Journal of Computational and Applied Mathematics 226 (2009) 114–124

Contents lists available at ScienceDirect

Journal of Computational and Applied Mathematics

journal homepage: www.elsevier.com/locate/cam

Preconditioned AOR iterative methods for M-matrices[☆]

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ARTICLE INFO

Article history:
Received 19 June 2007
Received in revised form 14 February 2008

MSC:
65F10

Keywords:
Linear system
Preconditioner
Convergence
AOR iterative method
M-matrix

ABSTRACT

Linear systems with M-matrices often appear in a wide variety of areas. In this paper, we give general preconditioners for solving the systems with nonsingular M-matrix. We show that our preconditioners increase the convergence rate of AOR iterative methods. Numerical results for corresponding preconditioned *GMRES* methods are also given.

1. Introduction

Consider the linear system

$$Ax = b, \tag{1.1}$$

where $A = (a_{i,j}) \in \mathcal{R}^{n\times n}$ is nonsingular and $b \in \mathcal{R}^n$.

For simplicity, in this paper we assume that A has unit diagonal entries and consider the usual splitting

$$A = I - L - U,$$

where $-L$ and $-U$ are strictly lower and strictly upper triangular parts of A, respectively.

The standard AOR (accelerated overrelaxation) iterative method given in [11] is defined as

$$x^{(i+1)} = \mathcal{L}_{\gamma,\omega}x^{(i)} + (I - \gamma L)^{-1}\omega b, \quad i = 0, 1, 2, \ldots$$

with the iteration matrix

$$\mathcal{L}_{\gamma,\omega} = (I - \gamma L)^{-1}[(1 - \omega)I + (\omega - \gamma)L + \omega U],$$

where ω and γ are real parameters with $\omega \neq 0$.

It is well known that for certain values of the parameters ω and γ, we can obtain successive overrelaxation (SOR), Gauss–Seidel, JOR and Jacobi methods. To improve the convergence rate of the basic iterative methods, several

☆ This work was supported by the National Natural Science Foundation of China under grant 10371056, the Foundation for the Authors of the National Excellent Doctoral Thesis Award of China under grant 200720, the Natural Science Foundation of Jiangsu Province of China under grant BK2006725 and the College Natural Science Foundation of Jiangsu Province under grant 05KJB110062.

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doi:10.1016/j.cam.2008.05.022

preconditioned iterative methods have been proposed in [8,10,12–16,20–22,24]. The main idea of these preconditioned iterative methods is to transform the original system into the preconditioned form

$$PAx = Pb, \tag{1.2}$$

where $P \in R^{n\times n}$ is nonsingular and nonnegative, and has unit diagonal entries. We call the basic iterative methods corresponding to the preconditioned system (1.2) the preconditioned iterative methods, such as the preconditioned Jacobi method, the preconditioned Gauss–Seidel method, etc.

For convenience, some notations, definitions and results that will be used in the following parts are given below. A matrix A is called nonnegative, semi-positive and positive if each entry of A is nonnegative, nonnegative but at least a positive entry and positive, respectively. We denote them by $A \geq 0$, $A > 0$ and $A \gg 0$. Similarly, for n-dimensional vectors, by identifying them with $n \times 1$ matrices, we can also define $x \geq 0$, $x > 0$ and $x \gg 0$. Additionally, we denote the spectral radius of A by $\rho(A)$.

Definition 1.1. A matrix $A = (a_{i,j}) \in \mathcal{R}^{n\times n}$ is called a Z-matrix if for any $i \neq j$, $a_{i,j} \leq 0$; an L-matrix if it is a Z-matrix with $a_{i,i} > 0$, $i = 1, \ldots, n$; a nonsingular M-matrix if $A = sI - B$, $B \geq 0$ and $s > \rho(B)$.

An equivalent definition for M-matrix is given in [23, Definition 2-7.3], where an M-matrix is defined as a Z-matrix with nonnegative inverse.

Definition 1.2. Let $A = (a_{i,j}) \in \mathcal{R}^{n\times n}$. We call $\langle A\rangle = (\tilde{a}_{i,j})$ its comparison matrix if $\tilde{a}_{ii} =\mid a_{ii} \mid$ and $\tilde{a}_{i,j} = - \mid a_{i,j} \mid$ for $i \neq j$. If $\langle A\rangle$ is a nonsingular M-matrix, then A is called an H-matrix.

Definition 1.3. Let $A \in \mathcal{R}^{n\times n}$. The splitting $A = M - N$ is called:

(a) weak regular if $M^{-1} \geq 0$ and $M^{-1}N \geq 0$;
(b) regular if $M^{-1} \geq 0$ and $N \geq 0$;
(c) M-splitting if M is a nonsingular M-matrix and $N \geq 0$;
(d) H-compatible splitting if $\langle A\rangle = \langle M\rangle - \mid N \mid$.

