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Numerical solution of a two-dimensional hyperbolic transmission problem

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ABSTRACT

An initial boundary value problem for a two-dimensional hyperbolic equation in two disjoint rectangles is investigated. The existence and uniqueness and a priori estimates for weak solutions in appropriate Sobolev-like spaces are proved. Few finite difference schemes approximating this problem are proposed and analyzed.

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1. Introduction

In recent years, important progress has been made in the numerical methods for the solution of fixed or moving interface problems. Reviews of some finite difference methods for interface problems are given in [1,2]. For interface problems the convergence theory of the numerical methods is difficult; we refer the readers to [3,1,2] for some discussions. The techniques that have been developed to obtain convergence rate estimates depend closely on the nature of the equations and interface conditions. In the present paper we investigate the convergence of difference schemes for hyperbolic equations coupling on a multicomponent domain with nonlocal interface conditions on parts of the boundary of the components.

Weak solutions and numerical methods for 1*D* and 2*D* nonlocal transmission parabolic problems have been studied in [4–7]. Basic physical models consist of thermoelastic contact of two bodies (rods, beams, discs, etc. see [8–11,4,6]) that is very common and important in industrial applications. One approach for the investigation of the existence, uniqueness and continuous dependence of solutions upon data as well as numerical solutions of the equations is for the difficulties associated with this physical problem to be overcome via a transform of the mathematical problem to nonlocal nonlinear parabolic systems [8–10,6]. In papers of the present authors a direct approach was developed [12,13,3,14,15].

In the present paper we use the same approach to the problem consisting of two hyperbolic equations on disjoint rectangles. The corresponding one-dimensional problem was studied in [13]. In this case the physical setting consists of two thin rods, each of which is clamped at one end but which may contact at their free ends. It was assumed a non-perfect contact between the rods and as a result non-local internal boundary (conjugation) conditions are imposed, see [16,13]. The two-dimensional problem has a similar interpretation. In particular, non-local conjugation conditions (3)–(4) are linearized radiation type conditions [17]. Further applications and theoretical results are obtained in [18].

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The layout of the paper is as follows. In the next section the mathematical formulation of the problem is introduced, namely the hyperbolic initial boundary value problem (IBVP) (1)–(7). In Section 3 we briefly expose the properties of IBVP (1)–(7) and give some a priori estimates for its weak solution. In Section 4 we introduce meshes, finite-difference operators and discrete Sobolev-like norms. In Section 5 we define the explicit finite difference scheme (FDS) approximating IBVP (1)–(7) and investigate its properties. A convergence rate estimate is obtained. In Section 6 we define a factorized FDS and investigate its properties and convergence.

2. Formulation of the problem

As a model example, we consider the following initial-boundary-value problem (IBVP): Find functions $u_1(x, y, t)$ and $u_2(x, y, t)$ that satisfy the system of hyperbolic equations

$$\frac{\partial^2 u_1}{\partial t^2} - \frac{\partial}{\partial x}\left(p_1(x,y)\frac{\partial u_1}{\partial x}\right) - \frac{\partial}{\partial y}\left(q_1(x,y)\frac{\partial u_1}{\partial y}\right) + r_1(x,y)u_1 = f_1(x,y,t), \quad (x,y) \in \Omega_1 \equiv (a_1,b_1)\times(c,d),\ t>0, \tag{1}$$

$$\frac{\partial^2 u_2}{\partial t^2} - \frac{\partial}{\partial x}\left(p_2(x,y)\frac{\partial u_2}{\partial x}\right) - \frac{\partial}{\partial y}\left(q_2(x,y)\frac{\partial u_2}{\partial y}\right) + r_2(x,y)u_2 = f_2(x,y,t), \quad (x,y) \in \Omega_2 \equiv (a_2,b_2)\times(c,d),\ t>0, \tag{2}$$

where $-\infty < a_1 < b_1 < a_2 < b_2 < +\infty$, the internal conjugation conditions of non-local Robin–Dirichlet type

$$p_1(b_1,y)\frac{\partial u_1}{\partial x}(b_1,y,t) + \alpha_1(y)u_1(b_1,y,t) = \int_c^d \beta_1(y,y')u_2(a_2,y',t)\mathrm{d}y', \tag{3}$$

$$-p_2(a_2,y)\frac{\partial u_2}{\partial x}(a_2,y,t) + \alpha_2(y)u_2(a_2,y,t) = \int_c^d \beta_2(y,y')u_1(b_1,y',t)\mathrm{d}y', \quad y\in(c,d),\ t>0, \tag{4}$$

the simplest external Dirichlet boundary conditions

$$\begin{aligned} &u_1(a_1,y,t)=0, \quad y\in(c,d); \qquad u_1(x,c,t)=u_1(x,d,t)=0, \quad x\in(a_1,b_1),\\ &u_2(b_2,y,t)=0, \quad y\in(c,d); \qquad u_2(x,c,t)=u_2(x,d,t)=0, \quad x\in(a_2,b_2), \end{aligned} \tag{5}$$

and the initial conditions

$$u_1(x,y,0)=\phi_1(x,y), \quad (x,y)\in\Omega_1; \qquad u_2(x,y,0)=\phi_2(x,y), \quad (x,y)\in\Omega_2, \tag{6}$$

$$\frac{\partial u_1}{\partial t}(x,y,0)=\psi_1(x,y), \quad (x,y)\in\Omega_1; \qquad \frac{\partial u_2}{\partial t}(x,y,0)=\psi_2(x,y), \quad (x,y)\in\Omega_2. \tag{7}$$

Throughout the paper we assume that the input data satisfy the usual regularity and ellipticity conditions

$$p_i(x,y),\ q_i(x,y)\in L_\infty(\Omega_i), \qquad r_i(x,y)\in L_p(\Omega_i), \quad p>1,\ i=1,2, \tag{8}$$

$$0<p_{i0}\le p_i(x,y)\le p_{i1}, \qquad 0<q_{i0}\le q_i(x,y)\le q_{i1}, \quad \text{a.e. in } \Omega_i,\ i=1,2 \tag{9}$$

and

$$\alpha_i\in L_\infty(c,d), \qquad \beta_i\in L_\infty\left((c,d)\times(c,d)\right), \quad i=1,2. \tag{10}$$

In real physical problems (see [17]) we also often have

$$\alpha_i>0, \qquad \beta_i>0, \quad i=1,2.$$

By C, c_j and C_j we denote positive constants, independent of the solution of the IBVP and the mesh-sizes. C can take different values in the different formulas.

3. Weak solutions and a priori estimates

We introduce the product space

$$L = L_2(\Omega_1)\times L_2(\Omega_2) = \{u=(u_1,u_2)|u_i\in L_2(\Omega_i)\},$$

endowed with the inner product and the associated norm

$$(u,v)_L = (u_1,v_1)_{L_2(\Omega_1)} + (u_2,v_2)_{L_2(\Omega_2)}, \quad \|u\|_L = (u,u)_L^{1/2},$$

where

$$(u_i,v_i)_{L_2(\Omega_i)} = \int_{\Omega_i} u_i v_i \mathrm{d}x\mathrm{d}y, \quad i=1,2.$$

We also define the spaces

$$H^k = \{u = (u_1, u_2) | u_i \in H^k(\Omega_i)\}, \quad k = 1, 2, \ldots$$

endowed with the inner products and norms

$$(u, v)_{H^k} = (u_1, v_1)_{H^k(\Omega_1)} + (u_2, v_2)_{H^k(\Omega_2)}, \quad \|u\|_{H^k} = (u, u)_{H^k}^{1/2},$$

where

$$(u_i, v_i)_{H^k(\Omega_i)} = \sum_{j=0}^{k} \sum_{l=0}^{j} \Big(\frac{\partial^j u_i}{\partial x^l \partial y^{j-l}}, \frac{\partial^j v_i}{\partial x^l \partial y^{j-l}} \Big)_{L_2(\Omega_i)}, \quad i = 1, 2, \ k = 1, 2, \ldots$$

In particular, we set

$$H_0^1 = \{u = (u_1, u_2) \in H^1 | u_i = 0 \text{ on } \Gamma_i, \ i = 1, 2\},$$

where $\Gamma_1 = \partial\Omega_1 \setminus \{(b_1, y) | y \in (c, d)\}$ and $\Gamma_2 = \partial\Omega_2 \setminus \{(a_2, y) | y \in (c, d)\}$.

