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Definitions, properties and applications of finite-part integrals

G. Monegato

Dipartimento di Matematica, Politecnico di Torino, Corso Duca degli Abruzzi 24, 10129 Torino, Italy

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ABSTRACT

By using standard calculus, we discuss some definitions and properties of finite-part integrals, point out the essence of this concept and show how these integrals naturally arise in some integral equation applications. A couple of new examples are also described.

1. Introduction

Although the concept of finite-part or hypersingular integral goes back to Hadamard (see [9,23]), in the last twenty years the number of papers devoted to hypersingular boundary integral equations, published mainly in engineering journals, has suddenly increased.

The boundary integral equation (BIE)/boundary element method has emerged as a powerful alternative tool to other numerical methods for many problems in engineering (see for example [1,28,21,14]). In particular, hypersingular BIE's have relevant applications, for example, in aerodynamics, elasticity, diffraction, acoustics, fluid mechanics. These equations are obtained by taking derivatives (gradients) of conventional integral equations.

Many papers and a few books (see for example [15]) have been published on definitions, properties and applications of hypersingular integrals. Nevertheless, many researchers seem to be still unaware of these, or have not fully understood these definitions and properties. For example, one current practical definition is the following one: *define the hypersingular integral*

$$\int\!\!\!\!\!\!=_a^b \frac{f(x)}{(x-y)^2}\,\mathrm{d}x, \quad -\infty < a < y < b < \infty,$$

by deleting first a symmetric neighborhood $(y-\epsilon, y+\epsilon)$ *and computing the expansion of the corresponding integral, for* $\epsilon \to 0$*; then ignore the diverging terms and take the limit of the remaining ones for* $\epsilon \to 0$. This limit value defines the finite-part (FP) integral. At first glance this definition looks like a practical trick, apparently with little mathematical foundation.

On the other hand in [26,15] definitions and properties of hypersingular integrals/equations are derived within the framework of pseudo-differential operators.

In this paper we present definitions and properties by using only standard calculus. In particular in the next section we describe the concept and some main properties of Cauchy principal value integrals, since these are of importance in

E-mail address: giovanni.monegato@polito.it.

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understanding finite-part integrals. Indeed, they are fundamental ingredients for the definitions and properties of finite-part integrals, which we present in Section 3. In the last section we describe four significant problems, whose solutions are obtained by reducing them to corresponding hypersingular integral equations. In particular we describe, with all needed details, the process that leads to these equations, and in particular to the previously defined finite-part integrals, that otherwise would appear as a mathematical expedient.

2. Cauchy principal value integrals

2.1. The one-dimensional case

The concept of one-dimensional (1D) Cauchy principal value (CPV) integral is well understood and widely used in applications. In the case of a bounded interval, the standard definition is:

$$\fint_a^b \frac{f(x)}{x-y}\mathrm{d}x = \lim_{\epsilon\to 0}\left[\int_a^{y-\epsilon} + \int_{y+\epsilon}^b\right]\frac{f(x)}{x-y}\mathrm{d}x, \quad a<y<b, \tag{1}$$

whenever this limit exists. A sufficient condition for its existence is that the function $f(x) \in L^1(a,b)$ is Hölder continuous at $x=y$. This condition has been slightly weakened in [20], where Martin and Rizzo have shown that if the even part of $f(x)$ is continuous at $x=y$, while the odd part is Hölder continuous at $x=y$, then the limit in (1) exists.

An alternative, but equivalent definition, is the following one:

$$\begin{aligned}\fint_a^b \frac{f(x)}{x-y}\mathrm{d}x &= \int_a^b \frac{f(x)-f(y)}{x-y}\mathrm{d}x + f(y)\lim_{\epsilon\to 0}\left[\int_a^{y-\epsilon} + \int_{y+\epsilon}^b\right]\frac{\mathrm{d}x}{x-y}\\ &= \int_a^b \frac{f(x)-f(y)}{x-y}\mathrm{d}x + f(y)\log\frac{b-y}{y-a}.\end{aligned} \tag{2}$$

Remark 1. It can be easily shown that

$$\fint_a^b \frac{f(x)}{x-y}\mathrm{d}x = \frac{\mathrm{d}}{\mathrm{d}y}\int_a^b \log|x-y| f(x)\mathrm{d}x.$$

This is indeed the analytic operation that in the applications often gives rise to one-dimensional Cauchy principal value integrals.

Remark 2. If in definition (1) we delete a non-symmetric neighborhood (ϵ_1, ϵ_2) of y, with $\epsilon_i = \epsilon_i(\epsilon)$ and $\epsilon_i \to 0$ as $\epsilon \to 0$, then we have (see (2))

$$\lim_{\epsilon\to 0}\left[\int_a^{y-\epsilon_1} + \int_{y+\epsilon_2}^b\right]\frac{f(x)}{x-y}\mathrm{d}x = \int_a^b \frac{f(x)-f(y)}{x-y}\mathrm{d}x \tag{3}$$

$$+ f(y)\log\frac{b-y}{y-a} + f(y)\lim_{\epsilon\to 0}\log\frac{\epsilon_1}{\epsilon_2}. \tag{4}$$

Obviously, the limit exists only if

$$\lim_{\epsilon\to 0}\frac{\epsilon_1}{\epsilon_2} = c \neq 0, \infty.$$

In particular if $c=1$ we have the same value given by the symmetric neighborhood; *otherwise the new definition would differ from the standard one by a* $f(y)\log c$ *additive term*. Therefore, in (1) the deleted neighborhood need not be symmetric. What is required is that it is *ultimately* symmetric, that is:

$$\lim_{\epsilon\to 0}\frac{\epsilon_1}{\epsilon_2} = 1.$$

Remark 3. An important consequence of the previous remark is that if in (1) we introduce a change of variable $x=\phi(t)$, $y=\phi(s)$, with $\phi'(t)$ Hölder continuous, then we have

$$\fint_a^b \frac{f(x)}{x-y}\mathrm{d}x = \fint_{t_a}^{t_b} \frac{f(\phi(t))}{\phi(t)-\phi(s)}\phi'(t)\mathrm{d}t,$$

where in the second integral, instead of deleting the pre-image of $(y-\epsilon, y+\epsilon)$ we delete a symmetric neighborhood of s.

Remark 4. Consider the following CPV integral

$$\fint_{-c_1}^{c_2} \frac{f(x)}{x}\mathrm{d}x = \int_{-c_1}^{c_2} \frac{f(x)-f(0)}{x}\mathrm{d}x + f(0)\log\frac{c_2}{c_1},$$

where $c_i = c_i(h) > 0$ and $c_i \to 0$ as $h \to 0$. Then

$$\lim_{h\to 0} \fint_{-c_1}^{c_2} \frac{f(x)}{x}\mathrm{d}x = f(0) \lim_{h\to 0} \log \frac{c_2}{c_1} = \begin{cases} 0 & \text{if } \lim_{h\to 0} \frac{c_2}{c_1} = 1 \\ f(0)\log c & \text{if } \lim_{h\to 0} \frac{c_2}{c_1} = c \neq 1 \\ \pm\infty & \text{if } \lim_{h\to 0} \frac{c_2}{c_1} = 0, \infty. \end{cases} \tag{5}$$

In boundary integral equations, for example in potential theory, CPV integrals, defined as in (1), arise from a limiting process of a regular integral which depends upon an outer variable. For example:

$$I = \lim_{t\to 0} \int_a^b \frac{f(x)(x-y)}{(x-y)^2 + t^2}\mathrm{d}x, \quad a < y < b. \tag{6}$$

In this case we have

$$\begin{aligned} I &= \int_a^b \frac{f(x) - f(y)}{x - y} + f(y) \lim_{t\to 0} \int_a^b \frac{x-y}{(x-y)^2 + t^2}\mathrm{d}x \\ &= \int_a^b \frac{f(x) - f(y)}{x - y} + f(y) \log \frac{b-y}{y-a} \equiv \fint_a^b \frac{f(x)}{x-y}\mathrm{d}x. \end{aligned}$$

This means that in (6) we can interchange the limit and the integration operations, as long as the final integral is defined as the Cauchy principal value (1).

