

Let

$$0 < \alpha \le \alpha_k^{f,1} := \frac{-(1-\rho)(\nabla\varphi_k^\mu)^T d_k}{\left(L_{df} + \frac{\mu}{(1-\tau)\min_i (x_k^{(i)})^2} \right) \|d_k\|^2}.$$

Then it follows from (30) that

$$\frac{|\varphi_k^\mu - \varphi^\mu(x_k + \alpha d_k) - (-\alpha(\nabla\varphi_k^\mu)^T d_k)|}{-\alpha(\nabla\varphi_k^\mu)^T d_k} \le 1 - \rho,$$

which yields (20).

Next, we consider (21). Using Lemma 3.4 and (15), we have

$$\begin{aligned}
&\|c(x_k + \alpha d_k)\| \\
\le&(1-\alpha)\|c_k\| + \alpha\|c_k + \nabla c_k^T v_k\| + \alpha\|\nabla c_k^T t_k\| + \frac{1}{2}\alpha^2 L_{dc}\|d_k\|^2 \\
\le&(1-\alpha)h_k^{\max} + \alpha\|c_k + \nabla c_k^T v_k\| \\
&+ \alpha\kappa_1(h_k^{\max} - \|c_k + \nabla c_k^T v_k\|) + \frac{1}{2}\alpha^2 L_{dc}\|d_k\|^2 \\
=&h_k^{\max} - \alpha(1-\kappa_1)(h_k^{\max} - \|c_k + \nabla c_k^T v_k\|) + \frac{1}{2}\alpha^2 L_{dc}\|d_k\|^2.
\end{aligned} \tag{31}$$

Then, from (31), it follows that (21) is satisfied if

$$0 < \alpha \le \alpha_k^{f,2} := \frac{2(1-\kappa_1)(h_k^{\max} - \|c_k + \nabla c_k^T v_k\|)}{L_{dc}\|d_k\|^2}.$$

Summarizing all the arguments, both (20) and (21) are satisfied for any $\alpha \in (0, \alpha_k^f]$, where

$$\alpha_k^f := \min\{\alpha_k^{f,1}, \alpha_k^{f,2}\}. \quad \square \tag{32}$$

Lemma 3.6. *Suppose that ∇f_k and ∇c_k are Lipschitz continuous on a bounded open convex set Ω containing all the iterates generated by Algorithm 3, that $\{H_k\}$ is bounded, and that the Algorithm 3 does not terminate at x_k. Suppose also that t_k satisfies (16) but does not satisfy (14). Then there exists a positive constant α_k^h such that for any $\alpha \in (0, \alpha_k^h]$, the condition (22) is satisfied.*

Proof. Since (14) does not hold and the algorithm does not terminate at x_k, we have $v_k \ne 0$, which implies $h_k > 0$.

Using Lemma 3.4, (16) and convexity, we have

$$\begin{aligned}
\|c(x_k+\alpha d_k)\| &\le \|c_k+\alpha\nabla c_k^T d_k\| + \frac{1}{2}\alpha^2 L_{dc}\|d_k\|^2 \\
&= (1-\rho)\|c_k+\alpha\nabla c_k^T d_k\| + \rho\|c_k+\alpha\nabla c_k^T d_k\| + \frac{1}{2}\alpha^2 L_{dc}\|d_k\|^2 \\
&\le (1-\rho)((1-\alpha)\|c_k\| + \alpha\|c_k+\nabla c_k^T d_k\|) + \rho\|c_k+\alpha\nabla c_k^T d_k\| \\
&\quad + \frac{1}{2}\alpha^2 L_{dc}\|d_k\|^2 \\
&= (1-\rho)\|c_k\| - (1-\rho)\alpha(\|c_k\| - \|c_k+\nabla c_k^T d_k\|) \\
&\quad + \rho\|c_k+\alpha\nabla c_k^T d_k\| + \frac{1}{2}\alpha^2 L_{dc}\|d_k\|^2 \\
&\le (1-\rho)\|c_k\| - (1-\rho)\alpha(\|c_k\| - \|c_k+\nabla c_k^T v_k\| - \|\nabla c_k^T t_k\|) \\
&\quad + \rho\|c_k+\alpha\nabla c_k^T d_k\| + \frac{1}{2}\alpha^2 L_{dc}\|d_k\|^2 \\
&\le (1-\rho)\|c_k\| - (1-\rho)(1-\kappa_2)\alpha(\|c_k\| - \|c_k+\nabla c_k^T v_k\|) \\
&\quad + \rho\|c_k+\alpha\nabla c_k^T d_k\| + \frac{1}{2}\alpha^2 L_{dc}\|d_k\|^2 \\
&\le (1-\rho)\|c_k\| + \rho\|c_k+\alpha\nabla c_k^T d_k\| \\
&\quad - (1-\rho)(1-\kappa_2)\alpha(\|c_k\| - \|c_k+\nabla c_k^T v_k\|) + \frac{1}{2}\alpha^2 L_{dc}\|d_k\|^2.
\end{aligned}$$

Define

$$\alpha_k^h := \begin{cases} \dfrac{(1-\rho)(1-\kappa_2)\|c_k\|^{\delta-1}\|v_k\|^2}{L_{dc}\|d_k\|^2}, & \text{if } \nabla c_k \text{ is rank deficient,} \\ \dfrac{(1-\rho)(1-\kappa_2)\|c_k\|}{L_{dc}\|d_k\|^2}, & \text{if } \nabla c_k \text{ is of full rank.} \end{cases} \tag{33}$$

Then by the above inequalities and Lemma 3.3, (22) is satisfied for all $\alpha \in (0, \alpha_k^h]$. □

Hence, we conclude the well-definedness of the algorithms.

Theorem 3.7. *Suppose that ∇f_k and ∇c_k are Lipschitz continuous on a bounded open convex set Ω containing all the iterates generated by Algorithm 3, that $\{H_k\}$ is bounded. Then both Algorithm 2 and Algorithm 3 are well-defined.*

4. Global convergence analysis

Let

$$\mathcal{A}(x) = \{i \mid x^{(i)} = 0\}$$

be the set of active inequality constraints and $\bar{\mathcal{A}}(x) = \{1, 2, \ldots, n\} \setminus \mathcal{A}(x)$. Denote $n_a = |\mathcal{A}(x)|$, $n_l = |\bar{\mathcal{A}}(x)|$.

To establish the global convergence theories for Algorithm 3, we need the following standard assumptions.

