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JOURNAL OF  
COMPUTATIONAL AND  
APPLIED MATHEMATICS

Journal of Computational and Applied Mathematics 169 (2004) 33–44

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# A computer-assisted proof on the stability of the Kolmogorov flows of incompressible viscous fluid

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Received 22 July 2002; received in revised form 16 September 2003

## Abstract

There exists a great number of references of bifurcations and stability for the Navier–Stokes equations. Only a few, however, provide a rigorous result which guarantees stability or instability. Our aim is to present a rigorous theorem which proves the stability of certain solutions arising in what is called the Kolmogorov problem. We accomplish this by the verified computation. The eigenvalue problem arising in the Kolmogorov problem is not self-adjoint and, accordingly, it is quite difficult to treat theoretically. Our method is a numerical approach to deal with this difficulty and numerical examples are given as a demonstration.

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*MSC:* 35P15; 65N25; 70K50

*Keywords:* Numerical verification; Kolmogorov flows; Stability; Infinite dimensional eigenvalue problems

## 1. Introduction

We consider the following Navier–Stokes equations:

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = \nu \Delta u - \frac{1}{\rho} \frac{\partial p}{\partial x} + \gamma \sin\left(\frac{\pi y}{b}\right), \quad (1.1)$$

$$\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} = \nu \Delta v - \frac{1}{\rho} \frac{\partial p}{\partial y}, \quad (1.2)$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \quad (1.3)$$

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where  $(u, v)$ ,  $\rho$ ,  $p$  and  $\nu$  are velocity vector, mass density, pressure and kinematic viscosity, respectively (cf. [1]) and  $\gamma$  is a constant representing the strength of the sinusoidal outer force. The flow region is a rectangle  $[-a, a] \times [-b, b]$  and the periodic boundary condition is imposed in both directions. We define the aspect ratio  $\alpha$  as  $b/a$ .

There exists a basic solution which is written as  $(u, v, p) = (k \sin(\pi y/b), 0, d)$ , where  $k = b^2 \gamma / (\pi^2 \nu)$  and  $d$  is any constant. This is a trivial solution but it is known that nontrivial solutions bifurcate from the basic solution at a certain Reynolds number, which is defined below, if and only if  $0 < \alpha < 1$ , see [1]. Stability of the bifurcating solutions are only partly known rigorously, although the computation in [7] strongly suggests stability for all  $0 < \alpha < 1$ . If  $\alpha$  is small enough, then the stability is proved mathematically (see [7] and the references therein). Stability of the bifurcating solutions is also proved, if  $\alpha$  is close to unity [3]. However, stability in the intermediate range of  $\alpha$  is very difficult to prove, as is easily guessed by reading [3,7]. We therefore take a new approach to this stability problem by employing the theory of verified computation. Our result shows that the stability is rigorously verified for the cases of  $\alpha = 0.4, 0.7$ , and  $0.8$ . Our method can be applied, in principle, to any  $\alpha \in (0, 1)$ .

The eigenvalue problem arising in this paper is not self-adjoint and presents us a considerably more difficult problem, if we compared it with simpler cases of scalar elliptic equations considered by [5,6]. However, we will show that, at least for bifurcation of steady states, basically the same idea as in [5,6] can be used even in the Navier–Stokes equations. Our result may be a mere mathematical curiosity, but it can be viewed as a proto-type of the stability problem, and we believe that our idea, not our results themselves, will pave the way to numerical verification of hydrodynamic stability problems arising in more practical, or even industrial problems.

In Section 2, we introduce a nondimensional form of our problem and derive the linearized eigenvalue problem. In Section 3, we formulate verification method by computer to enclose an eigenfunction corresponding to the zero eigenvalue as well as the Reynolds number which attain the eigenvalue “zero”. We prepare a theoretical set-up for the stability of the bifurcating solution in Section 4. We obtain some enclosure results for zero eigenvalues by quite narrow intervals and using the results we prove the stability of a bifurcating solution, which is presented in Section 5.

## 2. Nondimensionalization and linearized eigenvalue problem

In this section we transform (1.1)–(1.3) into a nondimensional form. We first introduce a stream function  $\psi$  satisfying  $u = \partial\psi/\partial y$ ,  $v = -\partial\psi/\partial x$  by the incompressibility condition (1.3). Using this function we can rewrite the Eqs. (1.1)–(1.3) as

$$\frac{\partial}{\partial t} \Delta \psi - \nu \Delta^2 \psi - J(\psi, \Delta \psi) = \frac{\gamma \pi}{b} \cos\left(\frac{\pi y}{b}\right), \quad (2.1)$$

where  $J$  is a bilinear form defined by  $J(u, v) = (\partial u / \partial x)(\partial v / \partial y) - (\partial u / \partial y)(\partial v / \partial x)$ . This equation was obtained by cross-differentiating Eqs. (1.1) and (1.2) and eliminating the pressure  $p$ .

