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A personal history of the *m*-coefficient

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Abstract

This is a short personal history of the *m*-coefficient for Sturm–Liouville differential and difference equations, now associated historically with the names of Weyl, Hellinger, Nevanlinna and Titchmarsh.

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1. Introduction

This is a personal, and thereby selective, history of the *m*-coefficient as connected with the Sturm–Liouville differential equation, and the corresponding coefficient for symmetric, second-order difference equations. The account is based on my years of contact with E.C. Titchmarsh (1899–1963) in Oxford, as undergraduate student (1949–52), doctoral student (1952–54), and at the Titchmarsh seminars (1954–63).

In places in the text I have quoted from Titchmarsh as best as my memory serves, but I have been careful not to attribute definite statements to him if I am uncertain of their authority.

Here are the main items in the development of this theory and which are treated in this account:

(1) The limit-point and limit-circle classification of Weyl.
(2) The existence of integrable-square solutions of Weyl.
(3) The Hellinger–Nevanlinna *w*-coefficient for difference equations.
(4) The Titchmarsh–Weyl *m*-coefficient.

I have drawn up a list of references which I hope has some degree of completeness. Volume II of *Eigenfunction Expansions*, see [23], is quoted in this list since although this book is largely

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concerned with partial differential equations, it contains the Titchmarsh list of references for the second edition of volume I, see [24].

In referring to the Sturm–Liouville differential equation the general notation used in this paper is

$$-(py')' + qy = \lambda wy \quad \text{on } I \subseteq \mathbb{R}, \tag{1.1}$$

for some interval I of the real line $\mathbb{R}$, and for the spectral parameter $\lambda \in \mathbb{C}$. Here the coefficients p, q, w have different conditions from reference to reference; however the right-definite conditions are implied in all cases, i.e., the weight coefficient $w \geqslant 0$ on the interval I.

In all the references referred to below the basic requirements are, for specific choices of the coefficients p, q, w and the interval I:

(i) $p, q, w : I \to \mathbb{R}$,
(ii) $w > 0$ on I or $w > 0$ almost everywhere (Lebesgue) on I,
(iii) properties of solutions of the differential equation are considered in the Hilbert function space $L^2(I; w)$.

Here are more specific conditions in some of the references quoted below; however, note that not all these references concern the m-coefficient but they do plot the development of the form of the Sturm–Liouville differential equation:

$$\begin{cases} 1910 & \text{Weyl} & [25\text{–}27] & p, q \in C[0,\infty),\ p > 0 \text{ and } w = 1 \text{ on } [0,\infty), \\ 1912 & \text{Dixon} & [10] & p^{-1}, q, w \in L^1[a,b], \\ 1932 & \text{Stone} & [18] & p^{-1}, q \in L^1_{\mathrm{loc}}(I) \text{ and } w = 1 \text{ on } I \subseteq \mathbb{R}, \\ 1941 & \text{Titchmarsh} & [19\text{–}24] & p = q = 1 \text{ on } [0,\infty) \text{ and } q \in C[0,\infty), \\ 1955 & \begin{cases}\text{Coddington}\\ \text{Levinson}\end{cases} & [9] & p, p', q \in C[0,\infty) \text{ and } p > 0 \text{ on } [0,\infty), \\ 1969 & \begin{cases}\text{Chaudhuri}\\ \text{Everitt}\end{cases} & [8] & p, p', q \in C[0,\infty) \text{ and } p > 0 \text{ on } [0,\infty), \\ 1983 & \text{Bennewitz} & [4] & p^{-1}, q, w \in L^1_{\mathrm{loc}}(I), \\ 1994 & \text{Everitt} & [13] & p^{-1}, q, w \in L^1_{\mathrm{loc}}(I). \end{cases} \tag{1.2}$$

2. The Titchmarsh–Weyl m-coefficient

At the beginning of this historical account it is well to record the significance of the m-coefficient; for this purpose we take, as an example, the Titchmarsh differential equation, see (1.2) above. For $q : [0,\infty) \to \mathbb{R}$ with $q \in C[0,\infty)$, we have

$$-y''(x) + q(x)y(x) = \lambda y(x) \quad \text{for all } x \in [0,\infty) \text{ and all } \lambda \in \mathbb{C}. \tag{2.1}$$