Assumption 1. Let $\{x_k\}$ be the sequence of the iterates generated by algorithm 3.

(A1) There is a bounded open convex set Ω containing $\{x_k\}$.
(A2) The gradients ∇f_k and ∇c_k are Lipschitz continuous on Ω.
(A3) The symmetric matrix H_k is uniformly bounded.
(A4) The linear independence constraint qualification is satisfied at any accumulation point $\widetilde{x}$ of $\{x_k\}$.
(A5) At any accumulation point $\widetilde{x}$ of $\{x_k\}$, $\widetilde{H}$ is positive definite on the tangent space of active constraints, i.e., there is a positive scalar $b > 0$ such that

$$d^T\widetilde{H}d \ge b\|d\|^2 \tag{34}$$

for all $d \in \{p \mid \nabla c(\widetilde{x})^T p = 0,\ \text{and } p^{(i)} = 0,\ i \in \mathcal{A}(\widetilde{x})\}$.

Assume that the algorithm does not terminate finitely. The following lemma shows the convexity of the quasi-tangential subproblem near a limit point. This result is a modification of [28, Theorem 17.5] in the context of interior point methods.

Lemma 4.1. *Suppose that Assumption 1 hold, and that $\widetilde{x}$ is an accumulated point of the iterate sequence $\{x_k\}$. Then there exist a neighborhood $\widetilde{\mathcal{N}}$ of $\widetilde{x}$ and a positive scalar $\bar{\nu}$, such that for all $x_k \in \widetilde{\mathcal{N}}$ and $\nu \in (0, \bar{\nu}]$, the symmetric matrix $\widetilde{W}_k + \frac{1}{\nu}\nabla c_k \nabla c_k^T$ is positive definite.*

Proof. For simplicity, we abbreviate $\mathcal{A}(\widetilde{x})$ to $\mathcal{A}$, and $\bar{\mathcal{A}}(\widetilde{x})$ to $\bar{\mathcal{A}}$ from now on. We denote with x_k^a the vector composed of $\{x_k^{(i)} \mid i \in \mathcal{A}\}$, and x_k^l the vector composed of remaining components. If $n_a = 0$ then this lemma reduces to [28, Theorem 17.5]. So we assume that $n_a > 0$. Without loss of generality, we suppose that

$$x_k = (x_k^a, x_k^l). \tag{35}$$

Similarly, we define ∇c_k^a, ∇c_k^l etc. Let

$$A_k = \begin{pmatrix} \nabla c_k^a & I^a \\ \nabla c_k^l & 0 \end{pmatrix}.$$

Then, by continuity, there exists a neighbor $\widetilde{\mathcal{N}}$ of $\widetilde{x}$ such that A_k has full column rank at all $x_k \in \widetilde{\mathcal{N}}$. For any $d \in R^n$, we can partition it into components in $\mathrm{Null}(A_k^T)$ and $\mathrm{Range}(A_k)$, and write

$$d = t + A_k w \tag{36}$$

with $w \in R^{m+n_a}$. Let us consider the lower bound of

$$d^T \left(H_k + \Sigma_k + \frac{1}{\nu} \nabla c_k \nabla c_k^T \right) d. \tag{37}$$

First, we consider $d^T H_k d$. Using the decomposition (36), we have

$$d^T H_k d = (t + A_k w)^T H_k (t + A_k w) \geq t^T H_k t + 2 w^T A_k^T H_k t + w^T A_k^T H_k A_k w.$$

By Assumption 1, there are positive constants a_1, a_2 and a_3, such that

$$\begin{aligned} & t^T H_k t \geq a_1 \|t\|^2, \\ & w^T A_k^T H_k t \geq -a_2 \|w\| \|t\|, \\ & w^T A_k^T H_k A_k w \geq -a_3 \|w\|^2, \end{aligned}$$

which yield

$$d^T H_k d = (t + A_k w)^T H_k (t + A_k w) \geq a_1 \|t\|^2 - a_2 \|w\| \|t\| - a_3 \|w\|^2.$$

Then we consider the harder block

$$d^T \left(\Sigma_k + \frac{1}{\nu} \nabla c_k \nabla c_k^T \right) d.$$

By the safeguard technique (24), we have

$$d^T \left(\Sigma_k + \frac{1}{\nu} \nabla c_k \nabla c_k^T \right) d \geq d^T \left(\frac{\mu}{\kappa_\sigma} X_k^{-2} + \frac{1}{\nu} \nabla c_k \nabla c_k^T \right) d.$$

Define $x_k^{a,\max} = \max_{i \in \mathcal{A}} \{x_k^{(i)}\}$ and use the partition (35), we have that

$$\begin{aligned} & d^T \left(\frac{\mu}{\kappa_\sigma} X_k^{-2} + \frac{1}{\nu} \nabla c_k \nabla c_k^T \right) d \\ & = d^T \left(\frac{\mu}{\kappa_\sigma} \begin{pmatrix} (X_k^a)^{-2} & \\ & (X_k^l)^{-2} \end{pmatrix} + \frac{1}{\nu} \begin{pmatrix} \nabla c_k^a (\nabla c_k^a)^T & \nabla c_k^a (\nabla c_k^l)^T \\ \nabla c_k^l (\nabla c_k^a)^T & \nabla c_k^l (\nabla c_k^l)^T \end{pmatrix} \right) d \\ & \geq b_\nu^\mu d^T \left(\begin{pmatrix} I^a & \\ & (x_k^{a,\max})^2 (X_k^l)^{-2} \end{pmatrix} + \begin{pmatrix} \nabla c_k^a (\nabla c_k^a)^T & \nabla c_k^a (\nabla c_k^l)^T \\ \nabla c_k^l (\nabla c_k^a)^T & \nabla c_k^l (\nabla c_k^l)^T \end{pmatrix} \right) d \\ & = b_\nu^\mu d^T \left(\begin{pmatrix} \nabla c_k^a (\nabla c_k^a)^T + I^a & \nabla c_k^a (\nabla c_k^l)^T \\ \nabla c_k^l (\nabla c_k^a)^T & \nabla c_k^l (\nabla c_k^l)^T \end{pmatrix} + \begin{pmatrix} 0 & \\ & (x_k^{a,\max})^2 (X_k^l)^{-2} \end{pmatrix} \right) d \\ & = b_\nu^\mu d^T \left(A_k A_k^T + \begin{pmatrix} 0 & \\ & (x_k^{a,\max})^2 (X_k^l)^{-2} \end{pmatrix} \right) d \\ & \geq b_\nu^\mu d^T A_k A_k^T d, \end{aligned}$$