Let $H^{-1} = (H_0^1)^*$ be the dual space for H_0^1. The spaces H_0^1, L and H^{-1} form a Gelfand triple $H_0^1 \subset L \subset H^{-1}$ ([19]), with continuous and dense embeddings.

With $u = (u_1, u_2)$ and $v = (v_1, v_2)$ we define the bilinear form:

$$\begin{aligned} a(u, v) = & \int_{\Omega_1} \Big(p_1 \frac{\partial u_1}{\partial x} \frac{\partial v_1}{\partial x} + q_1 \frac{\partial u_1}{\partial y} \frac{\partial v_1}{\partial y} + r_1 u_1 v_1 \Big) \mathrm{d}x\mathrm{d}y + \int_{\Omega_2} \Big(p_2 \frac{\partial u_2}{\partial x} \frac{\partial v_2}{\partial x} + q_2 \frac{\partial u_2}{\partial y} \frac{\partial v_2}{\partial y} + r_2 u_2 v_2 \Big) \mathrm{d}x\mathrm{d}y \\ & + \int_c^d \alpha_1(y) u_1(b_1, y) v_1(b_1, y) \mathrm{d}y + \int_c^d \alpha_2(y) u_2(a_2, y) v_2(a_2, y) \mathrm{d}y \\ & - \int_c^d \int_c^d \beta_1(y, y') u_2(a_2, y') v_1(b_1, y) \mathrm{d}y\mathrm{d}y'. - \int_c^d \int_c^d \beta_2(y', y) u_1(b_1, y) v_2(a_2, y') \mathrm{d}y\mathrm{d}y'. \end{aligned} \tag{11}$$

The following assertion holds true (see [14]):

Lemma 1. *Under the conditions (8) and (10) the bilinear form a, defined by (11), is bounded on $H^1 \times H^1$. If besides it the conditions (9) are fulfilled, this form satisfies the Gårding's inequality on H_0^1, i.e. there exist positive constants m and κ such that*

$$a(u, u) + \kappa \|u\|_L^2 \geq m \|u\|_{H^1}^2, \quad \forall\, u \in H_0^1.$$

If the condition

$$\beta_1(y, y') = \beta_2(y', y) = \beta(y, y') \tag{12}$$

is satisfied, the bilinear form a is symmetric and generates a selfadjoint linear operator $A : H_0^1 \to H^{-1}$ by the formula

$$(Au, v)_L = a(u, v), \quad \forall v \in H_0^1.$$

Let I be the identity operator. From Lemma 1 follows that operator $A_0 = A + \kappa I$ is positive definite. The corresponding energy norm $\|u\|_{A_0} = (A_0 u, u)_L^{1/2}$ is equivalent to $\|u\|_{H^1}$ for $u \in H_0^1$.

Let us consider additional assumptions on input data:

$$\alpha_i(y) \geq 0 \quad \text{a.e. in } (c, d), \ i = 1, 2, \tag{13}$$

$$|\beta(y, y')| \leq \frac{\sqrt{\alpha_1(y)\alpha_2(y')}}{d - c}, \tag{14}$$

$$r_i(x, y) \geq 0 \quad \text{a.e. in } \Omega_i, \ i = 1, 2. \tag{15}$$

Lemma 2. *Under the conditions (8)–(10) and (12)–(15) the bilinear form a, defined by (11), is symmetric on $H^1 \times H^1$ and coercive on H_0^1, i.e. there exists a positive constant c_0 such that*

$$a(u, u) \geq c_0 \|u\|_{H^1}^2, \quad \forall u \in H_0^1.$$

Proof. The first assertion is obvious while the second follows from Lemma 1 and inequalities

$$\int_{\Omega_1} r_i u_i^2 \mathrm{d}x\mathrm{d}y \geq 0, \quad i = 1, 2$$

and

$$\int_c^d \alpha_1(y)u_1^2(b_1,y)\mathrm{d}y + \int_c^d \alpha_2(y)u_2^2(a_2,y)\mathrm{d}y - 2\int_c^d\int_c^d \beta(y,y')u_1(b_1,y)u_2(a_2,y')\mathrm{d}y\mathrm{d}y'$$
$$= \int_c^d\int_c^d \left[\frac{\alpha_1(y)}{d-c}u_1^2(b_1,y) - 2\beta(y,y')u_1(b_1,y)u_2(a_2,y') + \frac{\alpha_2(y')}{d-c}u_2^2(a_2,y')\right]\mathrm{d}y\mathrm{d}y' \geq 0. \quad \square$$

In a standard manner (see [20]) we define the spaces $L_p((0,T),V)$, $p \geq 1$, of functions $u(t)$ mapping the real interval $(0,T)$ into a Hilbert space V, endowed with the norm

$$\|u\|_{L_p((0,T),V)} = \left(\int_0^T \|u(t)\|_V^p \mathrm{d}t\right)^{1/p},$$

with the standard modification for $p = \infty$. We also define Sobolev spaces of vector-valued functions $H^k((0,T),V)$, endowed with the inner product

$$(u,v)_{H^k((0,T),V)} = \int_0^T \sum_{j=0}^k (u^{(j)}(t), v^{(j)}(t))_V \mathrm{d}t, \quad k = 0,1,2,\ldots$$

In particular, $L_2((0,T),V) = H^0((0,T),V)$.

The weak form of (1)–(5) is:

$$(u''(\cdot,t),v)_L + a(u(\cdot,t),v) = (f(\cdot,t),v)_L, \quad \forall v \in H_0^1, \tag{16}$$

where we denoted $(\cdot)' = \partial/\partial t$, or, in operator form:

$$u'' + Au = f \quad \text{in } H^{-1}. \tag{17}$$

We define the weak solution of IBVP (1)–(7) for $t \in (0,T)$ as a function $u \in L_2((0,T),H_0^1) \cap H^1((0,T),L)$, equal to ϕ for $t = 0$ and

$$\int_0^T \left[-(u'(\cdot,t),v'(\cdot,t))_L + a(u(\cdot,t),v(\cdot,t))\right]\mathrm{d}t = (\psi, v(\cdot,0))_L + \int_0^T (f(\cdot,t),v(\cdot,t))_L \mathrm{d}t,$$
$$\forall v \in L_2((0,T),H_0^1) \cap H^1((0,T),L),\ v = 0 \text{ for } t = T. \tag{18}$$

The problem (16) fits in the general theory of hyperbolic differential operators in Hilbert spaces (see [19]). Applying Theorem 29.1 from [19] to (16) we obtain the following assertion:

Theorem 1. *Let the assumptions (8)–(10) and (12) hold and suppose that $\phi = (\phi_1,\phi_2) \in H_0^1$, $\psi = (\psi_1,\psi_2) \in L$, $f = (f_1,f_2) \in L_2((0,T),L)$. Then for $0 < T < +\infty$ the* IBVP (1)–(7) *has a unique weak solution $u \in L_2((0,T),H_0^1) \cap H^1((0,T),L)$, and it depends continuously on f, ϕ and ψ. The solution satisfies the a priori estimate*

$$\|u\|^2_{L_\infty((0,T),H^1)} + \left\|\frac{\partial u}{\partial t}\right\|^2_{L_\infty((0,T),L)} \leq C(T)\left(\|\phi\|^2_{H^1} + \|\psi\|^2_L + \|f\|^2_{L_2((0,T),L)}\right).$$

Here $C(T) = C_1 e^{C_2 T}$ is a computable constant depending on T.
Under stronger assumptions a more precise a priori estimate can be derived.