Lemma 1.4 (*[7, Theorem 2-1.11], [10, Theorem 2.2], [18]*). *Let $A \geq 0$.*

(a) *If $\alpha x \leq Ax$ for some $x > 0$, then $\alpha \leq \rho(A)$.*
(b) *If $Ax \leq \beta x$ for some $x \gg 0$, then $\rho(A) \leq \beta$. Moreover, if A is irreducible and if $Ax \leq \beta x$ for some $x > 0$, then $\rho(A) \leq \beta$ and $x \gg 0$.*

Lemma 1.5 (*[19, Theorem 2.20]*). *Let $A \geq 0$. Then:*

(a) *A has a nonnegative real eigenvalue equal to its spectral radius $\rho(A)$;*
(b) *to $\rho(A)$, there corresponds an eigenvector $x > 0$, which is called the Perron vector;*
(c) *$\rho(A)$ does not decrease when any entry of A is increased.*

Lemma 1.6 (*[7, Corollary 2-3.15]*). *If $A \geq 0$ is irreducible, then $\rho(A) > 0$ is a simple eigenvalue and A has an eigenvector $x \gg 0$ corresponding to $\rho(A)$.*

Lemma 1.7 (*[19, Theorem 3.16]*). *Let $A \geq 0$. Then $\alpha > \rho(A)$ if and only if $\alpha I - A$ is nonsingular and $(\alpha I - A)^{-1} \geq 0$.*

Lemma 1.8 (*[17, 2.4.10]*). *Let A_1 be an M-matrix with diagonal part D_1 and off-diagonal part $-B_1 = A_1 - D_1$. If $D_2 \geq 0$ is any diagonal matrix and $B_2 \geq 0$ any matrix with zero diagonal satisfying $B_2 \leq B_1$, then $A = D_1 + D_2 - (B_1 - B_2)$ is an M-matrix and $A^{-1} \leq A_1^{-1}$.*

Lemma 1.9 (*[7, Theorem 6-2.3]*). *Let A be a Z-matrix. Then the following statements are equivalent:*

(a) *A is a nonsingular M-matrix.*
(b) *There is a vector $x \gg 0$ such that $Ax \gg 0$.*
(c) *Any weak regular splitting is convergent.*

Lemma 1.10 (*[9, Theorem 3.4]*). *Let A be an H-matrix. If $A = M - N$ is an H-compatible splitting, then $\rho(M^{-1}N) < 1$, i.e., the splitting is convergent.*

This paper is organized as follows. In Section 2 we will propose new preconditioners and discuss the convergence and comparison of the preconditioned AOR iterative methods. In Section 3 numerical results are presented to show the efficiency of the preconditioned *GMRES* methods.

2. Preconditioned AOR iterative methods

Applying P on (1.1) we obtain the equivalent linear system

$$\tilde{A}x = \tilde{b} \tag{2.1}$$

with $\tilde{A} = PA$ and $\tilde{b} = Pb$. We split $\tilde{A}$ as

$$\tilde{A} = \tilde{D} - \tilde{L} - \tilde{U} \tag{2.2}$$

with $\tilde{D}, \tilde{L}$ and $\tilde{U}$ being diagonal, strictly lower and strictly upper triangular matrices, respectively.

The elements $\tilde{a}_{i,j}$ of $\tilde{A}$ are given by the expressions:

$$\tilde{a}_{i,j} = \begin{cases} 1 + \sum_{k=1,k\neq i}^{n} p_{i,k}a_{k,i}, & 1 \le i = j \le n, \\ p_{i,j} + \sum_{k=1,k\neq j}^{n} p_{i,k}a_{k,j}, & 1 \le i \neq j \le n. \end{cases} \tag{2.3}$$

When A is a Z-matrix, in order to preserve the nonpositivity of all the off-diagonal elements and the Z-matrix character of $\tilde{A}$, in the following we assume that

$$p_{i,j} + \sum_{k=1,k\neq j}^{n} p_{i,k}a_{k,j} \le 0, \quad 1 \le i \neq j \le n.$$

Then, we define

$$\begin{aligned} &\tilde{D} = \text{diag}\left(1 + \sum_{k=1,k\neq i}^{n} p_{i,k}a_{k,i}\right), \\ &P = I + P_1 + P_2, \\ &P_1U = E_1 + F_1 + G_1, \\ &P_2L = E_2 + F_2 + G_2, \end{aligned}$$

where E_1 and E_2 are diagonal matrices, F_1, F_2 and P_1 are strictly lower triangular matrices, while G_1, G_2 and P_2 are strictly upper triangular matrices, so that P_i, E_i, F_i and G_i, $i = 1, 2$, are all nonnegative. Then the three matrices on the right-hand side of (2.2) are given by

$$\begin{aligned} &\tilde{D} = I - E_1 - E_2, \\ &\tilde{L} = L - P_1 + P_1L + F_1 + F_2 \ge 0, \\ &\tilde{U} = U - P_2 + P_2U + G_1 + G_2 \ge 0. \end{aligned}$$

The preconditioned AOR iterative method of (1.1), i.e., the AOR iterative method of (2.1), is defined as

$$x^{(i+1)} = \tilde{\mathscr{L}}_{\gamma,\omega}x^{(i)} + (\tilde{D} - \gamma\tilde{L})^{-1}\omega b, \quad i = 0, 1, 2, \ldots,$$

where

$$\tilde{\mathscr{L}}_{\gamma,\omega} = (\tilde{D} - \gamma\tilde{L})^{-1}[(1-\omega)\tilde{D} + (\omega-\gamma)\tilde{L} + \omega\tilde{U}]$$

is the preconditioned AOR iteration matrix.

In this section, we will show that, when A is an M-matrix, the preconditioned AOR iterative method is asymptotically faster convergent than the original AOR iterative method for system (1.1). Some results about H-matrices are also given.

We first review and give the following lemmas which are used in the paper.