where $b_\nu^\mu = \min\left\{\frac{\mu}{\kappa_\sigma(x_k^{a,\max})^2}, \frac{1}{\nu}\right\}$. Using (36) and the continuity, there exists a positive constant $a_4 > 0$ such that

$$d^T A_k A_k^T d = w^T A_k^T A_k A_k^T A_k w \geq a_4^2 \|w\|^2.$$

To sum up these arguments, we get a lower bound of (37)

$$\begin{aligned} & d^T\left(H_k + \Sigma_k + \frac{1}{\nu}\nabla c_k \nabla c_k^T\right) d \\ & \geq a_1\|t\|^2 - 2a_2\|w\|\|t\| - a_3\|w\|^2 + a_4^2 b_\nu^\mu \|w\|^2 \\ & \geq a_1\left(\|t\| - \frac{a_2}{a_1}\|w\|\right)^2 + \left(a_4^2 b_\nu^\mu - a_3 - \frac{a_2^2}{a_1}\right)\|w\|^2. \end{aligned}$$

Hence, the symmetric matrix $H_k + \Sigma_k + \frac{1}{\nu}\nabla c_k \nabla c_k^T$ is positive definite provided that we choose $\bar{\nu}$ be a small positive constant such that

$$\left(a_4^2 \min\left\{\frac{\mu}{\kappa_\sigma(x_k^{a,\max})^2}, \frac{1}{\bar{\nu}}\right\} - a_3 - \frac{a_2^2}{a_1}\right) > 0$$

and choose $\nu \in (0, \bar{\nu}]$. □

A key point for establishing global convergence results of interior point method is to show that, for a given barrier parameter μ, the sequence of iterate $\{x_k\}$ is componentwise bounded away from 0. In [35, Theorem 3], Wächter and Biegler pointed out that, under reasonable assumptions, this is true for the primal iterate sequence $\{x_k\}$ generated by Newton–Lagrange methods for barrier problem. Here, we present the similar result in the context of our algorithm and give a more detailed proof, which follows the basic ideas of Wächter and Biegler's proof and uses the classical perturbation theory for linear system [36].

Lemma 4.2. *Under Assumption 1, suppose that $\widetilde{x}$ is an accumulation point of $\{x_k\}$. Then there is a neighborhood $\widetilde{\mathcal{N}}$ of $\widetilde{x}$ such that $d_k^{(i)} > 0$ for any $x_k \in \widetilde{\mathcal{N}}$ and $i \in \mathcal{A}$.*

Proof. If $n_a = 0$, then all the constraints $x^{(i)} \geq 0$ are inactive, in other words, are bounded away from 0. Hence, we can assume without loss of generality that $n_a \geq 1$. Define $\widetilde{\delta} = 0.5\min\{\widetilde{x}^{(i)} \mid i \in \bar{\mathcal{A}}\}$. Then there is a neighborhood $\widetilde{\mathcal{N}}$ such that $x_k^{(i)} \geq \widetilde{\delta}$ for all $x_k \in \widetilde{\mathcal{N}}$ and $i \in \bar{\mathcal{A}}$.

By Assumption 1, there is a (smaller) neighborhood $\widetilde{\mathcal{N}}$ of $\widetilde{x}$ such that A_k has full rank and H_k is positive definite on the null space of A_k^T in the sense of (34) for any $x_k \in \widetilde{\mathcal{N}}$. Then from the definition of v_k, we have

$$c_k + \nabla c_k^T v_k = 0. \tag{38}$$

By Lemma 4.1 and Step 1 of Algorithm 2, $H_k + \frac{1}{\nu_k}\nabla c_k \nabla c_k^T$ will be positive definite with $\nu_k = \max\{\min\{\nu_{k-1}, h_k\}, \underline{\nu}\}$ for sufficiently large k, which indicates that $\zeta_k = 0$. It follows from (26) that

$$(H_k + X_k^{-1}Z_k)^T d_k + \nabla c_k\left(\frac{1}{\nu_k}\nabla c_k^T t_k\right) = -\nabla f_k + \mu X_k^{-1} e. \tag{39}$$

By (15) and (16), the sequence $\{\nabla c_k^T t_k\}$ is bounded. From (18), (38) and (39), we obtain the following linear system

$$\begin{pmatrix} H_k + X_k^{-1}Z_k & \nabla c_k \\ \nabla c_k^T & 0 \end{pmatrix}\begin{pmatrix} d_k \\ \lambda_{k+1} \end{pmatrix} = -\begin{pmatrix} \nabla f_k - \mu X_k^{-1} e \\ c_k - \nabla c_k^T t_k \end{pmatrix}. \tag{40}$$

Partition (40) with respect to (35), we have

$$\begin{aligned} & \begin{pmatrix} H_k^{aa} + (X_k^a)^{-1}Z_k^a & H_k^{al} & \nabla c_k^a \\ H_k^{la} & H_k^{ll} + (X_k^l)^{-1}Z_k^l & \nabla c_k^l \\ \nabla(c_k^a)^T & \nabla(c_k^l)^T & 0 \end{pmatrix}\begin{pmatrix} d_k^a \\ d_k^l \\ \lambda_{k+1} \end{pmatrix} \\ & = -\begin{pmatrix} \nabla f_k^a - \mu(X_k^a)^{-1} e \\ \nabla f_k^l - \mu(X_k^l)^{-1} e \\ c_k - \nabla c_k^T t_k \end{pmatrix}. \end{aligned} \tag{41}$$

