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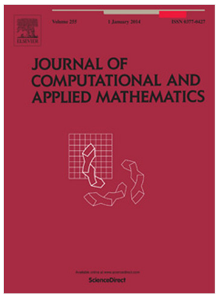

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Convergence of a Stirling-like method for fixed points in Banach spaces

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Abstract

We have very few literature on fixed point iterative methods for solving nonlinear equations. We consider the Stirling method given by Rall [2]. Based on stirling method, In this paper, we propose a third order stirling-like method for finding fixed point for nonlinear equations in Banach spaces. We study the local and semilocal convergence of this method for finding the fixed points of nonlinear equations in Banach spaces. The convergence established under the assumption that the first order Fréchet derivative satisfies the Lipschitz continuity condition. The existence and uniqueness theorem that establishes the convergence balls of these methods is obtained. We consider the numerical examples for local and semilocal convergence case and calculate the existence and uniqueness region of convergence balls even we fail to apply the results in [6, 11–13] due to the G is not contraction on Ω.

Keywords:Local convergence, Semilocal convergence, Fréchet derivative, Fixed points, Banach spaces

1 Introduction

One of the most important problems in computational mathematics is to solve nonlinear equations. The problem of solving these systems of nonlinear equations arises in diverse areas of engineering, mathematics, physics, chemistry and biology for systems which model various phenomena. In many situations, the nonlinear problems naturally appear in the form of nonlinear equations or systems of nonlinear equations in Banach spaces. Finding the solution of nonlinear equations is actually enough motivation for researchers to develop new computationally efficient iterative methods. A main reason is of course that analytical solutions are often not available for most types of nonlinear equations and hence numerical iterative methods are best suited for this purpose. We discuss the development of iterative schemes for scalar nonlinear equations, for computing the generalized inverse of a matrix, for general classes of systems of nonlinear equations and for specific systems of nonlinear equations associated with ordinary and partial differential equations. We have experience with many higher order iterative methods for solving nonlinear equations. The convergence analysis of iterative methods in Banach spaces is usually divided into two categories, namely semilocal and local convergence analysis. The local convergence analysis is based on

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the information in the vicinity of the solution, and the semilocal convergence is based on assumptions at initial approximations and on the domain. There are two different types of approaches to discuss the convergence analysis of iterative methods. The majorizing sequence approach and the recurrence relations approach. Using these approaches, we assume different types of Kantorovich assumptions to discuss the convergence analysis.

The development of higher order one point and multi point iterative methods without involving higher order derivative have important role to solve nonlinear equations. Many researchers developed higher order iterative methods for solving nonlinear equations (see [7,9]). The local and semilocal convergence of higher order iterative methods developed by [8, 10]

The purpose of this study is computation of a fixed point of nonlinear operator equation of the form

$$x = F(x) \tag{1.1}$$

where, $F : \Omega \subseteq X \to Y$ be a nonlinear Fréchet differentiable on a convex subset Ω of X with values in Y. We motivated from this approximating a fixed point x^* of nonlinear operator equation (1.1). This is important and challenging problems extensively studied in numerical analysis and many applied scientific areas. Bartle [1] studied the following quadratically convergence iterative method for finding fixed points of nonlinear equation

$$x_{n+1} = x_n - (I - F'(y_n))^{-1}(x_n - F(x_n)) \tag{1.2}$$

where, F has a Fréchet derivative F' at least at the required point. From this, for $y_n = x_n$ it is reduced to Newton's method and for $y_n = x_0$ modified Newton's method. The special case for $y_n = F(x_n), n = 0, 1, 2 \ldots$ we obtained the stirling's method (see Ref [2]). Consider the Stirling method defined in [3]. The semilocal convergence of a Stirling method under the assumption that the first order Fréchet derivative satisfies the Lipschitz continuity condition established by [4]. The local and semilocal convergence of the stirling method given in [5, 6, 11–13] under the restrictive assumption that $\|F'(x)\|$ is strictly bounded above by 1.The local and semilocal convergence provided without making use of $\|F'(x)\| < 1$ established in [6]. Inspired by this, we propose stirling like method

$$\begin{cases} y_n = F(x_n), \\ z_n = x_n - (I - F'(y_n))^{-1}(x_n - F(x_n)) \\ x_{n+1} = z_n - (I - F'(y_n))^{-1}(z_n - F(z_n)), \end{cases} \tag{1.3}$$

The main focus of this paper is that the the local and semilocal convergence of Stirlings method(1.3) is established, even if $\|F'(x)\| < 1$ is not satisfied. The convergence established under the assumption that the first order Fréchet derivative of the involved operator satisfies the Lipschitz continuity condition. The existence and uniqueness region of the solution for the method established. We have considered the number of numerical examples and computed radii of the convergence balls even we fail to apply the results in [5, 11–13] due to the F is not contraction on Ω.