Lemma 2.1. *Let $A = (a_{i,j}) \in R^{n\times n}$ be a nonsingular M-matrix. If $P = (p_{i,j}) \ge 0$ is a nonsingular matrix such that $p_{i,i} = 1$, $i = 1, 2, \ldots, n$, and*

$$p_{i,j} + \sum_{k=1,k\neq j}^{n} p_{i,k}a_{k,j} \le 0, \quad 1 \le i \neq j \le n,$$

then PA is also a nonsingular M-matrix.

Proof. Since A is a nonsingular M-matrix, by Lemma 1.9, there exists $x \gg 0$ such that $Ax \gg 0$. Then $PAx \gg 0$. From (2.3), it is easy to see that PA is a Z-matrix. The result is directly obtained from Lemma 1.9. $\square$

Lemma 2.2. *Let $A = (a_{i,j}) \in R^{n\times n}$ be a nonsingular M-matrix. Then there exists $\epsilon_0 > 0$ such that, for any $0 < \epsilon \le \epsilon_0$, $A(\epsilon) = (a_{i,j}(\epsilon))$ is also a nonsingular M-matrix, where*

$$a_{i,j}(\epsilon) = \begin{cases} a_{i,j}, & \text{if } a_{i,j} \neq 0, \\ -\epsilon, & \text{if } a_{i,j} = 0. \end{cases} \tag{2.4}$$

Proof. Since A is a nonsingular M-matrix, $A(\epsilon)$ is a Z-matrix. Further, by Lemma 1.9, there exists a vector $x \gg 0$ such that $Ax \gg 0$, i.e., it holds

$$\sum_{k=1}^{n} a_{i,k}x_k > 0, \quad i = 1, \dots, n.$$

Denote

$$\delta = \sum_{k=1}^{n} x_k.$$

Then $\delta > 0$. We can choose ϵ_0 such that

$$\epsilon_0 = \frac{1}{\delta} \min\left\{ \sum_{k=1}^{n} a_{i,k}x_k, i = 1, \dots, n \right\}.$$

Then $\epsilon_0 > 0$ and

$$\sum_{k=1}^{n} a_{i,k}x_k - \delta\epsilon_0 \ge 0, \quad i = 1, \dots, n.$$

Now, for any $0 < \epsilon \le \epsilon_0$, we have

$$\sum_{k=1}^{n} a_{i,k}(\epsilon)x_k > \sum_{k=1}^{n} a_{i,k}x_k - \delta\epsilon \ge \sum_{k=1}^{n} a_{i,k}x_k - \delta\epsilon_0 \ge 0, \quad i = 1, \dots, n,$$

which implies

$$A(\epsilon)x \gg 0.$$

It follows from Lemma 1.9 that $A(\epsilon)$ is a nonsingular M-matrix. $\square$

Lemma 2.3. *Let $A = (a_{i,j}) \in R^{n\times n}$ be an H-matrix. If $P = (p_{i,j}) \ge 0$ is a nonsingular matrix with $p_{i,i} = 1$, $i = 1, 2, \dots, n$, and*

$$p_{i,j} + \sum_{k=1,k\neq j}^{n} p_{i,k}\frac{a_{k,j} - |a_{k,j}|}{2} \le 0, \quad 1 \le i \neq j \le n, \tag{2.5}$$

then PA is also an H-matrix.

Proof. From (2.5) we obtain

$$p_{i,j} - \sum_{k=1,k\neq j}^{n} p_{i,k}|a_{k,j}| \le 0, \quad 1 \le i \neq j \le n.$$

By Lemma 2.1, it is easy to see that $P\langle A\rangle$ is a nonsingular M-matrix.

On the other hand, from (2.5), we have

$$1 + \sum_{k=1,k\neq i}^{n} p_{i,k}a_{k,i} \ge 1 - \sum_{k=1,k\neq i}^{n} p_{i,k}|a_{k,i}| \ge 0,$$

and for $1 \le i \neq j \le n$,

$$p_{i,j} - \sum_{k=1,k\neq j}^{n} p_{i,k}|a_{k,j}| \le p_{i,j} + \sum_{k=1,k\neq j}^{n} p_{i,k}a_{k,j} \le \sum_{k=1,k\neq j}^{n} p_{i,k}|a_{k,j}| - p_{i,j}.$$

By Lemma 1.8 it follows that PA is an H-matrix. $\square$

Theorem 2.4. *Assume $A = (a_{i,j}) \in \mathcal{R}^{n\times n}$ is an H-matrix. If $P = (p_{i,j}) \geq 0$ is a nonsingular matrix with $p_{i,i} = 1$ for $i = 1, 2, \dots, n$, $p_{i,j} = 0$ whenever $a_{i,j} \geq 0$, and $0 \leq p_{i,j} \leq |a_{i,j}|$ whenever $a_{i,j} < 0$, for $1 \leq i \neq j \leq n$, then PA is also an H-matrix.*

Proof. For $1 \leq i \neq j \leq n$, if $a_{i,j} \geq 0$, then we have

$$p_{i,j} + \sum_{k=1,k\neq j}^{n} p_{i,k}\frac{a_{k,j} - |a_{k,j}|}{2} = \sum_{k=1,k\neq i,j}^{n} p_{i,k}\frac{a_{k,j} - |a_{k,j}|}{2} \leq 0.$$

While if $a_{i,j} < 0$, then

$$p_{i,j} + \sum_{k=1,k\neq j}^{n} p_{i,k}\frac{a_{k,j} - |a_{k,j}|}{2} = p_{i,j} + a_{i,j} + \sum_{k=1,k\neq i,j}^{n} p_{i,k}\frac{a_{k,j} - |a_{k,j}|}{2} \leq 0.$$