The rest of the proof follows the ideas of Wächter and Biegler but with some necessary modifications and more details. Rewrite the linear system (41) by scaling the first rows and columns by X_k^a:

$$\begin{pmatrix} X_k^a H_k^{aa} X_k^a + X_k^a Z_k^a & X_k^a H_k^{al} & X_k^a \nabla c_k^a \\ H_k^{la} X_k^a & H_k^{ll} + (X_k^l)^{-1} Z_k^l & \nabla c_k^l \\ (\nabla c_k^a)^T X_k^a & (\nabla c_k^l)^T & 0 \end{pmatrix} \begin{pmatrix} \widetilde{d}_k^a \\ d_k^l \\ \lambda_{k+1} \end{pmatrix} = - \begin{pmatrix} X_k^a \nabla f_k^a - \mu e \\ \nabla f_k^l - \mu (X_k^l)^{-1} e \\ c_k - \nabla c_k^T t_k \end{pmatrix}, \tag{42}$$

where $\widetilde{d}_k^a = (X_k^a)^{-1} d_k^a$. For convenience, we write the coefficient matrix of this linear system into

$$\begin{pmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{pmatrix},$$

where

$$W_{11} = X_k^a H_k^{aa} X_k^a + X_k^a Z_k^a, \qquad W_{12} = (X_k^a H_k^{al} \; X_k^a \nabla c_k)$$
$$W_{21} = \begin{pmatrix} H_k^{la} X_k^a \\ (\nabla c_k^a)^T X_k^a \end{pmatrix}, \quad W_{22} = \begin{pmatrix} H_k^{ll} + (X_k^l)^{-1} Z_k^l & \nabla c_k^l \\ (\nabla c_k^l)^T & 0 \end{pmatrix}.$$

By assumption (A4), the matrix

$$\begin{pmatrix} \nabla c^a(\widetilde{x}) & I^a \\ \nabla c^l(\widetilde{x}) & 0 \end{pmatrix}$$

has full rank, from which it follows that $\nabla c^l(\widetilde{x})$ has full rank. Then there is a (smaller) neighborhood $\mathcal{N}$ of $\widetilde{x}$ such that ∇c_k^l has full rank. For any d^l in Null($(\nabla c_k^l)^T$), the vector $\bar{d} = (0, d^l)$ lies in Null(∇c_k^T). Thus, from assumption (A5), we get

$$(d_k^l)^T H_k^{ll} d_k^l = \bar{d}^T H_k \bar{d} \geq b \|\bar{d}\|^2 = b \|d^l\|^2,$$

i.e., H_k^{ll} is positive definite in Null($(\nabla c_k^l)^T$). Then W_{22} is nonsingular in $\widetilde{\mathcal{N}}$. By Assumption 1 and using $x_k^l \geq \widetilde{\delta} e$ and (24), their is a positive constant M_W such that $\|W_{22}^{-1}\| \leq M_W$. By eliminating d_k^l and λ_{k+1} in (42), we obtain

$$(W_{11} - W_{12} W_{22}^{-1} W_{21}) \widetilde{d}_k^a = -(X_k^a \nabla f_k^a - \mu e - W_{12} W_{22}^{-1} g_k), \tag{43}$$

where

$$g_k = \begin{pmatrix} \nabla f_k^l - \mu (X_k^l)^{-1} e \\ c_k - \nabla c_k^T t_k \end{pmatrix}.$$

Using $x_k^l \geq \widetilde{\delta} e$, there is $M_g > 0$ such that $\|g_k\| \leq M_g$ for all $x_k \in \widetilde{\mathcal{N}}$. Consider the linear system

$$X_k^a Z_k^a \bar{d}_k^a = \mu e. \tag{44}$$

System (43) can be viewed as a perturbed system for (44) with the coefficient matrix perturbed by

$$G_k = X_k^a H_k^{aa} X_k^a - W_{12} W_{22}^{-1} W_{21}$$

and the right hand side by

$$r_k = -(X_k^a \nabla f_k^a - W_{12} W_{22}^{-1} g_k).$$

By perturbation theory for linear system, see, for instance, [36], we have

$$\frac{\|\widetilde{d}_k^a - \bar{d}_k^a\|_\infty}{\|\bar{d}_k^a\|_\infty} \leq \frac{\kappa(X_k^a Z_k^a)}{1 - \kappa(X_k^a Z_k^a) \frac{\|G_k\|_\infty}{\|X_k^a Z_k^a\|_\infty}} \left(\frac{\|G_k\|_\infty}{\|X_k^a Z_k^a\|_\infty} + \frac{\|r_k\|_\infty}{\mu} \right), \tag{45}$$

where $\kappa(X_k^a Z_k^a)$ refers to the condition number of $X_k^a Z_k^a$, provided that

$$\|(X_k^a Z_k^a)^{-1}\|_\infty \|G_k\|_\infty < 1.$$

Using (24), we have that

$$\frac{\mu}{\kappa_\Sigma} \leq x_k^{(i)} z_k^{(i)} \leq \kappa_\Sigma \mu$$

for $i \in \mathcal{A}$, which implies that

$$\frac{\mu}{\kappa_\Sigma} \leq \|X_k^a Z_k^a\|_\infty \leq \kappa_\Sigma \mu, \;\; \|(X_k^a Z_k^a)^{-1}\|_\infty \leq \kappa_\Sigma/\mu, \tag{46}$$

and that the solution $\bar{d}_k^a$ of (44) satisfies

$$\frac{1}{\kappa_\Sigma} \leq \bar{d}_k^{(i)} \leq \kappa_\Sigma, \;\; \forall i \in \mathcal{A}. \tag{47}$$

Denote

$$\delta_\mu = \max\{x_k^{(i)} \mid i \in \mathcal{A}\}.$$

From Assumption 1 and using $x_k^I \geq \widetilde{\delta} e$, there are positive constants $M_H, M_{12}, M_{21}, M_{df}$ and M_g such that

$$\begin{aligned}\|G_k\|_\infty &\leq \|X_k^a\|_\infty \|H_k^{aa}\|_\infty + \|W_{12}\|_\infty \|W_{22}^{-1}\|_\infty \|W_{21}\|_\infty \\ &\leq M_H \delta_\mu + M_{12} M_W M_{21} \delta_\mu^2\end{aligned}$$

and

$$\|r_k\|_\infty \leq M_{df}\delta_\mu + M_{12} M_W M_g \delta_\mu.$$

From (46) and noting that $\delta_\mu \to 0$ as $x_k \to \widetilde{x}$, there is a (smaller) neighborhood $\widetilde{\mathcal{N}}$ of $\widetilde{x}$ such that

$$1 - \kappa(X_k^a Z_k^a)\frac{\|G_k\|_\infty}{\|X_k^a Z_k^a\|_\infty} = 1 - \|(X_k^a Z_k^a)^{-1}\|\|G_k\|_\infty \geq \frac{1}{2}.$$