2 Local convergence analysis

In this section, we discuss the local convergence analysis based on some Lipschitz constants and some contraction consideration. For prove the local convergence study of (1.3), Let $\Gamma_0 = (I - F'(y_0))^{-1} \in BL(X, X)$, exists at some point $x_0, y_0 \in \Omega$, where, $BL(X, X)$ is bounded linear operator defined from X to X and the following assumptions hold on the operator F such that

$$A_* = I - F'(F(x^*)) \tag{2.1}$$

is invertible, and for $\alpha \geq 0$, $\beta \in (0, 1)$ and $\gamma \geq 0$ and for all $x, y \in \Omega$ such that

$$\|A_*^{-1}[F'(F(x^*)) - F'(F(x))]\| \leq \alpha\|F(x^*) - F(x)\| \tag{2.2}$$

$$\|F(x^*) - F(x)]\| \leq \beta\|x^* - x\| \tag{2.3}$$

$$\|A_*^{-1}[F'(x) - F'(y)]\| \leq \gamma\|x - y\| \tag{2.4}$$

Now, define function $g_1(t)$ on the interval $[0, \frac{1}{\alpha\beta})$ by

$$g_1(t) = \frac{\gamma(1 + 2\beta)t}{2(1 - \alpha\beta t)} \tag{2.5}$$

$$g_2(t) = \frac{\gamma(g_1(t)(1 + \beta) + \beta g_1(t))g_1(t)t}{2(1 - \alpha\beta t)} \tag{2.6}$$

Throughout this study, we denote $\overline{\mathcal{B}}(x^*, \rho)$ and $\mathcal{B}(x^*, \rho)$ as closed and open ball at centered at x^* and radius ρ respectively. Now, we present the local convergence theorem of (1.3) followed by the above defined functions.

Theorem 1 *Let $F : \Omega \subseteq X \to X$ be a Fréchet differentiable operator. Suppose that there exist $x^* \in \Omega$ such that (2.1), (2.2),2.3 and (2.4) are satisfied and $\overline{B}(x^*, r) \subseteq \Omega$, where, the radius r is to be determined. The sequence $\{x_n\}$ generated by (1.3) for $x_0 \in B(x^*, r)$ is well defined for $n = 0, 1, 2 \ldots$ remains in $B(x^*, r)$ for all $n \geq 0$, and converges to x^*. Moreover, the following hold for $n = 0, 1, 2, \ldots,$*

$$\|y_n - x^*\| \leq \beta\|x_n - x^*\| < \|x_n - x^*\| \tag{2.7}$$

$$\|z_n - x^*\| \leq g_1(\|x_n - x^*\|)\|x_n - x^*\| < \|x_n - x^*\| \tag{2.8}$$

and

$$\|x_{n+1} - x^*\| \leq g_2(\|x_n - x^*\|)\|x_n - x^*\| < \|x_n - x^*\|, \tag{2.9}$$

where g_i, $i = 1, 2$ are defined above.

Proof We shall prove (2.7)-(2.9) using Mathematical induction. Using (2.1), (2.2), we get

$$\begin{aligned}
\|A_*^{-1}(A_* - (I - F'(F(x))))\| &\leq \|A_*^{-1}(I - F'(F(x^*)) - I + F'(F(x)))\| \\
&\leq \|A_*^{-1}(F'(F(x)) - F'(F(x^*)))\| \\
&= \alpha\|F(x) - F(x^*)\| = \alpha\beta\|x - x^*\| \leq \alpha\beta r^* < 1 \qquad (2.10)
\end{aligned}$$

and,

$$\|F(x) - x^*\| = \|F(x) - F(x^*)\| \leq \beta\|x - x^*\| \leq \beta r^* \leq r^* \qquad (2.11)$$

Using Banach Lemma on invertible functions, $(I - F'(F(x)))^{-1}$ exists and

$$|(I - F'(F(x)))^{-1}A_*| \leq \frac{1}{1 - \alpha\beta|x_0 - x^*|}. \qquad (2.12)$$

Thus, y_0 is well defined and using (1.3) for $n = 0$, we get

$$\begin{aligned}
\|y_0 - x^*\| &= \|F(x_0) - F(x^*)\| \\
&= \beta\|x_0 - x^*\| < \|x_0 - x^*\|
\end{aligned} \qquad (2.13)$$

this shows that (2.7) holds for $n = 0$ and $y_0 \in \mathcal{B}(x^*, t)$. Now to proceed further, From the second step of the method, we get

$$\begin{aligned}
z_0 - x^* &\leq (I - F'(F(x_0)))^{-1}(x_0 - F(x_0))\| \qquad (2.14)\\
&\leq (I - F'(F(x_0)))^{-1}((I - F'(F(x_0)))(x_0 - x^*) - (x_0 - F(x_0))) \\
&\leq (I - F'(F(x_0)))^{-1}A_*A_*^{-1}\Big(\int_0^1 F'(\theta x_0 + (1-\theta)x^*) - F'(\theta F(x_0) + (1-\theta)F(x_0))\Big)(x_0 - x^*)d\theta
\end{aligned}$$

