

On the approximation of nonlinear singular self-adjoint second order boundary value problems

M.A. El-Gebeily^a, K.M. Furati^a, Donal O'Regan^b, Ravi Agarwal^{c,*}

^a Department of Mathematical Sciences, King Fahd University of Petroleum and Minerals, Dhahran 31261, Saudi Arabia

^b Department of Mathematics, National University of Ireland, Galway, Ireland

^c Department of Mathematical Sciences, Florida Institute of Technology, 150 West University Blvd, Melbourne, FL 32901-6975, USA

ARTICLE INFO

Article history:

Received 25 September 2007

Received in revised form 7 May 2008

MSC:

65L10

Keywords:

Singular differential equations

Self-adjoint operators

Deficiency index

Avoiding singularity

ABSTRACT

We investigate the approximation of the solutions of a class of nonlinear second order singular boundary value problems with a self-adjoint linear part. Our strategy involves two ingredients. First, we take advantage of certain boundary condition functions to obtain well behaved functions of the solutions. Second, we integrate the problem over an interval that avoids the singularity. We are able to prove a uniform convergence result for the approximate solutions. We describe how the approximation is constructed for the various values of the deficiency index associated with the differential equation. The solution of the nonlinear problem is obtained by a globally convergent iterative method.

© 2008 Elsevier B.V. All rights reserved.

1. Introduction

We investigate the approximation of the solutions of the nonlinear singular ordinary differential equation

$$\ell y = Fy \quad (1)$$

where ℓ is a formally self-adjoint differential expression of the form

$$\ell(y) = \frac{1}{w} \left(-(py')' + qy \right). \quad (2)$$

We work with the Hilbert space $H = L_w^2(J)$ of square integrable functions with respect to the nonnegative weight w on the interval $J = (a, b)$, $-\infty \leq a < b \leq \infty$ and $F : H \rightarrow H$ is a gradient mapping of class C^1 . We assume that $1/p$ and q are integrable on any compact subinterval of J . The problem (1) is regular if J is finite and $1/p$ and q are integrable on J . Otherwise, it is singular. Singular problems have solutions that may become infinite or oscillate near the endpoints. In any case, the singularity of the problem poses a difficult challenge to numerical methods originally designed for regular problems, even if the solutions of the singular problem are well behaved.

Our strategy is to integrate (1) over a subinterval $J' \subset J$ which avoids the singularity. This is why this approach is called “avoiding the singularity”. Furthermore, certain boundary condition functions of the solutions remain well behaved, even if the original solutions are not. By taking these boundary condition functions into account we further regularize the numerical behavior of the solutions. In theory, these functions work well even if the integration is carried over the whole interval J .

* Corresponding author.

E-mail address: agarwal@fit.edu (R. Agarwal).

Basically, we begin by finding a fundamental system for

$$\ell(y) = 0$$

by solving initial value problems with initial conditions set at a regular point $c \in J$. This has two advantages: first, the regularity of the problem around c enables the use of efficient numerical solution routines, and, second, for a number of important types of problems, e.g., the Legendre or the Bessel equations, these solutions are explicitly known. Once these solutions are found, solving the boundary value problem is obtained by performing simple integration. The exact description of the method will depend on the *deficiency index* of the problem (see the next section). The deficiency index d can be 0, 1 or 2. We describe how to implement the approximation in all three cases and prove a related uniform convergence theorem.

This article consists of four sections beside the Introduction. In Section 2 we give the necessary definitions and investigate certain questions regarding the structure of self-adjoint operators associated with the expression ℓ . In Section 3 and its subsections we investigate the approximation method for the cases $d = 2, 1$ and 0. A main idea involved in this section is the replacement of the self-adjoint boundary conditions with suitable “initial conditions”. In Section 4 we consider the nonlinear problem. In the first subsection we give an existence and uniqueness theorem for the class of nonlinear problems to be dealt with here. The proof of the existence-uniqueness theorem is constructive and is then used in the second subsection to iteratively solve the nonlinear boundary value problem. Finally, in Section 5 we provide three illustrative examples.

2. Preliminaries

In this section we describe the class of self-adjoint operators S to be dealt with in this work. For convenience we list here some standard terminology and facts associated with linear singular differential expressions. The theoretical exposition of the subject can be found in [13,16].

The norm and inner product in H will be denoted by $\|\cdot\|$ and $\langle \cdot, \cdot \rangle$, respectively. The formally self-adjoint differential expression $\ell(y)$ of (2) induces the following operators in H .

The differential expression ℓ generates the following operators in H .

1. The maximal operator L defined by

$$D(L) = D = \{y \in L_w^2(J) : y, y^{[1]} \in AC(J), w\ell(y) \in L_w^2(J)\}, \\ Ly = \ell(y).$$

Here, and elsewhere, the notation $z^{[1]}$ is used to indicate the pseudo-derivative $z^{[1]} = pz'$.

2. The minimal operator $L_0 = L^*$. The domain of definition of L_0 is denoted by D_0 . It is known [13] that L_0 is a symmetric operator.

For $y, z \in D$ define the Lagrange bracket

$$[y, z]_x = (-y^{[1]}\bar{z} + y\bar{z}^{[1]})(x)$$

and the Wronskian

$$W_x(y, z) = (-y^{[1]}z + yz^{[1]})(x).$$

We assume throughout this paper that

$$\rho_\sigma(L_0) \cap \mathbb{R} \neq \emptyset$$

where $\rho_\sigma(L_0)$ denotes the set of points of regular type for L_0 (see [9]).