Now, by Lemma 2.3, the result can be obtained immediately. □

Theorem 2.5. *Let $A = (a_{i,j}) \in \mathcal{R}^{n\times n}$ be an H-matrix and $0 \leq \gamma \leq \omega \leq 1$, $\omega \neq 0$. If $P = (p_{i,j}) \geq 0$ is a nonsingular matrix with $p_{i,i} = 1$ for $i = 1, 2, \dots, n$, $p_{i,j} = 0$ whenever $a_{i,j} > 0$ and $0 \leq p_{i,j} \leq |a_{i,j}|$ whenever $a_{i,j} \leq 0$, for $1 \leq i \neq j \leq n$, then the splitting*

$$PA = \tilde{M} - \tilde{N} = \frac{1}{\omega}(\tilde{D} - \gamma\tilde{L}) - \frac{1}{\omega}[(1-\omega)\tilde{D} + (\omega-\gamma)\tilde{L} + \omega\tilde{U}]$$

is an H-compatible splitting and $\rho(\tilde{\mathscr{L}}_{\gamma,\omega}) < 1$.

Proof. By Theorem 2.4, PA is an H-matrix. Let $\langle PA\rangle = (\tilde{a}_{i,j})$ and $\langle\tilde{M}\rangle - |\tilde{N}| = (b_{i,j})$.

For $i = 1, 2, \dots, n$, we get

$$\tilde{a}_{i,i} = \left|1 + \sum_{k=1,k\neq i}^{n} p_{i,k}a_{k,i}\right|$$

and

$$\begin{aligned} b_{i,i} &= \left|\frac{1}{\omega}\left(1 + \sum_{k=1,k\neq i}^{n} p_{i,k}a_{k,i}\right)\right| - \left|\frac{1}{\omega}(1-\omega)\left(1 + \sum_{k=1,k\neq i}^{n} p_{i,k}a_{k,i}\right)\right| \\ &= \left|1 + \sum_{k=1,k\neq i}^{n} p_{i,k}a_{k,i}\right| = \tilde{a}_{i,i}. \end{aligned}$$

For $n \geq i > j \geq 1$, we have

$$\tilde{a}_{i,j} = -\left|p_{i,j} + \sum_{k=1,k\neq j}^{n} p_{i,k}a_{k,j}\right|$$

and

$$b_{i,j} = -\frac{1}{\omega}\gamma\left|p_{i,j} + \sum_{k=1,k\neq j}^{n} p_{i,k}a_{k,j}\right| - \frac{1}{\omega}(\omega-\gamma)\left|p_{i,j} + \sum_{k=1,k\neq j}^{n} p_{i,k}a_{k,j}\right| = \tilde{a}_{i,j}.$$

For $1 \leq i < j \leq n$, we have

$$\tilde{a}_{i,j} = -\left|p_{i,j} + \sum_{k=1,k\neq j}^{n} p_{i,k}a_{k,j}\right|$$

and

$$b_{i,j} = -\frac{1}{\omega}\omega\left|p_{i,j} + \sum_{k=1,k\neq j}^{n} p_{i,k}a_{k,j}\right| = \tilde{a}_{i,j}.$$

Now, we can conclude that

$$\langle PA\rangle = \langle\tilde{M}\rangle - |\tilde{N}|,$$

which implies

$$PA = \tilde{M} - \tilde{N}$$

is an H-compatible splitting. Then, by Lemma 1.10, we obtain $\rho(\tilde{\mathscr{L}}_{\gamma,\omega}) < 1$. $\square$

Now we give comparison results about the asymptotic convergence rates of the original and the preconditioned matrix splittings; see [1,6,2] about this kind of results for parallel matrix multisplitting and parallel decomposition-type relaxation methods for large sparse systems of linear equations.

Theorem 2.6. *Let $A = (a_{i,j}) \in \mathscr{R}^{n\times n}$ be a nonsingular Z-matrix. Assume that $0 \le \gamma \le \omega \le 1$, $\omega \ne 0$ and $P = (p_{i,j}) \ge 0$ is a nonsingular preconditioner with $p_{i,i} = 1$ for $1 \le i \le n$, and*

$$p_{i,j} + \sum_{k=1,k\ne j}^{n} p_{i,k}a_{k,j} \le 0, \quad 1 \le i \ne j \le n. \tag{2.6}$$

(a) *If $\rho(\mathscr{L}_{\gamma,\omega}) < 1$, then*

$$\rho(\tilde{\mathscr{L}}_{\gamma,\omega}) \le \rho(\mathscr{L}_{\gamma,\omega}) < 1.$$

(b) *If $\rho(\mathscr{L}_{\gamma,\omega}) > 1$ and P satisfies*

$$1 + \sum_{k=1,k\ne i}^{n} p_{i,k}a_{k,i} > 0, \quad 1 \le i \le n, \tag{2.7}$$

then

$$\rho(\tilde{\mathscr{L}}_{\gamma,\omega}) \ge \rho(\mathscr{L}_{\gamma,\omega}) > 1.$$

Proof. Consider the splittings

$$A = M - N = \frac{1}{\omega}(I - \gamma L) - \frac{1}{\omega}[(1-\omega)I + (\omega-\gamma)L + \omega U]$$

and

$$\tilde{A} = E - F = \frac{1}{\omega}(\tilde{D} - \gamma\tilde{L}) - \frac{1}{\omega}[(1-\omega)\tilde{D} + (\omega-\gamma)\tilde{L} + \omega\tilde{U}].$$

Since A is a nonsingular Z-matrix, $0 \le \gamma \le \omega \le 1$ and $\omega \ne 0$, then L, U and N are nonnegative matrices. Thus the splitting $A = M - N$ is an M-splitting.