We then introduce our nondimensional form using the following change of variables and the definition of the Reynolds number:

$$(x', y') = \left(\frac{\pi x}{b}, \frac{\pi y}{b}\right), \quad t' = \frac{\gamma b}{\nu \pi} t, \quad \psi'(t', x', y') = \frac{\nu \pi^3}{\gamma b^3} \psi(t, x, y), \quad R \equiv \frac{\gamma b^3}{\nu^2 \pi^3}.$$

After dropping the primes, we have

$$\frac{\partial}{\partial t} \Delta \psi - \frac{1}{R} \Delta^2 \psi - J(\psi, \Delta \psi) = \frac{1}{R} \cos(y). \quad (2.2)$$

This equation is satisfied in  $\mathbf{T}_\alpha \equiv [-\pi/\alpha, \pi/\alpha] \times [-\pi, \pi]$ . Since nonexistence of bifurcation is proved for  $1 \leq \alpha$  in [1], we assume henceforth that  $0 < \alpha < 1$ .

We assume that  $\psi$  is periodic in  $x$  and  $y$ . Noting that  $\psi(t, x, y) \equiv -\cos y$  is a solution in (2.2) for any  $R > 0$ , we call it a basic solution. Defining as  $\phi \equiv \psi + \cos y$ , we write (2.2) as follows:

$$\frac{\partial}{\partial t} \Delta \phi - \frac{1}{R} \Delta^2 \phi + \sin y (\Delta + I) \frac{\partial \phi}{\partial x} - J(\phi, \Delta \phi) = 0, \quad (2.3)$$

where  $I$  is the identity operator.

We here show that Eq. (2.3) has a certain symmetry, which helps us to better understand the bifurcation structure. We define an operator  $U$  by

$$Uf(x, y) = f(-x, -y).$$

If the left-hand side of (2.3) is denoted by  $Q(\phi)$ , then it holds that

$$Q(U\phi) = UQ(\phi),$$

for all functions  $\phi$  for which  $Q(\phi)$  can be defined. We call this property equivariance of the mapping  $Q$  with respect to the operator  $U$  on the functions defined in  $\mathbf{T}_\alpha$ . In the present paper we consider solutions which reserves this symmetry.

In order to study the stationary bifurcation from the basic flow  $\phi \equiv 0$ , we consider the following linearized eigenvalue problem of (2.3):

$$\sigma \Delta \phi - \frac{1}{R} \Delta^2 \phi + \sin y (\Delta + I) \frac{\partial \phi}{\partial x} = 0, \quad (2.4)$$

where  $\sigma$  is an eigenvalue,  $\phi$  an eigenfunction. We define the linearized operator  $L_{R,\alpha}$  by

$$L_{R,\alpha} \phi \equiv -\frac{1}{R} \Delta^2 \phi + \sin y (\Delta + I) \frac{\partial \phi}{\partial x}.$$

As is well-known, the basic flow is stable if all eigenvalues  $\sigma$  have negative real parts. This is known to be the case if the Reynolds number  $R$  is small enough. Meshalkin and Sinai [4] proved the following principle of exchange of stability:

**Proposition 2.1.** *If (2.4) has an eigenvalue with  $\text{Re}[\sigma] \geq 0$ , then it must be a real number.*

Noting that the eigenvalue is continuous for the Reynolds number, this proposition indicates that the eigenvalues can cross the imaginary axis only at the origin, and that only stationary solutions can bifurcate from the basic flow.

### 3. Fixed point formulation and error estimates

Since we need the “zero” eigenvalue and the corresponding eigenfunction, we restrict  $\phi$  as *real* and consider the following system of equations to find a function  $\phi$  and a real number  $R$ :

$$\begin{aligned} \frac{1}{R} \Delta^2 \phi - \sin y (\Delta + I) \frac{\partial \phi}{\partial x} &= 0, \\ \int_{\mathbf{T}_\alpha} \phi^2 \, dx &= 1. \end{aligned} \quad (3.1)$$

Note that (3.1) can be regarded as a kind of eigenvalue problem, i.e., eigenvalue  $R$  and corresponding eigenfunction  $\phi$ . In what follows, for some integer  $m$ , let  $H^m(\mathbf{T}_\alpha)$  denote the  $L^2$ -Sobolev space of order  $m$  on  $\mathbf{T}_\alpha$ .