Let $\lambda \in \mathbb{R}$ and θ, φ be the two linearly independent solutions of

$$(\ell - \lambda)y = 0 \tag{3}$$

satisfying the initial conditions

$$\theta(c, \lambda) = 1, \quad \varphi^{[1]}(c, \lambda) = 1, \quad \theta^{[1]}(c, \lambda) = \varphi(c, \lambda) = 0 \tag{4}$$

for some $c \in J$. Such solutions exist [13], and $\theta, \varphi, \theta^{[1]}, \varphi^{[1]} \in AC(J)$. It can easily be shown that

$$W_x(\theta, \varphi) = 1 \quad \text{for all } x \in J.$$

We extend this identity to \bar{J} by setting $W_a(\theta, \varphi) = W_b(\theta, \varphi) = 1$. The operator L_0 is said to have deficiency index d where:

1. $d = 2$ if θ, φ (and hence all solutions of (3)) belong to H .
2. $d = 1$ if at least one of the functions θ and φ is not in H but a linear combination of them belongs to H .
3. $d = 0$ otherwise.

We say that the endpoint $a(b)$ is a regular point for the expression ℓ if it is finite and $1/p, q, w$ are integrable in a neighborhood of $a(b)$. Otherwise, $a(b)$ is singular. If $a(b)$ is singular, then it is further classified as:

1. Limit circle point (LC) for the expression ℓ if θ, φ are square integrable with respect to the weight w near $a(b)$.

2. Limit point (LP) for the expression ℓ if either θ or φ is not square integrable with respect to the weight w near a (b) but some linear combination of them is.

We have the following characterization [1,17] of the deficiency index.

1. $d = 2$ if and only if both a and b are either regular or LC.
2. $d = 1$ if and only if one endpoint is regular or LC and the other is LP.
3. $d = 0$ if and only if both endpoints are LP.

Note that if a is regular or LC then, for any $y \in D$,

$$W_a(y, \theta) = \lim_{x \rightarrow a^+} W_x(y, \theta),$$

$$W_a(y, \varphi) = \lim_{x \rightarrow a^+} W_x(y, \varphi)$$

both exist and are finite [13,17]. A similar statement holds at b . The most general boundary conditions to be assumed in connection with the expression ℓ are stated in terms of the above Wronskians (see [10]).

All self-adjoint operators S generated by the expression ℓ in the space H are d -dimensional extensions of the minimal operator L_0 (restrictions of the maximal operator L). i.e., $L_0 \subseteq S \subseteq L$.

In the case $d = 0$, $L_0 = S = L$ and no boundary conditions are needed.

In the case $d = 1$ a self-adjoint extension S of L_0 is characterized by the existence of a $\gamma \in [0, \pi)$ such that (assuming a to be LC and b to be LP)

$$D(S) = \{y \in D : [\cos \gamma \quad \sin \gamma] Y(a) = 0\}, \quad (5)$$

where

$$Y(x) = \begin{bmatrix} W_x(y, \theta) \\ W_x(y, \varphi) \end{bmatrix}. \quad (6)$$

Suppose $\lambda \in \rho_\sigma(L_0) \cap \mathbb{R}$. Then there is precisely one function $\psi \in D$ such that

$$(L - \lambda I) \psi = 0. \quad (7)$$

In general, we are interested in self-adjoint extensions S of L_0 that avoid having λ as an eigenvalue. The following lemma makes the connection with the boundary condition parameter (5).

Lemma 1. Suppose $\lambda \in \rho_\sigma(L_0) \cap \mathbb{R}$ and S is a self-adjoint extension of L_0 . Let ψ be defined by (7) and $\Psi(x)$ by (6). Then, for a given $\gamma \in [0, \pi)$, $\lambda \in \sigma_p(S)$ if and only if

$$[\cos \gamma \quad \sin \gamma] \Psi(a) = 0. \quad (8)$$

Proof. The assumption that λ is a point of regular type for the operator L_0 means that λ is not in the essential spectrum of any self-adjoint extension of L_0 . Therefore, λ could either be an eigenvalue or a resolvent point of any such an extension. If $\lambda \in \sigma_p(S)$ then there exists a function $\psi_1 \in H$ such that $(S - \lambda I) \psi_1 = 0$. Hence, $(L - \lambda I) \psi_1 = 0$. Since $d = 1$, we may assume without loss of generality that $\psi_1 = \psi$. Therefore, $\psi \in D(S)$ and the boundary condition defining $D(S)$ must be satisfied. i.e., (8) holds. On the other hand, if (8) holds, then $\psi \in D(S)$ and we have $(L - \lambda I) \psi = (S - \lambda I) \psi = 0$. i.e., $\lambda \in \sigma_p(S)$. ■

In the case $d = 2$ the characterization of real self-adjoint extensions of L_0 in terms of boundary values is described in one of two ways depending on whether we have coupled or separated boundary conditions. In the case of coupled boundary conditions a self-adjoint extension S of L_0 is characterized by the existence of a real 2×2 matrix A such that $\det(A) = 1$ and an $\alpha \in [0, 2\pi)$ such that

$$D(S) = \{y \in D : Y(b) = e^{i\alpha} A Y(a)\}. \quad (9)$$

We emphasize that the matrix A depends on λ because $Y(a)$ and $Y(b)$ do. In the case of separated boundary conditions, a real [11] self-adjoint extension S of L_0 is characterized by the existence of two numbers $\gamma, \delta \in [0, \pi)$ such that

$$D(S) = \{y \in D : [\cos \gamma \quad \sin \gamma] Y(a) = [\cos \delta \sin \delta] Y(b) = 0\}. \quad (10)$$

The following lemma is the counterpart of Lemma 1 in the case $d = 2$.

Lemma 2. Suppose $\lambda \in \rho_\sigma(L_0) \cap \mathbb{R}$ and S is a self-adjoint extension of L_0 .

1. In the case of coupled boundary conditions, $\lambda \in \rho(S)$ if and only if $\det(I - e^{i\alpha} A) \neq 0$.
2. In the case of separated boundary conditions, $\lambda \in \rho(S)$ if and only if $\delta = \gamma$.