For case (a), if $\rho(\mathscr{L}_{\gamma,\omega}) < 1$, by Lemma 1.9, A is a nonsingular M-matrix. By Lemma 2.1, $\tilde{A}$ is also a nonsingular M-matrix. So $\tilde{D} \ge 0$ is invertible, $\tilde{L} \ge 0$, $\tilde{U} \ge 0$ and $\tilde{D}^{-1} \ge 0$.

On the other hand, since $\gamma \ge 0$ and $\tilde{L} \ge 0$, it is obvious that the diagonal elements of E are positive and its off-diagonal elements are nonpositive, i.e., E is an L-matrix. Since $\gamma\tilde{D}^{-1}\tilde{L} \ge 0$ is a strictly lower triangular matrix and $\rho(\gamma\tilde{D}^{-1}\tilde{L}) = 0 < 1$, by Lemma 1.7, we have $(I - \gamma\tilde{D}^{-1}\tilde{L})^{-1} \ge 0$. Then

$$E^{-1} = (I - \gamma\tilde{D}^{-1}\tilde{L})^{-1}\tilde{D}^{-1} \ge 0,$$

which means that E is an nonsingular M-matrix. Furthermore, $F \ge 0$. This shows that $\tilde{A} = E - F$ is also an M-splitting.

For case (b), from (2.6), (2.7) and the above analysis, we can conclude that $\tilde{A} = E - F$ is an M-splitting.

Now, according to Lemma 1.5, there exists a Perron vector $x > 0$ such that

$$\mathscr{L}_{\gamma,\omega}x = \rho(\mathscr{L}_{\gamma,\omega})x. \tag{2.8}$$

We denote $\rho(\mathscr{L}_{\gamma,\omega})$ by λ. From (2.8) and the expression of $\mathscr{L}_{\gamma,\omega}$, we obtain the following equality

$$[(1-\omega)I + (\omega-\gamma)L + \omega U]x = \lambda(I - \gamma L)x,$$

which is equivalent to

$$[(1-\omega-\lambda)I + (\omega-\gamma+\lambda\gamma)L + \omega U]x = 0.$$

From these we get

$$\omega(L + U - I)x = (\lambda - 1)(I - \gamma L)x$$

and

$$Lx = \frac{(-1+\omega+\lambda)I - \omega U}{\omega - \gamma + \lambda\gamma} x, \quad \textit{if} \quad \omega - \gamma + \lambda\gamma \neq 0.$$

It is easy to see that for $\rho(\mathscr{L}_{\gamma,\omega}) > 0$ we have $\omega - \gamma + \lambda\gamma \neq 0$. Then, we have

$$\begin{aligned}
\tilde{\mathscr{L}}_{\gamma,\omega}x - \lambda x &= (\tilde{D} - \gamma\tilde{L})^{-1}[(1-\omega)\tilde{D} + (\omega-\gamma)\tilde{L} + \omega\tilde{U} - \lambda(\tilde{D} - \gamma\tilde{L})]x \\
&= (\tilde{D} - \gamma\tilde{L})^{-1}[(1-\omega-\lambda)\tilde{D} + (\omega-\gamma+\lambda\gamma)\tilde{L} + \omega\tilde{U}]x \\
&= (\tilde{D} - \gamma\tilde{L})^{-1}[(1-\omega-\lambda)(I - E_1 - E_2) \\
&\quad + (\omega-\gamma+\lambda\gamma)(L - P_1 + P_1L + F_1 + F_2) + \omega(U - P_2 + P_2U + G_1 + G_2)]x \\
&= (\tilde{D} - \gamma\tilde{L})^{-1}\{[(1-\omega-\lambda)I + (\omega-\gamma+\lambda\gamma)L + \omega U] + [-(1-\omega-\lambda)(E_1+E_2) \\
&\quad + (\omega-\gamma+\lambda\gamma)(-P_1 + P_1L + F_1 + F_2) + \omega(-P_2 + P_2U + G_1 + G_2)]\}x \\
&= (\tilde{D} - \gamma\tilde{L})^{-1}[-(1-\omega-\lambda)(E_1+E_2) + (\omega-\gamma+\lambda\gamma)(-P_1 + P_1L + F_1 + F_2) \\
&\quad + \omega(-P_2 + P_2U + G_1 + G_2)]x \\
&= (\tilde{D} - \gamma\tilde{L})^{-1}[(\lambda-1)(E_1+E_2) + \gamma(\lambda-1)(-P_1 + P_1L + F_1 + F_2) + \omega(P_1+P_2)(L+U-I)]x \\
&= (\tilde{D} - \gamma\tilde{L})^{-1}[(\lambda-1)(E_1+E_2) + \gamma(\lambda-1)(-P_1 + P_1L + F_1 + F_2) + (\lambda-1)(P_1+P_2)(I-\gamma L)]x \\
&= (\lambda-1)(\tilde{D} - \gamma\tilde{L})^{-1}[E_1 + E_2 + \gamma(F_1+F_2) + (1-\gamma)P_1 + P_2 - \gamma P_2 L]x \\
&= (\lambda-1)(\tilde{D} - \gamma\tilde{L})^{-1}\left[E_1 + E_2 + \gamma(F_1+F_2) + (1-\gamma)P_1 + P_2 - \gamma P_2 \frac{(-1+\omega+\lambda)I - \omega U}{\omega-\gamma+\lambda\gamma}\right]x \\
&= (\lambda-1)(\tilde{D} - \gamma\tilde{L})^{-1}\left[E_1 + E_2 + \gamma(F_1+F_2) + (1-\gamma)P_1 + \omega P_2 \frac{(1-\gamma)I + \gamma U}{\omega-\gamma+\lambda\gamma}\right]x.
\end{aligned} \tag{2.9}$$