Theorem 2. *Let the assumptions (8)–(10) and (12)–(15) hold and let $\phi = (\phi_1,\phi_2) \in H_0^1$, $\psi = (\psi_1,\psi_2) \in L$, $f = (f_1,f_2) \in L_1((0,T),L)$. Then the solution of the* IBVP (1)–(7) *satisfies the a priori estimate*

$$\|u\|^2_{L_\infty((0,T),H^1)} + \left\|\frac{\partial u}{\partial t}\right\|^2_{L_\infty((0,T),L)} \leq C\left(\|\phi\|^2_{H^1} + \|\psi\|^2_L + \|f\|^2_{L_1((0,T),L)}\right). \tag{19}$$

Proof. Let $\{w^i \mid i \in \mathbb{N}\}$ be a basis for H_0^1. We seek an approximate solution of the IBVP (1)–(7) in the form

$$u^m = \sum_{j=1}^m \sigma_m^j(t) w^j$$

which satisfies the conditions

$$\begin{aligned} &((u^m)'', w^i)_L + a(u^m, w^i) = (f, w^i)_L, \quad t \in (0,T),\ i = 1,2,\ldots,m, \\ &u^m = \phi^m, \qquad (u^m)' = \psi^m, \quad t = 0, \end{aligned} \tag{20}$$

where

$$\phi^m = \sum_{j=1}^{m} \xi_m^j w^j \to \phi \quad \text{in } H_0^1, \qquad \psi^m = \sum_{j=1}^{m} \eta_m^j w^j \to \psi \quad \text{in } L.$$

Problem (20) reduces to a system of linear ODEs for the coefficients $\sigma_m^j(t)$:

$$\sum_{j=1}^{m}\Big[(w^j, w^i)_L(\sigma_m^j(t))'' + a(w^j, w^i)\sigma_m^j(t)\Big] = (f(\cdot,t), w^i)_L, \quad i = 1, 2, \ldots, m,$$
$$\sigma_m^i(0) = \xi_m^i, \qquad (\sigma_m^i)'(0) = \eta_m^i, \quad i = 1, 2, \ldots, m.$$

The matrix of coefficients multiplying second derivatives is positive definite, while the right hand sides belong to $L_1(0, T)$. Therefore the system of ODEs has a unique solution $(\sigma_m^1(t), \sigma_m^2(t), \ldots, \sigma_m^m(t))$.

Multiplying Eq. (20) with $(\sigma_m^i)'$ and summing from $i = 1$ to $i = m$ we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\left\|\frac{\partial u^m}{\partial t}(\cdot,t)\right\|_L^2 + a(u^m(\cdot,t), u^m(\cdot,t))\right) = 2\Big(f(\cdot,t), \frac{\partial u^m}{\partial t}(\cdot,t)\Big)_L.$$

Integrating this equality from 0 to t, $0 < t < T$, we find

$$\left\|\frac{\partial u^m}{\partial t}(\cdot,t)\right\|_L^2 + a(u^m(\cdot,t), u^m(\cdot,t)) = \|\psi^m\|_L^2 + a(\phi^m, \phi^m) + 2\int_0^t \Big(f(\cdot,t'), \frac{\partial u^m}{\partial t}(\cdot,t')\Big)_L \mathrm{d}t'.$$

Using Lemma 2 and exploiting the convergence of ϕ^m and ψ^m, we obtain

$$\begin{aligned}\left\|\frac{\partial u^m}{\partial t}(\cdot,t)\right\|_L^2 + c_0\|u^m(\cdot,t)\|_{H^1}^2 &\le \|\psi^m\|_L^2 + c_1\|\phi^m\|_{H^1}^2 + 2\|f\|_{L_1((0,T),L)}\left\|\frac{\partial u^m}{\partial t}\right\|_{L_\infty((0,T),L)} \\ &\le \|\psi\|_L^2 + c_1\|\phi\|_{H^1}^2 + \varepsilon + 2\|f\|_{L_1((0,T),L)}^2 + \frac{1}{2}\left\|\frac{\partial u}{\partial t}\right\|_{L_\infty((0,T),L)}^2\end{aligned}$$

and taking the supremum over $t \in (0, T)$:

$$\|u^m\|_{L_\infty((0,T),H^1)}^2 + \left\|\frac{\partial u^m}{\partial t}\right\|_{L_\infty((0,T),L)}^2 \le C\Big(\|\phi\|_{H^1}^2 + \|\psi\|_L^2 + \varepsilon + \|f\|_{L_1((0,T),L)}^2\Big).$$

Hence the sequence u^m contains a subsequence which converges weakly-* in $L_\infty((0, T), H^1)$ while its time derivative converges weakly-* in $L_\infty((0, T), L)$. Its limit u satisfies a priori estimate (19).

Let us show that u is the weak solution of the IBVP (1)–(7) in the sense of (18). Let $\{\tilde{w}^k \mid k \in \mathbb{N}\}$ be a basis of the space $H_*^1(0, T) = \{\tilde{w} \in H^1(0, T) \mid w(T) = 0\}$. Multiplying (20) with $\tilde{w}^k(t)$, integrating over $(0, T)$ and performing partial integration, we obtain

$$\int_0^T \big[-((u^m)'(\cdot,t), v'(\cdot,t))_L + a(u^m(\cdot,t), v(\cdot,t))\big]\mathrm{d}t = (\psi^m, v(\cdot,0))_L + \int_0^T (f(\cdot,t), v(\cdot,t))_L \mathrm{d}t,$$

where denoted $v(\cdot,t) = \tilde{w}^k(t)w^i$. Letting $m \to \infty$ we conclude that function u satisfies (19) for $v(\cdot,t) = \tilde{w}^k(t)w^i$. But the set of such v is dense in the space $V = \{v \in L_2((0,T), H_0^1) \cap H^1((0,T), L) \mid v = 0 \text{ for } t = T\}$.

Uniqueness of the weak solution follows immediately from (19). It is enough to show that $u = 0$ if $\phi = 0$, $\psi = 0$ and $f = 0$. For a fixed $t \in [0, T]$ the function

$$v(\cdot,s) = \begin{cases} -\displaystyle\int_s^t u(\cdot,s')\mathrm{d}s', & 0 \le s \le t \\ 0, & t < s \le T \end{cases}$$

has sufficient regularity to be inserted in (19). This yields

$$\int_0^t \big[-(u,u)_L + a(v,v)\big]'\mathrm{d}s = 0$$

and

$$\|u(\cdot,t)\|_L^2 + a(v(\cdot,0), v(\cdot,0)) = 0$$

whereby $\|u(\cdot,t)\|_L = 0$ and $u = 0$. □

A priori estimate in L_2 norm holds under weaker assumptions on input data:

Theorem 3. *Let the assumptions (8)–(10) and (12) hold and suppose that $\phi \in L$, $\psi \in H^{-1}$, $f \in L_2((0,T),H^{-1})$. Then the solution of the* IBVP *(1)–(7) satisfies the a priori estimate*

$$\|u\|^2_{L_\infty((0,T),L)} + \left\|\frac{\partial u}{\partial t}\right\|^2_{L_\infty((0,T),H^{-1})} \le C(T)\Big(\|\phi\|^2_L + \|\psi\|^2_{H^{-1}} + \|f\|^2_{L_2((0,T),H^{-1})}\Big).$$

Proof. Let us define an approximate solution u^m of the IBVP (1)–(7) in the same way as in the proof of Theorem 2. Since $f \in L_2((0,T),H^{-1})$ the right-hand sides in (20) belong to $L_2(0,T)$ and its solution exists. We introduce the function

$$v^m = \sum_{j=1}^{m} \tau^j_m(t) w^j$$

such that

$$a_0(v^m, w^i) \equiv a(v^m, w^i) + \kappa(v^m, w^i)_L = (u^m, w^i)_L, \quad i = 1, 2, \ldots, m.$$

The least condition reduces to a system of linear equations for $\tau^j_m(t)$ with a positive definite matrix, so v^m is well defined. From definition of v^m follows

$$a_0(v^m, u^m) = (u^m, u^m)_L \quad \text{and} \quad a_0(v^m, v^m) = (u^m, v^m)_L.$$

Further

$$(u^m, v^m)_L \le \sup_{\varphi \in H^1_0} \frac{(u^m, \varphi)_L}{\|\varphi\|_{H^1}} \|v^m\|_{H^1}$$

whereby

$$a_0(v^m, v^m) \le \frac{1}{m}\|u^m\|^2_{H^{-1}}.$$

From (20) follows

$$((u^m)'', (v^m)')_L + a_0(u^m, (v^m)') = \kappa(u^m, (v^m)')_L + (f, (v^m)')_L.$$