Proof. 1. For $y \in D(S)$ the equation $(S - \lambda I)y = 0$ is equivalent to $y = c_1\theta + c_2\varphi$. Applying the coupled boundary conditions (9) we see that this is equivalent to

$$(I - e^{i\alpha}A) \begin{bmatrix} -c_2 \\ c_1 \end{bmatrix} = 0$$

which has the trivial solution if and only if $\det(I - e^{i\alpha}A) \neq 0$. Hence, this last condition is equivalent to the condition that λ is not an eigenvalue of S . Since in the case $d = 2$, the spectrum of S consists of separated eigenvalues [13], it follows that the condition is equivalent to $\lambda \in \rho(S)$.

2. Following an argument similar to the first part we get that

$$\begin{bmatrix} \cos \gamma & \sin \gamma \\ \cos \delta & \sin \delta \end{bmatrix} \begin{bmatrix} -c_2 \\ c_1 \end{bmatrix} = 0$$

has the trivial solution if and only if $\sin(\gamma - \delta) \neq 0$, and since $\gamma, \delta \in [0, \pi)$ this is equivalent to $\gamma \neq \delta$. ■

3. Avoiding singularity

In this section we discuss the approximation of the singular problem

$$(\ell - \lambda)y = f$$

on the interval J by the regular problem on an interval $J' = (a', b')$ with $a' \geq a$ and $b' \leq b$. This approach is called avoiding singularity (see [8,14]). Assume that S is a self-adjoint realization of ℓ in H . Then we want to solve

$$(S - \lambda I)y = f. \quad (11)$$

3.1. The Case $d = 2$

Here, S is determined by either coupled or separated boundary conditions. Define the function G by

$$G(x) = \int_a^x \begin{bmatrix} \theta f \\ \varphi f \end{bmatrix}. \quad (12)$$

Lemma 3. Assume $d = 2$ and $\lambda \in \rho(S)$. For a given $f \in H$, the solution y of (11) satisfies

$$Y(x) = (I - e^{i\alpha}A)^{-1}G(b) + G(x) \quad (13)$$

if S is defined by coupled boundary conditions and

$$Y(x) = -B^{-1}\tilde{B}G(b) + G(x) \quad (14)$$

where

$$B = \begin{bmatrix} \cos \gamma & \sin \gamma \\ \cos \delta & \sin \delta \end{bmatrix}$$

and

$$\tilde{B} = \begin{bmatrix} 0 & 0 \\ \cos \delta & \sin \delta \end{bmatrix}$$

if S is defined by separated boundary conditions.

Proof. Consider the initial value problem

$$\begin{aligned} (\ell - \lambda)y &= f, \\ Y(a) &= \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}. \end{aligned} \quad (15)$$

Note that

$$\int_a^x \theta(L - \lambda I)yw - \int_a^x y(L - \lambda I)\theta w = W_x(y, \theta) - W_a(y, \theta)$$

where θ, φ are the solutions of (3) and (4). This equation reduces to

$$W_x(y, \theta) = c_1 + \int_a^x \theta f w.$$

Similarly,

$$W_x(y, \varphi) = c_2 + \int_a^x \varphi f w.$$

Hence,

$$Y(x) = Y(a) + G(x). \quad (16)$$

For the operator S with coupled boundary conditions, a simple calculation gives

$$Y(a) = (I - e^{i\alpha} A)^{-1} G(b). \quad (17)$$

This, together with (16), gives (13).

For the operator S with separated boundary conditions, another simple calculation gives

$$Y(a) = -B^{-1} \widetilde{B} G(b). \quad (18)$$

This, together with (16), gives (14). ■

Now let $a' \geq a$ and $b' \leq b$ (with strict inequality in the case of singularity) and let $J' = (a', b')$. For a given $f \in H$ consider the regularized problem

$$(S' - \lambda I)y = f, \quad (19)$$

where S' is the self-adjoint operator generated in $L_w^2(J')$ by the same boundary conditions determining S . For example, if S is determined by the coupled boundary conditions $Y(b) = e^{i\alpha} A Y(a)$ then S is determined by the boundary conditions

$$Y(b') = e^{i\alpha} A Y(a').$$

Theorem 4. Assume $d = 2$ and $\lambda \in \rho(S)$. For a', b' sufficiently close to a, b , respectively, $\lambda \in \rho(S')$. Furthermore, the solution y' of (19) is such that $Y' \rightarrow Y$ uniformly as $(a', b') \rightarrow (a, b)$ (it is understood here that $Y'(x)$ is extended by zero outside J').

Proof. The proof that $\lambda \in \rho(S')$ can be found in [1]. Denoting by G' the counterpart of G for the interval J' , extended by zero outside J' , we have

$$|G'(x) - G(x)| \leq \sqrt{2} \left(\int_a^{a'} + \int_{b'}^b \right) (|\theta f| + |\varphi f|) w. \quad (20)$$

The result follows from this, Eqs. (13) and (14) and the absolute continuity of the Lebesgue integral. ■

Thus, in order to approximate the solution of (11) it suffices to solve (19) on a suitable truncation J' of the interval J . This in turn means that we need only construct the functions θ and φ (numerically) on the interval J' . Once this is done, the solution of (19) is obtained from (13) or (14) restricted of course to J' . Observe also that Eq. (20) provides an error estimate for the truncation and may be used to determine J' if θ and φ are known on J .

3.2. The Case $d = 1$

Assume, that the endpoint a is regular or LC and the endpoint b is LP. Here, the self-adjoint operator S in (11) is determined by a boundary condition of the form

$$[\cos \gamma \quad \sin \gamma] Y(a) = 0 \quad (21)$$

and, as usual, we assume that λ is a resolvent point of S . We begin by finding an explicit expression of $Y(a)$. This will be the content of Theorem 8 below.