We come to the same conclusion also by following a different route, which is often used to derive boundary integral equations (see also Section 4). Write

$$I = \left(\int_a^{y-\epsilon_1} + \int_{y+\epsilon_2}^b \right) \frac{f(x)}{x-y}\mathrm{d}x + \lim_{t\to 0} \int_{y-\epsilon_1}^{y+\epsilon_2} \frac{f(x)(x-y)}{(x-y)^2 + t^2}\mathrm{d}x.$$

Here, ϵ_1, ϵ_2 are independent one from the other. Proceeding further we obtain

$$\begin{aligned} I &= \left(\int_a^{y-\epsilon_1} + \int_{y+\epsilon_2}^b \right) \frac{f(x)}{x-y}\mathrm{d}x + \int_{y-\epsilon_1}^{y+\epsilon_2} \frac{f(x) - f(y)}{x-y}\mathrm{d}x + \frac{f(y)}{2} \lim_{t\to 0} \int_{y-\epsilon_1}^{y+\epsilon_2} \frac{2(x-y)}{(x-y)^2 + t^2}\mathrm{d}x \\ &= \left(\int_a^{y-\epsilon_1} + \int_{y+\epsilon_2}^b \right) \frac{f(x)}{x-y}\mathrm{d}x + \int_{y-\epsilon_1}^{y+\epsilon_2} \frac{f(x) - f(y)}{x-y}\mathrm{d}x + f(y) \log \frac{\epsilon_2}{\epsilon_1} \\ &= \int_a^b \frac{f(x) - f(y)}{x-y}\mathrm{d}x + f(y) \log \frac{b-y}{y-a} \equiv \fint_a^b \frac{f(x)}{x-y}\mathrm{d}x, \end{aligned}$$

where the CPV integral is defined as in (1), i.e., by deleting a symmetric neighborhood. This is in spite of the fact that above ϵ_1, ϵ_2 are arbitrary.

In BIE applications one may have to deal with CPV integrals defined on closed or open plane curves ℓ as follows:

$$\fint_\ell \frac{f(x,y)}{r}\mathrm{d}\ell_x = \lim_{\epsilon\to 0} \int_{\ell-\ell_\epsilon} \frac{f(x,y)}{r}\mathrm{d}\ell_x, \quad y \in \ell, \tag{7}$$

where $r = \|x - y\|$ and ℓ_ϵ is the symmetric neighborhood of y obtained by intersecting the line ℓ with a circle of radius ϵ centered at y. The function $f(x, y)$ is such that the above limit exists: for example, Hölder continuous. A classical case is

$$\frac{f(x,y)}{r} = u(x) \frac{\partial}{\partial t_y} \log r, \tag{8}$$

where $u(x)$ is a Hölder continuous function and $\frac{\partial}{\partial t_y}$ is the tangential derivative at y.

Assuming that the curve ℓ has a smooth, let us say for simplicity C^2, parametrization

$$x = \phi(t), \qquad y = \phi(s)$$

integral (7) can be rewritten in the form

$$\fint_\ell \frac{f(x,y)}{r}\mathrm{d}\ell_x = \fint_a^b \frac{g(t,s)}{t-s}\mathrm{d}t, \tag{9}$$

where, because of Remark 3, in both integrals we delete a symmetric neighborhood.

If the parametrization is not C^2 at y, for example y is a corner point between two C^2 curves joined at y, after having introduced the parametrization of each curve in the integral on the right of (7), one should delete a neighborhood (ϵ_1, ϵ_2), in general non-symmetric, coinciding with the pre-image of ℓ_ϵ. If one deletes a symmetric neighborhood also in the transformed integral, to maintain the equality one has to add an extra term which is due to the non-zero coefficient $\log \frac{\epsilon_1}{\epsilon_2}$ (see Remark 2), which depends upon the tangents of the two curves at y.

To verify this statement, assume that the curve has a C^2 parametric representation $x = \phi(t),\ a \le t \le t_0$ to the left of the point $y = \phi(t_0)$, and a (different) C^2 representation $x = \psi(t)$ for $t_0 \le t \le b$, having

$$\phi(t_0) = \psi(t_0), \qquad |\phi'(t_0)| \ne |\psi'(t_0)|.$$

Introducing this representation in (7) we obtain

$$\int_{\ell-\ell_\epsilon} \frac{f(x,y)}{r} \mathrm{d}\ell_x = \int_a^{t_0-\epsilon_1} \frac{g_-(t,t_0)}{t_0-t} \mathrm{d}t + \int_{t_0+\epsilon_2}^{b} \frac{g_+(t,t_0)}{t-t_0} \mathrm{d}t, \tag{10}$$

where we have set

$$g_-(t,t_0) = \frac{f(\phi(t),\phi(t_0))}{\left\{\sum_{i=1}^{2}\left[\frac{\phi_i(t)-\phi_i(t_0)}{t-t_0}\right]^2\right\}^{\frac{1}{2}}} |\phi'(t)|,$$

$$g_+(t,t_0) = \frac{f(\psi(t),\psi(t_0))}{\left\{\sum_{i=1}^{2}\left[\frac{\psi_i(t)-\psi_i(t_0)}{t-t_0}\right]^2\right\}^{\frac{1}{2}}} |\psi'(t)|.$$

Now if we let $\epsilon \to 0$ (recall that $\epsilon_i = \epsilon_i(\epsilon)$), we have the standard Cauchy principal value. If however on the right-hand side of (10) we replace ϵ_1, ϵ_2 by ϵ, we have to add the following extra term

$$\int_{t_0-\epsilon}^{t_0-\epsilon_1} \frac{g_-(t,t_0)}{t_0-t} \mathrm{d}t + \int_{t_0+\epsilon_2}^{t_0+\epsilon} \frac{g_+(t,t_0)}{t-t_0} \mathrm{d}t. \tag{11}$$

When we let $\epsilon \to 0$ the only non-negligible terms arising from (11) are

$$g_-(t_0^-,t_0) \int_{t_0-\epsilon}^{t_0-\epsilon_1} \frac{1}{t_0-t} \mathrm{d}t + g_+(t_0^+,t_0) \int_{t_0+\epsilon_2}^{t_0+\epsilon} \frac{1}{t-t_0} \mathrm{d}t$$

i.e.,

$$g_-(t_0^-,t_0) \log \frac{\epsilon_1}{\epsilon} + g_+(t_0^+,t_0) \log \frac{\epsilon}{\epsilon_2}. \tag{12}$$

Notice that

$$\lim_{t\to t_0} g_\pm(t,t_0) = f(y,y).$$

Therefore the extra term (12) reduces to

$$f(y,y) \log \frac{\epsilon_1}{\epsilon_2}.$$

When $\epsilon \to 0$ this term takes the value

$$f(y,y) \log \frac{|\psi'(t_0)|}{|\phi'(t_0)|}.$$

2.2. *The two-dimensional case*

Two-dimensional (2D) CPV integrals have been defined and examined in [27] and later in [8,22].