Case I: $\lambda > 1$. Then (2.9) gives

$$\tilde{\mathscr{L}}_{\gamma,\omega}x \geq \lambda x.$$

By Lemma 1.4, we have

$$\rho(\tilde{\mathscr{L}}_{\gamma,\omega}) \geq \rho(\mathscr{L}_{\gamma,\omega}) > 1$$

and (b) is proved.

Case II: $0 \leq \lambda < 1$.

We first consider the case when A is irreducible.

Note that

$$\begin{aligned}
\mathscr{L}_{\gamma,\omega} &= (I - \gamma L)^{-1}[(1-\omega)I + (\omega-\gamma)L + \omega U] \\
&= [I + \gamma L + (\gamma L)^2 + \cdots + (\gamma L)^{n-1}][(1-\omega)I + (\omega-\gamma)L + \omega U] \\
&\geq (1-\omega)I + (\omega-\gamma)L + \omega U + \gamma(1-\omega)L \\
&= (1-\omega)I + \omega(1-\gamma)L + \omega U.
\end{aligned} \tag{2.10}$$

When $0 \leq \gamma < 1$, from (2.10), we can see that $\mathscr{L}_{\gamma,\omega}$ is also irreducible. So, by Lemma 1.6, $\lambda > 0$ and we can assume that the Perron vector x given in (2.8) is positive. Now, from (2.9), we have

$$\tilde{\mathscr{L}}_{\gamma,\omega}x \leq \lambda x.$$

By Lemma 1.4, we obtain

$$\rho(\tilde{\mathscr{L}}_{\gamma,\omega}) \leq \lambda = \rho(\mathscr{L}_{\gamma,\omega}).$$

When $\gamma = 1$, then $\omega = \gamma = 1$ and we have

$$\rho(\tilde{\mathscr{L}}_{1,1}) = \lim_{\gamma \to 1^-} \rho(\tilde{\mathscr{L}}_{\gamma,1}) \leq \lim_{\gamma \to 1^-} \rho(\mathscr{L}_{\gamma,1}) = \rho(\mathscr{L}_{1,1}) < 1.$$

Now, we consider the case when A is reducible. According to Lemma 2.2, for any sufficiently small positive number ϵ, $A(\epsilon)$ given in (2.6) is also a nonsingular M-matrix and irreducible. By the proof above, we have

$$\rho(\tilde{\mathscr{L}}_{\gamma,\omega}) = \lim_{\epsilon \to 0^+} \rho(\tilde{\mathscr{L}}_{\gamma,\omega}(\epsilon)) \leq \lim_{\epsilon \to 0^+} \rho(\mathscr{L}_{\gamma,\omega}(\epsilon)) = \rho(\mathscr{L}_{\gamma,\omega}) < 1.$$

The proof of (a) is completed. $\square$

Theorem 2.7. *Let $A = (a_{i,j}) \in \mathcal{R}^{n\times n}$ be a nonsingular M-matrix. Assume that $0 \le \gamma \le \omega \le 1$, $\omega \ne 0$, $P = (p_{i,j}) \ge 0$ is a nonsingular preconditioner with $p_{i,j} = -\alpha_{i,j}a_{i,j}$, $0 \le \alpha_{i,j} \le 1$, for $1 \le i \ne j \le n$, and $p_{i,i} = 1$ for $1 \le i \le n$. Then we have*

$$\rho(\tilde{\mathscr{L}}_{\gamma,\omega}) \le \rho(\mathscr{L}_{\gamma,\omega}) < 1.$$

Proof. By Lemma 1.9, we get $\rho(\mathscr{L}_{\gamma,\omega}) < 1$.

On the other hand, for $1 \le i \ne j \le n$, we have

$$\begin{aligned} p_{i,j} + \sum_{k=1,k\ne j}^{n} p_{i,k}a_{k,j} &= p_{i,j} + a_{i,j} + \sum_{k=1,k\ne i,j}^{n} p_{i,k}a_{k,j} \\ &= (1-\alpha_{i,j})a_{i,j} - \sum_{k=1,k\ne i,j}^{n} \alpha_{i,k}a_{i,k}a_{k,j} \\ &\le 0, \end{aligned}$$

i.e., (2.6) holds.