Apply the norm on both side, we get

$$\begin{aligned}
\|z_0 - x^*\| &\leq \frac{1}{(1 - \alpha\beta\|x_0 - x^*\|)}\gamma\int_0^1 \|\theta x_0 + (1-\theta)x^* - (\theta F(x_0) - (1-\theta)F(x_0))\|\|x_0 - x^*\|d\theta \\
&\leq \frac{\gamma}{2(1 - \alpha\beta\|x_0 - x^*\|)}\Big(\|x_0 - F(x_0)\| + \|F(x_0) - F(x^*)\|\Big)\|x_0 - x^*\| \qquad (2.15)
\end{aligned}$$

Since,

$$\|x_0 - F(x_0)\| \leq \|x_0 - x^*\| + \|F(x_0) - F(x^*)\| \leq (1 + \beta)\|x_0 - x^*\| \qquad (2.16)$$

We get,

$$\begin{aligned}
\|z_0 - x^*\| &\leq \frac{\gamma((1+\beta)\|x_0 - x^*\| + \beta\|x_0 - x^*\|)}{2(1 - \alpha\beta\|x_0 - x^*\|)}\|x_0 - x^*\| \\
&\leq g_1(\|x_0 - x^*\|)\|x_0 - x^*\| \qquad (2.17)
\end{aligned}$$

Consider the function $h_1(t) = g_1(t) - 1$. Since, $h_1(0) = -1 < 0$ and $h_1(1/\alpha\beta) \to \infty$. Therefore, by intermediate value theorem, $h_1(t)$ has at least one root in $(0, 1/\alpha\beta)$. let r_1 be the smallest root of $h_1(t)$ in $(0, 1/\alpha\beta)$. then we get $0 < r_1 < 1/\alpha\beta$ and

$$0 \leq g_1(t) < 1 \tag{2.18}$$

Therefore, by using (2.17) and (2.18), we get

$$\|z_0 - x^*\| \leq g_1(\|x_0 - x^*\|)\|x_0 - x^*\| < \|x_0 - x^*\| \tag{2.19}$$

Since

$$\begin{aligned} \|z_0 - F(z_0)\| &\leq \|z_0 - x^*\| + \|F(z_0) - F(x^*)\| \\ &\leq \|z_0 - x^*\| + \beta\|z_0 - x^*\| \\ &\leq (1+\beta)\|z_0 - x^*\| \\ &\leq g_1(\|x_0 - x^*\|)(1+\beta)\|x_0 - x^*\| \end{aligned} \tag{2.20}$$

From the third step of the iterative method,

$$\begin{aligned} \|x_1 - x^*\| &\leq \|(I - F'(F(x_0)))^{-1}\Big((I - F'(F(x_0)))(z_0 - x^*) - (z_0 - F(z_0))\Big)\| \\ &\leq \|(I - F'(F(x_0)))^{-1}A_*\|\|A_*^{-1}\int_0^1 F'(\theta z_0 + (1-\theta)x^*) - F'(\theta F(z_0) + (1-\theta)F(z_0))\|\|z_0 - x^*\|d\theta \\ &\leq \frac{\gamma}{1 - \alpha\beta\|x_0 - x^*\|}\int_0^1 \|\theta z_0 + (1-\theta)x^* - \theta F(z_0) - (1-\theta)F(z_0)\|\|z_0 - x^*\|d\theta \\ &\leq \frac{\gamma}{2(1 - \alpha\beta\|x_0 - x^*\|)}(\|z_0 - F(z_0)\| + \|F(x^*) - F(z_0)\|)\|z_0 - x^*\| \end{aligned} \tag{2.21}$$

Using (2.20), we get

$$\begin{aligned} \|x_1 - x^*\| &\leq \frac{\gamma}{2(1 - \alpha\beta\|x_0 - x^*\|)}\Big(g_1(\|x_0 - x^*\|)(1+\beta)\|x_0 - x^*\| \\ &\quad +\beta g_1(\|x_0 - x^*\|)\|x_0 - x^*\|)g_1(\|x_0 - x^*\|\Big)\|x_0 - x^*\| \qquad (2.22) \\ &\leq g_2(\|x_0 - x^*\|)\|x_0 - x^*\| \end{aligned}$$