Integrating this equality over $(0,t) \subset (0,T)$ we find

$$\begin{aligned} a_0((v^m)'(\cdot,t), (v^m)'(\cdot,t)) + \|u^m(\cdot,t)\|^2_L &= a_0((v^m)'(\cdot,0), (v^m)'(\cdot,0)) \\ &+ \|u^m(\cdot,0)\|^2_L + 2\int_0^t \big(f(\cdot,s), (v^m)'(\cdot,s)\big)_L ds + 2\kappa \int_0^t \big(u^m(\cdot,s), (v^m)'(\cdot,s)\big)_L ds. \end{aligned}$$

After obvious majorations we get

$$\begin{aligned} a_0((v^m)'(\cdot,t), (v^m)'(\cdot,t)) + \|u^m(\cdot,t)\|^2_L \le C_1 &\left(\|\psi^m\|^2_{H^{-1}} + \|\phi^m\|^2_L + \int_0^T \|f(\cdot,s)\|^2_{H^{-1}} ds\right) \\ &+ C_2 \int_0^t \Big(a_0((v^m)'(\cdot,s), (v^m)'(\cdot,s)) + \|u^m(\cdot,s)\|^2_L\Big) ds \end{aligned}$$

and result follows applying the Grönwall lemma (see e.g. [19]) and passing to the limit when $m \to \infty$. □

Theorem 4. *Let the assumptions of Theorem 3 hold and let $f = g' = \partial g/\partial t$, $g \in L_2((0,T),L) \cap L_\infty((0,T),H^{-1})$. Then the solution of the* IBVP *(1)–(7) satisfies the a priori estimate*

$$\|u\|^2_{L_\infty((0,T),L)} \le C(T)\Big(\|\phi\|^2_L + \|\psi - g(\cdot,0)\|^2_{H^{-1}} + \|g\|^2_{L_2((0,T),L)}\Big).$$

Proof. Let us define u^m and v^m in the same way as in the proof of Theorem 3. Analogously we define the functions

$$g^m = \sum_{j=1}^{m} \gamma^j_m(t) w^j \quad \text{and} \quad \tilde{g}^m = \sum_{j=1}^{m} \tilde{\gamma}^j_m(t) w^j$$

such that

$$(g^m, w^i)_L = (g, w^i)_L, \quad i = 1, 2, \ldots, m,$$

and

$$a_0(\tilde{g}^m, w^i) = (g^m, w^i)_L, \quad i = 1, 2, \ldots, m.$$

From (20) follows

$$\left(((u^m)' - g^m)', (v^m)' - \tilde{g}^m\right)_L + a_0(u^m, (v^m)') = a_0(u^m, \tilde{g}^m) + \kappa(u^m, (v^m)' - \tilde{g}^m)_L.$$

Integrating this equality over $(0, t) \subset (0, T)$ we find

$$\begin{aligned} a_0((v^m)'(\cdot, t) - \tilde{g}^m(\cdot, t), (v^m)'(\cdot, t) - \tilde{g}^m(\cdot, t)) + \|u^m(\cdot, t)\|_L^2 &= a_0((v^m)'(\cdot, 0) - \tilde{g}^m(\cdot, 0), (v^m)'(\cdot, 0) - \tilde{g}^m(\cdot, 0)) \\ &+ \|u^m(\cdot, 0)\|_L^2 + 2\int_0^t \left(u^m(\cdot, s), g^m(\cdot, s)\right)_L \mathrm{d}s + 2\kappa \int_0^t \left(u^m(\cdot, s), (v^m)'(\cdot, s) - \tilde{g}^m(\cdot, s)\right)_L \mathrm{d}s \end{aligned}$$

and, after obvious majoration

$$\begin{aligned} &a_0((v^m)'(\cdot, t) - \tilde{g}^m(\cdot, t), (v^m)'(\cdot, t) - \tilde{g}^m(\cdot, t)) + \|u^m(\cdot, t)\|_L^2 \\ &\le C_1\left(\|\psi^m - g^m(\cdot, o)\|_{H^{-1}}^2 + \|\phi^m\|_L^2 + \int_0^T \|g^m(\cdot, s)\|_L^2 \mathrm{d}s\right) \\ &\quad + C_2 \int_0^t \left(a_0((v^m)'(\cdot, s) - \tilde{g}^m(\cdot, s), (v^m)'(\cdot, s) - \tilde{g}^m(\cdot, s)) + \|u^m(\cdot, s)\|_L^2\right) \mathrm{d}s. \end{aligned}$$

The result follows applying the Grönwall lemma and passing to the limit. □

Remark 1. If more regularity of the data and appropriate compatibility conditions at $t = 0$ are assumed, one obtains more regular solutions (see [19] for details). For the influence of corner singularities and the corresponding compatibility conditions ensuring the existence of a smooth solution we refer readers to [21].

4. Meshes, finite differences and discrete norms

Let $\bar{\omega}_{i,h_i}$ be a uniform mesh in $[a_i, b_i]$ with the step-size $h_i = (b_i - a_i)/n_i$, $i = 1, 2$. We denote $\omega_{i,h_i} := \bar{\omega}_{i,h_i} \cap (a_i, b_i)$, $\omega_{i,h_i}^- := \omega_{i,h_i} \cup \{a_i\}$, $\omega_{i,h_i}^+ := \omega_{i,h_i} \cup \{b_i\}$. Analogously we define a uniform mesh $\bar{\omega}_k$ in $[c, d]$ with the step-size $k = (d - c)/n_3$ and its submeshes $\omega_k := \bar{\omega}_k \cap (c, d)$, $\omega_k^- := \omega_k \cup \{c\}$, $\omega_k^+ := \omega_k \cup \{d\}$. Finally, we introduce a uniform mesh $\bar{\omega}_\tau$ in $[0, T]$ with the step-size $\tau = T/n$ and set $\omega_\tau := \bar{\omega}_\tau \cap (0, T)$, $\omega_\tau^- := \omega_\tau \cup \{0\}$, $\omega_\tau^+ := \omega_\tau \cup \{T\}$. We will consider vector-functions of the form $v = (v_1, v_2)$ where v_i is a mesh function defined on $\bar{\omega}_{i,h_i} \times \bar{\omega}_k$ or on $\bar{\omega}_{i,h_i} \times \bar{\omega}_k \times \bar{\omega}_\tau$, $i = 1, 2$. We define difference quotients in the usual way (see [2]):

$$\begin{aligned} v_{i,x}(x, y, t) &= \frac{v_i(x + h_i, y, t) - v_i(x, y, t)}{h_i} = v_{i,\bar{x}}(x + h_i, y, t), \\ v_{i,y}(x, y, t) &= \frac{v_i(x, y + k, t) - v_i(x, y, t)}{k} = v_{i,\bar{y}}(x, y + k, t), \\ v_{i,t}(x, y, t) &= \frac{v_i(x, y, t + \tau) - v_i(x, y, t)}{\tau} = v_{i,\bar{t}}(x, y, t + \tau). \end{aligned}$$

We shall use the following notational conventions:

$$v_i = v_i(x, y, t), \qquad \widehat{v}_i(x, y, t) = \frac{1}{2}\left[v_i(x, y, t) + v_i(x, y, t + \tau)\right] = \breve{v}_i(x, y, t + \tau).$$

We define the Steklov averaging operators

$$\begin{aligned} T_x f_i(x, y, t) &= \frac{1}{h_i} \int_{x - h_i/2}^{x + h_i/2} f_i(x', y, t) \mathrm{d}x', \quad x \in \omega_{i,h_i},\ i = 1, 2, \\ T_y f_i(x, y, t) &= \frac{1}{k} \int_{y - k/2}^{y + k/2} f_i(x, y', t) \mathrm{d}y', \qquad T_t f_i(x, y, t) = \frac{1}{\tau} \int_{t - \tau/2}^{t + \tau/2} f_i(x, y, t') \mathrm{d}t'. \end{aligned}$$