By Theorem 2.6, we can prove the result. □

Theorem 2.8. *Let $A = (a_{i,j}) \in \mathcal{R}^{n\times n}$ be an H-matrix. Assume that $0 \le \gamma \le \omega \le 1$, $\omega \ne 0$, and $P = (p_{i,j}) \ge 0$ is a preconditioner with*

$$p_{i,j} = \alpha_{i,j}\left|\frac{a_{i,j} - |a_{i,j}|}{2}\right|, \quad 0 \le \alpha_{i,j} \le 1,$$

for $1 \le i \ne j \le n$ and $p_{i,i} = 1$ for $1 \le i \le n$. Then, we have

$$\rho(\tilde{\mathscr{L}}_{\gamma,\omega}) \le \rho(\mathscr{L}_{\gamma,\omega}(P\langle A\rangle)) \le \rho(\mathscr{L}_{\gamma,\omega}(\langle A\rangle)) < 1, \tag{2.11}$$

where $\mathscr{L}_{\gamma,\omega}(B)$ represents the AOR iteration matrix corresponding to the matrix B.

Proof. Since $\langle A\rangle$ is an M-matrix, by Lemma 1.9, it holds

$$\rho(\mathscr{L}_{\gamma,\omega}(\langle A\rangle)) < 1. \tag{2.12}$$

For $1 \le i \ne j \le n$, we have

$$\begin{aligned} p_{i,j} - \sum_{k=1,k\ne j}^{n} p_{i,k}|a_{k,j}| &= \alpha_{i,j}\left|\frac{a_{i,j} - |a_{i,j}|}{2}\right| - |a_{i,j}| - \sum_{k=1,k\ne i,j}^{n} \alpha_{i,k}\left|\frac{a_{i,k} - |a_{i,k}|}{2}\right||a_{k,j}| \\ &\le 0. \end{aligned}$$

By Lemma 2.1, $P\langle A\rangle$ is a nonsingular M-matrix. Furthermore, by Theorem 2.6, we derive

$$\rho(\mathscr{L}_{\gamma,\omega}(P\langle A\rangle)) \le \rho(\mathscr{L}_{\gamma,\omega}(\langle A\rangle)) < 1.$$

Note that

$$\begin{aligned} p_{i,j} + \sum_{k=1,k\ne j}^{n} p_{i,k}\frac{a_{k,j} - |a_{k,j}|}{2} &= \alpha_{i,j}\left|\frac{a_{i,j} - |a_{i,j}|}{2}\right| + \frac{a_{i,j} - |a_{i,j}|}{2} + \sum_{k=1,k\ne i,j}^{n} p_{i,k}\frac{a_{k,j} - |a_{k,j}|}{2} \\ &= \frac{|a_{i,j}| - a_{i,j}}{2}(\alpha_{i,j} - 1) + \sum_{k=1,k\ne i,j}^{n} p_{i,k}\frac{a_{k,j} - |a_{k,j}|}{2} \\ &\le 0. \end{aligned}$$

By Lemma 2.3, PA is an H-matrix.

Then, it can be derived that

$$1 + \sum_{k=1,k\ne i}^{n} p_{i,k}a_{k,i} \ge 1 - \sum_{k=1,k\ne i}^{n} p_{i,k}|a_{k,i}| > 0$$

and

$$p_{i,j} - \sum_{k=1,k\ne j}^{n} p_{i,k}|a_{k,j}| \le p_{i,j} + \sum_{k=1,k\ne j}^{n} p_{i,k}a_{k,j} \le -p_{i,j} + \sum_{k=1,k\ne j}^{n} p_{i,k}|a_{k,j}|.$$

Table 1
CPU time and the number of iterations

N	$GMRES(100)$		$PKGMRES(100)$		$PRGMRES(100)$	
	IT	CPU	IT	CPU	IT	CPU
32	85	0.453	79	0.359	43	0.219
40	108	1.219	99	1.062	53	0.547
48	139	3.094	125	2.703	63	1.5
56	181	7.031	175	6.547	73	2.922

For convenience, we denote the split by

$$P\langle A\rangle = \hat{D} - \hat{L} - \hat{U}$$

with $\hat{D}$, $\hat{L}$ and $\hat{U}$ being diagonal, strictly lower and strictly upper triangular matrices, respectively, so that $\hat{D} \geq 0$, $\hat{L} \geq 0$ and $\hat{U} \geq 0$.

So, we have

$$\begin{aligned}
|\tilde{\mathscr{L}}_{\gamma,\omega}| &= |(\tilde{D} - \gamma\tilde{L})^{-1}[(1-\omega)\tilde{D} + (\omega-\gamma)\tilde{L} + \omega\tilde{U}]| \\
&= |(I - \gamma\tilde{D}^{-1}\tilde{L})^{-1}[(1-\omega)I + (\omega-\gamma)\tilde{D}^{-1}\tilde{L} + \omega\tilde{D}^{-1}\tilde{U}]| \\
&\leq |(I - \gamma\tilde{D}^{-1}\tilde{L})^{-1}||(1-\omega)I + (\omega-\gamma)\tilde{D}^{-1}\tilde{L} + \omega\tilde{D}^{-1}\tilde{U}| \\
&\leq (I - \gamma|\tilde{D}^{-1}\tilde{L}|)^{-1}[(1-\omega)I + (\omega-\gamma)|\tilde{D}^{-1}\tilde{L}| + \omega|\tilde{D}^{-1}\tilde{U}|] \\
&\leq (I - \gamma\hat{D}^{-1}\hat{L})^{-1}[(1-\omega)I + (\omega-\gamma)\hat{D}^{-1}\hat{L} + \omega\hat{D}^{-1}\hat{U}] \\
&= \mathscr{L}_{\gamma,\omega}(P\langle A\rangle),
\end{aligned}$$

which implies

$$\rho(\tilde{\mathscr{L}}_{\gamma,\omega}) \leq \rho(\mathscr{L}_{\gamma,\omega}(P\langle A\rangle))$$

and the proof is completed. □

Remark 2.9. When we choose $\alpha_{i,j}$ suitably, we can obtain many known preconditioners. For instance, if $\alpha_{i,j} = 0, j \neq 1$, then P is reduced to the preconditioner proposed first by Milaszewicz and generalized in [12]. If $\alpha_{i,j} = 0, j \neq i+1$, and $j \neq i$, then P is reduced to the preconditioner presented in [10] and parameterized in [13]. If $\alpha_{i,j} = 0$, for $i > j$, then P is the preconditioner proposed in [14], etc.