Let $D \subset R^2$ be a bounded (open and convex) region, and $y \in D$ a given fixed point. The boundary of D is defined by the function $A(\theta),\ 0 \le \theta < 2\pi$, taking as (polar coordinates) origin the given point $y \in D$. To assume D to be convex is not a restriction, since the definition we are going to present concerns a (fixed) neighborhood of y that of course we can always take of convex form. Consider a function $F(y;x),\ x \in D$, having the form

$$F(y;x) = \frac{f(y;\theta)}{r^2} + F_1(y;x), \tag{13}$$

where $r = \|x-y\|,\ x = y + re, e = (\cos\theta, \sin\theta)^{\mathrm{T}}, f(y;\theta)$ is assumed integrable with respect to θ and $F_1 \in L^1(D)$. Delete a neighborhood $\sigma \subset D$ of the point y, defined by a contour $\alpha(\epsilon;\theta)$, such that $\alpha(\epsilon;\theta) \to 0$ as $\epsilon \to 0$ uniformly with respect to θ.

Then, consider the integral

$$I_\sigma = \int_{D-\sigma} F(y;x)\mathrm{d}x = \int_{D-\sigma} F_1(y;x)\mathrm{d}x + \int_0^{2\pi} f(y;\theta) \int_{\alpha(\epsilon;\theta)}^{A(\theta)} \frac{\mathrm{d}r}{r} \mathrm{d}\theta. \tag{14}$$

We have

$$\lim_{\epsilon\to 0} I_\sigma = \int_D F_1(y;x)\mathrm{d}x + \int_0^{2\pi} f(y;\theta)\log A(\theta)\mathrm{d}\theta - \lim_{\epsilon\to 0}\int_0^{2\pi} f(y;\theta)\log\alpha(\epsilon;\theta)\mathrm{d}\theta. \quad (15)$$

If

$$\lim_{\epsilon\to 0}\frac{\alpha(\epsilon;\theta)}{\epsilon} = \alpha_0(\theta), \quad (16)$$

then

$$\lim_{\epsilon\to 0}\int_0^{2\pi} f(y;\theta)\log\frac{\alpha(\epsilon;\theta)}{\epsilon}\mathrm{d}\theta = \int_0^{2\pi} f(y;\theta)\log\alpha_0(\theta)\mathrm{d}\theta. \quad (17)$$

A necessary and sufficient condition for the existence of the limit in (15) is

$$\int_0^{2\pi} f(y;\theta)\mathrm{d}\theta = 0. \quad (18)$$

In this case we have

$$\lim_{\epsilon\to 0} I_\sigma = \int_D F_1(y;x)\mathrm{d}x + \int_0^{2\pi} f(y;\theta)\log\frac{A(\theta)}{\alpha_0(\theta)}\mathrm{d}\theta. \quad (19)$$

Definition 1. Under the above assumptions, taking in (19) $\alpha_0(\theta) = 1$, we define

$$⨍_D F(y;x)\mathrm{d}x = \lim_{\epsilon\to 0} I_\sigma. \quad (20)$$

Remark 5. In the standard definition of two-dimensional CPV integrals one takes $\alpha(\epsilon;\theta) = \epsilon$; however, the value of (20) will not change if one chooses $\alpha(\epsilon;\theta) \neq \epsilon$, as long as relation (16) holds with $\alpha_0(\theta) \equiv 1$.

Remark 6. Let

$$D \equiv C_\delta = \{(r,\theta),\ 0 < r \le \lambda_\delta(\theta) \le \delta,\ 0 \le \theta < 2\pi\}$$

and assume that the CPV integral

$$⨍_{C_\delta} F(y;x)\mathrm{d}x \quad (21)$$

defined above exists, and therefore (18) holds. Notice that the latter implies

$$\int_0^{2\pi}\int_\epsilon^{\lambda_\delta(\theta)} \frac{f(y;\theta)}{r}\mathrm{d}r\mathrm{d}\theta = \int_0^{2\pi} f(y;\theta)\log\lambda_\delta(\theta)\mathrm{d}\theta, \quad (22)$$

where $f(y;\theta)$ is defined in (13). Thus

$$\lim_{\delta\to 0} ⨍_{C_\delta} F(y;x)\mathrm{d}x = \int_0^{2\pi} f(y;\theta)\log\lambda_\delta(\theta)\mathrm{d}\theta.$$

If $\lambda_\delta(\theta) = \delta$ we have

$$\lim_{\delta\to 0} ⨍_{C_\delta} F(y;x)\mathrm{d}x = 0,$$

while in the case $\lambda_\delta(\theta) = \delta g(\theta)$

$$\lim_{\delta\to 0} ⨍_{C_\delta} F(y;x)\mathrm{d}x = \int_0^{2\pi} f(y;\theta)\log g(\theta)\mathrm{d}\theta.$$

Definition 2. Two-dimensional CPV integrals can also be defined on closed or open surfaces S in R^3. In this case, the neighborhood of the point $y \in S$ we delete is the intersection S_ϵ of the surface S with the unit sphere of radius ϵ, centered at the singular point y, i.e.

$$⨍_S F(y;x)\mathrm{d}S_x = \lim_{\epsilon\to 0}\int_{S-S_\epsilon} F(y;x)\mathrm{d}S_x.$$

Remark 7. If we have, for example, a C^2 parametrization $x = \phi(t),\ y = \phi(s)$ of the surface S, with $t, s \in D \subset R^2$, then we may write

$$\int_S F(y; x)\mathrm{d}S_x = \int_D F(\phi(s); \phi(t))|\phi'(t)|\mathrm{d}t,$$

(where, according to (20), in the definition of the integral on the right we delete a circular neighborhood of radius ϵ). If however, the surface S is not C^2 at y, but it is only piecewise C^2 and globally continuous, that is y is a "corner" point, then the integral on the right coincides with the one on the left only if the neighborhood σ we delete in the former one, defined by (19), is the pre-image of that we consider in the latter. If we delete a "circular" neighborhood in both integrals, then, as in the case of a non-smooth curve, an extra term has to be added to the integral on the right.

3. Finite-part integrals

3.1. Integrals over bounded intervals — p integer

Let us first consider the following elementary CPV integral

$$\int_a^b \frac{\mathrm{d}x}{x-y} = \log\frac{b-y}{y-a}, \quad -\infty < a < y < b < \infty. \tag{23}$$

The right-hand side is certainly a C^∞ function at any point $y \in (a, b)$; therefore we can compute its derivatives. In particular we have

$$\frac{\mathrm{d}}{\mathrm{d}y}\int_a^b \frac{\mathrm{d}x}{x-y} = -\left[\frac{1}{b-y} + \frac{1}{y-a}\right]. \tag{24}$$

Now exchange the derivative and the integral operators and formally apply the standard integration rule, ignoring the behavior of the integrand at the singular point $x = y$. We identify this particular integration technique by the symbol $\oint_a^b$:

$$\oint_a^b \frac{\mathrm{d}x}{(x-y)^2} = -\left[\frac{1}{b-y} + \frac{1}{y-a}\right]. \tag{25}$$

The same definition can be obtained by a different route. Delete a neighborhood $(y-\epsilon_1, y+\epsilon_2)$, not necessarily symmetric, of the singular point y and consider the integral

$$\left(\int_a^{y-\epsilon_1} + \int_{y+\epsilon_2}^b\right)\frac{\mathrm{d}x}{(x-y)^2} = -\left[\frac{1}{b-y} + \frac{1}{y-a}\right] + \frac{1}{\epsilon_1} + \frac{1}{\epsilon_2}. \tag{26}$$

If we ignore the terms diverging as $\epsilon_i \to 0$, we obtain the previous integral (23). This integral is called "finite-part" value.