Consider the function $h_2 = g_2(t) - 1$, Since, $h_2(0) = -1 < 0$ and $h_2(r_1) > 0$. Therefore, by intermediate value theorem, $h_2(t)$ has at least one root in $(0, r_1)$. let r be the smallest root of $h_2(t)$ in $(0, r_1)$. then we get $r < r_1 < 1/\alpha\beta$ and

$$0 \leq g_2(t) < 1 \forall t \in [0, r) \tag{2.23}$$

By using (2.22) and (2.23) we have

$$\|x_1 - x^*\| \le g_2(\|x_0 - x^*\|)\|x_0 - x^*\| < \|x_0 - x^*\| < r \tag{2.24}$$

Thus $x_1 \in \mathcal{B}(x^*, r)$ and (2.9) holds for $n = 0$. Now, interchange $x_0,\ y_0,\ z_0$ and x_1 by $x_n,\ y_n,\ z_n$ and x_{n+1} in the similar way. We arrive to the estimate (2.7)-(2.9) for $n = 0, 1, 2, \ldots$ and $y_n,\ z_n$ and $x_{n+1} \in \mathcal{B}(x^*, r)$. Now,

$$\|x_{n+1} - x^*\| \le g_2(\|x_n - x^*\|)\|x_n - x^*\| < g_2(\|x_n - x^*\|)g_2(\|x_{n-1} - x^*\|)\|x_{n-1} - x^*\| < \ldots < g_2(t)^{n+1}(\|x_0 - x^*\|)$$

This shows that $x_n \to x^*$ as $n \to \infty$.

3 Semilocal convergence analysis

In this section, semilocal convergence analysis of (1.3) is developed. Let $\eta > 0, a_0 > 0, b > 0, l > 0$ and l_0 are some positive constants. We say that F belongs to the class $C(\eta, a_0, b, l, l_0)$ if

1. $\Gamma_0 = (I - F'(y_0))^{-1} \in BL(X, X)$.
2. $\|\Gamma_0(x_0 - F(x_0))\| \le \eta$
3. $\|F(x) - F(x_0)\| \le a_0\|x - x_0\|$
4. $\|F'(F(x))\| \le b$
5. $\|\Gamma_0(F'(x) - F'(y))\| \le l\|x - y\|$
6. $\|\Gamma_0(F'(F(x)) - F'(F(x_0)))\| \le l_0\|F(x) - F(x_0)\|$

In earlier studies $[1, 3, 5]$, the condition $b < 1$ is used which restricts the applicability of (1.3). This condition is omitted here and in place of this condition, we have taken $a_0 < 1$. Now, for the semilocal convergence of (1.3), we present a lemma which will utilize further for the main result. This lemma shows the convergence of sequences $\{s_n\}$ and $\{t_n\}$, starting with $s_0 = \eta,\ t_0 = 0,\ t_1 = s_0 + l\left(\frac{1+3b}{2}\right)(s_0 - t_0)^2$ and for $n \ge 1$

$$s_n = t_n + \frac{l\left(b(s_{n-1} - t_{n-1}) + \frac{1}{2}(t_n - s_{n-1})\right)(t_n - s_{n-1})}{1 - l_0 a_0 t_n} \tag{3.1}$$

$$t_{n+1} = s_n + l\left(\frac{1+3b}{2}\right)\frac{(s_n - t_n)^2}{1 - l_0 a_0 t_n}. \tag{3.2}$$

Let us define two functions $h_1(t)$ and $h_2(t)$, by

$$h_1(t) = l_0 a_0 t^4 + \left(l_0 a_0 + \frac{1}{2}l\right)t^3 + lbt^2 - \frac{1}{2}lb - \frac{1}{2}l \tag{3.3}$$

$$h_2(t) = \left(l\left(\frac{1+3b}{2}\right) + l_0 a_0\right)t^2 + l_0 a_0 t - l\left(\frac{1+3b}{2}\right) \tag{3.4}$$

We can easily get that $h_1(0) = -\frac{1}{2}lb - \frac{1}{2}l < 0$ and $h_1(1) = 2l_0a_0 + \frac{lb\eta}{2} > 0$. Using intermediate value theorem, there exist at least one root of $h_1(t)$ in $(0,1)$, denote δ_1 as the such smallest positive root. Now, $h_2(0) = -l\left(\frac{1+3b}{2}\right) < 0$ and $h_2(1) = 2l_0a_0 > 0$. Using intermediate value theorem, there exist at least one root of $h_2(t)$ in $(0,1)$, denote δ_2 as the such smallest positive root. Let us define another quantities δ_3 and δ_4, by

$$\begin{aligned}\delta_3 &= \frac{l\left(b\eta + \frac{1}{2}l\left(\frac{1+3b}{2}\right)\eta^2\right)}{1 - l_0a_0t_1}\\ \delta_4 &= l\left(\frac{1+3b}{2}\right)\eta\end{aligned}$$