For $x = b_1, x = a_2$ and $t = 0$ we need the following asymmetric averaging operators

$$\begin{aligned} T_x^- f_1(b_1, y, t) &= \frac{2}{h_1} \int_{b_1 - h_1/2}^{b_1} f_1(x', y, t) \mathrm{d}x', \qquad T_x^+ f_2(a_2, y, t) = \frac{2}{h_2} \int_{a_2}^{a_2 + h_2/2} f_2(x', y, t) \mathrm{d}x', \\ T_t^+ f_i(x, y, 0) &= \frac{2}{\tau} \int_0^{t + \tau/2} f_i(x, y, t') \mathrm{d}t'. \end{aligned}$$

With the notational conventions $\hbar_i = h_i, x \in \omega_{i,h_i}, i = 1, 2, \hbar_1(b_1) = h_1/2, \hbar_2(a_2) = h_2/2$, we introduce the discrete inner products and the associated norms

$$(v, w)_{L_h} = k \sum_{x\in\omega^+_{1,h_1}} \sum_{y\in\omega_k} v_1 w_1 \hbar_1 + k \sum_{x\in\omega^-_{2,h_2}} \sum_{y\in\omega_k} v_2 w_2 \hbar_2, \quad \|v\|_{L_h} = (v, v)^{1/2}_{L_h},$$
$$(v, w)_{L_{h'}} = h_1 k \sum_{x\in\omega^+_{1,h_1}} \sum_{y\in\omega_k} v_1 w_1 + h_2 k \sum_{x\in\omega^+_{2,h_2}} \sum_{y\in\omega_k} v_2 w_2, \quad \|v\|_{L_{h'}} = (v, v)^{1/2}_{L_{h'}},$$
$$(v, w)_{L_{h''}} = k \sum_{x\in\omega^+_{1,h_1}} \sum_{y\in\omega^+_k} v_1 w_1 \hbar_1 + k \sum_{x\in\omega^-_{2,h_2}} \sum_{y\in\omega^+_k} v_2 w_2 \hbar_2, \quad \|v\|_{L_{h''}} = (v, v)^{1/2}_{L_{h''}}.$$

We also define the following discrete norms

$$\|v\|_{H^1_h} = \left(\|v_{\bar{x}}\|^2_{L_{h'}} + \|v_{\bar{y}}\|^2_{L_{h''}} + \|v\|^2_{L_h}\right)^{1/2} \quad \text{and} \quad \|v\|_{H^{-1}_h} = \sup_{w\in H^1_{h,0}} \frac{(v, w)_{L_h}}{\|w\|_{H^1_h}},$$

where $H^1_{h,0}$ is the space of mesh functions $v = (v_1, v_2)$ such that $v_i = 0$ on $\Gamma_i \cap \left(\bar{\omega}_{i,h_i} \times \bar{\omega}_k\right), i = 1, 2$. For the mesh functions of one variable $v(y) = (v_1(y), v_2(y))$ defined on ω_k we set

$$(v, w)_{L_2(\omega_k)} = k \sum_{y\in\omega_k} (v_1 w_1 + v_2 w_2), \qquad \|v\|_{L_2(\omega_k)} = (v, v)^{1/2}_{L_2(\omega_k)}.$$

Finally, we introduce the discrete counterparts of norms $L_2((0, T), V)$ and $L_\infty((0, T), V)$:

$$\|v\|_{L_2(\tilde{\omega}_\tau, V_h)} = \left(\tau \sum_{t\in\tilde{\omega}_\tau} \|v(\cdot, t)\|^2_{V_h}\right)^{1/2}, \qquad \|v\|_{L_\infty(\tilde{\omega}_\tau, V_h)} = \max_{t\in\tilde{\omega}_\tau} \|v(\cdot, t)\|_{V_h},$$

where $\tilde{\omega}_\tau = \omega_\tau, \bar{\omega}_\tau, \omega^-_\tau$ or ω^+_τ and V_h is some of the previously defined spaces.

5. Explicit finite difference scheme and its convergence

In this and subsequent sections we shall assume that the solution u_i and input data are sufficiently smooth, such that its values and values of its derivatives appearing in formulas are well defined. We also assume that

$$h_1 \asymp h_2 \asymp k \asymp \tau. \tag{21}$$

We approximate the IBVP (1)–(7) and (12) with the following explicit FDS:

$$v_{1,\bar{t}t} - (\bar{p}_1 v_{1,\bar{x}})_x - (\bar{q}_1 v_{1,\bar{y}})_y + \bar{r}_1 v_1 = \bar{f}_1, \quad x \in \omega_{1,h_1},\ y \in \omega_k,\ t \in \omega_\tau, \tag{22}$$
$$v_{1,\bar{t}t}(b_1, y, t) + \frac{2}{h_1}\Big[\bar{p}_1(b_1, y) v_{1,\bar{x}}(b_1, y, t) + \alpha_1(y) v_1(b_1, y, t) - k \sum_{y'\in\omega_k} \beta(y, y') v_2(a_2, y', t)\Big]$$
$$- (\bar{q}_1 v_{1,\bar{y}})_y(b_1, y, t) + \bar{r}_1(b_1, y) v_1(b_1, y, t) = \bar{f}_1(b_1, y, t), \quad y \in \omega_k,\ t \in \omega_\tau, \tag{23}$$
$$v_{2,\bar{t}t} - (\bar{p}_2 v_{2,\bar{x}})_x - (\bar{q}_2 v_{2,\bar{y}})_y + \bar{r}_2 v_2 = \bar{f}_2, \quad x \in \omega_{2,h_2},\ y \in \omega_k,\ t \in \omega_\tau, \tag{24}$$
$$v_{2,\bar{t}t}(a_2, y, t) - \frac{2}{h_2}\Big[\bar{p}_2(a_2 + h_2, y) v_{2,x}(a_2, y, t) - \alpha_2(y) v_2(a_2, y, t) + k \sum_{y'\in\omega_k} \beta(y, y') v_1(b_1, y', t)\Big]$$
$$- (\bar{q}_2 v_{2,\bar{y}})_y(a_2, y, t) + \bar{r}_2(a_2, y) v_2(a_2, y, t) = \bar{f}_2(a_2, y, t), \quad y \in \omega_k,\ t \in \omega_\tau, \tag{25}$$
$$\begin{aligned} &v_1(a_1, y, t) = 0, \qquad v_2(b_2, y, t) = 0, \quad y \in \omega_k,\ t \in \bar{\omega}_\tau, \\ &v_1(x, c, t) = 0, \qquad v_1(x, d, t) = 0, \quad x \in \bar{\omega}_{1,h_1},\ t \in \bar{\omega}_\tau, \\ &v_2(x, c, t) = 0, \qquad v_2(x, d, t) = 0, \quad x \in \bar{\omega}_{2,h_2},\ t \in \bar{\omega}_\tau, \end{aligned} \tag{26}$$
$$v_i(x, y, 0) = \phi_i(x, y), \quad x \in \omega^\pm_{i,h_i},\ y \in \omega_k,\ i = 1, 2, \tag{27}$$
$$v_i(x, y, \tau) = \phi_i(x, y) + \tau\psi_i(x, y) + \frac{\tau^2}{2}\left[\frac{\partial}{\partial x}\left(p_i \frac{\partial \phi_i}{\partial x}\right) + \frac{\partial}{\partial y}\left(q_i \frac{\partial \phi_i}{\partial y}\right) - r_i(x, y)\phi_i + f_i(x, y, 0)\right],$$
$$x \in \omega^\pm_{i,h_i},\ y \in \omega_k,\ i = 1, 2, \tag{28}$$

where

$$\begin{aligned} &\bar{p}_i(x, y) = \frac{1}{2}[p_i(x, y) + p_i(x - h_i, y)], \qquad \bar{r}_i(x, y) = r_i(x, y), \\ &\bar{q}_i(x, y) = \frac{1}{2}[q_i(x, y) + q_i(x, y - k)], \qquad \bar{f}_i(x, y, t) = f_i(x, y, t). \end{aligned} \tag{29}$$