It also follows from (46) that $\kappa(X_k^a Z_k^a) \leq \kappa_\Sigma^2$. Having all these facts in mind, the right hand side of the inequality (45) tends to 0 as x_k approaching $\widetilde{x}$. It follows that, for any $i \in \mathcal{A}$,

$$\begin{aligned}\widetilde{d}_k^{(i)} &\geq \bar{d}_k^{(i)} - |\widetilde{d}_k^{(i)} - \bar{d}_k^{(i)}| \geq \bar{d}_k^{(i)} - \|\widetilde{d}_k^a - \bar{d}_k^a\|_\infty \\ &\geq \bar{d}_k^{(i)} - \|\widetilde{d}_k^a - \bar{d}_k^a\|_\infty \|\bar{d}_k^a\|_\infty \\ &\geq \bar{d}_k^{(i)} - \frac{\kappa(X_k^a Z_k^a)}{1 - \kappa(X_k^a Z_k^a)\frac{\|G_k\|_\infty}{\|X_k^a Z_k^a\|_\infty}}\left(\frac{\|G_k\|_\infty}{\|X_k^a Z_k^a\|_\infty} + \frac{\|r_k\|_\infty}{\mu}\right)\kappa_\Sigma \\ &\geq \frac{1}{\kappa_\Sigma} - \frac{\kappa(X_k^a Z_k^a)}{1 - \kappa(X_k^a Z_k^a)\frac{\|G_k\|_\infty}{\|X_k^a Z_k^a\|_\infty}}\left(\frac{\|G_k\|_\infty}{\|X_k^a Z_k^a\|_\infty} + \frac{\|r_k\|_\infty}{\mu}\right)\kappa_\Sigma \\ &\to \frac{1}{\kappa_\Sigma} > 0,\end{aligned}$$

where (45), (47) are used. Therefore, by shrinking $\widetilde{\mathcal{N}}$ if necessary, we have $\widetilde{d}_k^a > 0$ for all $x_k \in \widetilde{\mathcal{N}}$, which yields $d_k^a > 0$ because $d_k^a = X_k^a \widetilde{d}_k^a$. □

Theorem 4.3. *Under Assumption 1, there is a positive constant $\widetilde{\epsilon} > 0$, such that $x_k \geq \widetilde{\epsilon} e$ for any index k.*

Proof. By Lemma 4.2, for any accumulation point $\widetilde{x}$ of $\{x_k\}$, there is a neighborhood $\widetilde{\mathcal{N}}$ of it, such that $d_k^{(i)} > 0, i \in \mathcal{A}$ for all $x_k \in \widetilde{\mathcal{N}}$. By compactness, there are finite number of such neighborhoods, denoting by $\mathcal{N}_1, \ldots, \mathcal{N}_q$, of different accumulation points $\widetilde{x}_1, \ldots, \widetilde{x}_q$, whose union covers all the accumulation points. Define $\widetilde{\delta}_p = 0.5\min\{\widetilde{x}_p^{(i)} \mid i \notin \mathcal{A}(\widetilde{x}_p)\}, p = 1, \ldots, q$. Then $x_k^{(i)} \geq \widetilde{\delta}_p$, for any $i \notin \mathcal{A}(\widetilde{x}_p)$ and $x_k \in \mathcal{N}_p$. Denote $\mathcal{N}_S = \bigcup_{p=1}^q \mathcal{N}_p$. Then there are only finite iterates which are not covered by $\mathcal{N}_S$. Let

$$\widetilde{\epsilon} := (1-\tau)\min\{\min_{p=1,\ldots,q} \widetilde{\delta}_p, \min_{x_k \notin \mathcal{N}_S, i=1,\ldots,n} x_k^{(i)}\}.$$

Suppose, without loss of generality, that $x_0 \geq \widetilde{\epsilon} e$. Assume that $x_k \geq \widetilde{\epsilon} e$. Let us consider x_{k+1}. If $x_k \notin \mathcal{N}_S$, then the lemma is trivially true. Now we consider the case where $x_k \in \mathcal{N}_S$. Note that $x_k^{(i+1)} < x_k^{(i)}$ occurs only if i is not active constraints. Then, if i belongs to some $\mathcal{A}(\widetilde{x}_l), l = 1, 2, \ldots, p$, then $x_k^{(i+1)} \geq x_k^{(i)} \geq \widetilde{\epsilon}$. Otherwise,

$$x_k^{(i+1)} \geq (1-\tau)x_k^{(i)} \geq (1-\tau)\min_{p=1,\ldots,q} \widetilde{\delta}_p \geq \widetilde{\epsilon}. \quad \square$$

From Assumption 1, Theorem 4.3 and (24), it is easy to conclude the boundedness of search direction $\{d_k\}$. From now on, we denote with M_d the upper bound of $\{\|d_k\|\}$. The boundedness of the Lagrange multipliers $\{\lambda_k\}$ follows from (18).

The remainder of this section gives global convergence results. We shall use the following two index sets

$$\mathcal{K}_h=\{k \mid h_{k+1}^{\max}<h_k^{\max}\},\ \bar{\mathcal{K}}_h=\{k \mid h_{k+1}^{\max}=\kappa_h h_k^{\max}\},$$

which contain h-iterations. First, we consider the case where $\mathcal{K}_h$ is infinite.

Lemma 4.4. *Suppose that* $|\mathcal{K}_h|=+\infty$. *Then* $\lim_{k\to\infty} h_k=0$.

Proof. Without loss of generality, we assume that ∇c_k has full rank for all $k\geq 0$. Suppose that $k\in\mathcal{K}_h$. By the updating rule of $h_k^{\max}$ (25), we have

$$h_k^{\max}-h_{k+1}^{\max}\geq \min\{(1-\kappa_h)h_k^{\max},(1-\bar{\kappa}_h)(h_k-h_{k+1})\}\geq \min\{(1-\kappa_h)h_k,(1-\bar{\kappa}_h)\rho\alpha_k h_k\},$$

where Lemma 3.1, (10), (16) and (22) are used. Since $\alpha_k\geq 0.5\alpha_k^h$, where α_k^h given by (33), the previous inequalities give

$$h_k^{\max}-h_{k+1}^{\max}\geq \min\left\{(1-\kappa_h)h_k,\frac{(1-\bar{\kappa}_h)(1-\kappa_2)(1-\rho)\rho h_k}{2L_{dc}M_d^2}\right\},$$

where the upper bound of $\{\|d_k\|\}$ is used. From the non-increasing property of the sequence $\{h_k^{\max}\}$ (see Lemma 3.1) and the above inequality, it follows that

$$\lim_{k\in\mathcal{K}_h} h_k=0. \tag{48}$$

Consider the cardinal number of $\bar{\mathcal{K}}_h$. If $|\bar{\mathcal{K}}_h|=\infty$, then it is a direct consequence of the definition of $\bar{\mathcal{K}}_h$ that $\lim_{k\to\infty}h_k^{\max}=0$. Otherwise, there is an index $k_1>0$ such that

$$h_{k+1}^{\max}=\kappa_h h_k+(1-\kappa_h)h_{k+1}\leq h_k$$

for all $k\geq k_1$ and $k\in\mathcal{K}_h$. Then we have by this inequality and (48) that

$$\lim_{k\in\mathcal{K}_h} h_{k+1}^{\max}=0.$$

Using non-increasing property of $\{h_k^{\max}\}$, we have $\lim_{k\to\infty}h_k^{\max}=0$, which implies, by Lemma 3.1, that $\lim_{k\to\infty}h_k=0$. □

Lemma 4.5. *Suppose that* $|\mathcal{K}_h|=+\infty$. *Then* $\lim_{k\in\mathcal{K}_h} t_k=0$.