We now define the function space

$$Y^k = \{\phi \in H^k(\mathbf{T}_\alpha)/\mathbf{R} \mid U\phi = \phi\}, \quad (k \geq 0),$$

where the symbol  $/\mathbf{R}$  implies that only those functions with zero spatial mean are collected. And we suppose that  $\psi \in H^k(\mathbf{T}_\alpha)/\mathbf{R}$  ( $k \geq 0$ ) is expanded in the Fourier series as

$$\psi = \sum a_{m,n} e^{im\alpha x + iny},$$

where the summation is taken over all the pairs of integers but  $(m,n)=(0,0)$ . Since  $\psi$  is real-valued, it holds that  $\overline{a_{m,n}} = a_{-m,-n}$ . If  $\psi$  satisfies  $U\psi = \psi$ , the relation  $a_{m,n} = a_{-m,-n}$  holds. These relations enable the function space  $Y^k$  to have the following orthogonal decomposition:  $Y^k = Y_0^k \oplus Y_1^k \oplus Y_2^k \oplus \cdots$ , where

$$\begin{aligned} Y_0^k &\equiv \left\{ \sum_{n=1}^{\infty} a_n \cos ny \mid a_n \in \mathbf{R}, \sum_{n=1}^{\infty} (1+n^{2k}) a_n^2 < \infty \right\}, \\ Y_m^k &\equiv \left\{ \sum_{n=-\infty}^{\infty} a_n \cos(m\alpha x + ny) \mid a_n \in \mathbf{R}, \sum_{n=-\infty}^{\infty} (1+n^{2k}) a_n^2 < \infty \right\} \quad (k \geq 0, m \geq 1) \end{aligned}$$

and  $Y_m^k \subset H^k(\mathbf{T}_\alpha)$  ( $k, m \geq 0$ ) holds. Moreover,  $L_{R,\alpha}$  maps  $Y_m^4$  into  $Y_m^0$  for each  $m \geq 0$ . In the following we denote  $Y_1^3$  as  $V$ .

It is known that the null space  $N(L_{R,\alpha}; V) = \{u \in V; L_{R,\alpha} u = 0\}$  is nontrivial if and only if  $R$  is a critical Reynolds number, and that if this is the case,  $N(L_{R,\alpha}; V)$  is one-dimensional (see [7]). Therefore, we look for an eigenfunction of  $L_{R,\alpha}$  in  $V$ .

Now we also define the following function spaces:

$$X^k \equiv Y_1^k \quad (k \geq 1).$$

Note that  $X^k \subset H^k(\mathbf{T}_\alpha)$ , especially  $X^3 = V$  holds, and that the topology in each  $X^k$  is the same one in  $H^k(\mathbf{T}_\alpha)$ .

By direct computations, we find that for any  $g \in X^0$  there exists a unique solution  $\psi \in X^4$  satisfying

$$\Delta^2 \psi = g. \quad (3.2)$$

For  $g \in X^0$ , let  $\tilde{K}g$  be the solution  $\psi \in X^4$  of Eq. (3.2),  $\iota$  the embedding from  $X^4$  into  $X^3$  and define  $K \equiv \iota \tilde{K}$ , then  $K$  is a compact operator from  $X^0$  to  $V$ .

Let  $V_N$  be the finite-dimensional subspace of  $V$ , which depends on an integer parameter  $N$ , defined by

$$V_N \equiv \left\{ \psi_N \mid \psi_N = \sum_{n=-N}^N v_n \cos(\alpha x + ny) \right\}.$$

We also define the projection  $P_N \psi \in V_N$  for the solution  $\psi$  in (3.2) by

$$(\Delta^2(\psi - P_N \psi), \psi_N)_{L^2(\mathbf{T}_\alpha)} = 0 \quad \forall \psi_N \in V_N,$$

where  $(\cdot, \cdot)_{L^2(\mathbf{T}_\alpha)}$  means the usual  $L^2$  norm in  $\mathbf{T}_\alpha$ . Note that for  $\psi = \sum_{n=-\infty}^{\infty} a_n \cos(\alpha x + ny)$  the projection  $P_N$  satisfies

$$P_N \psi = \sum_{n=-N}^N a_n \cos(\alpha x + ny) \in V_N.$$

**Lemma 1.** For  $\psi \in V$ , if  $\Delta^2 \psi \in L^2(\mathbf{T}_\alpha)$  holds, then we have

$$|\psi - P_N \psi|_{H^3(\mathbf{T}_\alpha)} \leq \sqrt{\frac{1}{\alpha^2 + (N+1)^2}} \|\Delta^2 \psi\|_{L^2(\mathbf{T}_\alpha)}, \quad (3.3)$$

$$\|\psi - P_N \psi\|_{L^2(\mathbf{T}_\alpha)} \leq \sqrt{\frac{1}{\{\alpha^2 + (N+1)^2\}^3}} |\psi - P_N \psi|_{H^3(\mathbf{T}_\alpha)}. \quad (3.4)$$