By iterating these simple ideas we obtain the following definition of finite-part integral in the case of any integer $p \geq 2$:

$$\oint_a^b \frac{\mathrm{d}x}{(x-y)^{p+1}} = \frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}y}\oint_a^b \frac{\mathrm{d}x}{(x-y)^p}$$

or, compute

$$\left(\int_a^{y-\epsilon_1} + \int_{y+\epsilon_2}\right)\frac{\mathrm{d}x}{(x-y)^{p+1}} = -\frac{1}{p}\left[\frac{1}{(b-y)^p} + \frac{(-1)^{p+1}}{(y-a)^p}\right] + \frac{1}{p}\left[\frac{(-1)^{p+1}}{\epsilon_1^p} + \frac{1}{\epsilon_2^p}\right]$$

and ignore the diverging terms (as $\epsilon_i \to 0$).

For a more general integrand function, of the type

$$\frac{f(x)}{(x-y)^2}$$

with $f \in C^{1,\mu}(a, b)$ (see however [20]), we define

$$\oint_a^b \frac{f(x)}{(x-y)^2}\mathrm{d}x = \frac{\mathrm{d}}{\mathrm{d}y}\int_a^b \frac{f(x)}{x-y}\mathrm{d}x \tag{27}$$

$$= \frac{\mathrm{d}}{\mathrm{d}y}\left[\int_a^b \frac{f(x)-f(y)}{x-y}\mathrm{d}x + f(y)\log\frac{b-y}{y-a}\right]$$

$$= \int_a^b \frac{f(x)-f(y)}{(x-y)^2}\mathrm{d}x + f(y)\oint_a^b \frac{\mathrm{d}x}{(x-y)^2}, \tag{28}$$

where, we recall that, the CPV integral is defined as in (1), i.e., by deleting a symmetric neighborhood of y. From expression (28), and recalling Remark 4, it is a trivial task to find out the behavior of

$$\oint_{y-c_1}^{y+c_2} \frac{f(x)}{(x-y)^2}\mathrm{d}x,$$

where $c_i = c_i(h) > 0$, for $h \to 0$.

Alternatively, we could consider the integral

$$\left(\int_a^{y-\epsilon} + \int_{y+\epsilon}^b\right) \frac{f(x)}{(x-y)^2}\mathrm{d}x = \left(\int_a^{y-\epsilon} + \int_{y+\epsilon}^b\right) \frac{f(x)-f(y)}{(x-y)^2}\mathrm{d}x + f(y)\left(\int_a^{y-\epsilon} + \int_{y+\epsilon}^b\right) \frac{\mathrm{d}x}{(x-y)^2}$$
$$= \left(\int_a^{y-\epsilon} + \int_{y+\epsilon}^b\right) \frac{f(x)-f(y)}{(x-y)^2}\mathrm{d}x - f(y)\left[\frac{1}{b-y} + \frac{1}{y-a} - \frac{2}{\epsilon}\right], \tag{29}$$

ignore the diverging term and let $\epsilon \to 0$ in the remaining ones. Once again we obtain (28).

Remark 8. If in (29) we delete a non-symmetric neighborhood $(y-\epsilon_1, y+\epsilon_2)$, $\epsilon_i = \epsilon_i(\epsilon)$, then the above procedure would lead to a corresponding CPV integral, whose value would differ from that given by the previous definition by an additive term of the type $f'(y)\log c$, where $c = \lim_{\epsilon\to 0}\frac{\epsilon_1}{\epsilon_2}$. Therefore, the associated finite-part integral value would differ by the same quantity. This happens whenever we have a non-constant function $f(x)$, and it is due to the CPV integral component.

The above definitions easily extend to finite-part integrals of the form

$$\oint_a^b \frac{f(x)}{(x-y)^{p+1}}\mathrm{d}x, \quad a < y < b, \tag{30}$$

where $p \geq 1$ is an integer and $f(x)$ is assumed to have a Hölder continuous p-derivative (for a milder sufficient condition see [20]). These definitions justify the following practical rule, that otherwise would appear like a trick: *delete a symmetric neighborhood* $(y-\epsilon, y+\epsilon)$ *and compute the expansion of the corresponding integral, for* $\epsilon \to 0$*; then ignore the diverging terms and take the limit of the remaining ones for* $\epsilon \to 0$. This limit value is the finite-part integral (30).

3.2. Integrals over bounded intervals — p real

Next we consider the improper integral

$$\int_y^b \frac{\mathrm{d}x}{(x-y)^\alpha} = \frac{(b-y)^{1-\alpha}}{1-\alpha}, \quad 0 < \alpha < 1$$

and define

$$\oint_y^b \frac{\mathrm{d}x}{(x-y)^{\alpha+1}} = \frac{1}{\alpha}\frac{\mathrm{d}}{\mathrm{d}y}\int_y^b \frac{\mathrm{d}x}{(x-y)^\alpha} = -\frac{1}{\alpha(b-y)^\alpha}.$$

This definition formally allows the exchange between differentiation and integration. Alternatively, we could consider the integral

$$\int_{y+\epsilon}^b \frac{\mathrm{d}x}{(x-y)^{\alpha+1}} = -\frac{1}{\alpha}\left[\frac{1}{(b-y)^\alpha} - \frac{1}{\epsilon^\alpha}\right]$$

and, as in the case of Section 3.1, ignore the diverging terms.

In the more general case, where we assume that $f \in C^1$, we have

$$\oint_y^b \frac{f(x)}{(x-y)^{\alpha+1}}\mathrm{d}x = \frac{1}{\alpha}\frac{\mathrm{d}}{\mathrm{d}y}\int_y^b \frac{f(x)}{(x-y)^\alpha}\mathrm{d}x$$

or, we compute the integral

$$\int_{y+\epsilon}^b \frac{f(x)}{(x-y)^{\alpha+1}}\mathrm{d}x = \int_{y+\epsilon}^b \frac{f(x)-f(y)}{(x-y)^{\alpha+1}}\mathrm{d}x + f(y)\int_{y+\epsilon}^b \frac{\mathrm{d}x}{(x-y)^{\alpha+1}} \tag{31}$$

and ignore the diverging terms (as $\epsilon \to 0$). We obtain

$$\oint_y^b \frac{f(x)}{(x-y)^{\alpha+1}}\mathrm{d}x = \int_y^b \frac{f(x)-f(y)}{(x-y)^{\alpha+1}}\mathrm{d}x + f(y)\oint_y^b \frac{\mathrm{d}x}{(x-y)^{\alpha+1}}.$$

These definitions can be generalized to define the finite-part value of integrals of the form

$$\oint_y^b \frac{f(x)}{(x-y)^{p+1}}\mathrm{d}x \tag{32}$$

for any real $p \geq 0$, where f is assumed to have a Hölder continuous $\lfloor p \rfloor$-derivative. This integral is sometimes called a *one-sided* finite-part integral, while (30) is called a *two-sided* finite-part integral.

Notice that when p is an integer

$$\neq\!\!\!\!\!\!\int_a^b \frac{f(x)}{(x-y)^{p+1}}\mathrm{d}x = \left(\neq\!\!\!\!\!\!\int_a^y + \neq\!\!\!\!\!\!\int_y^b\right)\frac{f(x)}{(x-y)^{p+1}}\mathrm{d}x.$$

We also have

$$\int_a^b \frac{f(x)}{x-y}\mathrm{d}x = \left(\neq\!\!\!\!\!\!\int_a^y + \neq\!\!\!\!\!\!\int_y^b\right)\frac{f(x)}{x-y}\mathrm{d}x.$$

3.3. Integrals over curves and surfaces

The previous finite-part definitions can be extended to integrals over open or closed lines in R^2, or even on open or closed surfaces in R^3.