Since $d = 1$, there exists a function $\psi \in D$ such that

$$(L - \lambda I)\psi = 0.$$

As a matter of fact, we can write

$$\psi = \theta + m\varphi \quad (22)$$

for some real scalar m . To see this let ρ be the spectral function for S (see [13]). Then ρ is related to the Titchmarsh–Weyl m -function by

$$m(\lambda') - m(\lambda_0) = \int_{-\infty}^{\infty} \left(\frac{1}{\lambda' - \mu} - \frac{1}{\lambda_0 - \mu} \right) d\rho(\mu) + c(\lambda' - \lambda_0),$$

where c is a constant and $\text{Im}(\lambda_0) \neq 0$ (see [7]). Since λ is a resolvent point for S , ρ is a constant in a neighborhood of λ and $d\rho(\mu)$ vanishes in that neighborhood. Therefore, the right-hand side of the above equation can be continued across the real line at $\lambda' = \lambda$. We set $m = m(\lambda)$. It follows from the properties of the Titchmarsh–Weyl m -function that m is real. For more on the theory of the Titchmarsh–Weyl m -function and its relation to the spectral function see [2,6,6,15] and for the computational effort on its approximation see [3–5] and the references therein.

The following lemma gives a convenient representation of D as a 2-dimensional extension of D_0 .

Lemma 5. *There exist two functions ψ_1, ψ_2 in D such that:*

1. $\psi_1 = \theta, \psi_2 = \varphi$ near a and $\psi_1 = \psi_2 = 0$ near b ,
2. ψ_1, ψ_2 are linearly independent modulo D_0 ,
3. $D = D_0 + \text{span}[\psi_1, \psi_2]$.

Proof. See [1]. ■

Now suppose that we want to solve the initial value problem

$$(\ell - \lambda)y = f, \tag{23}$$

$$Y(a) = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}. \tag{24}$$

Since not all solutions of (23) are in H , we wish to determine the restrictions on the initial values c_1, c_2 in order to obtain solutions in D .

Lemma 6. *The solution of (23) and (24) is in D if and only if*

$$c_1 + mc_2 = -\langle f, \psi \rangle.$$

Proof. The solution y of (23) and (24) is in D if and only if we can write

$$y = y_0 + r\psi_1 + s\psi_2$$

for some $y_0 \in D_0$ and where ψ_1, ψ_2 are the functions constructed in the previous lemma. Then

$$Y(a) = \begin{bmatrix} -s \\ r \end{bmatrix}.$$

Hence, $r = c_2$ and $s = -c_1$. On the other hand, Eq. (23) can be split into

$$(L_0 - \lambda I)y_0 + P(c_2f_1 - c_1f_2) = Pf,$$

$$Q(c_2f_1 - c_1f_2) = Qf,$$

where $(L - \lambda I)\psi_1 = f_1, (L - \lambda I)\psi_2 = f_2, P$ is the orthogonal projection on $R(L_0 - \lambda I)$ and Q is the orthogonal projection on $R(L_0 - \lambda I)^\perp$. Since the latter space is one dimensional, the second equation is equivalent to

$$c_2\langle f_1, \psi \rangle - c_1\langle f_2, \psi \rangle = \langle Qf, \psi \rangle,$$

or

$$c_2\langle f_1, \psi \rangle - c_1\langle f_2, \psi \rangle = \langle f, \psi \rangle.$$

Using the definitions of f_1, f_2 and integrating by parts, we get

$$c_1W_a(\varphi, \psi) - c_2W_a(\theta, \psi) = \langle f, \psi \rangle.$$

Finally, taking into account that

$$\psi = \theta + m\varphi$$

we obtain

$$-c_1 - mc_2 = \langle f, \psi \rangle. \quad \blacksquare$$

The solution of (11) is now obtained as follows. Since $y \in D$, then the following system must be satisfied

$$\begin{bmatrix} \cos \gamma & \sin \gamma \\ 1 & m \end{bmatrix} Y(a) = \begin{bmatrix} 0 \\ -\langle f, \psi \rangle \end{bmatrix}.$$

Lemma 7. $\lambda \in \rho(S)$ if and only if the matrix

$$C = \begin{bmatrix} \cos \gamma & \sin \gamma \\ 1 & m \end{bmatrix}$$

is invertible.

Proof. If C is not invertible, then its rows are linearly dependent. Assume that $\begin{bmatrix} \cos \gamma & \sin \gamma \end{bmatrix} = \alpha \begin{bmatrix} 1 & m \end{bmatrix}$, then boundary condition (21) gives

$$Y(a) = \tau \begin{bmatrix} -m \\ 1 \end{bmatrix} = \tau \Psi(a).$$

Hence, ψ satisfies the boundary conditions determining S . Therefore, $\psi \in D(S)$. We also have

$$(S - \lambda I) \psi = (L - \lambda I) \psi = 0.$$

This means that λ is an eigenvalue of S . On the other hand, if C is invertible, then the system

$$\begin{bmatrix} \cos \gamma & \sin \gamma \\ 1 & m \end{bmatrix} Y(a) = 0$$

has only the trivial solution and so does the equation

$$(S - \lambda I) y = 0.$$

Hence, λ is not an eigenvalue of S . Since λ is also a point of regular type of L_0 , then $\lambda \in \rho(S)$. ■

Thus, we have the following theorem.

Theorem 8. Assume $d = 1$, $\lambda \in \rho(S)$ and m is as determined by (22). Then y is the solution of (11) if and only if

$$Y(a) = \begin{bmatrix} \cos \gamma & \sin \gamma \\ 1 & m \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ -\langle f, \psi \rangle \end{bmatrix}. \quad (25)$$

Now that $Y(a)$ has been determined, the solution of (11) satisfies the equation

$$Y(x) = Y(a) + G(x),$$

where $G(x)$ is given by (12). The regularized problem is constructed by choosing $b_0 < b$ and $a' \geq a$ and solving the “initial value problem”

$$\begin{aligned} (\ell - \lambda) u(x) &= f(x), \quad x \in J' \\ U(a') &= Y(a), \end{aligned} \quad (26)$$

where $J' = (a', b_0)$. The solution y' of (23) satisfies

$$Y'(x) = Y(a) + G'(x),$$

where G' is given by (12) with a replaced by a' .