Suppose that $\delta = \max(\delta_1, \delta_2, \delta_3, \delta_4) < 1$ and choose the quantity $\beta = \min(\beta_1, \beta_2)$, where

$$\beta_1 = 1 - l_0a_0\eta \quad \text{and} \quad \beta_2 = \frac{1}{1 + l_0a_0\eta}$$

Lemma 1 *Let $\{s_n\}$ and $\{t_n\}$ be two sequences defined by (3.1) and (3.2), respectively. Assume $\delta \leq \beta$ and $\eta < \frac{1}{l_0a_0}$, then sequences $\{s_n\}$ and $\{t_n\}$ defined by (3.1) and (3.2), respectively are well defined and bounded from above by $l^{**} = \frac{\eta}{1-\delta}$. Moreover, it converge to their unique least upper bound $l^* \in [0, l^{**})$. Furthermore, the following hold for all $n \geq 0$.*

$$t_n \leq s_n \leq t_{n+1} \leq s_{n+1},$$

and

$$\begin{cases} s_{n+1} - t_{n+1} \leq \delta(t_{n+1} - s_n) \\ \quad t_{n+1} - s_n \leq \delta(s_n - t_n), \end{cases} \tag{3.5}$$

δ is defined above.

Proof Using mathematical induction on 'n', we shall prove this lemma. For this, we first show that

$$0 \leq \frac{l\left(b(s_n - t_n) + \frac{1}{2}(t_{n+1} - s_n)\right)}{1 - l_0a_0t_{n+1}} \leq \delta, \tag{3.6}$$

$$0 \leq l\left(\frac{1+3b}{2}\right)\frac{(s_n - t_n)}{1 - l_0a_0t_n} \leq \delta \text{ and} \tag{3.7}$$

$$0 \leq l_0a_0t_{n+1} < 1 \tag{3.8}$$

For n=0, it directly follows from the definition of δ and assumption of the lemma. Suppose that (3.5)-(3.8) is true for some $k \leq n$. Using induction hypothesis, we get

$$\begin{aligned} s_k &\leq t_k + \delta(t_k - s_{k-1})\\ &\leq s_{k-1} + \delta(s_{k-1} - t_{k-1}) + \delta^2(s_{k-1} - t_{k-1})\\ &\leq \ldots\ldots\\ &\leq (1 + \delta + \ldots + \ldots + \delta^{2k})\eta = \frac{1 - \delta^{2k+1}}{1 - \delta}\eta \end{aligned} \tag{3.9}$$

and in a similar way, we have

$$t_{k+1} \leq s_k + \delta(s_k - t_k) \leq (1 + \delta + \delta^2 + \ldots + \delta^{2k+1})\eta. \tag{3.10}$$

Now, in order to show (3.6), with the help of (3.9) and (3.10), it will be sufficient to show that

$$l\left(b(s_k - t_k) + 1/2(t_{k+1} - s_k)\right) + \delta l_0 a_0 t_{k+1} - \delta \leq 0 \text{ or}$$
$$\delta l_0 a_0 \frac{1 - \delta^{2k+2}}{1 - \delta}\eta + l(b\delta^{2k} + \frac{1}{2}\delta^{2k+1})\eta - \delta \leq 0.$$

This motivates us to introduce the recurrent function h_k^1 on $(0, 1)$ by

$$h_k^1(t) = l_0 a_0 (1 + t + \ldots + t^{2k+1})\eta + lb\eta t^{2k-1} + \frac{1}{2} l\eta t^{2k} - 1 \tag{3.11}$$

In order to establish between two consecutive terms of $h_k^1(t)$, we replace 'k' by '$k+1$' in (3.11), which gives

$$\begin{aligned} h_{k+1}^1(t) &= l_0 a_0 (1 + t + \ldots + t^{2k+3})\eta + lb\eta t^{2k+1} + \frac{1}{2} l\eta t^{2k+2} - 1, \\ &= h_k^1(t) + t^{2k-1} h_1(t)\eta, \end{aligned} \tag{3.12}$$

where $h_1(t)$ is defined in (3.3). Since, δ_1 is the smallest positive root of $h_1(t)$ and

$$h_k^1(0) = l_0 a_0 \eta - 1 < 0 \tag{3.13}$$

for sufficiently large $t > 0$, we have

$$h_k^1(t) > 0 \tag{3.14}$$

using (3.13) and (3.14) and the intermediate value theorem, it assures the existence of at least one root of $h_k^1(t)$ for each $k \geq 1$. Let us denote them by δ_1^k. We can assert that δ_1^k are unique as ${h_k^1}'(t) > 0$ for $t > 0$. Let us define the function $h_\infty^1(t)$ by