Remark 2.10. Throughout the paper, by choosing special parameters, similar results about SOR, JOR, Gauss–Seidel and Jacobi methods can be copied word by word from the above theorems. Here we omit them.

3. A numerical example

Consider the two dimensional convection–diffusion equation

$$-(u_{xx} + u_{yy}) + 2exp(x+y)(xu_x + yu_y) = f(x, y)$$

on the square domain $\Omega = [0, 1] \times [0, 1]$, with the homogeneous Dirichlet boundary conditions.

When the central difference scheme on a uniform grid with $N \times N$ interior nodes ($N^2 = n$) is applied to this equation, we can obtain a system of linear equations (1.1) with the five diagonal coefficient matrix; see [3–5].

We test for the preconditioned *GMRES* methods.

In the experiment, we choose two classes of preconditioners. They are Kotakemori et al.'s preconditioner and the preconditioner $P = I + L + U$. The initial approximation of $x^{(0)}$ is taken as a zero vector, and the right-hand-side vector is chosen so that $e = [1, 1, \ldots, 1]^T$ is the solution of the considered system. Here $\|Ax^{(k)} - Ae\|_2/\|Ae\|_2 \leq 10^{-6}$ is used as the stopping criterion.

All experiments were executed on a PC using MATLAB programming package.

In Table 1, we report the CPU time (T) and the number of iterations (IT) for the corresponding preconditioned *GMRES* methods.

Here $GMRES(100)$ represents the restarted $GMRES(100)$ method, the preconditioned restarted $GMRES(100)$ method with preconditioner $P = I + L + U$ is denoted by $PRGMRES(100)$, while $PKGMRES(100)$ corresponds to Kotakemori et al.'s preconditioner.

From the table, we can see that the preconditioned *GMRES* methods are superior to the basic *GMRES* methods. The table also shows that the preconditioned *GMRES* method associated with $P = I + L + U$ is the best.

The observation can be further illustrated by the spectrum pictures plotted in Figs. 1–3. Clearly, the spectrum of the preconditioned matrices $(I + L)A$ and $(I + L + U)A$ are more clustered than those of the original matrix A. Figs. 1–3 also show that the spectral distribution associated with $P = I + L + U$ is more tightly bounded than those associated with Kotakemori et al.'s preconditioner.

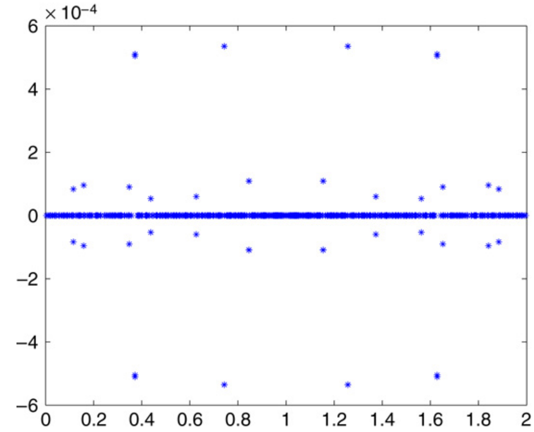

Fig. 1. Spectra of the matrix A.

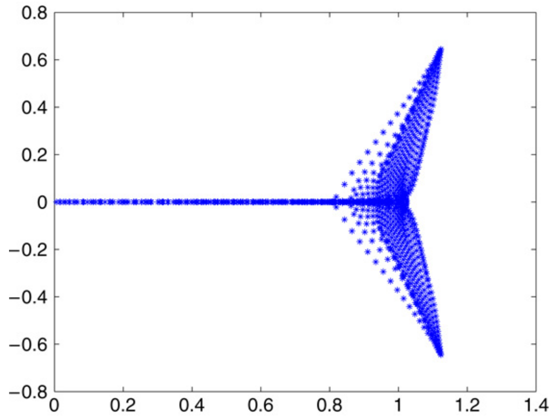

Fig. 2. Spectra of the preconditioned matrix $(I + L)A$.

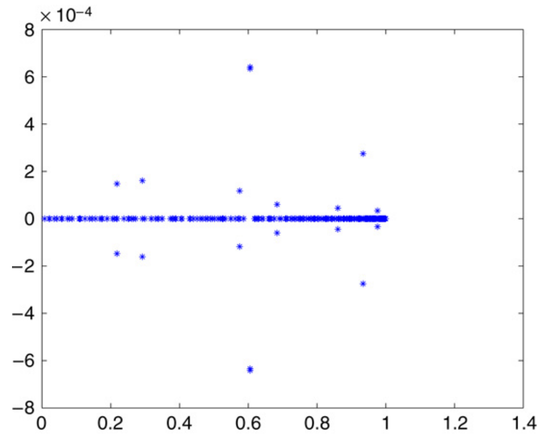

Fig. 3. Spectra of the preconditioned matrix $(I + L + U)A$.

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