Eqs. (22)–(25) can be written in compact form

$$v_{\bar{t}t} + A_h v = \bar{f} \tag{30}$$

where, as usual, $v = (v_1, v_2), \bar{f} = (\bar{f}_1, \bar{f}_2)$ and $A_h v = (A_{h,1} v, A_{h,2} v)$ is the following difference operator

$$A_{h,i} v = -(\bar{p}_i v_{i,\bar{x}})_x - (\bar{q}_i v_{i,\bar{y}})_y + \bar{r}_i v_i, \quad x \in \omega_{i,h_i},\ y \in \omega_k,$$

$$A_{h,1} v = \frac{2}{h_1}\Big[\bar{p}_1 v_{1,\bar{x}} + \alpha_1 v_1 - k \sum_{y' \in \omega_k} \beta(y, y') v_2(a_2, y', t)\Big] - (\bar{q}_1 v_{1,\bar{y}})_y + \bar{r}_1 v_1, \quad x = b_1,\ y \in \omega_k,$$

$$A_{h,2} v = -\frac{2}{h_2}\Big[\bar{p}_2(a_2 + h_2, y) v_{2,x} - \alpha_2 v_2 + k \sum_{y' \in \omega_k} \beta(y, y') v_1(b_1, y', t)\Big] - (\bar{q}_2 v_{2,\bar{y}})_y + \bar{r}_2 v_2, \quad x = a_2,\ y \in \omega_k.$$

Operator A_h is selfadjoint on $H^1_{h,0}$:

$$(A_h v, w)_{L_h} = a_h(v, w) = (\bar{p} v_{\bar{x}}, w_{\bar{x}})_{L_{h'}} + (\bar{q} v_{\bar{y}}, w_{\bar{y}})_{L_{h''}} + (\bar{r} v, w)_{L_h}$$
$$+k \sum_{y \in \omega_k} \big[\alpha_1(y) v_1(b_1, y) w_1(b_1, y) + \alpha_2(y) v_2(a_2, y) w_2(a_2, y)\big]$$
$$-k^2 \sum_{y \in \omega_k} \sum_{y' \in \omega_k} \beta(y, y') \big[v_1(b_1, y) w_2(a_2, y') + w_1(b_1, y) v_2(a_2, y')\big] = (v, A_h w)_{L_h}.$$

The discrete analog of Lemma 1 holds.

Lemma 3. *The bilinear form a_h is bounded on $H^1_h \times H^1_h$. If $\bar{p}_i$ and $\bar{q}_i$, $i = 1, 2$, are positive this form satisfies Gårding's inequality on $H^1_{h,0}$, i.e. there exist positive constants m and κ such that*

$$a_h(v, v) + \kappa \|v\|^2_{L_h} \geq m \|v\|^2_{H^1_h}, \quad \forall v \in H^1_{h,0}.$$

FDS (22)–(28) *is computationally efficient. It follows from the general theory of difference schemes,* [22], *that the* FDS (22)–(28) *is stable under condition*

$$\tau \leq c_3 \min\{h_1, h_2, k\}, \tag{31}$$

where c_3 is a computable constant depending on the maximums of $p_i, q_i, |r_i|, |\alpha_i|$ and $|\beta|$. In this case, the following discrete analog of Theorems 1, 3 and 4 holds.

Theorem 5. *Let the input data $\bar{p}_i, \bar{q}_i, \bar{r}_i, \bar{f}_i, \alpha_i$ and β be well defined mesh functions and let $\bar{p}_i(x, y) \geq \bar{p}_{i0} > 0$ and $\bar{q}_i(x, y) \geq \bar{q}_{i0} > 0$. Let also the condition (31) be satisfied. Then the solution of* FDS (22)–(28) *satisfies a priori estimates*

$$\|\widehat{v}\|^2_{L_\infty(\omega^-_\tau, H^1_h)} + \|v_t\|^2_{L_\infty(\omega^-_\tau, L_h)} \leq C(T)\Big(\|\widehat{v}(\cdot, 0)\|^2_{H^1_h} + \|v_t(\cdot, 0)\|^2_{L_h} + \|\bar{f}\|^2_{L_2(\omega_\tau, L_h)}\Big)$$

and

$$\|\widehat{v}\|^2_{L_\infty(\omega^-_\tau, L_h)} + \|v_t\|^2_{L_\infty(\omega^-_\tau, H^{-1}_h)} \leq C(T)\Big(\|\widehat{v}(\cdot, 0)\|^2_{L_h} + \|v_t(\cdot, 0)\|^2_{H^{-1}_h} + \|\bar{f}\|^2_{L_2(\omega_\tau, H^{-1}_h)}\Big).$$

If in (22)–(28) $\bar{f} = \bar{g}_{\bar{t}}$ then

$$\|\widehat{v}\|^2_{L_\infty(\omega^-_\tau, L_h)} \leq C(T)\Big(\|\widehat{v}(\cdot, 0)\|^2_{L_h} + \|v_t(\cdot, 0) - \bar{g}(\cdot, 0)\|^2_{H^{-1}_h} + \tau^4 \|v_t(\cdot, 0)\|^2_{H^1_h} + \|\bar{g}\|^2_{L_2(\omega^-_\tau, L_h)}\Big).$$

Let $u = (u_1, u_2)$ be the solution of the IBVP (1)–(7) and (12) and $v = (v_1, v_2)$ the solution of the FDS (22)–(29). Then the error $z = u - v$ satisfies the following FDS:

$$z_{\bar{t}t} + A_h z = \varphi, \tag{32}$$

$$z_1(a_1, y, t) = 0, \quad z_2(b_2, y, t) = 0, \quad y \in \omega_k,\ t \in \bar{\omega}_\tau,$$
$$z_1(x, c, t) = 0, \quad z_1(x, d, t) = 0, \quad x \in \bar{\omega}_{1,h_1},\ t \in \bar{\omega}_\tau, \tag{33}$$
$$z_2(x, c, t) = 0, \quad z_2(x, d, t) = 0, \quad x \in \bar{\omega}_{2,h_2},\ t \in \bar{\omega}_\tau,$$

$$z_i(x, y, 0) = 0, \quad x \in \omega^{\pm}_{i,h_i},\ y \in \omega_k,\ i = 1, 2, \tag{34}$$

$$z_i(x, y, \tau) = \chi_i(x, y), \quad x \in \omega^{\pm}_{i,h_i},\ y \in \omega_k,\ i = 1, 2, \tag{35}$$

where

$$\varphi = (\varphi_1, \varphi_2), \qquad \varphi_i = \xi_i + \eta_i + \zeta_i, \quad i = 1, 2,$$

$$\xi_i = u_{i,\bar{t}t} - \frac{\partial^2 u_i}{\partial t^2}, \quad x \in \omega^{\mp}_{i,h_i},\ y \in \omega_k,\ t \in \omega_\tau,\ i = 1, 2,$$

$$\eta_i = \frac{\partial}{\partial x}\Big(p_i \frac{\partial u_i}{\partial x}\Big) - (\bar{p}_i u_{i,\bar{x}})_x, \quad x \in \omega_{i,h_i},\ y \in \omega_k,\ t \in \omega_\tau,\ i = 1, 2,$$

$$\eta_1 = \frac{\partial p_1}{\partial x}\Big(\frac{\partial u_1}{\partial x} - u_{1,\bar{x}}\Big) + \Big(\frac{\partial p_1}{\partial x} - p_{1,\bar{x}}\Big)u_{1,\bar{x}} + p_1\Big[\frac{\partial^2 u_1}{\partial x^2} - \frac{2}{h_1}\Big(\frac{\partial u_1}{\partial x} - u_{1,\bar{x}}\Big)\Big]$$
$$- \frac{2}{h_1}\Big[\int_c^d \beta(y, y')u_2(a_2, y', t)\mathrm{d}y' - k\sum_{y' \in \omega_k} \beta(y, y')u_2(a_2, y', t)\Big], \quad x = b_1,$$