Proof. Since, t_k solves (12) and does not satisfy (14), we have that for all $k\in\mathcal{K}_h$

$$\begin{aligned}\frac{1}{2}b_1\|t_k\|^2 &\leq \frac{1}{2}t_k^T\left(\widetilde{W}_k+\frac{1}{\nu}\nabla c_k\nabla c_k^T+\zeta_k I\right)t_k\\ &\leq -(\nabla\varphi_k^{\mu})^T t_k-(\widetilde{W}_k v_k)^T t_k\\ &= -(\nabla\varphi_k^{\mu})^T d_k+(\nabla\varphi_k^{\mu})^T v_k-(\widetilde{W}_k v_k)^T t_k\\ &\leq \sigma_1 h_k^{\sigma_2}+(\nabla\varphi_k^{\mu})^T v_k-(\widetilde{W}_k v_k)^T t_k.\end{aligned}$$

Note that by Theorem 4.3 and (28), the sets $\{\nabla\varphi_k^{\mu}\}$, $\{\widetilde{W}_k\}$ and $\{t_k\}$ are bounded. Then it follows from Lemma 4.4 and the above inequalities that $\lim_{k\in\mathcal{K}_h}t_k=0$. □

Next, we consider the case where $\mathcal{K}_h$ is a finite set.

Lemma 4.6. *Suppose that* $|\mathcal{K}_h|<+\infty$. *Then* $\lim_{k\to+\infty} h_k=0$.

Proof. Finiteness of the set $\mathcal{K}_h$ implies, using the updating rule of $h_k^{\max}$, that there is an index $k_2>0$ such that for all $k\geq k_2$, x_k is an f-iterate, i.e., the inequality (14) holds. Then following the acceptance rules for f-steps and (14), we have

$$\varphi_k^{\mu}-\varphi_{k+1}^{\mu}\geq \rho\sigma_1\alpha_k h_k^{\sigma_2}. \tag{49}$$

Without loss of generality, we still assume that ∇c_k has full rank for all $k\geq 0$. By Lemma 3.5, we have $\alpha_k\geq 0.5\alpha_k^f$, where α_k^f is given in (32). Noting that $c_k+\nabla c_k v_k=0$ and the algorithm does not update $h_k^{\max}$ for $k\geq k_2$, (49) yields

$$\varphi_k^{\mu}-\varphi_{k+1}^{\mu}\geq \frac{1}{2}\rho\sigma_1\min\left\{\frac{(1-\rho)\sigma_1 h_k^{\sigma_2}}{\left(L_{df}+\frac{\mu}{(1-\tau)\widetilde{\epsilon}^2}\right)M_d^2},\frac{(1-\kappa_1)(h_{k_0}^{\max})}{L_{dc}M_d^2}\right\}h_k^{\sigma_2} \tag{50}$$

for all $k\geq k_2$, where Theorem 4.3, (14) and the upper bound of $\{\|d_k\|\}$ are used. Note that Theorem 4.3 ensures the boundedness of $\{\varphi_k^{\mu}\}$ and the acceptance criteria for f-steps imply the non-increasing property for $\{\{\varphi_k^{\mu}\}\}_{k\geq k_2}$. It follows from (50) that $\lim_{k\to\infty}h_k=0$. □

Lemma 4.7. *Suppose that* $|\mathcal{K}_h| < +\infty$. *Then* $\lim_{k\to+\infty} t_k = 0$.

Proof. By similar arguments as the proof of Lemma 4.6, we have

$$\varphi_k^\mu - \varphi_{k+1}^\mu \geq -\frac{1}{2}\rho \min\left\{ \frac{-(1-\rho)(\nabla\varphi_k^\mu)^T d_k}{\left(L_{df} + \frac{\mu}{(1-\tau)\tilde{\epsilon}^2}\right) M_d^2}, \frac{(1-\kappa_1)(h_{k_0}^{\max})}{L_{dc} M_d^2} \right\} (\nabla\varphi_k^\mu)^T d_k,$$

for all $k \geq k_2$. It follows that $\lim_{k\to\infty} \|(\nabla\varphi_k^\mu)^T d_k\| = 0$, where the non-increasing property and lower boundedness of $\{\varphi_k^\mu\}$ are used. By Lemma 4.6, we have $\lim_{k\to\infty} \|(\nabla\varphi_k^\mu)^T t_k\| = 0$. Therefore, by (13) and (27), we have $\lim_{k\to\infty} t_k = 0$. □

Lemma 4.8. *Suppose that Assumption 1 hold, that there is an index set* $\mathcal{K} \subset \{0, 1, 2, \ldots\}$ *such that*

$$\lim_{k\in\mathcal{K}} h_k = 0, \quad \lim_{k\in\mathcal{K}} \|t_k\| = 0,$$

and that $\tilde{x}$ *is an accumulation point of* $\{x_k \mid k \in \mathcal{K}\}$. *Then* $\tilde{x}$ *is a KKT point for (2).*

Proof. Without loss of generality, we assume that $\lim_{k\in\mathcal{K}} x_k = \tilde{x}$. By Theorem 4.3, $\tilde{x}^{(i)} > 0$ for all $i \in \{1, 2, \ldots, n\}$. By the rule of updating ν_k, Assumption 1 and Lemma 4.1, the matrix $\widetilde{W}_k + \frac{1}{\nu}\nabla c_k \nabla c_k^T$ is positive definite for sufficiently large $k \in \mathcal{K}$, which implies $\zeta_k = 0$. From (18) and (26), it follows that, for sufficiently large $k \in \mathcal{K}$,

$$\lambda_{k+1} = -(\nabla c_k^T \nabla c_k)^{-1} \nabla c_k^T (\nabla\varphi_k^\mu + \widetilde{W}_k(v_k + t_k)).$$

Note that the sequence

$$\{\lambda_{k+1} \mid k \in \mathcal{K} \text{ and is sufficiently large}\}$$

is convergent. Denote by $\tilde{\lambda}$ its limit. Then by taking limit on (26) with respect to $k \in \mathcal{K}$, we get

$$\nabla\varphi^\mu(\tilde{x}) + \nabla c(\tilde{x})\tilde{\lambda} = 0.$$

This, combined with the fact that

$$h(\tilde{x}) = \lim_{k\in\mathcal{K}} h_k = 0,$$

shows that $\tilde{x}$ is a KKT point for (2). □

To sum up Lemmas 4.4–4.7, we get our global convergence theorem.