$$h_\infty^1(t) = \lim h_k(t) \tag{3.15}$$

Using (3.11), we have

$$h_\infty^1(t) = \frac{l_0 a_0 \eta}{1 - t} - 1 \tag{3.16}$$

Using the definition of (3.15) and δ_1, we have that $h_{k+1}^1(t) = h_k^1(t)$ for each k. As ${h_k^1}'(t) > 0$ for each k this shows that $h_{k+1}^1(t)$ is an increasing sequence and $h_\infty^1(\delta_1) \leq 0$. Thus, (3.6) holds true. Now, we shall show (3.7), we adopt the same procedure as for (3.6). For convenience, we take $L = l\left(\frac{1+3b}{2}\right)$ and then (3.7) can be rewritten as

$$L(s_k - t_k) \leq \delta(1 - l_0 a_0 t_k)$$

which gives

$$L\delta^{2k}\eta + l_0 a_0(\delta + \ldots + \delta^{2k})\eta - \delta \leq 0$$

Now, repeating the same way as above. We can easily establish that sequences $\{s_n\}$, $\{t_n\}$ are nondecreasing, bounded from above by l^{**} and such that they converge to their common limit l^*.

Theorem 2 *Let $F \in \mathcal{C}$ as defined above. Suppose that Lemma 1 hold true and $\overline{\mathcal{B}}(x_0, l^*) \subseteq \Omega$. Then starting with x_0, sequence $\{x_n\}$ generated by (1.3) is well defined and converge to the fixed point $x^* \in \overline{\mathcal{B}}(x_0, l^*)$ of (1.1). Furthermore, the following estimates hold for all $n \geq 0$*

$$\|z_n - x_n\| \leq s_n - t_n \tag{3.17}$$

$$\|x_{n+1} - z_n\| \leq t_{n+1} - s_n, \tag{3.18}$$

where s_k, t_k and l^* are defined in Lemma 1 and definitions therein. Finally, if there exist R such that $\frac{la_0}{2}(t^* + R) < 1$ then x^* is unique in $\overline{\mathcal{B}}(x_0, R)$.

Proof Using induction on n, we show (3.17) and (3.18). First we show that the estimates hold for $n = 0$.

$$\|z_0 - x_0\| = \|\Gamma_0(x_0 - F(x_0))\| \leq \eta = s_0 - t_0$$

$$\begin{aligned} z_0 - y_0 &= z_0 - F(x_0) \\ &= z_0 - x_0 - (I - F'(y_0))(z_0 - x_0) = F'(y_0)(z_0 - x_0) \end{aligned}$$

This gives

$$\|z_0 - y_0\| \leq b\|(z_0 - x_0)\|. \tag{3.19}$$

Now to proceed further, we need some estimation

$$\begin{aligned} z_0 - F(z_0) &= x_0 - F(z_0) - \Gamma_0(x_0 - F(x_0)) \\ &= \Gamma_0((I - F'(y_0))(x_0 - F(z_0)) - (x_0 - F(x_0))) \\ &= \Gamma_0(F(x_0) - F(z_0) - F'(y_0)(x_0 - z_0) - F'(y_0)(z_0 - F(z_0))) \end{aligned} \tag{3.20}$$

and

$$(I + (I + F'(y_0))^{-1}F'(y_0))(z_0 - F(z_0)) = (I + F'(y_0))^{-1}\left(F(x_0) - F(z_0) - F'(y_0)(x_0 - z_0)\right) \tag{3.21}$$

Using (3.20) and (3.21), we get

$$(z_0 - F(z_0)) = F(x_0) - F(z_0) - F'(y_0)(x_0 - z_0) \tag{3.22}$$

Now, from (1.3), we have

$$
\begin{aligned}
x_1 - z_0 &= -(I - F'(y_0))^{-1}(z_0 - F(z_0)) \\
&= \int_0^1 F'(z_0 + t(x_0 - y_0)) - F'(y_0)(x_0 - z_0) \qquad (3.23)
\end{aligned}
$$

On taking norm both sides and using (3.19)-(3.23),

$$
\begin{aligned}
\|x_1 - z_0\| &\leq l\left(\|z_0 - y_0\| + \frac{1}{2}\|x_0 - y_0\|\right)\|z_0 - x_0\| \\
&\leq l\left(b\|z_0 - x_0\| + \frac{1}{2}(1+b)\|z_0 - x_0\|\right)\|z_0 - x_0\| \\
&\leq l\left(\frac{1+3b}{2}\right)\|z_0 - x_0\|^2 = l\left(\frac{1+3b}{2}\right)(s_0 - t_0)^2 = t_1 - s_0. \qquad (3.24)
\end{aligned}
$$