Theorem 9. Assume $d = 1$ and $\lambda \in \rho(S)$. For a' sufficiently close to a , the solution y' of (19) is such that $Y' \rightarrow Y$ uniformly on $(a, b_0]$ as $a' \rightarrow a$ (it is understood here that $Y'(x)$ is extended by zero outside J').

3.3. The case $d = 0$

In this case both endpoints are LP and the operator L_0 is self-adjoint. The domain D_0 of L_0 is then described by

$$D_0 = \{y \in H : \ell y \in H\}.$$

It follows naturally that, for any pair $y, z \in D_0$, $[y, z]_a^b = 0$. As in the previous subsection, the same type of difficulty arises also in this case: Which “initial values” guarantee that the corresponding solution of (23) and (24) is in H ? To work out this problem first we need to obtain a more convenient representation for \mathcal{D}_0 .

Lemma 10. *There exist two functions $\varphi_a, \varphi_b \in D_0$ such that:*

1. φ_a is identically zero near b ,
2. φ_b is identically zero near a ,
3. $D_0 = \{y \in H : \ell y \in H, [y, \varphi_a](a) = [y, \varphi_b](b) = 0\}$.

Proof. Let $c \in J$ and denote by L_0^-, L_0^+ the minimal operators induced by ℓ in $L_w^2(a, c)$, $L_w^2(c, b)$, respectively. The deficiency index of L_0^\pm is 1 since c is a regular point for both operators. Also, let A' be the restriction of L_0' to functions y satisfying the conditions

$$y^{[k]}(c) = 0, \quad k = 0, 1$$

and A be its closure. Clearly

$$\dim_{D(A)} D_0 = 2$$

and

$$A = L_0^- \oplus L_0^+.$$

Hence, if $\psi_1, \psi_2 \in D_0$ are linearly independent modulo $D(A)$ then any $y \in D_0$ can be written as

$$y = y^- + y^+ + \alpha_1 \psi_1 + \alpha_2 \psi_2,$$

where y^\pm is the extension by 0 of some function in D_0^\pm . The functions ψ_1, ψ_2 can actually be chosen in D_0' . Now there are real numbers m^\pm (see the previous subsection) such that $\psi^\pm = \theta + m^\pm \varphi \in \mathcal{D}^\pm$. Choose $\varphi_a, \varphi_b \in D_0$ such that

$$\varphi_a = \begin{cases} \psi^- & \text{on } (a, c) \\ 0 & \text{near } b \end{cases}$$

and

$$\varphi_b = \begin{cases} \psi^+ & \text{on } [c, b) \\ 0 & \text{near } a \end{cases}$$

then φ_a, φ_b satisfy the first two properties of the lemma. Moreover, if $y \in D_0$ then

$$[y, \varphi_a](a) = [y, \varphi_b](b) = 0.$$

This shows that D_0 is contained in the right-hand side of Part 3 of the lemma. On the other hand, it is clear that the right-hand side of Part 3 is a subset of D_0 . ■

In order to solve the boundary value problem

$$(S - \lambda I)y = f \tag{27}$$

in the case $d = 0$ we split the domain at a point $c \in (a, b)$. Then we have two problems, each with $d = 1$ on the intervals (a, c) and (c, b) since c is a regular point for ℓ . Using the approach of the previous subsection we can then solve the equations

$$(S^\pm - \lambda I^\pm)y^\pm = f^\pm$$

where the superscript $+$ denotes restriction to (c, b) and $-$ denotes restriction to (a, c) . Here the self-adjoint operators S^\pm satisfy boundary conditions of the form

$$[\cos \gamma^\pm \quad \sin \gamma^\pm] Y^\pm(c) = 0$$

and γ^\pm are to be chosen among those values for which λ is a resolvent point for S^\pm (see Lemma 1). That λ is a point of regular type for L_0^\pm follows from the assumption that λ is a point of regular type for $L_0 (= S)$ since any function in $D(L_0^\pm)$ can be extended by zero to a function in D_0 . Now by applying Eq. (25) to this case, we have

$$Y^\pm(c) = \begin{bmatrix} \cos \gamma^\pm & \sin \gamma^\pm \\ 1 & m^\pm \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ -(f^\pm, \psi^\pm)_\pm \end{bmatrix}.$$

Hence, the solution over the interval (a, b) is obtained by matching $Y^-(c)$ with $Y^+(c)$. i.e., by choosing γ^\pm to satisfy

$$\begin{bmatrix} \cos \gamma^- & \sin \gamma^- \\ 1 & m^- \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ -\langle f^-, \psi^- \rangle_- \end{bmatrix} = \begin{bmatrix} \cos \gamma^+ & \sin \gamma^+ \\ 1 & m^+ \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ -\langle f^+, \psi^+ \rangle_+ \end{bmatrix}.$$

It is not difficult to see that this system gives $\gamma^+ = \gamma^- = \gamma$ where γ satisfies

$$\left(\langle f^-, \psi^- \rangle_- - \langle f^+, \psi^+ \rangle_+ \right) \sin \gamma = \left(m^+ \langle f^-, \psi^- \rangle_- - m^- \langle f^+, \psi^+ \rangle_+ \right) \cos \gamma. \quad (28)$$

Since $[y^-, \psi^-]_a = [y^+, \psi^+]_b = 0$, it follows from Lemma 10 that the function $y = y^- \oplus y^+$ belongs to D_0 . Hence y is the desired solution of (27).

To summarize, the numerical method for the case $d = 0$ reduces to solving the two initial value problems

$$(\ell - \lambda)y^\pm = f^\pm, \\ Y^\pm(c) = \begin{bmatrix} \cos \gamma & \sin \gamma \\ 1 & m^\pm \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ -\langle f^\pm, \psi^\pm \rangle_\pm \end{bmatrix}$$

where $c \in (a, b)$ and γ is chosen to satisfy Eq. (28). The solution y of (27) is formed given by $y = y^- \oplus y^+$.