Here,  $|\cdot|_{H^3(\mathbf{T}_\alpha)}$  denotes the usual  $H^3$ -seminorm on  $\mathbf{T}_\alpha$ , i.e.,

$$|\psi|_{H^3(\mathbf{T}_\alpha)}^2 \equiv \sum_{\beta_1 + \beta_2 = 3 (\beta_1, \beta_2 \in \mathbb{N} \cup \{0\})} \binom{3}{\beta_1} \left\| \frac{\partial^3}{\partial x^{\beta_1} \partial y^{\beta_2}} \psi \right\|_{L^2(\mathbf{T}_\alpha)}^2.$$

**Proof.** We have

$$|\psi - P_N \psi|_{H^3(\mathbf{T}_\alpha)}^2 = \frac{2\pi^2}{\alpha} \left( \sum_{n=-\infty}^{-N-1} + \sum_{n=N+1}^{\infty} \right) (\alpha^2 + n^2)^3 a_n^2$$

$$\begin{aligned}
&\leq \frac{2\pi^2}{\alpha} \left( \sum_{n=-\infty}^{-N-1} + \sum_{n=N+1}^{\infty} \right) \frac{a_n^2}{\alpha^2 + (N+1)^2} (\alpha^2 + n^2)^4 \\
&\leq \frac{1}{\alpha^2 + (N+1)^2} \|\Delta^2 \psi\|_{L^2(\mathbf{T}_x)}^2,
\end{aligned}$$

thus (3.3) is proved.

The (3.4) is also proved in a similar way as follows:

$$\begin{aligned}
\|\psi - P_N \psi\|_{L^2(\mathbf{T}_x)}^2 &= \frac{2\pi^2}{\alpha} \left( \sum_{n=-\infty}^{-N-1} + \sum_{n=N+1}^{\infty} \right) a_n^2 \\
&\leq \frac{1}{\{\alpha^2 + (N+1)^2\}^3} \|\psi - P_N \psi\|_{H^3(\mathbf{T}_x)}^2. \quad \square
\end{aligned}$$

The error estimates in Lemma 1 will be used to evaluate the infinite dimensional part of the fixed point equation which is introduced later.

Now, we calculate an approximate solution  $w_N = (\phi_N, R_N) \in V_N \times \mathbf{R}$  of (3.1) satisfying

$$\begin{aligned}
&\left( \frac{1}{R_N} \Delta^2 \phi_N - \sin y(\Delta + I) \frac{\partial \phi_N}{\partial x}, v_N \right)_{L^2(\mathbf{T}_x)} = 0, \\
&\int_{\mathbf{T}_x} \phi_N^2 dx = 1
\end{aligned} \tag{3.5}$$

for all  $v_N \in V_N$ . We used the library PROFIL [2] to enclose it by very small intervals.

In order to verify the solution  $(\phi, R)$  of (3.1) near  $(\phi_N, R_N)$ , we set

$$\tilde{\phi} \equiv \phi - \phi_N, \quad \tilde{R} \equiv R - R_N,$$

and rewrite (3.1) as

$$\begin{aligned}
&\Delta^2(\phi_N + \tilde{\phi}) - (R_N + \tilde{R}) \sin y(\Delta + I) \frac{\partial(\phi_N + \tilde{\phi})}{\partial x} = 0, \\
&\int_{\mathbf{T}_x} (\phi_N + \tilde{\phi})^2 dx = 1.
\end{aligned} \tag{3.6}$$

Since the term  $\{(R_N + \tilde{R}) \sin y(\Delta + I) \frac{\partial(\phi_N + \tilde{\phi})}{\partial x}\}$  is in  $X^0$  for  $\phi_N$  and  $\tilde{\phi} \in V$ , using the following compact map on  $V \times \mathbf{R}$ :

$$F(\tilde{\phi}, \tilde{R}) \equiv \begin{pmatrix} K \left\{ (R_N + \tilde{R}) \sin y(\Delta + I) \frac{\partial(\phi_N + \tilde{\phi})}{\partial x} \right\} - \phi_N \\ \tilde{R} + \int_{\mathbf{T}_x} (\phi_N + \tilde{\phi})^2 dx - 1 \end{pmatrix}, \tag{3.7}$$

we have the following fixed point equation:

$$w = F(w) \quad \text{in} \quad V \times \mathbf{R} \quad (3.8)$$

for  $w = (\tilde{\phi}, \tilde{R})$ .