Thus,

$$
\|x_1 - x_0\| = \|x_1 - z_0 + z_0 - x_0\| \leq t_1 - s_0 + s_0 - t_0 = t_1 \in \mathcal{B}(0, l^*)
$$

the induction holds for $n = 0$. Suppose this is true for some $k \geq 1$.

$$
\|x_k - x_0\| \leq \|x_k - z_{k-1}\| + \ldots + \|z_0 - x_0\| \leq t_k - s_{k-1} + \ldots + s_0 - t_0 = t_k \in \mathcal{B}(0, l^*)
$$

This shows that $x_k \in \mathcal{B}(x_0, l^*)$ and it is easy to show that $y_k \in \mathcal{B}(x_0, l^*)$. Now,

$$
\begin{aligned}
\|I - \Gamma_0(I - F'(y_k))^{-1}\| &= \|\Gamma_0(F'(y_k) - F'(y_0))\| \\
&\leq l_0\|F(x_k) - F(x_0)\| \leq l_0 a_0\|x_k - x_0\| \leq l_0 a_0 t_k < 1.
\end{aligned}
$$

Using Banach lemma on invertible operators, we get

$$
\|(I - F'(y_k))^{-1}\Gamma_0\| \leq \frac{1}{1 - l_0 a_0 t_k}.
$$

Using some algebraic manipulations, we can show that

$$
x_k - F(x_k) = \left(F'(y_{k-1}) - \int_0^1 F'(z_{k-1} + t(x_k - z_{k-1}))dt\right)(x_k - z_{k-1}) \qquad (3.25)
$$

With the help of (1.3), (3.25) and definition of $\mathcal{C}$, we get

$$
\begin{aligned}
\|z_k - x_k\| &\leq \|(I - F'(y_k))^{-1}\Gamma\|\|\int_0^1 \Gamma_0(F'(y_{k-1}) - F'(z_{k-1} + t(x_k - z_{k-1})))dt\|\|(x_k - z_{k-1})\| \\
&\leq \|(I - F'(y_k))^{-1}\Gamma\| l\left(\|y_{k-1} - z_{k-1}\| + \frac{1}{2}\|x_k - z_{k-1}\|\right)(x_k - z_{k-1}) \\
&\leq \|(I - F'(y_k))^{-1}\Gamma\| l\left(b\|z_{k-1} - x_{k-1}\| + \frac{1}{2}\|x_k - z_{k-1}\|\right)(x_k - z_{k-1}) \\
&\leq l\left(b(s_{k-1} - t_{k-1}) + \frac{1}{2}(t_k - s_{k-1})\right)\frac{t_k - s_{k-1}}{1 - l_0 a_0 t_k} = s_k - t_k.
\end{aligned}
$$

Similarly, along the same lines as (3.24), it can also be shown that

$$
\|x_{k+1} - z_k\| \leq t_{k+1} - s_k.
$$

Now, we have to show the existence of x^*. Using (3.25) and $\mathcal{C}$ we have

$$\|x_k - F(x_k)\| \leq 2b(t_k - s_{k-1}) \leq 2b\delta^{2k} t_1 - s_0 \to 0 \text{ as } k \to \infty$$

This shows that x^* is the unique fixed point of (1.1). It remains to show the uniqueness part. Suppose y^* be another fixed point of (1.1) in $\mathcal{B}(x_0, R)$, then we have

$$0 = x^* - F(x^*) - y^* + F(y^*) = x^* - y^* - \int_0^1 F'(y^* + t(x^* - y^*))dt(x^* - y^*).$$

In order to show $x^* = y^*$, it will be sufficient to show that $\int_0^1 I - F'(y^* + t(x^* - y^*))$ is invertible. Let $T = \int_0^1 I - F'(y^* + t(x^* - y^*))$ and

$$\begin{aligned} \|I - \Gamma_0 T\| &\leq \int_0^1 \|\Gamma_0(F'(y^* + t(x^* - y^*)) - F'(y_0))\|dt \\ &\leq l\int_0^1 (1-t)\|y_0 - y^*\| + t\|y_0 - x^*\| \\ &\leq \frac{la_0}{2}(\|x_0 - y^*\| + \|x_0 - x^*\|) \\ &\leq \frac{la_0}{2}(R + t^*) < 1. \end{aligned}$$

So, using Banach lemma, it gives that T is invertible and it proves the theorem.

4 Numerical Experiment

In this section, we workout Numerical examples to find the radius of convergence using local and semilocal convergence of iterative method. We conclude that our results can be used to solve nonlinear equations but earlier ones (see [6, 11–13]) using even stronger contractivity type hypotheses cannot be used.