For instance, in the case of a C^2 curve ℓ in R^2 we define the FP integral

$$\neq\!\!\!\!\!\!\int_\ell \frac{f(x,y)}{r^2}\mathrm{d}\ell_x, \quad r = \|x-y\|,\ y \in \ell, \tag{33}$$

by considering first the integral over $\ell - \ell_\epsilon$, where ℓ_ϵ is the symmetric neighborhood of the singular point y obtained by intersecting the curve ℓ with the circle of radius ϵ centered at y. Then, after computing this integral, we ignore the diverging terms (as $\epsilon \to 0$) and take the limit, for $\epsilon \to 0$, of the remaining ones.

The calculation is actually performed as follows. Since $f(x,y)/r^2$ is integrable over $\ell - \ell_\epsilon$, we introduce the curve parametrization and reduce the integral to the form

$$\left(\int_a^{s-\epsilon_1} + \int_{s+\epsilon_2}^b\right)\frac{g(t,s)}{(t-s)^2}\mathrm{d}t,$$

where $(s-\epsilon_1, s+\epsilon_2)$ is the pre-image of ℓ_ϵ. Then we proceed as described in Section 3.1. We remark once more that when the curve ℓ is smooth (C^2), we can replace the neighborhood $(s-\epsilon_1, s+\epsilon_2)$ by the symmetric one $(s-\epsilon, s+\epsilon)$; the final answer will not change. In this case we have

$$\neq\!\!\!\!\!\!\int_\ell \frac{f(x,y)}{r^2}\mathrm{d}\ell_x = \int_a^b \frac{g(t,s)-g(s,s)}{(t-s)^2}\mathrm{d}t + g(s,s)\neq\!\!\!\!\!\!\int_a^b \frac{\mathrm{d}t}{(t-s)^2}. \tag{34}$$

Notice that in both the FP integrals above we delete a symmetric neighborhood.

If ℓ is only piecewise smooth, and y is a corner point (in this case see [20]), then if we replace $(s-\epsilon_1, s+\epsilon_2)$ by $(s-\epsilon, s+\epsilon)$ we have to add an extra term (evaluated at y) to the right-hand side in (34) (see also [12]). This is due to the CPV term, which should have been defined by using the original non-symmetric neighborhood (see (12)). An analogous extra term appears also in the case of a one-sided FP integral.

Similarly, in the case of an integral defined on a surface, for example,

$$\neq\!\!\!\!\!\!\int_S \frac{f(x,y)}{r^3}\mathrm{d}S_x, \quad r = \|x-y\|,\ y \in S$$

we start by considering the integral defined over $S - S_\epsilon$, where S_ϵ is the neighborhood defined by the intersection of S with the sphere of radius ϵ centered at y. Then we proceed as in the previous case of a curve. After having introduced the surface parametrization $x = \phi(t),\ t \in D \subset R^2$, in the case of a C^2 surface we can replace the (pre-image) neighborhood of S_ϵ of the new singular point by a circle D_ϵ, and the final FP integral value will not change. That is, in both the integrals we delete a "circular" neighborhood. In particular we write

$$\int_{S-S_\epsilon} \frac{f(x,y)}{r^3}\mathrm{d}S_x = \int_{D-D_\epsilon} \frac{g(t,s)-g(s,s)}{\bar{r}^3}\mathrm{d}t + g(s,s)\int_{D-D_\epsilon}\frac{\mathrm{d}t}{\bar{r}^3},$$

where we have set $\bar{r} = \|t-s\|$, examine the behavior of the last integral as $\epsilon \to 0$, ignore the divergent terms, and finally let $\epsilon \to 0$. If this limit exists then it defines the FP value.

If S is only piecewise smooth and y is a "corner" point, then, if we replace the above pre-image by a circle, an extra term generated by the CPV integral ought to be added (see also [12]).

These are the definitions that naturally spring from the applications (see the next section).

Remark 9. In [24] and [17] one-dimensional and two-dimensional finite-part integrals have been defined by means of analytic continuation, performed using integration by parts. In particular in [17], two-dimensional finite-part integrals have been defined also in the case of line hypersingularities.

4. Some significant applications of FP integrals

After having defined finite-part integrals, a natural question is: why are these integrals needed? How do they stem from applications?

In this section we briefly describe four examples of engineering problems whose solutions have been reduced to those of corresponding hypersingular boundary integral equations. In particular we will carefully analyze the limiting processes that give rise to the definitions of FP integrals given in the previous section. Some (hopefully illuminating) remarks are also presented.

4.1. Problem 1

The first example is taken from [11]. There, a two-dimensional elasticity problem of an infinite (plane) strip $S = \{(x, y),\ 0 < x < h,\ -\infty < y < \infty\}$ containing a crack $0 < a < x < b < h,\ y = 0$ perpendicular to its boundary, under a prescribed surface traction $p(x)$, is described by a system of two second-order elliptic partial differential equations, subject to certain boundary conditions. This system defines the displacement vector u, and in particular its x-, y-components u_x, u_y.

The solution of this problem is reduced (see [11]) to the determination of the crack opening displacement

$$V(x) = u_y(x, 0^+) - u_y(x, 0^-), \quad a < x < b,$$

which is related to the stress tensor component σ_{yy} by means of the following integral transform

$$\sigma_{yy}(x, y) = \frac{2\mu}{\pi(\kappa + 1)} \int_a^b [K(|x - t|; y) + K_0(t, x; y)]V(t)\mathrm{d}t, \tag{35}$$

where μ and κ are physical constants, K_0 is continuous, while $K(|x - t|; y)$ is given by

$$K(|x - t|; y) = \frac{1}{r^2} - 4\frac{y^2}{r^4} + 8\frac{(x - t)^2 y^2}{r^6}, \qquad r^2 = (x - t)^2 + y^2. \tag{36}$$

To solve the original problem, one has to enforce the boundary condition $\sigma_{yy}(x, 0) = p(x),\ a < x < b$, to determine the new unknown $V(x)$. Here it is of key importance to realize that the function σ_{yy} has a limit for $y \to 0$; thus, in spite of the behavior of the kernel K when $y = 0$, the integral in (35) exists for $y \neq 0$ and has a finite limit for $y \to 0$. This limit exists only if we consider all three terms in K. If we drop one of them, or modify some of the coefficients, the limit would not exist. This is a property of the kernel, i.e., of the original elliptic PDE.

Since we are only interested in showing how the FP integral is generated, for simplicity we consider only the integral transform

$$A(x, y) = \int_a^b K(|x - t|; y)V(t)\mathrm{d}t. \tag{37}$$

Then, for any sufficiently small $\epsilon > 0$ we rewrite $A(x, y)$ as follows:

$$\begin{aligned} A(x, y) &= \left(\int_a^{x-\epsilon} + \int_{x+\epsilon}^b\right) K(|x - t|; y)V(t)\mathrm{d}t + \int_{x-\epsilon}^{x+\epsilon} K(|x - t|; y)V(t)\mathrm{d}t \\ &= \int_{I-I_\epsilon} K(|x - t|; y)V(t)\mathrm{d}t + \int_{I_\epsilon} K(|x - t|; y)[V(t) - V(x)]\mathrm{d}t + V(x)\int_{I_\epsilon} K(|x - t|; y)\mathrm{d}t, \end{aligned} \tag{38}$$

where we have set $I = (a, b),\ I_\epsilon = (x - \epsilon, x + \epsilon)$. We have

$$\lim_{y\to 0^+} A(x, y) = \int_{I-I_\epsilon} K(|x - t|; 0)V(t)\mathrm{d}t + \lim_{y\to 0}\left[\int_{I_\epsilon} K(|x - t|; y)[V(t) - V(x)]\mathrm{d}t + V(x)\int_{I_\epsilon} K(|x - t|; y)\mathrm{d}t\right].$$