$$\eta_2 = \frac{\partial p_2}{\partial x}\Big(\frac{\partial u_2}{\partial x} - u_{2,x}\Big) + \Big(\frac{\partial p_2}{\partial x} - p_{2,x}\Big)u_{2,x} + p_2\Big[\frac{\partial^2 u_2}{\partial x^2} + \frac{2}{h_2}\Big(\frac{\partial u_2}{\partial x} - u_{2,x}\Big)\Big]$$
$$+ \frac{2}{h_2}\Big[\int_c^d \beta(y, y')u_1(b_1, y', t)\mathrm{d}y' - k\sum_{y' \in \omega_k} \beta(y, y')u_1(b_1, y', t)\Big], \quad x = a_2,$$

$$\zeta_i = \frac{\partial}{\partial y}\Big(q_i \frac{\partial u_i}{\partial y}\Big) - (\bar{q}_i u_{i,\bar{y}})_y, \quad x \in \omega^{\mp}_{i,h_i},\ y \in \omega_k,\ t \in \omega_\tau,\ i = 1, 2,$$

$$\chi_i = \int_0^\tau \int_0^{t'} \Big[\frac{\partial^2 u_i}{\partial t^2}(x, y, t'') - \frac{\partial^2 u_i}{\partial t^2}(x, y, 0)\Big]\mathrm{d}t''\mathrm{d}t', \quad x \in \omega^{\mp}_{i,h_i},\ y \in \omega_k,\ i = 1, 2.$$

From the Theorem 5 we immediately obtain the following a priori estimate for the error $z = u - v$:

$$\|\widehat{z}\|_{L_\infty(\omega_\tau^-, H_h^1)} + \|z_t\|_{L_\infty(\omega_\tau^-, L_h)} \le C(T)\Big(\|\chi\|_{H_h^1} + \frac{1}{\tau}\|\chi\|_{L_h} + \|\xi\|_{L_2(\omega_\tau, L_h)} + \|\eta\|_{L_2(\omega_\tau, L_h)} + \|\zeta\|_{L_2(\omega_\tau, L_h)}\Big). \tag{36}$$

Therefore, in order to determine the convergence rate of the FDS (22)–(29), it is enough to estimate the right-hand side terms in the inequality (36).

The value of ξ_i in the node $(x, y, t) \in \omega_{i,h_i} \times \omega_k \times \omega_\tau$ is the bounded linear functional of $u_i \in H^s(e_i)$, where $e_i = (x - h_i, x + h_i) \times (y - k, y + k) \times (t - \tau, t + \tau)$ and $s > 3.5$, which vanishes on polynomials of degree ≤ 3. Using the Bramble–Hilbert lemma [23,24] we obtain

$$|\xi_i(x, y, t)| \le C(h)|u_i|_{H^s(e_i)}, \quad 3.5 < s \le 4,$$

where $h = \max\{h_1, h_2, k, \tau\}$ and $C(h) = Ch^{s-3.5}$. An analogous result holds for $x = b_1$ (and $x = a_2$) with adequate modification of e_i. By summation over the mesh we obtain the estimate

$$\|\xi\|_{L_2(\omega_\tau, L_h)} \le Ch^{s-2}\|u\|_{H^s(Q)}, \quad 3.5 < s \le 4, \tag{37}$$

where $\|u\|^2_{H^s(Q)} = \|u_1\|^2_{H^s(Q_1)} + \|u_2\|^2_{H^s(Q_2)}$, $Q_i = \Omega_i \times (0, T)$, $i = 1, 2$.

Similarly, using procedure proposed in [25] one obtains:

$$\|\zeta\|_{L_2(\omega_\tau, L_h)} \le Ch^{s-2} \max_i \|q_i\|_{H^{s-1}(\Omega_i)}\|u\|_{H^s(Q)}, \quad 3.5 < s \le 4, \tag{38}$$

$$\Big(\sum_{i=1}^2 h_i \sum_{x \in \omega_{h_i}} k \sum_{y \in \omega_k} \tau \sum_{t \in \omega_\tau} \eta_i^2\Big)^{1/2} \le Ch^{s-2} \max_i \|p_i\|_{H^{s-1}(\Omega_i)}\|u\|_{H^s(Q)}, \quad 3.5 < s \le 4, \tag{39}$$

and

$$\|\chi\|_{H_h^1} \le Ch^2\|u\|_{H^s(Q)}, \quad s > 3.5. \tag{40}$$

Let us denote

$$\widetilde{\chi}_1 = \begin{cases} T_x T_y \chi_1, & x \in \omega_{1,h_1}, \\ T_x^- T_y \chi_1, & x = b_1, \end{cases} \qquad \widetilde{\chi}_2 = \begin{cases} T_x T_y \chi_2, & x \in \omega_{2,h_2}, \\ T_x^+ T_y \chi_2, & x = a_2. \end{cases}$$

Then, in the same manner as above, we obtain

$$\|\chi - \widetilde{\chi}\|_{L_h} \le Ch^3\|u\|_{H^s}, \quad s > 3.5$$

and

$$\|\widetilde{\chi}\|_{L_h} \le Ch^{s-1/2}\Big(\|u_1\|^2_{H^s(\Omega_1 \times (0,\tau))} + \|u_2\|^2_{H^s(\Omega_2 \times (0,\tau))}\Big)^{1/2}, \quad 2.5 < s \le 3.$$

From the last formula using inequality (see [26])

$$\|g\|_{L_2(0,\varepsilon)} \le C\varepsilon^{1/2}\|g\|_{H^\sigma(0,1)}, \quad 0 < \varepsilon < 1,\ 0.5 < \sigma \le 1 \tag{41}$$

we obtain

$$\|\widetilde{\chi}\|_{L_h} \le Ch^3\|u\|_{H^s}, \quad s > 3.5,$$

whereby follows

$$\|\chi\|_{L_h} \le Ch^3\|u\|_{H^s}, \quad s > 3.5. \tag{42}$$

For $x = b_1$ we decompose term η_1 in the following way:

$$\eta_1 = \eta_{11} + \eta_{12} + \eta_{13} + \eta_{14}, \quad \text{where}$$
$$\eta_{11} = \frac{\partial p_1}{\partial x}\Big(\frac{\partial u_1}{\partial x} - u_{1,\bar{x}}\Big),$$
$$\eta_{12} = \Big(\frac{\partial p_1}{\partial x} - p_{1,\bar{x}}\Big)u_{1,\bar{x}},$$
$$\eta_{13} = p_1\Big[\frac{\partial^2 u_1}{\partial x^2} - \frac{2}{h_1}\Big(\frac{\partial u_1}{\partial x} - u_{1,\bar{x}}\Big)\Big],$$
$$\eta_{14} = \frac{2}{h_1}\Big(k\sum_{y'\in\omega_k}\beta(y,y')u_2(a_2,y',t) - \int_c^d \beta(y,y')u_2(a_2,y',t)\mathrm{d}y'\Big).$$

Term η_{11} can be estimated directly

$$\Big(\hbar_1 k\sum_{y\in\omega_k}\tau\sum_{t\in\omega_\tau}\eta_{11}^2(b_1,x,t)\Big)^{1/2} \le Ch^{3/2}\|p_1\|_{H^{s-1}(\Omega_1)}\|u_1\|_{H^s(Q_1)}, \quad s > 3.5. \tag{43}$$

Using the Bramble–Hilbert lemma and inequality (41), for $j = 2, 3$, we obtain

$$\Big(\hbar_1 k\sum_{y\in\omega_k}\tau\sum_{t\in\omega_\tau}\eta_{1j}^2(b_1,x,t)\Big)^{1/2} \le Ch^{3/2}\|p_1\|_{H^{s-1}(\Omega_1)}\|u_1\|_{H^s(Q_1)}, \quad s > 3.5. \tag{44}$$

Applying an error estimate for the trapezoidal quadrature formula, we get

$$\Big(\hbar_1 k\sum_{y\in\omega_k}\tau\sum_{t\in\omega_\tau}\eta_{14}^2(b_1,x,t)\Big)^{1/2} \le Ch^{3/2}\|\beta\|_{H^{s-1}((c,d)^2)}\|u_2\|_{H^s(Q_2)}, \quad s > 3.5. \tag{45}$$

From (39), (43)–(45) and analogous estimates of $\eta_2(a_2,\cdot)$ we obtain

$$\|\eta\|_{L_2(\omega_\tau,L_h)} \le Ch^{3/2}(\max_i\|p_i\|_{H^{s-1}(\Omega_i)} + \|\beta\|_{H^{s-1}((c,d)^2)})\|u\|_{H^s}, \quad s > 3.5. \tag{46}$$

Finally, from (36)–(38), (40), (42) and (46) one obtains the next assertion.