#### 4. Method of verification by a computer

Now, as in [5], we decompose (3.8) into the finite and infinite dimensional parts:

$$\begin{aligned} P_N w &= P_N F(w), \\ (I - P_N)w &= (I - P_N)F(w). \end{aligned} \quad (4.1)$$

Since we apply a Newton-like method only for the former part of (4.1), we define the following operator:

$$\mathcal{N}(w) \equiv P_N w - [I - F'(0)]_N^{-1} (P_N w - P_N F(w)),$$

where  $I$  is the identity map on  $V \times \mathbf{R}$  and  $F'(0)$  stands for the Fréchet derivative of  $F$  at 0. And we assumed that the restriction to  $V_N \times \mathbf{R}$  of the operator  $P_N[I - F'(0)] : V \times \mathbf{R} \rightarrow V_N \times \mathbf{R}$  has the inverse  $[I - F'(0)]_N^{-1}$ . The validity of this assumption can be numerically checked in actual computations.

We next define the operator  $T : V \times \mathbf{R} \rightarrow V \times \mathbf{R}$  by

$$T(w) \equiv \mathcal{N}(w) + (I - P_N)F(w).$$

Then  $T$  becomes a compact map on  $V \times \mathbf{R}$  and we have the following equivalence relation:

$$w = T(w) \Leftrightarrow w = F(w).$$

We can write an arbitrary element  $w \in V \times \mathbf{R}$  as

$$w = (w_N + w_\perp, \mu), \quad w_N = \sum_{n=-N}^N v_n \cos(\alpha x + ny) \in V_N, \quad w_\perp \in V_N^\perp, \quad \mu \in \mathbf{R},$$

where  $V_N^\perp$  means the orthogonal complement of  $V_N$  in  $V \subset H^3(\mathbf{T}_\alpha)$ .

For the above  $w$  we define the following notation:

$$(w)_i \equiv |v_{i-N-1}|, \quad i = 1, \dots, 2N+1,$$

$$(w)_{2N+2} \equiv |v_\perp|_{H^3(\mathbf{T}_\alpha)},$$

$$(w)_{2N+3} \equiv |\mu|.$$

Our purpose is to find a fixed point of (3.8) in a certain set  $W \subset V \times \mathbf{R}$ , which is called a “candidate set”. Given a vector  $(W_1, \dots, W_{2N+3})^t$  such that  $W_i > 0$  ( $i = 1, \dots, 2N+3$ ), we define the corresponding candidate set  $W$  by

$$W \equiv \{w \in V \times \mathbf{R} \mid (w)_i \leq W_i (i = 1, \dots, 2N+3)\}. \quad (4.2)$$

Now let  $T'$  be the Fréchet derivative of  $T$  and choose two vectors  $(Y_1, \dots, Y_{2N+3})^t, Y_i > 0 (i = 1, \dots, 2N+3)$  and  $(Z_1, \dots, Z_{2N+3})^t, Z_i > 0 (i = 1, \dots, 2N+3)$  such that

$$(T(0))_i \leq Y_i, \quad i = 1, \dots, 2N+3, \quad (4.3)$$

$$(T'(w_1)w_2)_i \leq Z_i, \quad i = 1, \dots, 2N+3, \quad \text{for all } w_1, w_2 \in W. \quad (4.4)$$

Here, notice that, in order to estimate or to bound the infinite dimensional parts in the left-hand side of (4.3) and (4.4), namely  $(T(0))_{2N+2}$  and  $(T'(w_1)w_2)_{2N+2}$  for a given candidate set  $W$ , we use the error estimates obtained in Lemma 1. We omit the detailed procedures, because they are similar to those in [5].

We now derive the following theorem in which the verification condition is described.

**Theorem 1.** *If a candidate set  $W$  defined by (4.2) satisfies*

$$Y_i + Z_i < W_i (i = 1, \dots, 2N+3), \quad (4.5)$$

*then there exists a fixed point of  $T$  in*

$$S \equiv \{v \in V \times \mathbf{R} \mid (v)_i \leq Y_i + Z_i (i = 1, \dots, 2N+3)\}. \quad (4.6)$$

*Moreover, this fixed point is unique within the set  $W$ .*

This theorem is proved by Banach's fixed point theorem. Since the procedures of proof are almost same as those given in [5], we omit it.

Now, we describe an iterative algorithm for finding a vector  $(W_1, \dots, W_{2N+3})^t$  which satisfies the verification condition (4.5) (cf. [5]). Since  $(Z_i)_{i=1}^{2N+3}$  depends on  $W$ , we write  $Z_i$  as  $Z_i(W)$ . We use the following iteration method.