4. The nonlinear problem

In this section we investigate the application of the numerical method developed in the last section to a class of nonlinear problems of the form

$$\ell y = Fy,$$

where F is a nonlinear mapping. The exact assumptions on F will be stated later. In the next subsection we prove an existence and uniqueness theorem for the class of nonlinear singular problems to be considered. The treatment there goes along the same lines as in [12]. In the subsection that follows we briefly consider the numerical approximations discussed above to this class of problems.

4.1. Existence and uniqueness theorem

The following assumptions are made on F .

1. F is a gradient mapping, i.e., there exists a mapping $N : H \rightarrow H$ such that $F(y) = DN(y)$ where D denotes the Fréchet derivative.
2. There exist real numbers α, β , with $\alpha \leq \beta$ such that $[\alpha, \beta] \subset \rho(S)$, and for all $y, z \in H$, one has

$$\alpha \|y - z\|^2 \leq \langle Fy - Fz, y - z \rangle \leq \beta \|y - z\|^2. \quad (29)$$

Let $(\lambda^-, \lambda^+) \subset \rho(S)$ be the largest interval containing $[\alpha, \beta]$. Let $\lambda \in (\lambda^-, \lambda^+)$ and define the nonlinear operator $M : H \rightarrow H$ by

$$My = (F - \lambda I)y.$$

It is easy to check that M is a gradient operator and

$$(\alpha - \lambda) \|f - g\|^2 \leq \langle Mf - Mg, f - g \rangle \leq (\beta - \lambda) \|f - g\|^2.$$

Lemma 11. For all $y, z \in H$ we have

$$\|My - Mz\| \leq \max(|\alpha - \lambda|, |\beta - \lambda|) \|y - z\|.$$

Proof. See [12]. ■

Under these assumptions one has the following theorem.

Theorem 12. Let $(\lambda^-, \lambda^+) \subset \rho(S)$ be the largest interval containing $[\alpha, \beta]$. For each $f \in H$:

1. the equation

$$Sy - F(y) = f \quad (30)$$

has a unique solution,

2. the mapping $f \mapsto y$ is Lipschitzian in f , and

3. the solution y may be obtained by successive approximation using the iterations

$$(S - \lambda I) y_{k+1} = (F - \lambda I) y_k + f, \quad k = 0, 1, \dots \quad (31)$$

where $y_0 \in D(S)$ is arbitrary and

$$\frac{\lambda^- + \alpha}{2} < \lambda < \frac{\lambda^+ + \beta}{2}. \quad (32)$$

Proof. For $\lambda \in (\lambda^-, \lambda^+)$, Eq. (30) is equivalent to the fixed point problem

$$y = R(S, \lambda) (My + f). \quad (33)$$

Now

$$\begin{aligned} \|R(S, \lambda) (My - Mz)\| &\leq \|R(S, \lambda)\| \|My - Mz\| \\ &\leq \frac{1}{\text{dist}(\lambda, \sigma(S))} \max(|\alpha - \lambda|, |\beta - \lambda|) \|y - z\| \\ &= \chi(\lambda) \|y - z\|, \end{aligned}$$

where

$$\chi(\lambda) = \frac{\max(|\alpha - \lambda|, |\beta - \lambda|)}{\min(\lambda - \lambda^-, \lambda^+ - \lambda)}.$$

It is easy to check that $\chi(\lambda) < 1$ if and only if λ is chosen such that

$$\frac{\lambda^- + \alpha}{2} < \lambda < \frac{\lambda^+ + \beta}{2}.$$

The Banach fixed point theorem then implies that Eq. (33) has a unique solution. Also, if y, z are solutions of (33) corresponding to f, f' respectively, then

$$\begin{aligned} \|y - z\| &= \|R(S, \lambda) (My - Mz + f - f')\| \\ &\leq \|R(S, \lambda)\| (\|My - Mz\| + \|f - f'\|) \\ &\leq \chi(\lambda) \|y - z\| + \frac{1}{\min(\lambda - \lambda^-, \lambda^+ - \lambda)} \|f - f'\|, \end{aligned}$$

so that

$$\|y - z\| \leq \frac{1}{\min(\lambda - \lambda^-, \lambda^+ - \lambda) - \max(|\alpha - \lambda|, |\beta - \lambda|)} \|f - f'\|.$$

Hence, the mapping $f \mapsto y$ is Lipschitzian in f . ■

4.2. The numerical method for the nonlinear problem

The numerical method for the nonlinear problem (1) is implemented iteratively with the help of Eq. (31) and the results of Section 3. Namely, starting with an initial guess y^0 we use the shooting method of either one of the previous subsections to find a solution for the linear equation

$$(S - \lambda I) y = (F - \lambda I) y_k + f, \quad k = 0, 1, \dots$$

and then set $y_{k+1} = y$. Theorem 12 guarantees convergence to the solution of the nonlinear problem.

5. Examples

In this section we give three examples to illustrate the foregoing theoretical treatment. The first example involves a limit circle case ($d = 2$), the second, a limit point case ($d = 1$), and the third, a nonlinear problem. In all examples, we can use Lemmas 1 and 2 to verify the solvability of the equation in question.