Moreover,

$$\lim_{y\to 0}\int_{I_\epsilon} K(|x - t|; y)\mathrm{d}t = -\frac{2}{\epsilon} \tag{39}$$

and

$$\lim_{y\to 0}\int_{I_\epsilon} K(|x - t|; y)[V(t) - V(x)]\mathrm{d}t = ⨍_{x-\epsilon}^{x+\epsilon} \frac{V(t) - V(x)}{(t - x)^2}\mathrm{d}t.$$

Finally, we consider

$$\int_{I-I_\epsilon} K(|x - t|; 0)V(t)\mathrm{d}t = \int_{I-I_\epsilon} \frac{V(t) - V(x)}{(t - x)^2}\mathrm{d}t + V(x)\int_{I-I_\epsilon} \frac{\mathrm{d}t}{(t - x)^2} \tag{40}$$

and remark that

$$\int_{I-I_\epsilon} \frac{dt}{(t-x)^2} = \frac{2}{\epsilon} + \oint_a^b \frac{dt}{(t-x)^2}.$$

Notice that the above diverging term cancels the one that we have in (39). Thus

$$\lim_{y\to 0} A(x,y) = \oint_a^b \frac{V(t)-V(x)}{(t-x)^2}dt + V(x)\oint_a^b \frac{dt}{(t-x)^2} \equiv \oint_a^b \frac{V(t)}{(t-x)^2}dt. \tag{41}$$

The unknown function $V(x)$ ought to satisfy the following hypersingular integral equation:

$$\oint_a^b \frac{V(t)}{(t-x)^2}dt + \int_a^b V(t)K_0(t,x)dt = -\pi\frac{1+\kappa}{2\mu}p(x), \quad a < x < b. \tag{42}$$

Remark 10. In (38) we could have considered any non-symmetric neighborhood $(x-\epsilon_1, x+\epsilon_2)$ of x, since $A(x,y)$ does not depend on it. The final result would have been the same, i.e. (41), where, in spite of this choice, the FP integral is defined by deleting a symmetric neighborhood. The choice of a symmetric interval simplifies the calculation.

4.2. Problem 2

The second application (see [25]) refers to the study of a time-harmonic electromagnetic scattering problem associated with a T-junction between two rectangular waveguides. This junction is composed of an infinite (primary) waveguide

$$D^I = \{(x,y,z),\ 0 < x < a,\ 0 < y < b, -\infty < z < \infty\}$$

together with a semi-infinite (secondary) waveguide

$$D^{II} = \{(x,y,z), -\infty < x < 0,\ 0 < y < b,\ 0 < z < a'\}$$

having the same height; these are coupled through a common aperture $(0,b)\times(0,a')$.

In this particular case, standard calculation reduces the Maxwell equation which models the problem, to a simpler scalar two-dimensional non-homogeneous Helmholtz equation, to which a domain decomposition technique is then applied. In each waveguide the scattered electric field satisfies a non-homogeneous Helmholtz equation with homogeneous boundary conditions. An integral representation for this field is then obtained. For the primary waveguide we have:

$$E_y^{Is}(x,z) = -\int_0^{a'} \left.\frac{\partial g^I(x',z';x,z)}{\partial x'}\right|_{x'=0^+} M(z')dz', \quad 0 < x \le a \tag{43}$$

$$E_y^{Is}(0,z) = 0,$$

where g^I is the Green function for the Helmholtz equation defined on this waveguide (see [10]). It is to be remarked that $E_y^{Is}(x,z)$ is discontinuous at $x=0$, for $0 \le z \le a'$. Recalling that

$$H_z^{Is}(x,z) = -\frac{1}{j\omega\mu}\left[\frac{\partial E_y^{Is}(x,z)}{\partial x} + M(z)\delta(x-0^+)\right] \tag{44}$$

$\delta(x-a)$ being the Dirac delta function centered at $x=a$, $j=\sqrt{-1}$, and ω, μ two physical constants, the corresponding expression for the scattered magnetic field follows:

$$H_z^{Is}(0^+,z) = \frac{1}{j\omega\mu}\lim_{x\to 0^+}\int_0^{a'} \left.\frac{\partial^2 g^I(x',z';x,z)}{\partial x\partial x'}\right|_{x'=0^+} M(z')dz'. \tag{45}$$

Note that the singular term in (44) is exactly cancelled by the one that appears when the derivative of the discontinuous function $E_y^{Is}(x,z)$ is evaluated. The tangential magnetic field $H_z^{Is}(x,z)$ is analytic for $x>0,\ 0<z<a'$, and has a finite limit for $x\to 0^+$.

Proceeding as in the case of the primary waveguide, we obtain the following integral representation for the scattered magnetic field $H_z^{IIs}(0^-,z)$ on the secondary waveguide:

$$H_z^{IIs}(0^-,z) = -\frac{1}{j\omega\mu}\lim_{x\to 0^-}\int_0^{a'} \left.\frac{\partial^2 g^{II}(x',z';x,z)}{\partial x\partial x'}\right|_{x'=0^-} M(z')dz'. \tag{46}$$

In (45) and (46) g^I and g^{II} are the Green functions associated with the primary and secondary waveguides, respectively. For both of them we have a representation (see [10]) of the type:

$$g(x',z';x,z) = -\frac{j}{4}H_0^{(2)}(k\|\rho-\rho'\|) + g_0(\rho,\rho'), \tag{47}$$

where

$$\rho = (x, z)^{\mathrm{T}}, \qquad \rho' = (x', z')^{\mathrm{T}}$$

$$H_0^{(2)}(s) = -j\frac{2}{\pi} \log s\, J_0(s) + h_0(s)$$

and g_0 is analytic in its (x, z) (open) domain of definition. In these expressions $H_0^{(2)}$ is the Hankel function of second kind of order 0, J_0 is the Bessel function of first kind of order 0, and h_0 is an entire function. Here and in the following we drop the upper index I, II of the functions g and H_z, since the analysis we perform holds in both cases. Notice that the kernels of the integral representations (45) and (46) become hypersingular as $x \to 0$, since

$$H_0^{(2)\prime}(s) = -H_1^{(2)}(s) \tag{48}$$

$$H_1^{(2)}(s) = j\frac{2}{\pi s} - j(s \log s)h_1(s^2) + h_2(s^2), \quad s \to 0 \tag{49}$$

where $H_1^{(2)}(s)$ is the Hankel function of second kind and order one, and $h_1(t)$ and $h_2(t)$ are entire functions. In particular for $z \neq z'$ we have

$$\frac{\partial^2 g}{\partial x \partial x'}|_{x'=0} = -\frac{j}{2}\frac{\partial^2}{\partial x \partial x'} H_0^{(2)}(k\sqrt{x^2 + (z - z')^2}) + k_0(x; z, z'), \tag{50}$$

where k_0 is at most weakly singular when $x = 0$, otherwise it is analytic. Moreover

$$\frac{\partial^2}{\partial x \partial x'} H_0^{(2)}(k\sqrt{x^2 + (z - z')^2}) = \frac{2j}{\pi}\left\{\frac{2x^2}{[x^2 + (z - z')^2]^2} - \frac{1}{x^2 + (z - z')^2}\right\} + k_1(x; z, z'), \tag{51}$$

where k_1 is at most weakly singular when $x = 0$, otherwise it is analytic.