#### Algorithm.

1. Fix a maximum iteration number.
2. Find a vector  $(Y_1, \dots, Y_{2N+3})^t$  satisfying (4.3).
3. Set  $W_i \leftarrow Y_i (i = 1, \dots, 2N+3)$ .
4. Find a vector  $(Z_1(W), \dots, Z_{2N+3}(W))^t$  satisfying (4.4).
5. Check the verification condition (4.5);

$$Y_i + Z_i(W) < W_i \quad (i = 1, \dots, 2N+3).$$

If the condition is satisfied, then the verification has succeeded.

If not, set

$$W_i \leftarrow (1 + \delta)(Y_i + Z_i) \quad (i = 1, \dots, 2N+3), \quad (4.7)$$

where  $\delta$ , which is positive and sufficiently small, is the so-called inflation parameter (cf. [8,9]), increase the iteration number by 1, and return to step 4.

6. If the maximum iteration number is exceeded without (4.5) being satisfied, the verification has failed.



**Remark 1.** By virtue of our formulation of the Newton-like method, if an approximate eigenpair  $(\phi_N, R_N)$  is sufficiently accurate, namely  $N$  is sufficiently large, and inflation parameter  $\delta$  is properly chosen, then the above algorithm is well expected to be successful.

## 5. The stability of the bifurcating solution

In this section we discuss the stability of the bifurcating solution which depends on the aspect ratio. We put

$$G(R, \phi) = -\frac{1}{R} \Delta^2 \phi + \sin y(\Delta + I) \frac{\partial \phi}{\partial x} - J(\phi, \Delta \phi),$$

where the aspect ratio  $\alpha \in (0, 1)$  is fixed. As is often done, we expand the bifurcating solution in a small parameter  $\varepsilon$ :

$$R = R^* + c\varepsilon^2 + O(\varepsilon^3), \quad \phi = \varepsilon\phi_1 + \varepsilon^2\phi_2 + \varepsilon^3\phi_3 + O(\varepsilon^4), \quad (5.1)$$

where  $c$  is a constant,  $R^*$  is the Reynolds number for which the operator  $L_{R,\alpha}$  has the zero eigenvalue, and  $\phi_1$  is the corresponding eigenfunction. Note that the bifurcation is known to be a pitchfork so that  $O(\varepsilon)$  term does not appear in the expansion of  $R$  (see [7]).

Substituting Eq. (5.1) into  $G(R, \phi) = 0$  and computing the coefficients of  $\varepsilon^2$  and  $\varepsilon^3$ , we obtain

$$L_{R^*,\alpha}\phi_2 - J(\phi_1, \Delta\phi_1) = 0 \quad (5.2)$$

and

$$L_{R^*,\alpha}\phi_3 + \frac{c}{(R^*)^2} \Delta^2 \phi_1 - J(\phi_1, \Delta\phi_2) - J(\phi_2, \Delta\phi_1) = 0. \quad (5.3)$$

Since  $L_{R^*,\alpha} : Y_0^4 \oplus Y_2^4 \rightarrow Y_0^0 \oplus Y_2^0$  is invertible and  $J(\phi_1, \Delta\phi_1) \in Y_0^0 \oplus Y_2^0$ , the linear equation (5.2) defines the solution  $\phi_2 \in Y_0^4 \oplus Y_2^4$  uniquely. We enclosed the solution  $\phi_2$  using our verification method.

We also find the solution  $(\phi_1^*, R) \in V \times \mathbf{R}$  of the following system:

$$\begin{aligned} L_{R,\alpha}^* \phi_1^* &= 0, \\ \int_{\mathbf{T}_\alpha} (\phi_1^*)^2 dx &= 1, \end{aligned} \quad (5.4)$$

where  $L_{R,\alpha}^*$  stands for the adjoint operator of  $L_{R,\alpha}$ .

**Remark 2.** In general, if an operator  $L$  has a zero eigenvalue at a Reynolds number then the adjoint operator  $L^*$  also has a zero eigenvalue at the same Reynolds number. In actual calculations we decided  $R^*$  as the intersection of two intervals which were enclosed in (3.1) and (5.4), but those upper bounds and lower bounds were equal. In our verification method the local uniqueness of the Reynolds number in the enclosed interval are assured (see [5]), so the enclosed eigenfunction  $\phi_1$  and  $\phi_1^*$  are rigorous ones for the critical Reynolds number  $R^*$ .