Theorem 4.9. *Under Assumption 1, suppose that Algorithm 3 does not terminate finitely.*

(i) *If* $|\mathcal{K}_h| = \infty$, *then any accumulation point of* $\{x_k\}_{k\in\mathcal{K}_h}$ *is a KKT point for (2).*
(ii) *If* $|\mathcal{K}_h| < \infty$, *then any accumulation point of* $\{x_k\}$ *is a KKT point for (2).*

Theorem 4.9 indicates that the inner loop of Algorithm 1, i.e., Step 2 will terminate finitely under Assumption 1. Hence, by the mechanism of the algorithm, we get the global convergence of the whole algorithm.

Theorem 4.10. *Under Assumption 1, if Algorithm 1 does not terminate finitely, then at least one of the accumulation points of the iterate sequence is a KKT point for problem (1).*

5. Applications

In this section, we solve some degenerate problems for test purpose. We choose those problems rather than non-degenerate ones because we note that the term $\frac{1}{\nu}\nabla c \nabla c^T$ can improve the regularity of the subproblem. These problems do not satisfy MFCQ because of not having strict interior region. Hence, the set of dual variables near the solution set is unbounded, which may causes serious numerical problems in practice. Our test shows that our method still works on these problems.

However, we should point out that we aim to show the probable applications of this method to degenerate problems. We do not modify the algorithmic framework before application. To get better numerical performance, the special structure of the degenerate problems should be carefully considered, which is beyond the scope that this paper discusses.

The first problem is a bilevel programming problem which goes back to [37]:

$$\begin{aligned} &\min x_1^2 - 2x_1 + x_2^2 - 2x_2 + y_1^2 + y_2^2 \\ &\quad \text{s.t. } 0 \leq x \leq 2, \end{aligned}$$

where $y = (y_1, y_2)$ solves problem

$$E(x) \quad \begin{aligned} &\min\,(x_1-y_1)^2+(x_2-y_2)^2\\ &\text{s.t. } 0.25-(x_1-1)^2\ge 0,\\ &\qquad 0.25-(x_2-1)^2\ge 0. \end{aligned}$$

Problem $E(x)$ has a point of nondifferentiability at the optimal solution $(x_1^*, x_2^*) = (0.5, 0.5)$. This problem was used in [38,39] for test purpose. In the MacMPEC test suite [40], this problem is formulated into a mathematical programming with equilibrium constraints (MPEC):

$$\begin{aligned} &\min x_1^2-2x_1+x_2^2-2x_2+y_1^2+y_2^2\\ &\text{ s.t. } 2y_1-2x_1+2(y_1-1)l_1=0,\\ &\quad 2y_2-2x_2+2(y_2-1)l_2=0,\\ &\quad 0\le 0.25-(y_1-1)^2\perp l_1\ge 0,\\ &\quad 0\le 0.25-(y_2-1)^2\perp l_2\ge 0,\\ &\quad 0\le x\le 2. \end{aligned} \tag{51}$$

To apply our method, we write (51) into the form (2) by introduce slack variables, which reads

$$\begin{aligned} &\min x_1^2-2x_1+x_2^2-2x_2+y_1^2+y_2^2\\ &\text{ s.t. } 2y_1-2x_1+2(y_1-1)l_1=0,\\ &\quad 2y_2-2x_2+2(y_2-1)l_2=0,\\ &\quad 0.25-(y_1-1)^2-z_1=0,\\ &\quad 0.25-(y_2-1)^2-z_2=0,\\ &\quad z^T l=0,\\ &\quad 0\le x\le 2,\ \ l,z\ge 0. \end{aligned}$$

The second problem is a Stackelberg leader follower game studied by Henderson and Quandt [41] and was tested by [38,39,42]. It is stated as

$$\begin{aligned} &\max_{x_1\ge 0} \pi_1 = x_1F(x_1+x_2)-C_1(x_1)\\ &\text{ s.t. } \max_{x_2\ge 0}\ \pi_2 = x_2F(x_1+x_2)-C_2(x_2), \end{aligned}$$

where π_1 denotes the profit of the leader, π_2 denotes the profit of the follower,

$$F(x_1+x_2)=100-0.5(x_1+x_2)$$

is the invested demand function and

$$C_1(x_1)=5x_1,\ \ C_2(x_2)=0.5x_2^2$$

give the total costs of the leader and the follower, respectively. This problem can be written into a MPEC by using the KKT conditions of the lower-level programming:

$$\begin{aligned} &\min\ -x_1F(x_1+x_2)+C_1(x_1)\\ &\text{ s.t. } 0.5x_1+2x_2-100-y=0,\\ &\quad 0\le x_1\le 200,\\ &\quad 0\le x_2\perp y\ge 0. \end{aligned}$$

We reformulated this MPEC into the standard form of nonlinear programming

$$\begin{aligned} &\min\ -x_1F(x_1+x_2)+C_1(x_1)\\ &\text{ s.t. } 0.5x_1+2x_2-100-y=0,\\ &\quad x_2y=0,\\ &\quad 0\le x_1\le 200,\ \ x_2,y\ge 0. \end{aligned}$$

Table 1
The results for the tested problems.