Example 1 *Let $X = \Omega = \mathcal{B}(0,1)$ and define the function F on Ω by*

$$F(x) = e^x - x - 1 \tag{4.1}$$

Solution: From the assumptions, (2.2), (2.3) and (2.4) we get (for $x^* = 0$), $\alpha = e - 1, \beta = e - 2, \gamma = e$ and radius r is given by [illegible] $= 0.219981153$. We fail to apply the results in [6, 11–13] due to the F is not contraction on Ω.

Example 2 *Let $X = D = \mathcal{B}(0,1)$ and $x^* = (0,0,0)^T$. Define the function F on Ω for $u = (x,y,z)^T$ by*

$$F(u) = \left(e^x - 1, \frac{e-1}{2}y^2 + y, z\right)^T \tag{4.2}$$

Solution: From the assumptions, (2.2), (2.3) and (2.4) (for $x^* = (0,0,0)^T$), we get $\beta = e + 1, \alpha = e - 1$, $\gamma = e^{\frac{1}{\beta\gamma}}$ and the radius r is given by $r = 0.0883232$. We notice that the results in [6, 11–13] cannot apply due to the F is not contraction on Ω.

Example 3 *Let $X = D = \mathcal{B}(0,1)$ a. Define the function F on Ω by*

$$F(x) = \frac{x^2}{2} for all x \in \Omega = [-1,1] \tag{4.3}$$

Solution: From the assumptions (1)-(6), we have $x_0 = 0.1$, $a_0 = 0.505$, $\Gamma_0 = 1.00005$, $\eta = 0.0099504$, $l = 1.00005$, $l_0 = 1.00005$, $b = 1$. Since, all the conditions of the Lemma 1 are satisfied. Hence from the Theorem 2, we conclude that the the iterative method converge to fixed point $x^* \in \overline{\mathcal{B}}(x_0, l^*)$, where $l^* \in [0, l^{**})$ and $l^{**} = 0.04935$.

5 Conclusions

we discussed the local and semilocal convergence of third order Stirling-Like method under the assumption that the first order Fréchet derivative satisfies the lipschitz continuity condition. The existence and uniqueness region that is the convergence balls of these method is obtained. Numerical examples for both the local and semilocal convergence cases discussed with our work where earlier works cannot apply to solve equations.

References

[1] R.G. Bartle, Newtons method in Banach spaces, *Proc. Amer. Math. Soc.,* 6 (1955), 827-831.

[2] L.B. Rall, Computational Solution of Nonlinear Operator Equations, *E. Robert Krieger, New York, 1969.*

[3] J.Stirling, Methodus Differentialis:Sive tractatus de summatione et interpolatione serierum infinitarum, London, 1730.

[4] S.K. Parhi, D.K. Gupta, A third order method for fixed points in Banach spaces, Journal of Mathematical Analysis and Applications, 359 (2009), 642-652.

[5] I.K.Argyros, On the convergence and Application of Stirling's method, Applicationes Mathematicae, 30 (2003), 109–119.

[6] L. B. Rall, Convergence of Stirling's Method in Banach Spaces, Aequationes Math. 12(1975), 12-20

[7] Weihong Bi,Hongmin Ren, Qingbiao Wu, Three-step iterative methods with eighth-order convergence for solving nonlinear equations, J.Computational and Applied Mathematics, 225 (2009), 105–112.

[8] Á.Alberto Magreñán, Ioannis K.Argyros, On the local convergence and the dynamics of Chebyshev-Halley methods with six and eight order of convergence, J.Computational and Applied Mathematics, 298 (2016), 236–251.

[9] P. Maroju, R. Behl,M.G. Orakwelu,S.S. Motsa and V. Kanwar, An efficient family of two-point sixth-order methods suitable for non-convergence cases, Proceedings of 2015 RAECS UIET Punjab University Chandigarh, (2015), 1–6.

[10] E. Martinez, S. Singh, J.L. Hueso and D.K. Gupta, Enlarging convergence balls in local convergence studies for iterative methods in Banach Spaces. Applied Mathematics and Computation 281(2016), 252–265.

[11] Argyros, I.K., Stirlings method and fixed points of nonlinear operator equations in Banach spaces, Bull. Inst. Math. Acad. Sin. (N.S.) 23(1995), 13-20 .

[12] Parhi, S.K., Gupta, D.K, Semilocal convergence of Stirlings method under Hölder continuous first derivative in Banach spcaes, Int. J. Comput. Math. 87(2010), 2752-2759.

[13] Parhi, S.K., Gupta, D.K, Relaxing convergence conditions for Stirlings method, Math. Methods Appl. Sci. 33(2010), 224-232.