Multiplying (5.3) by  $\phi_1^*$  and integrating by parts, we have the following equation:

$$\frac{c}{(R^*)^2}(\Delta\phi_1, \Delta\phi_1^*)_{L^2(\mathbf{T}_x)} = (J(\phi_2, \Delta\phi_1), \phi_1^*)_{L^2(\mathbf{T}_x)} + (J(\phi_1, \Delta\phi_2), \phi_1^*)_{L^2(\mathbf{T}_x)}. \quad (5.5)$$

Note that the function  $\phi_3$  is eliminated in the process of the integration by parts.

It is known that if  $c > 0$  holds then the bifurcating solution is supercritical and it is stable in some neighborhood of the bifurcation point. Since  $(\Delta\phi_1, \Delta\phi_1^*)$  is positive under an appropriate normalization of  $\phi_1$  and  $\phi_1^*$  [1], we only have to check the sign of the right-hand side of (5.5) to prove the stability of the solution.

## 6. Some examples of numerical proof

We took  $N=128$  as a truncation number and  $\alpha$  as 0.4, 0.7 and 0.8. In calculations, we used interval arithmetic in order to avoid the effects of rounding errors in the floating-point computations. The computations were carried out on the DELL Precision WorkStation 530 (Intel Xeon 1.7 GHz) using the interval library PROFIL coded by Knüppel of the Technical University of Hamburg–Harburg [2].

The critical Reynolds number (when  $\alpha = 0.7$ ) was computed as 3.011193 in [7]. We, however, found that it was actually enclosed in the following interval:

$$[3.011528364444, 3.011528364446].$$

In the cases of  $\alpha = 0.4$  and  $\alpha = 0.8$ , the critical Reynolds numbers were enclosed in the interval

$$[1.792408580078417, 1.792408580078419]$$

and

$$[4.05753367090524, 4.05753367090526],$$

respectively. As for those eigenfunctions and the solution of (5.2), the verified results are in Tables 1–3. Figs. 1–3 illustrate the graphs of  $\phi_1, \phi_1^*$  and  $\phi_2$ . The positivity of  $c$  is guaranteed as is shown in Table 4.

Table 1  
Results of verification for  $\alpha = 0.4$

	$\phi_1$	$\phi_1^*$	$\phi_2$
Iteration number	2	2	2
Maximum absolute value of coefficients intervals in the enclosing for the finite part	4.97827E – 15	5.66712E – 15	3.80287E – 5
Error in $H^3(\mathbf{T}_x)$ semi-norm	3.8624E – 10	5.59887E – 10	6.73445E – 5

Table 2

Results of verification for  $\alpha = 0.7$ 

	$\phi_1$	$\phi_1^*$	$\phi_2$
Iteration number	2	2	3
Maximum absolute value of coefficients intervals in the enclosing for the finite part	$3.001312\text{E} - 14$	$3.064001\text{E} - 14$	$1.781004\text{E} - 4$
Error in $H^3(\mathbf{T}_z)$ semi-norm	$9.184929\text{E} - 10$	$2.667120\text{E} - 9$	$4.984014\text{E} - 4$

Table 3

Results of verification for  $\alpha = 0.8$ 

	$\phi_1$	$\phi_1^*$	$\phi_2$
Iteration number	2	2	3
Maximum absolute value of coefficients intervals in the enclosing for the finite part	$2.73114\text{E} - 14$	$2.57794\text{E} - 14$	$1.81907\text{E} - 4$
Error in $H^3(\mathbf{T}_z)$ semi-norm	$9.93808\text{E} - 10$	$1.95631\text{E} - 9$	$8.41844\text{E} - 4$