Now we consider the integral representations (45), (46), define $I = (0, a')$, $I_\epsilon = (z - \epsilon, z + \epsilon)$, and write

$$\begin{aligned} H_z^s(0, z) = \frac{1}{j\omega\mu}\Bigg\{&\int_{I - I_\epsilon} \frac{\partial^2 g(0, z'; 0, z)}{\partial x \partial x'} M(z')\mathrm{d}z' + \lim_{x \to 0} \int_{I_\epsilon} \frac{\partial^2 g(0, z'; x, z)}{\partial x \partial x'} [M(z') - M(z) - (z' - z)M'(z)]\mathrm{d}z' \\ &+ M(z) \lim_{x \to 0} \int_{I_\epsilon} \frac{\partial^2 g(0, z'; x, z)}{\partial x \partial x'} \mathrm{d}z' + M'(z) \lim_{x \to 0} \int_{I_\epsilon} \frac{\partial^2 g(0, z'; x, z)}{\partial x \partial x'} (z' - z)\mathrm{d}z'\Bigg\}. \end{aligned} \tag{52}$$

They hold for any real $\epsilon > 0$ sufficiently small, therefore also in the limit $\epsilon \to 0$. Taking into account the above expression (50) and (51) for the the kernel $\frac{\partial^2 g(0, z'; x, z)}{\partial x \partial x'}$, we notice that the second and the last terms in (52) vanish when we let $\epsilon \to 0$.

The value of the third term in (52) is

$$\frac{2M(z)}{\epsilon} + O(\epsilon).$$

Given $\epsilon > 0$, the limit in (52) exists because of the special form of the singular component on the right-hand side in (51). If, for example, we replace $2x^2$ by x^2 in the first quotient, the limit would be ∞. Furthermore, the analysis of the first integral in (52) shows that it contains a singular term, for $\epsilon \to 0$, whose value is

$$-\frac{2M(z)}{\epsilon}.$$

Therefore we have

$$H_z^s(0, z) \equiv \frac{1}{j\omega\mu} \,=\!\!\!\!\!\!\int_0^{a'} \frac{\partial^2 g(0, z'; 0, z)}{\partial x \partial x'} \mathrm{d}z'.$$

Remark 10 also applies to this case. This proof was not given in [25].

When we enforce the continuity condition on the tangential component of the total magnetic fields at the aperture $(0, a')$:

$$H_z^I(0^+, z) = H_z^{II}(0^-, z), \quad 0 < z < a'. \tag{53}$$

we obtain the following hypersingular integral equation defining the unknown magnetic current $M(z)$:

$$=\!\!\!\!\!\!\int_0^{a'} G^I(z - z')M(z')\mathrm{d}z' + =\!\!\!\!\!\!\int_0^{a'} G^{II}(z, z')M(z')\mathrm{d}z' \tag{54}$$

$$= j\omega\mu(H_z^{Ii}(0, z) - H_z^{IIi}(0, z)), \quad 0 < z < a', \tag{55}$$

where $H_z^{Ii}(0, z)$ and $H_z^{IIi}(0, z)$ are given incident magnetic fields and

$$G^I(z - z') = \lim_{x \to 0^+} \left.\frac{\partial^2 g^I(x', z'; x, z)}{\partial x \partial x'}\right|_{x'=0^+} \tag{56}$$

$$G^{II}(z, z') = \lim_{x \to 0^-} \left.\frac{\partial^2 g^{II}(x', z'; x, z)}{\partial x \partial x'}\right|_{x'=0^-}$$

4.3. Problem 3

This problem concerns the standard one-dimensional (linear) heat equation in the half-space for the temperature $T(x, t)$

$$\frac{\partial T(x,t)}{\partial t} = \alpha \frac{\partial^2 T(x,t)}{\partial x^2}, \quad x > 0, t > 0, \tag{57}$$

where α is the thermal diffusivity, subject to the initial condition

$$T(x, 0) = T_0, \; x \geq 0$$

and to boundary conditions.

Often one needs to compute the heat flux $q(x, t)$, related to the temperature $T(x, t)$ by the relationship

$$q(x,t) = -k \frac{\partial T(x,t)}{\partial x},$$

where k is the thermal conductivity.

Since the temperature may be expressed in terms of the heat flux by the following Abel integral transform (see [4])

$$T(x,t) = T_0 + \frac{1}{k}\sqrt{\frac{\alpha}{\pi}} \int_0^t q(x,u) \frac{\mathrm{d}u}{\sqrt{t-u}}$$

using the inversion formula given in ([16], Sect. 5.2) we obtain

$$q(x,t) = \beta \int_0^t \frac{\partial T(x,u)}{\partial u} \frac{\mathrm{d}u}{\sqrt{t-u}}, \tag{58}$$

where we have set $\beta = \frac{k}{\sqrt{\alpha\pi}}$.

Since $T(x, t)$ can only be approximated numerically, to avoid the computation of $\frac{\partial T(x,t)}{\partial u}$ we proceed as follows. Write

$$q(x,t) = \beta \lim_{\epsilon \to 0} \int_0^{t-\epsilon} \frac{\partial T(x,u)}{\partial u} \frac{\mathrm{d}u}{\sqrt{t-u}} \tag{59}$$

and, recalling that $T(x, 0) = T_0$, perform integration by parts; we obtain:

$$\begin{aligned} q(x,t) &= \beta \lim_{\epsilon\to 0} \left[\frac{T(x,t-\epsilon)}{\sqrt{\epsilon}} - \frac{T_0}{\sqrt{t}} - \frac{1}{2}\int_0^{t-\epsilon} \frac{T(x,y)}{(t-y)^{\frac{3}{2}}} \mathrm{d}y \right] \\ &= \beta \lim_{\epsilon\to 0} \left[\frac{T(x,t-\epsilon)}{\sqrt{\epsilon}} - \frac{T_0}{\sqrt{t}} - \frac{1}{2}\int_0^{t-\epsilon} \frac{T(x,y)-T(x,t)}{(t-y)^{\frac{3}{2}}} \mathrm{d}y - T(x,t)\left(\frac{1}{\sqrt{\epsilon}} - \frac{1}{\sqrt{t}}\right) \right] \\ &= -\beta \frac{T_0}{\sqrt{t}} - \frac{\beta}{2}\left[\int_0^t \frac{T(x,y)-T(x,t)}{(t-y)^{\frac{3}{2}}} \mathrm{d}y - 2\frac{T(x,t)}{\sqrt{t}} \right] \\ &\equiv -\beta \frac{T_0}{\sqrt{t}} - \frac{\beta}{2} \,\,{\large\neq}\!\!\!\!\!\!\int_0^t \frac{T(x,y)}{(t-y)^{\frac{3}{2}}} \mathrm{d}y, \quad x, t > 0. \end{aligned} \tag{60}$$

Since in the last integral change of variable is allowed, we finally have

$$q(x,t) = -\frac{\beta}{\sqrt{t}} \left(T_0 + \frac{1}{2} \,\,{\large\neq}\!\!\!\!\!\!\int_0^1 \frac{T(x,ts)}{(1-s)^{\frac{3}{2}}} \mathrm{d}s \right), \quad x, t > 0. \tag{61}$$

For the numerical evaluation of the last FP integral several quadrature formulas are available; see for example [23,24].

The above described reduction of the original problem to a hypersingular integral equation has been recently presented also in [5], Appendix B.