Prob.	Init	nf	ngf	nc	ngc	f^*
1	$x_0 = 0, y_0 = e$	21	14	9	14	−1.0000
2	$x_0 = 0, y_0 = 5e$	21	21	18	21	−3.2677
3	$x_0 = 5e, y_0 = 10$	18	17	10	17	3.2077
4	$x_0 = 5e, y_0 = 10$	29	23	15	23	3.4494
5	$x_0 = 5e, y_0 = 10$	25	21	15	21	4.6034
6	$x_0 = 5e, y_0 = 10$	24	21	11	21	6.5927

The following four problems are taken from [43]. The original problems are bilevel programming which share a same lower level programming. Since the lower programming is convex, they are reformulated into optimizations with variational inequality constraints [42,43] and into MPECs by Leyffer [40] in MacMPEC suite. In MacMPEC the constraints of these four problems are

$$\begin{aligned}
&0 \le (1 + 0.2y)x_1 - (3 + 1.333y) - 0.333x_3 + 2x_1x_4 \perp x_1 \ge 0,\\
&0 \le (1 + 0.1y)x_2 - y + x_3 + 2x_2x_4 \perp x_2 \ge 0,\\
&0 \le 0.333x_1 - x_2 + 1 - 0.1y \perp x_3 \ge 0,\\
&0 \le 9 + 0.1y - x_1^2 - x_2^2 \perp x_4 \ge 0.
\end{aligned}$$

The objective functions are different, which are, in turn,

$$\begin{aligned}
f_1 &= \frac{1}{2}((x_1 - 3)^2 + (x_2 - 4)^2);\\
f_2 &= \frac{1}{2}((x_1 - 3)^2 + (x_2 - 4)^2 + (x_3 - 1)^2);\\
f_3 &= \frac{1}{2}((x_1 - 3)^2 + (x_2 - 4)^2 + 10x_4^2);
\end{aligned}$$

and

$$f_4 = \frac{1}{2}((x_1 - 3)^2 + (x_2 - 4)^2 + (x_3 - 1)^2 + (x_4 - 1)^2 + y^2).$$

We also reformulate these problems into the standard form (2) in which the constraints read

$$\begin{aligned}
&(1 + 0.2y)x_1 - (3 + 1.333y) - 0.333x_3 + 2x_1x_4 - y_1 = 0,\\
&(1 + 0.1y)x_2 - y + x_3 + 2x_2x_4 \perp x_2 - y_2 = 0,\\
&0.333x_1 - x_2 + 1 - 0.1y \perp x_3 - y_3 = 0,\\
&9 + 0.1y - x_1^2 - x_2^2 \perp x_4 - y_4 = 0,\\
&x^T y = 0, x \ge 0, y \ge 0.
\end{aligned}$$

Our algorithm was implemented in Matlab and run on a Dell Inspiron 14-7437 personal computer with Intel Core i5 CPU, 1.60 GHz and a 6 G RAM. For all the tests, the algorithm is terminated if

$$E_0(x, \lambda, \mu) \le 10^{-5}.$$

The following parameter setting is used:

$$\begin{aligned}
&\mu_0 = 2,\ \tau_{\min} = 10^{-5},\ \kappa_\mu = 0.25,\ \theta_\mu = 2,\ \kappa_\sigma = 10^2,\\
&\kappa_\epsilon = 10, \psi = 10^{-3},\ \kappa_v = 0.25,\ M_v = 100,\ v_{\min} = 10^{-18},\\
&\kappa_1 = 0.01,\ \kappa_2 = 0.01,\ \alpha_{\min} = 10^{-5}, \tau_{\min} = 0.95,\ \kappa_\alpha = 0.5,\\
&\theta_\alpha = 2, \sigma_1 = 0.01,\ \sigma_2 = 2; \rho = 10^{-8}, \kappa_h = 0.5,\ \bar{\kappa}_h = 0.25.
\end{aligned}$$

We use exact Hessian during this test. When an inner loop is finished with an approximate solution, the barrier parameter is updated via the following rule

$$\mu_{j+1} = \min\{\kappa_\mu \mu, \mu^{\theta_\mu}\},$$

and τ is updated as $\tau_{j+1} = \max\{\tau_{\min}, 1 - \mu_{j+1}\}$.

The results are listed in Table 1. In this table, *nf*, *nc*, *ngf* and *ngc* refer to the times of evaluating objective function, constraints and their gradients, respectively. The optimal value of objective function are reported in column f^*. The column *Init*. gives the starting point. The symbol e denotes a vector of all ones with proper dimension.

6. Discussion

In this paper, we present a new interior point algorithm for nonlinear programming. This method uses Byrd–Omojokun's step decomposition idea, and are characterized by employing quasi-tangential subproblem. The idea of the quasi-tangential subproblem is to penalized the null space constraint to the objective function. Global convergence of this algorithm has been studied and test on a set of small test problems were done.

As we have shown in the paper, under reasonable assumptions, this algorithm compute trial steps by solving two unconstrained optimizations, which are easy to solve. However, penalizing null space constraints results a quadratic item as

$$d^T(H + \frac{1}{\nu}\nabla c \nabla c^T)d.$$

The symmetric matrix $H + \frac{1}{\nu}\nabla c \nabla c^T$ is more regular than H, but is always denser, which may lead to more expensive computation when dealing with large scale problems. In order to maintain the sparsity of the large scale problems, the quasi-tangential subproblem (12) can be formulated to an augmented sparse linear system. Using the first order necessary condition of (12), we get

$$\bar{W}_k t + \frac{1}{\nu_k}\nabla c_k \nabla c_k^T t = -\nabla \varphi_k^{\mu} - \bar{W}_k v_k,$$

where $\bar{W}_k = \widetilde{W}_k + \zeta_k I$. Let $\eta = \frac{1}{\nu_k}\nabla c_k^T t$ and by simple induction, we have the following linear system

$$\begin{pmatrix} \bar{W}_k & \nabla c_k \\ \nabla c_k^T & -\nu_k I \end{pmatrix} \begin{pmatrix} t \\ \eta \end{pmatrix} = -\begin{pmatrix} \nabla \varphi_k^{\mu} + \bar{W}_k v_k \\ 0 \end{pmatrix}. \tag{52}$$

By selecting proper parameter ζ_k, the coefficient matrix of (52) is quasi-definite [44], hence it can be effectively factorized. System (52) is symmetric and sparse, and hence can be efficiently solved by large scale sparse linear system solver. The augmented formulation (52) also indicate the connection of our algorithm with the regularized interior point methods (RIPM) (see, for instance, [45,46] etc.) and stabilized SQP (sSQP) methods (see [47–49] etc.). Given the considerable researches on RIPM and sSQP, there are lots of further work left to do, such as algorithm for large scale problems, convergence properties under degenerate assumptions, etc.

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