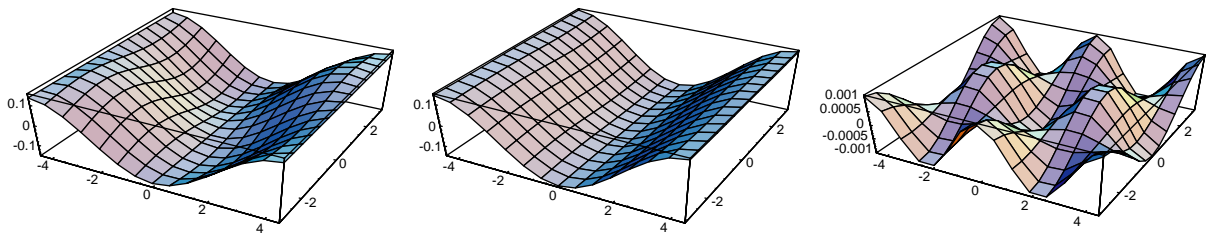
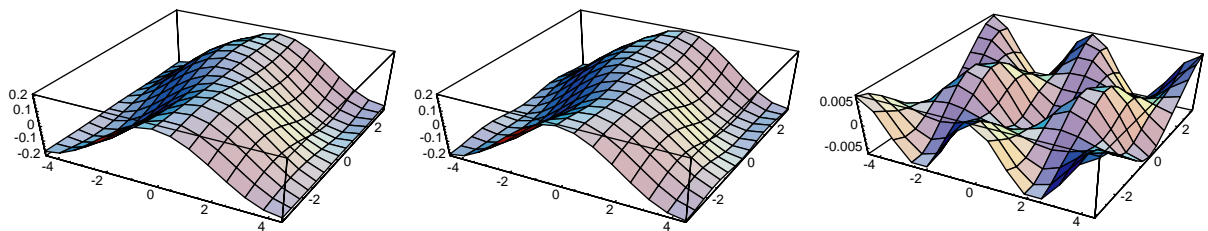
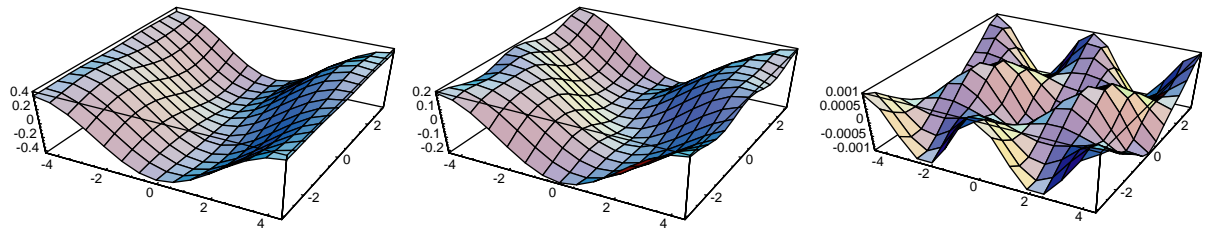
Fig. 1. Approximation of  $\phi_1, \phi_1^*$  and  $\phi_2$  from the left ( $\alpha = 0.4$ ).Fig. 2. Approximation of  $\phi_1, \phi_1^*$  and  $\phi_2$  from the left ( $\alpha = 0.7$ ).Fig. 3. Approximation of  $\phi_1, \phi_1^*$  and  $\phi_2$  from the left ( $\alpha = 0.8$ ).

Table 4

Inclusions of the constant “ $c$ ” in Eq. (5.5)

$\alpha$	0.4	0.7	0.8
$c$	[0.00127863, 0.00139123]	[0.00121662, 0.0018981]	[0.00113052, 0.00255719]

## Acknowledgements

The author would like to express her sincere gratitude to Professor M.T. Nakao and Professor H. Okamoto for their encouraging suggestions and useful comments throughout this work. The author also would like to express many thanks to Professor M. Plum for his useful comments on the formulation for the problem. And the author also thanks referees for their kind suggestions to revise this paper.

## References

- [1] V.I. Iudovich, Example of the generation of a secondary stationary or periodic flow when there is loss of stability of the laminar flow of a viscous incompressible fluid, *J. Appl. Math. Mech.* 29 (1965) 527–544.
- [2] O. Knüppel, PROFIL/BIAS—A fast interval library, *Computing* 53 (1994) 277–288.
- [3] M. Matsuda, S. Miyatake, Bifurcation curves on Kolmogorov flow, Preprint.
- [4] L.D. Meshalkin, Y.G. Sinai, Investigation of the stability of a stationary solutions of a system of equations for the plane movement of an incompressible viscous liquid, *J. Appl. Math. Mech.* 25 (1962) 1700–1705.
- [5] K. Nagatou, A numerical method to verify the elliptic eigenvalue problems including a uniqueness property, *Computing* 63 (1999) 109–130.
- [6] M.T. Nakao, N. Yamamoto, K. Nagatou, Numerical verifications for eigenvalues of second-order elliptic operators, *Japan J. Ind. Appl. Math.* 16 (1999) 307–320.
- [7] H. Okamoto, M. Shoji, Bifurcation diagrams in Kolmogorov’s problem of viscous incompressible fluid on 2-D flat tori, *Japan J. Ind. Appl. Math.* 10 (2) (1993) 191–218.
- [8] S.M. Rump, Solving algebraic problems with high accuracy, in: U. Kulisch, W.L. Miranker (Eds.), *A New Approach to Scientific Computation*, Academic Press, New York, 1983, pp. 51–120.
- [9] Y. Watanabe, M.T. Nakao, Numerical verifications of solutions for nonlinear elliptic equations, *Japan J. Ind. Appl. Math.* 10 (1993) 165–178.