Let the solutions of (2.1) $\theta, \varphi : [0, \infty) \times \mathbb{C} \to \mathbb{C}$ be defined by the initial conditions, for some $\alpha \in [0, \pi)$,

$$\theta(0, \lambda) = \sin(\alpha) \quad \theta'(0, \lambda) = \cos(\alpha),$$

$$\varphi(0, \lambda) = -\cos(\alpha) \quad \varphi'(0, \lambda) = \sin(\alpha)$$

and for all $\lambda \in \mathbb{C}$. Then the pair θ, φ forms a basis for solutions of (2.1), for all $\lambda \in \mathbb{C}$, and $\theta(x, \cdot), \theta'(x, \cdot), \varphi(x, \cdot), \varphi'(x, \cdot)$ are all entire (integral) functions on $\mathbb{C}$, for all $x \in [0, \infty)$.

Weyl, see [26], proved that either

(i) the limit-point case

$$\theta(\cdot, \lambda) \notin L^2(0, \infty) \quad \text{and} \quad \varphi(\cdot, \lambda) \notin L^2(0, \infty) \text{ for all } \lambda \in \mathbb{C} \setminus \mathbb{R}$$

or

(ii) the limit-circle case

$$\theta(\cdot, \lambda) \in L^2(0, \infty) \quad \text{and} \quad \varphi(\cdot, \lambda) \in L^2(0, \infty) \text{ for all } \lambda \in \mathbb{C}.$$

In both cases Titchmarsh showed, see [19], on the basis of an earlier result in [26], that there exists an analytic function (the m-coefficient) with the properties:

(i) m is regular on $\mathbb{C} \setminus \mathbb{R}$,
(ii) $\bar{m}(\lambda) = m(\bar{\lambda})$ for all $\lambda \in \mathbb{C} \setminus \mathbb{R}$,
(iii) $\mathrm{Im}(m(\lambda)) > 0$ for all λ with $\mathrm{Im}(\lambda) > 0$; $\mathrm{Im}(m(\lambda)) < 0$ for all λ with $\mathrm{Im}(\lambda) < 0$,
(iv) the solution of Eq. (2.1) $\psi(\cdot, \lambda)$ defined by

$$\psi(x, \lambda) := \theta(x, \lambda) + m(\lambda)\varphi(x, \lambda) \quad \text{for all } x \in [0, \infty) \text{ and all } \lambda \in \mathbb{C} \setminus \mathbb{R} \tag{2.2}$$

satisfies

$$\int_0^\infty |\psi(x, \lambda)|^2 \, dx = \frac{\mathrm{Im}(m(\lambda))}{\mathrm{Im}(\lambda)} < +\infty \quad \text{for all } \lambda \in \mathbb{C}. \tag{2.3}$$

The existence of this integrable-square solution ψ, and the analytic m-coefficient are fundamental to the Titchmarsh eigenfunction analysis as developed in the text [24].

Properties (i)–(iii) above imply that analytic coefficient $m(\cdot)$ is a Nevanlinna (Herglotz, Pick, Riesz) function and so has a representation of the form

$$m(\lambda) = \alpha + \beta\lambda + \int_{-\infty}^{+\infty} \left\{ \frac{1}{t - \lambda} - \frac{t}{t^2 + 1} \right\} d\rho(t) \quad \text{for all } \lambda \in \mathbb{C} \setminus \mathbb{R}. \tag{2.4}$$

Here the function $\rho : \mathbb{R} \to \mathbb{R}$ and is monotonic nondecreasing on $\mathbb{R}$; this is the spectral function for the m-coefficient. For additional details of Nevanlinna functions see Section 7 below.

3. The contributions of Hermann Weyl

The references concerned are [25–27]; I have given all three of these references because Titchmarsh quotes all of these papers in the five references to Hermann Weyl in [19–23].