4.4. Problem 4

This last example describes a classical application of FP integrals, which springs from the boundary integral representation given by Green's second formula applied to the Laplace equation, defined on a bounded domain D, with boundary S. In particular, if we consider the boundary representation of the potential $u(y)$ at $y \in D$:

$$u(y) = \int_S \left[\frac{\partial u(x)}{\partial n_x} G(x,y) - u(x) \frac{\partial G(x,y)}{\partial n_x} \right] \mathrm{d}S_x, \quad y \in D, \tag{62}$$

where $G(x, y) = \frac{1}{2\pi} \log r$ in the two-dimensional case, $G(x, y) = \frac{1}{4\pi r}$ for three-dimensional problems, and $r = \|x - y\|$.

In several applications it is convenient or mandatory to take the gradient of $u(y)$

$$\frac{\partial u(y)}{\partial y_k} = \int_S \left[\frac{\partial u(x)}{\partial n_x} \frac{\partial G(x,y)}{\partial y_k} - u(x) \frac{\partial^2 G(x,y)}{\partial y_k \partial n_x} \right] \mathrm{d}S_x, \quad y \in D, \tag{63}$$

and let $y \to \xi \in S$, in order to derive an (implicit) expression for the normal derivative at S. In this case we have to compute

$$\lim_{y\to\xi} \int_S u(x) \frac{\partial^2 G(x,y)}{\partial y_k \partial n_x} \mathrm{d}S_x. \tag{64}$$

To this end, we delete a neighborhood of ξ, obtained by intersecting the curve (surface) S by a circle (sphere) of radius ϵ centered at ξ. We denote this domain by S_ϵ. As remarked in Section 3, we could delete a neighborhood (of "size" ϵ) of arbitrary shape, but the one above simplifies the following (analytic) calculation. Then we write

$$\int_S = \int_{S-S_\epsilon} + \int_{S_\epsilon}.$$

Since we have

$$\lim_{y\to\xi} \int_{S-S_\epsilon} = \int_{S-S_\epsilon} u(x) \frac{\partial^2 G(x,\xi)}{\partial y_k \partial n_x} \mathrm{d}S_x, \tag{65}$$

the critical part is

$$\lim_{y\to\xi} \int_{S_\epsilon} u(x) \frac{\partial^2 G(x,y)}{\partial y_k \partial n_x} \mathrm{d}S_x, \tag{66}$$

since its kernel has a non-integrable, even in the CPV sense, singularity at $x = y = \xi$. The calculation of (66) can however be easily reduced to the case $u(x) \equiv 1$. Indeed we have:

$$\begin{aligned} \int_{S_\epsilon} u(x) \frac{\partial^2 G(x,y)}{\partial y_k \partial n_x} u(x) \mathrm{d}S_x &= \int_{S_\epsilon} \frac{\partial^2 G(x,y)}{\partial y_k \partial n_x} [u(x) - u(\xi) - u'(\xi)(x-\xi)] \mathrm{d}S_x \\ &\quad + u(\xi) \int_{S_\epsilon} \frac{\partial^2 G(x,y)}{\partial y_k \partial n_x} \mathrm{d}S_x + u'(\xi) \int_{S_\epsilon} \frac{\partial^2 G(x,y)}{\partial y_k \partial n_x} (x-\xi) \mathrm{d}S_x \\ &=: I_\epsilon^1(y) + u(\xi) I_\epsilon^2(y) + u'(\xi) I_\epsilon^3(y), \end{aligned} \tag{67}$$

where we have set $u'(\xi) = (\nabla u(\xi))^{\mathrm{T}}$.

At this point we remark that

$$\lim_{y\to\xi} I_\epsilon^1(y) = I_\epsilon^1(\xi),$$
$$\lim_{y\to\xi} I_\epsilon^3(y) = I_\epsilon^3(\xi),$$

where $I_\epsilon^3(\xi)$ is defined in the CPV sense. Moreover,

$$\lim_{\epsilon\to 0} I_\epsilon^1(\xi) = 0,$$

while for the computation of

$$\lim_{\epsilon\to 0} I_\epsilon^3(\xi)$$

we need to apply Remarks 6 and 7. This means that when S is smooth (let us say C^2) at ξ, we have

$$\lim_{\epsilon\to 0} I_\epsilon^3(\xi) = 0,$$

otherwise

$$\lim_{\epsilon\to 0} I_\epsilon^3(\xi) = a_k(\xi).$$

Thus the only difficult part is the evaluation of

$$\lim_{y\to\xi} \int_{S_\epsilon} \frac{\partial^2 G(x,y)}{\partial y_k \partial n_x} \mathrm{d}S_x. \tag{68}$$

This is not a simple task, even when S is C^2 at ξ (see for example [19,13]). For instance, in [13], to evaluate (68) in this case, the authors apply a clever formulation of Stokes theorem, to reduce the integral over S_ϵ in (68) to a line integral defined on the boundary of S_ϵ, whose value is of the type $b_k(\xi)/\epsilon$; however this term exactly cancels the corresponding one that springs from the integral (65). Therefore in this case we have

$$\lim_{y\to\xi}\int_S u(x)\frac{\partial^2 G(x,y)}{\partial y_k \partial n_x}\mathrm{d}S_x = \oint_S u(x)\frac{\partial^2 G(x,\xi)}{\partial y_k \partial n_x}\mathrm{d}S_x. \tag{69}$$

When ξ is a corner point of S, to our knowledge in no applications integral (68) has been evaluated.

An alternatively, and apparently simpler approach (see [7,18]) applies Green's formula to the modified domain $D_\epsilon = S - S_\epsilon + C_\epsilon$, where S, S_ϵ are defined as before, and C_ϵ is the part of the boundary of the circle (sphere) of radius ϵ, centered at $\xi \in S$, which belongs to the domain D. Thus for this domain we have:

$$\int_{S-S_\epsilon+C_\epsilon}\left[\frac{\partial u(x)}{\partial n_x}G(x,\xi) - u(x)\frac{\partial G(x,\xi)}{\partial n_x}\right]\mathrm{d}S_x = 0, \quad \epsilon > 0, \tag{70}$$

hence

$$\lim_{\epsilon\to 0}\int_{S-S_\epsilon+C_\epsilon}\left[\frac{\partial u(x)}{\partial n_x}\frac{\partial G(x,\xi)}{\partial \xi_k} - u(x)\frac{\partial^2 G(x,\xi)}{\partial \xi_k \partial n_x}\right]\mathrm{d}S_x = 0. \tag{71}$$

Since we assume that $u \in C^{1,\alpha},\ \alpha > 0$, the only term whose evaluation poses some difficulties is

$$\int_{C_\epsilon}\frac{\partial^2 G(x,\xi)}{\partial \xi_k \partial n_x}\mathrm{d}S_x. \tag{72}$$

The calculation of this integral, even when S is not smooth at ξ, turns out to be much simpler than that of (68), since it requires only basic calculus tools, after introducing polar coordinates centered at ξ. An expression of the form

$$\int_{C_\epsilon}\frac{\partial^2 G(x,\xi)}{\partial \xi_k \partial n_x}\mathrm{d}S_x = \frac{b_k(\xi)}{\epsilon} + c_k(\xi) + O(\epsilon), \tag{73}$$

with the coefficients $b_k(\xi)$, $c_k(\xi)$ given explicitly, can be obtained. Thanks to this approach, Guiggiani [6], Mantič and Paris [18], and Young [29], have been the first to evaluate (72), hence to point out that when ξ is a "corner" point and the two continuous elements, separated by it, are not linear, we have $c_k(\xi) \neq 0$. This implies that (71) will also produce the additive term $-c_k(\xi)u(\xi)$. For a more precise statement see [7]; see also [3].

Finally we recall that in these last years there has been an increasing use of higher-order hypersingular BIEs (see for example [2]). They are obtained by taking higher-order derivatives of Cauchy singular integral equations.

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