In the first two examples we consider the Euler equation

$$-y'' + \frac{c}{x^2} y = f(x), \quad 0 < x \leq 1. \quad (34)$$

If $v^2 = c + \frac{1}{4}$ then

$$\theta(x) = x^{v+1/2}, \quad \varphi(x) = \begin{cases} x^{1/2} \ln x, & v = 0 \\ x^{-v+1/2}, & v \neq 0 \end{cases}, \quad W(\theta, \varphi) = \begin{cases} 1, & v = 0 \\ -2v, & v \neq 0 \end{cases}.$$

The point $x = 1$ is regular and the point $x = 0$ is singular. The singular point $x = 0$ is:

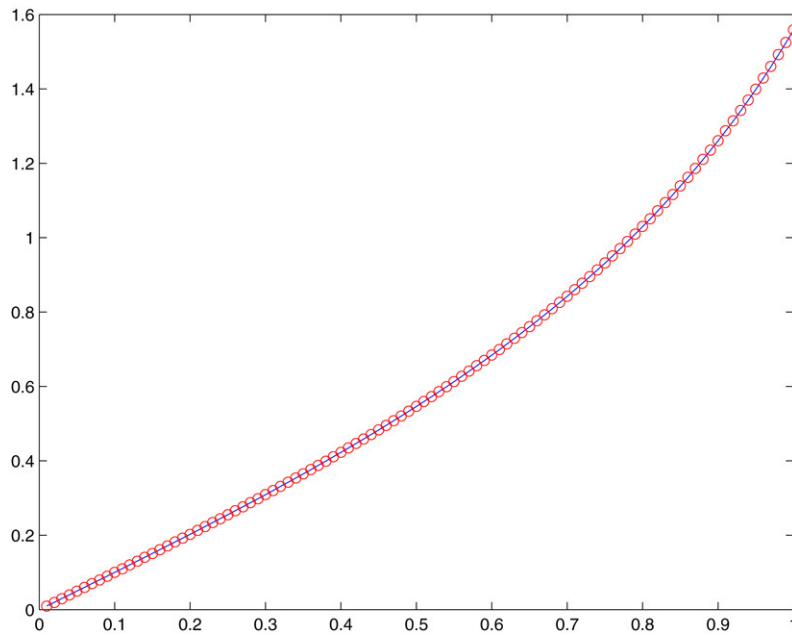


Fig. 1. Exact (solid) and approximate (circles) solutions for Example 1.

- LC ($d = 2$) if and only if $c < \frac{3}{4}$ ($\nu^2 < 1$),
- LP ($d = 1$) if and only if $c \geq \frac{3}{4}$ ($\nu^2 \geq 1$).

Example 1 (LC, $d = 2$). Consider the Euler equation (34) with $c = 5/16$, $f(x) = \frac{5}{16x^2} - 2 \sec^2 x \tan x$

The self-adjoint boundary conditions taken correspond to $\gamma_a = 0$ and $\gamma_b = \arctan(5 - \frac{48}{8 + \sin 2})$. The exact solution is $y(x) = \tan x$. To avoid the singular point at 0 we took $a' = .01$. The exact and approximate solutions are plotted in Fig. 1 with an error tolerance of 10^{-5} .

Example 2 (LP, $d = 1$). Consider the Euler equation (34) with $c = 15/4$ ($\nu = 2$), $f(x) = x^{-3/2}(3 + \ln x)$.

Here we impose the self-adjoint boundary conditions $[\cos \gamma \sin \gamma] Y(1) = 0$, $\gamma = \arctan 5/3$. The exact solution of this equation is $y(x) = \frac{1}{4} \sqrt{x}(3 - x^2 + \ln x)$. The functions θ and φ are $\theta(x) = x^{5/2}$ and $\varphi(x) = x^{-3/2}$. Therefore, $\psi(x) = \theta(x)$ ($m = 0$). Fig. 2 depicts the exact solution (solid curve) and the approximate solution (circles) with an error tolerance of 10^{-5} .

Example 3 (Nonlinear). Consider the nonlinear problem

$$-\frac{1}{x}(xy')' + e^y = 0, \quad y'(0) = 0, \quad y(1) = 0.$$

In this example, $\theta(x) = 1$, $\varphi(x) = \ln x$, therefore, 0 is LC for ℓ . The assumed boundary conditions are self-adjoint with $\gamma_a = 0$, $\gamma_b = \frac{\pi}{2}$. The linear operator $-(xy')'$ has positive eigenvalues while the function $f(x, y) = -e^y$ has negative range. This is a limiting case for Theorem 12 where $\lambda^- = -\infty$, $\alpha = -\infty$, $\lambda^+ = \lambda_1 > 0$ (the first eigenvalue of the operator $-(xy')'$) and $\beta \in (0, \lambda_1)$ is arbitrary. To see that we still obtain convergence of the iterations (31), we observe that we have (29) with a suitably chosen $\alpha < 0$ which is dependent on y, z . Now, if we choose $\lambda^- < \alpha$, we see that Eq. (32) is always satisfied with the choice of $\lambda = 0$ irrespective of the values of λ^- and α .

The exact solution of this problem is $y(x) = 2 \ln \left(\frac{1+\delta}{1+\delta x^2} \right)$, where $\delta = -5 + 2\sqrt{6}$. The iterative method starts with an initial guess $y(x) = 0$. After 10 iterations we reach the approximate solution depicted in Fig. 3, which is in agreement with the exact solution to a tolerance of 10^{-5} with 100 mesh points for integration taken in the interval $[0, 1]$.

Acknowledgement

This research project has been funded by King Fahd University of Petroleum and Minerals under Project number MS/Singular ODE/274.

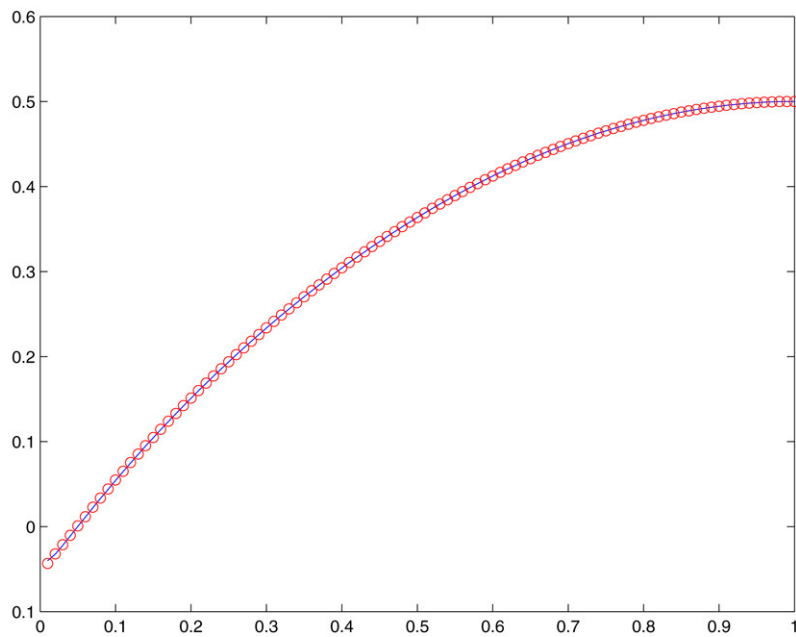


Fig. 2. Exact (solid) and approximate (circles) solutions for Example 2.