However it is the 1910 paper in *Mathematische Annalen*, see [26], that is the best source for the Weyl results. This paper now stands as one of the most enduring and outstanding contributions to mathematical analysis in the 20th century. The differential equation considered in this paper is

$$-(py')' + qy = \lambda y \quad \text{on } [0,\infty). \tag{3.1}$$

In addition, in 1950 towards the end of his long life in the vineyard of mathematics, Hermann Weyl made his only return to the early work on the Sturm–Liouville differential equation; this is his paper entitled "Ramifications, old and new, of the eigenvalue problem.", see [28]. In considering again the limit-point/limit-circle classification problem he wrote:

> The very first result by which I added my mite to our stock of mathematical knowledge had to do with the clarification of this issue.

In viewing the scene now after the passage of nearly 100 years we may consider ourselves justified in releasing Hermann Weyl from his notable modesty and replace the word 'mite' with 'might'.

For Eq. (3.1) Weyl worked on the interval $[0,\infty)$ and proved, see [26, Kapitel I]:

(a) global existence theorems on $[0,\infty)$ and for all $\lambda \in \mathbb{C}$,
(b) the existence of at least one nonnull solution in the space $L^2(0,\infty)$, for all $\lambda \in \mathbb{C} \setminus \mathbb{R}$,
(c) the existence of the limit-point/limit-circle classification of the singular endpoint at ∞,
(d) the results in [26, Kapitel I] can be extended to prove the independence of the limit-point/limit-circle classification from the spectral parameter λ.

The extension of these results for Eq. (3.1) to the weighted differential equation

$$-(py')' + qy = \lambda k y \quad \text{on } [0,\infty),$$

where the coefficient k is continuous and nonnegative on $[0,\infty)$ and the function space is

$$L^2([0,\infty); k)$$

is discussed in a closing remark; see [26, Schlussbemerkung, p. 268, (90)].

Remark 3.1. In respect of results (a)–(d) above we remark, respectively:

(i) There are now more general existence theorems based on the use of the Lebesgue integral; for suitable initial conditions imposed at a regular point of the differential equation, the solutions are entire (integral) Cauchy analytic functions of the spectral parameter λ.
(ii) The existence of this integrable-square solution is fundamental to the whole development of Sturm–Liouville theory.
(iii) In later years this classification is seen as connected with the introduction of deficiency indices for unbounded closed symmetric operators in abstract Hilbert spaces.
(iv) This result set in train the mathematical industry involved with the search for sufficiency conditions on the coefficients p, q, w to distinguish between the limit-point and limit-circle cases, at singular endpoints of the interval I.

Remark 3.2. The condition on the coefficient p in the paper [26, See Kapitel I, Section 1, p. 221] states only that $p \in C[0,\infty)$; there is no requirement on the existence of the derivative p', in particular there is no requirement that $p' \in C[0,\infty)$ as is used in the standard text of Coddington and Levinson [9, Chapter 9]. In general, the initial conditions for a solution at the regular endpoint 0 of (1.1), involve the determination not of the classical derivative y' but the combined function py'. In this sense the Weyl paper [26] introduced, although not explicitly, the later-termed quasi-derivative of the differential equation (1.1).

Remark 3.3. The list (i)–(iv), in Remark 3.1 above, covers the properties of the differential equation (3.1) required for the definition of the m-coefficient. However, in later sections of the paper [26, Kapitel II and Kapitel IIII] Weyl introduces his definitions of *Punktspektrum and Strekenspektrum*. Whilst his definition of *Strekenspektrum* has been overtaken by the later-named continuous spectrum, the Weyl spectral analysis remains a remarkable achievement made 20 years or so before the introduction of the spectral theory of unbounded self-adjoint operators in abstract Hilbert space.

4. The differential equation of Dixon

This paper of 1912, see [10], seems to have been written without the knowledge of the Weyl paper of 1910, see [26]. The main reason to regard this paper as significant in the development of the Sturm–Liouville differential equation is that it seems to be the first paper in which the continuity conditions on the coefficients p,q,w, see Remark 3.2, are replaced by the Lebesgue integrable conditions; these latter conditions are the minimal conditions to be satisfied by p,q,w within the environment provided by the Lebesgue integral.

The Dixon conditions are:

(i) The interval $[a,b]$ is compact and $p,q,w:[a,b]\to\mathbb{R}$.
(ii) The coefficients p,q,w satisfy the Lebesgue minimal conditions $p^{-1},q,w\in L^1[a,b]$, and both $p,w>0$ almost everywhere on $[a,b]$.
(iii) The Sturm–Liouville differential equation is studied in the Hilbert space $L^2((a,b);w)$.

The paper considers existence of solutions under these coefficient conditions; this is the significant interest of the paper in respect of the development of the analytical properties of the Sturm–Liouville differential equation. The paper contains a discussion of properties of regular boundary value problems on the compact interval $[a,b]$, with both separated and coupled boundary conditions.

5. The contributions of Hellinger–Nevanlinna

I failed to realise until some few years ago, that many of the ideas of the m-coefficient had been anticipated by the work of Hellinger [14] and Nevanlinna [17], both working in 1922 but independently. This work concerned not the Sturm–Liouville equation (1.1) but the difference equation

$$b_{k-1}y_{k-1}+a_k y_k+b_k y_{k+1}=\lambda y_k \quad \text{for all } k=\{1,2,3,\ldots\}, \tag{5.1}$$

where $a_k\in\mathbb{R}$, and $b_k>0$ for all $k=\{0,1,2,3,\ldots\}$, with initial condition

$$(a_0-\lambda)y_0+b_0y_1=0. \tag{5.2}$$

The solution $\{y_k: k=0,1,2,3,\ldots\}$ is then determined if the value y_0 is given.

Hellinger and Nevanlinna then developed properties of the solutions of this difference equation in the Hilbert sequence space ℓ^2.

The original papers [14,17] are cited in the references, but there is a clear account of the results therein to be found in the text [1] by Akhiezer; in particular:

(i) see [1, Chapter 1, Theorem 1.2.3] for the definition of the monotonic decreasing family of Hellenger–Nevanlinna circles, for each $\lambda \in \mathbb{C} \setminus \mathbb{R}$, with a footnote indicating that these circles are analogous to the Weyl circles in [26],
(ii) see [1, Chapter 1, Theorem 1.3.1] for the existence of solutions of the difference equation (5.1) in the Hilbert sequence space ℓ^2, with a footnote indicating that these solutions are analogous to the Weyl solutions in the Hilbert function space $L^2([0,\infty);w)$ which, taking notation differences into account, are given in [26],
(iii) see [1, Chapter 1, Theorem 1.3.2] for invariance, in the spectral parameter λ, of the cases when the circles collapse to a point or reduce to a circle, again with an appropriate footnote to the Weyl paper [26],
(iv) see [1, Chapter 1, Definition 1.3.2] for, in effect, the introduction of the limit-point/limit-circle classification for the difference equation (5.1),
(v) see [1, Chapter 1, Theorem 1.3.3] for the introduction of the Hellinger–Nevanlinna $w(\cdot)$ analytic function that is equivalent to the m-coefficient for the Sturm–Liouville differential equation (1.1); here there is no footnote to the Weyl paper [26] since, in spite of the remarkable results in [26] in other directions, there is no contribution to the m-coefficient in this landmark paper of Weyl.

There are recent studies of the Hellinger–Nevanlinna theory by Brown et al.; see [6,7]. In particular in [7] the authors introduce the Hellinger–Nevanlinna m-coefficient thereby renaming the $w(\cdot)$ analytic function referred to in item (v) above; for details see Section 10 below.

In effect the Hellinger–Nevanlinna work is the first introduction of the m-coefficient and precedes the results of Titchmarsh discussed in the next section. However, there is no entry in all of the Titchmarsh citations, included at the end of this paper, of either of the Hellinger paper [14] nor of the Nevanlinna paper [17]. In all my conversations with Titchmarsh about the m-coefficient I do not recall any mention of the Hellinger–Nevanlinna results; to the best of my knowledge he did not know of these results even at the time of his death in 1963.

I am bound to the personal conclusion that the later introduction of the Titchmarsh–Weyl m-coefficient in 1941, see the next section, was carried out quite independently by Titchmarsh with no knowledge of the earlier Hellinger–Nevanlinna results.

Remark 5.1. The Hellinger–Nevanlinna $w(\cdot)$ function is a Nevanlinna (Herglotz, Pick, Riesz) analytic function; see, in particular, [1, Chapter 3, Section 1, [3.3]], and also [2, Chapter 6, Section 69]

For a collected account of the references concerning the names of Herglotz, Nevanlinna, Pick, M. Riesz, for these particular analytic functions, see Chapters 1–3 and the bibliography of the text [1].