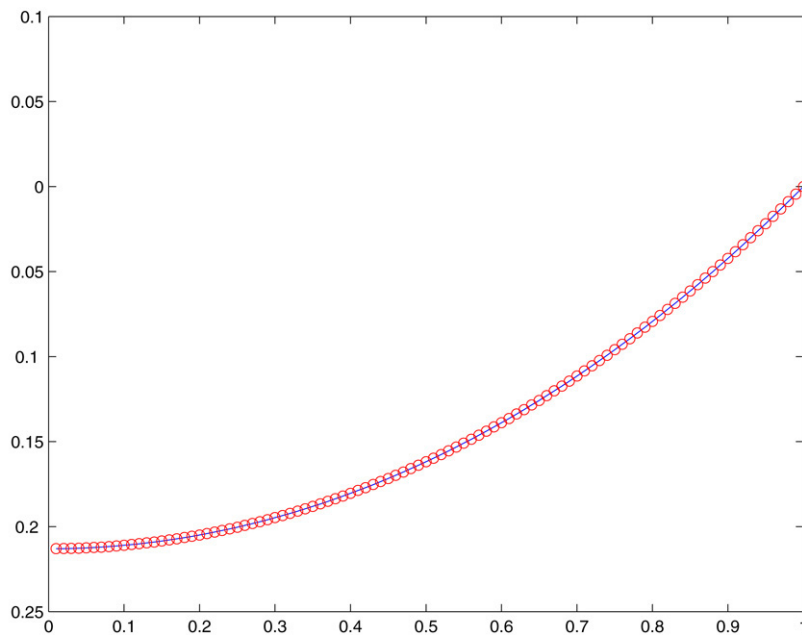


Fig. 3. Exact (solid) and approximate (circles) solution for Example 3.

References

- [1] P.B. Bailey, W.N. Everitt, J. Weidmann, A. Zettl, Regular approximation of singular Sturm–Liouville problems, *Results in Mathematics* 23 (1993) 1–20.
- [2] P.A. Binding, P.J. Browne, B.A. Watson, Recovery of the m -function from spectral data for generalized Sturm–Liouville problems, *Journal of Computational and Applied Mathematics* 171 (2004) 73–91.
- [3] M. Brown, W. Evans, The computation of the Titchmarsh–Weyl m -function, in: D. Hinton, P. Schaefer (Eds.), *Spectral Theory and Computational Methods of Sturm–Liouville Problems*, Marcel Dekker, New York, Basel, Hong Kong, 1997, pp. 197–210.
- [4] B.M. Brown, M.S.P. Eastham, W.D. Evans, V.G. Kirby, Repeated diagonalisation and the numerical computation of the Titchmarsh–Weyl m -function, *Proceedings of the Royal Society of London Series A* 445 (1994) 113–126.
- [5] B.M. Brown, W.D. Evans, V.G. Kirby, M. Plum, Safe numerical bounds for the Titchmarsh–Weyl $m(\lambda)$ -function, *Mathematical Proceedings of the Cambridge Philosophical Society* 113 (1993) 583–599.
- [6] J. Chaudhuri, W.N. Everitt, On the spectrum of ordinary second-order differential operators, *Proceedings of the Royal Society of Edinburgh A* 68 (1968) 95–119.

- [7] E. Coddington, L. Levinson, *Theory of Ordinary Differential Equations*, McGraw-Hill, New York, 1955.
- [8] W. Gautschi, The work of Philip Rabinowitz on numerical integration, *Numerical Algorithms* 9 (2) (1995).
- [9] M.L. Gorbachuk, V.I. Gorbachuk, M. G. Krein's lectures on entire operators, in: *Operator Theory, Advances and Applications*, vol. 97, Birkhauser Verlag, Basel, Boston, Berlin, 1997.
- [10] A.M. Krall, A. Zettl, Singular self-adjoint Sturm–Liouville problems, *Integral Equations* 1 (4) (1988) 423–432.
- [11] K.M. Furati, M.A. El-Gebeily, Real self-adjoint Sturm–Liouville problems, *Applicable Analysis* 83 (4) (2004) 377–387.
- [12] J. Mawhin, Semilinear equations of gradient type in Hilbert spaces and applications to differential equations, in: *Nonlinear Differential Equations Invariance, Stability and Bifurcation*, Academic Press, New York, 1981, pp. 269–282.
- [13] M.A. Naimark, *Linear Differential Operators*, vol. Part II, Ungar, New York, 1968.
- [14] P. Rabinowitz, On avoiding the singularity in the numerical integration of proper integrals, *BIT Num. Math.* 19 (1) (1979) 104–110.
- [15] E.C. Titchmarsh, *Eigenfunction Expansions*, Part I, 2nd ed., Oxford University Press, 1962.
- [16] J. Weidmann, Spectral theory of ordinary differential operators, in: *Lecture Notes in Mathematics*, vol. 1258, Springer-Verlag, Heidelberg, 1987.
- [17] A. Zettl, Sturm–Liouville problems, in: D. Hinton, P. Schaefer (Eds.), *Spectral Theory and Computational Methods of Sturm–Liouville Problems*, in: *Pure and Applied Mathematics*, Marcel Dekker, New York, 1997, pp. 1–104.