6. The Stone book

About the year 1927 J. von Neumann and M.H. Stone were independently developing the properties of unbounded symmetric operators in abstract Hilbert space.

The results of von Neumann were published in papers from Göttinger Nachrichten (1927) to Annals of Mathematics (1932), and in his book *Mathematische Grundlagen der Quatenmechanik* (1932).

The results of Stone were collected together into his now classic text *Linear Transformations in Hilbert Space* (1932); see [18].

Stone refers to the 1910 paper of Weyl [26] for the early results concerning properties of the Sturm–Liouville differential equation at singular endpoints of the interval I.

In the Stone book there is the first account of the spectral properties of Sturm–Liouville differential operators, considered in the right-definite Hilbert function space, under the local Lebesgue integrability conditions, see [18, Chapter X, Section 3, Theorem 10.11]. The setting for the Stone analysis is

$$p^{-1}, q \in L^1_{\mathrm{loc}}(I) \quad \text{and} \quad w = 1 \text{ on } I \subseteq \mathbb{R}. \tag{6.1}$$

Remark 6.1. We note

(i) $p, q : I \to \mathbb{R}$.
(ii) There is no sign restriction on the coefficient p.
(iii) The Hilbert function space is $L^2(I)$.
(iv) The Stone analysis can be extended to the case when $w(x) > 0$, for almost all $x \in I$, within the space $L^2(I; w)$.

The interest for this paper, however, is that the Stone theory does not make use of complex variable techniques; in particular there is no introduction of the m-coefficient. He works essentially with the now named minimal and maximal operators, generated by the Sturm–Liouville differential expression, in his space $L^2(I)$; then boundary conditions, in the form of linear functionals, are applied on the elements of the maximal domain, to give domains of self-adjoint operators.

7. The contributions of Titchmarsh

The relevant Titchmarsh 1941 papers are [19–21]. These papers reconsider and reprove the Weyl results listed as (a)–(d) in Section 3 above.

The Titchmarsh–Weyl l-coefficient is introduced in [19, Section 2]; the analytic properties of this coefficient are given in [19, Section 5], noting the use made of the Vitali convergence theorem. However, the notation here is quite different from the notation later introduced in the texts of 1946 [22] and 1962 [24]. In [19] the spectral parameter and the coefficient are denoted by w and $l(\cdot)$, respectively; this notation was changed to λ and $m(\cdot)$, respectively, for the two editions of the book [22, Chapters II and III; 24, Chapters II and III], and has since been widely, but not exclusively, used in later times.

In these 1941 papers Titchmarsh showed that the Weyl circle method, and thereby the limit-point/limit-circle classification, are connected with the l-coefficient; this coefficient is unique if and only if the differential equation (1.1) is in the limit-point case.

The eigenfunction expansion results in [19] are concerned with the special case when the interval for the differential equation is $[0, \infty)$ and the l-coefficient is a meromorphic function on the complex plane $\mathbb{C}$.

In the paper [20, Section 4, Lemma v] the interval is again $[0,\infty)$ but now the l-coefficient is used to define the function $k:\mathbb{R}\to\mathbb{R}$, which is monotonic non-decreasing on $\mathbb{R}$. This k-function is later defined in the book [24, Chapter III, Section 3, Lemma 3.3] by

$$k(\lambda) := -\lim_{\delta\to 0^+}\int_0^\lambda \mathrm{Im}(m(u+\mathrm{i}\delta))\,\mathrm{d}u \quad \text{for } \lambda\in\mathbb{R}. \tag{7.1}$$

In this respect it is interesting to quote the Titchmarsh statement from the references at the end of [24, Chapter III, p. 70]:

> So far as I know the function $k(\lambda)$ appeared for the first time in the first edition of this book. [1] It was rediscovered a little later by Kodaira, 1949; [2] see also Chapter VI. [3]

The general form of the eigenfunction expansion is given in [20, Sections 5 and 7], but see also [24, Chapters III and VI].

In the paper [21] the earlier results are extended to cover the case when the interval is $(-\infty,\infty)$ and requires the introduction of two l-coefficients, l_1 and l_2.

In the 1941 papers and, later in the texts [22, Chapter III; 24, Chapters III and VI], Titchmarsh introduces his definition of the spectrum of his boundary value problem. This definition, in the general case, depends on the continuity and discontinuity properties of the monotonic nondecreasing function $k(\cdot)$ generated by the particular boundary value problem, i.e. from the associated l-coefficient, equivalently the later defined m-coefficient. (In this respect see the remarks in Section 9 below on the Chaudhuri–Everitt paper [8].)

I add the following personal remarks:

(1) Sometime about 1958 I pointed out to Titchmarsh that his k-function, see above, gave him the Lebesgue–Stieltjes Hilbert space $L^2((-\infty,\infty);k)$ of Borel measurable functions $f:(-\infty,\infty)\to\mathbb{C}$, such that

$$\int_{-\infty}^{+\infty} |f(t)|^2\,\mathrm{d}k(t) < +\infty,$$

and that in this space the multiplication operator gives the canonical representation of his self-adjoint boundary value problem. My intention was to link his remarkable achievements in deriving results using only the methods of real and complex classical analysis, with the general theory of eigenfunction expansions for self-adjoint operators in abstract Hilbert spaces. However, he never expressed any enthusiasm for relating his results to abstract operator theory.

(2) About the same time I remarked to Titchmarsh that his m-coefficient is a Nevanlinna (Herglotz, Pick, Riesz) function, see [1, Chapter 3, Section 1; 2, Section 59], in that

$$\frac{\mathrm{Im}(-m(\lambda))}{\mathrm{Im}(\lambda)} > 0 \quad \text{for all } \lambda\in\mathbb{C}\setminus\mathbb{R}, \tag{7.2}$$

[1] See [22], but note also the above references to the earlier papers published in 1941.
[2] See [15].
[3] See [24, Chapter VI].

due to the formula, see [24, Chapter II, Section 2.1 (2.1.8) and Section 2 (2.5.2)],

$$\int_0^{+\infty} |\psi(x,\lambda)|^2 \, \mathrm{d}x \equiv \int_0^{\infty} |\theta(x,\lambda)+m(\lambda)\varphi(x,\lambda)|^2 \, \mathrm{d}x = \frac{\mathrm{Im}(-m(\lambda))}{\mathrm{Im}(\lambda)} \quad \text{for all } \lambda \in \mathbb{C}\setminus\mathbb{R}. \tag{7.3}$$

Here θ and φ denote the solutions of the Titchmarsh differential equation determined by real-valued initial conditions [24, Chapter II, Section 2.1, (2.1.4)] at the endpoint 0 of the interval $[0,\infty)$; see also Section 2 above.

For such analytic functions $m(\cdot)$, taking into account the negative sign in (7.2), there is the Nevanlinna representation, see see [1, Chapter 3, Section 1; 2, Section 59; 5, Section 4] and [11, Chapter 2],

$$-m(\lambda) = \alpha + \beta\lambda + \int_{-\infty}^{+\infty} \left\{ \frac{1}{t-\lambda} - \frac{t}{t^2+1} \right\} \mathrm{d}\rho(t) \quad \text{for all } \lambda \in \mathbb{C}\setminus\mathbb{R}, \tag{7.4}$$

where

(i) the function $\rho:\mathbb{R}\to\mathbb{R}$, is monotonic nondecreasing on $\mathbb{R}$, and is uniquely determined if made right-continuous on $\mathbb{R}$; it is normalised by requiring that $\rho(0)=0$

(ii) it follows from the existence of representation (7.4) that the growth of ρ at $\pm\infty$ is controlled such that

$$\int_{-\infty}^{+\infty} \frac{1}{t^2+1} \, \mathrm{d}\rho(t) < +\infty \tag{7.5}$$

(iii) α, β are uniquely determined real numbers with $\beta \geqslant 0$.

The integrals in (7.4) and (7.5) are best seen as Lebesgue–Stieltjes integrals with respect to the regular, nonnegative, Borel measure generated by the function ρ on the Borel σ-algebra of $\mathbb{R}$.

(3) I further mentioned that the Titchmarsh inverse formula (7.1) is equivalent to the Nevanlinna inverse formula for (7.4),

$$\rho(t) = -\lim_{\delta\to 0^+} \frac{1}{\pi} \int_0^t \mathrm{Im}(m(s+\mathrm{i}\delta)) \, \mathrm{d}s \quad \text{for all } t \in \mathbb{R}, \tag{7.6}$$

except for the factor involving π, i.e.,

$$k(t) = \pi\rho(t) \quad \text{for } t \in \mathbb{R}, \tag{7.7}$$

but I cannot recall his comment on this information.

(4) In retrospect it seems surprising that Titchmarsh made no use of the Nevanlinna representation (7.4); however there are similar results, which seem to be due to Titchmarsh himself, to be seen in [20,23, Section 1; Chapter XXII, Theorems 22.23 and 22.24].

(5) I discussed the content of the two references Kodaira, [15], and Stone, [18], with Titchmarsh but again, I seem to recall, he was reluctant to become too involved with the content and methods of, respectively, this paper and this volume. He gave the Kodaira paper [15] as a reference in the second edition of his book [24] but did not reference the Stone book [18].

All these remarks and discussions from 1958 have to be seen in the light of the changes that Titchmarsh made for the second edition of his text [24] which appeared in 1962; see in particular Chapter VI. However, note that (7.7) is equivalent to [24, Chapter VI, Section 6.7, (6.7.6)].

I played no part in the preparation of the 1962 second edition [24] and I have no memory of ever discussing the contents of Chapter VI with Titchmarsh. The Titchmarsh references for this Chapter involve the names of Levitan and Sears.

8. The Coddington-Levinson book

The well-known text on ordinary differential equations by Coddington and Levinson (1955), see [9], devotes a whole chapter to singular Sturm–Liouville boundary value problems, including an account of the properties of the m-coefficient; see [9, Chapter 9].

Boundary value problems are considered on the interval $[0,\infty)$ with $p, p', q \in C[0,\infty)$, with $p(x) > 0$ and $w(x) = 1$ for all $x \in [0,\infty)$. However there is a footnote [9, Chapter 9, p. 224] which states it suffices for the real-valued coefficients p, q to satisfy $p \in AC_{\mathrm{loc}}[0,\infty)$ and $q \in L^1_{\mathrm{loc}}[0,\infty)$.

The Weyl limit-point/limit-circle classification is introduced, together with independence upon the spectral parameter $\lambda \in \mathbb{C}$; see [9, Chapter 9, Section 2].

The Titchmarsh analysis is followed to give the existence of the m-coefficient and the Weyl solution in the integrable-square space $L^2[0,\infty)$.

Eigenfunction expansions are determined but not by the Titchmarsh contour integration method; instead, the expansion is obtained following the Levitan (1950) and Levinson (1951) analysis, using a limit process from boundary value problems on compact subintervals $[0,b] \subset [0,\infty)$.

The Nevanlinna representation of the m-coefficient, see (7.4)$-(py')' + qy = \lambda wy$ on, is introduced indirectly, see [9, Chapter 9, Section 3, (3.10)]; no call is made on the general Nevanlinna representation theorem.

9. The Chaudhuri–Everitt paper

The 1969 paper [8] seems to have been the first contribution to show the equivalence of the Titchmarsh definition of spectrum of the boundary value problem in terms of the $k(\cdot)$ function and the m-coefficient, see (7.1) and [23, Chapter III, Section 3.9], and of the abstract definition of the spectrum of the associated self-adjoint differential operator in the Hilbert function space, see [16,2, Chapter IV, Section 12.5; and Chapter 4, Section 48].

10. The Brown, Evans, Littlejohn papers

For Sturm–Liouville type difference equations the equivalent of Eq. (1.1) can be written in the Lagrange symmetric (formerly self-adjoint) form

$$-\Delta(p_{n-1}\Delta x_{n-1}) + q_n x_n = \lambda w_n x_n \quad \text{for all } n \in \mathbb{N}_0. \tag{10.1}$$

Here the difference operator Δ acting on any sequence of real or complex numbers $\{a_n\}$ is given by $\Delta a_n := a_{n+1} - a_n$ for any integer n; the spectral parameter $\lambda \in \mathbb{C}$.

The Hellinger/Nevanlinna form (5.1) of the difference equation can be recast into the symmetric form (10.1).

In considering the difference equation (10.1) we follow the notation of Brown and Evans, in [6, Section 1] and of Brown et al. [7, Section 1]; in turn these papers are based on the results of Atkinson in his now classic text [3, Chapters 2 and 5].

The sequence $\{x_n\}$ is the dependent variable of the difference equation; $\{p_n\}$, $\{q_n\}$ and $\{w_n\}$ are the coefficient sequences; these sequences satisfy the conditions:

(i) $x_n \in \mathbb{C}$ for $n = -1, 0, 1, 2 \cdots$
(ii) $p_n, w_n \in \mathbb{R}$, $p_n \neq 0$, and $w_n > 0$ for $n = -1, 0, 1, 2 \cdots$
(iii) $q_n \in \mathbb{R}$ for all $n \in \mathbb{N}_0$.

Eq. (10.1) is studied in the Hilbert sequence space ℓ^2_w defined by

$$\ell^2_w := \left\{ \{x_n \in \mathbb{C} : n = -1, 0, 1, 2 \cdots\} : \sum_{n=-1}^{\infty} w_n |x_n|^2 < +\infty \right\} \tag{10.2}$$

with norm $||\cdot||_w$ and inner-product $\langle \cdot, \cdot \rangle_w$.

As in the Weyl and then the Titchmarsh study of the Sturm–Liouville equation (1.1), the results of Hellinger [14] and Nevanlinna [17] were re-considered by Atkinson [3, Chapter 5] to prove that the difference equation (10.1) has at least one non-null solution $\{\psi_n : n = -1, 0, 1, 2 \cdots\}$ in ℓ^2_w for all $\lambda \in \mathbb{C} \setminus \mathbb{R}$, i.e.,

$$||\{\psi_n\}||^2_w = \sum_{n=-1}^{\infty} w_n |\psi_n(\lambda)|^2 < +\infty. \tag{10.3}$$

From this solution there follows the existence of the Hellinger/Nevanlinna $w(\cdot)$ analytic function, and then the Atkinson m-coefficient, see [3, Chapter 5]. In terms of recent notation see the account in [6, Section 2] and [7, Section 2]; in particular we have the identity, see [7, Section 2, item (ii)] and compare with (7.3),

$$\sum_{n=0}^{\infty} w_n |\psi_n(\lambda)|^2 = \frac{\operatorname{Im}(m(\lambda))}{\operatorname{Im}(\lambda)} \quad \text{for all } \lambda \in \mathbb{C} \setminus \mathbb{R}. \tag{10.4}$$

This last property yields the m-coefficient for the difference equation (10.1) as a Nevanlinna (Herglotz, Pick, Riesz) analytic function.

The Nevanlinna representation of this m-coefficient, i.e. following (7.4),

$$m(\lambda) = \alpha + \beta\lambda + \int_{-\infty}^{+\infty} \left\{ \frac{1}{t - \lambda} - \frac{t}{t^2 + 1} \right\} \mathrm{d}\rho(t) \quad \text{for all } \lambda \in \mathbb{C} \setminus \mathbb{R}. \tag{10.5}$$

This representation is used in the papers [6,7].

11. The Bennewitz paper

In 1983 Bennewitz, on a visit to the University of Birmingham, completed a draft manuscript [4] which, starting from the existence of an m-coefficient for a regular or singular boundary value problem and the associated Nevanlinna spectral function ρ, gives a complete proof of the Sturm–Liouville eigenfunction expansion using the contour integration method of Titchmarsh.

In order to claim the existence of the m-coefficient one point of the interval I is taken to be regular, whilst the other endpoint is regular, limit-point, or limit-circle; the one proof covers the expansion theorem in all cases.

The results are connected by the spectral function ρ with the abstract definition of the spectrum of the self-adjoint differential operators and so complement the earlier work of Chaudhuri and Everitt [8].

It is hoped to incorporate the results of this manuscript into a future publication.

12. The Everitt paper

This paper [13] is concerned with using the properties of the Sturm–Liouville self-adjoint differential operators to define the m-coefficient. The results provide an alternative method to show the equivalence of the Titchmarsh definition of the spectrum and the abstract operatic theoretic spectral definition; this gives another approach to the Chaudhuri–Everitt results given in Section 9.

13. Uncited reference

[12]

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