Theorem 6. *Let $u \in H^s(Q)$, $s > 3.5$, $p_i, q_i \in H^{s-1}(\Omega_i)$, $r_i \in H^{s-2}(\Omega_i)$, $\alpha_i \in H^{s-1}(c,d)$, $i = 1, 2$, $\beta \in H^{s-1}((c,d)^2)$, and let the assumptions (9) and (31) hold. Then the solution v of* FDS (22)–(29) *converges to the solution u of* IBVP (1)–(7) *and the following convergence rate estimate holds:*

$$\|\widehat{u} - \widehat{v}\|_{L_\infty(\omega_\tau^-,H_h^1)} + \|u_t - v_t\|_{L_\infty(\omega_\tau^-,L_h)} \le Ch^{3/2}\Big(\max_i\|p_i\|_{H^{s-1}(\Omega_i)} + \max_i\|q_i\|_{H^{s-1}(\Omega_i)} + \|\beta\|_{H^2((c,d)^2)} + 1\Big)$$
$$\times\|u\|_{H^s(Q)}, \quad s > 3.5. \tag{47}$$

Let us now estimate the convergence of FDS in discrete L_2 norm. In this case it is sufficient to assume that $u \in H^s(Q)$ for $s \le 3$. Consequently, coefficients r_i, right hand sides f_i and initial data ψ_i may be discontinuous functions. Therefore, we set in (22)–(25)

$$\begin{aligned}
&\bar{p}_i(x,y) = \frac{1}{2}[p_i(x,y) + p_i(x-h_i,y)], \quad x \in \omega_{i,h_i}^+,\ y \in \omega_k,\\
&\bar{q}_i(x,y) = \frac{1}{2}[q_i(x,y) + q_i(x,y-k)], \quad x \in \omega_{i,h_i},\ y \in \omega_k,\\
&\bar{q}_1(b_1,y) = \frac{1}{4}\Big[q_1(b_1,y) + q_1(b_1,y-k) + q_1\Big(b_1 - \frac{h_1}{2},y\Big) + q_1\Big(b_1 - \frac{h_1}{2},y-k\Big)\Big],\\
&\bar{q}_2(a_2,y) = \frac{1}{4}\Big[q_2(a_2,y) + q_2(a_2,y-k) + q_2\Big(a_2 + \frac{h_2}{2},y\Big) + q_2\Big(a_2 + \frac{h_2}{2},y-k\Big)\Big],\\
&\bar{r}_i = T_xT_yr_i, \quad x \in \omega_{i,h_i},\ y \in \omega_k,\\
&\bar{r}_1(b_1,y) = T_x^-T_yr_1(b_1,y), \qquad \bar{r}_2(a_2,y) = T_x^+T_yr_2(a_2,y),\\
&\bar{f}_i = T_xT_yT_tf_i, \quad x \in \omega_{i,h_i},\ y \in \omega_k,\ t \in \omega_\tau,\\
&\bar{f}_1(b_1,y,t) = T_x^-T_yT_tf_1(b_1,y,t), \qquad \bar{f}_2(a_2,y,t) = T_x^+T_yT_tf_2(a_2,y,t),
\end{aligned} \tag{48}$$

and replace (28) with

$$v(\cdot,\tau)=\phi+\tau\bar{\psi}+\frac{\tau^2}{2}\big[\bar{f}^{+}(\cdot,0)-A_h\phi\big], \tag{49}$$

where

$$\bar{\psi}_i=T_xT_y\psi_i,\quad x\in\omega_{i,h_i},\ y\in\omega_k,$$
$$\bar{\psi}_1(b_1,y)=T_x^{-}T_y\psi_1(b_1,y),\qquad \bar{\psi}_2(a_2,y)=T_x^{+}T_y\psi_2(a_2,y),$$
$$\bar{f}_i^{+}(x,y,0)=T_xT_yT_t^{+}f_i(x,y,0),\quad x\in\omega_{i,h_i},\ y\in\omega_k,$$
$$\bar{f}_1^{+}(b_1,y,0)=T_x^{-}T_yT_t^{+}f_1(b_1,y,0),\qquad \bar{f}_2^{+}(a_2,y,0)=T_x^{+}T_yT_t^{+}f_2(a_2,y,0).$$

Let $u=(u_1,u_2)$ be the solution of the IBVP (1)–(7) and (12) and $v=(v_1,v_2)$ the solution of the FDS (22)–(27), (48) and (49). Then the error $z=u-v$ satisfies FDS (32)–(35) where:

$$\varphi_i=\tilde{\xi}_{i,\bar{t}}+\tilde{\eta}_{i,x}+\tilde{\zeta}_{i,y}+\mu_i,\quad x\in\omega_{i,h_i},\ y\in\omega_k,\ t\in\omega_\tau,\ i=1,2,$$

$$\varphi_1=\tilde{\xi}_{1,\bar{t}}+\frac{2}{h_1}(-\tilde{\eta}_1+\nu_1)+\tilde{\zeta}_{1,y}+\mu_1,\quad x=b_1,\ y\in\omega_k,\ t\in\omega_\tau,$$

$$\varphi_2=\tilde{\xi}_{2,\bar{t}}+\frac{2}{h_2}\big(\tilde{\eta}_2(x+h_2,\cdot)+\nu_2\big)+\tilde{\zeta}_{2,y}+\mu_2,\quad x=a_2,\ y\in\omega_k,\ t\in\omega_\tau,$$

$$\tilde{\xi}_i=u_{i,t}-T_xT_y\frac{\partial u_i}{\partial t}\left(\cdot,t+\frac{\tau}{2}\right),\quad x\in\omega_{i,h_i},\ y\in\omega_k,\ t\in\omega_\tau^{-},\ i=1,2,$$

$$\tilde{\xi}_1=u_{1,t}-T_x^{-}T_y\frac{\partial u_1}{\partial t}\left(\cdot,t+\frac{\tau}{2}\right)-\frac{h_1}{4}\left(T_y\frac{\partial u_1}{\partial x}\right)_t,\quad x=b_1,$$

$$\tilde{\xi}_2=u_{2,t}-T_x^{+}T_y\frac{\partial u_2}{\partial t}\left(\cdot,t+\frac{\tau}{2}\right)+\frac{h_2}{4}\left(T_y\frac{\partial u_2}{\partial x}\right)_t,\quad x=a_2,$$

$$\tilde{\eta}_i=T_yT_t\left(p_i\frac{\partial u_i}{\partial x}\right)\left(x-\frac{h_i}{2},\cdot\right)-\bar{p}_iu_{i,\bar{x}},\quad x\in\omega_{i,h_i}^{+},\ y\in\omega_k,\ t\in\omega_\tau,\ i=1,2,$$

$$\tilde{\zeta}_i=T_xT_t\left(q_i\frac{\partial u_i}{\partial y}\right)\left(\cdot,y-\frac{k}{2},\cdot\right)-\bar{q}_iu_{i,\bar{y}},\quad x\in\omega_{i,h_i},\ y\in\omega_k^{+},\ t\in\omega_\tau,\ i=1,2,$$

$$\tilde{\zeta}_1=T_x^{-}T_t\left(q_1\frac{\partial u_1}{\partial y}\right)\left(\cdot,y-\frac{k}{2},\cdot\right)-\bar{q}_1u_{1,\bar{y}}+\frac{h_1}{4}\bar{q}_1\left(T_t\frac{\partial u_1}{\partial x}\right)_{\bar{y}},\quad x=b_1,$$

$$\tilde{\zeta}_2=T_x^{+}T_t\left(q_2\frac{\partial u_2}{\partial y}\right)\left(\cdot,y-\frac{k}{2},\cdot\right)-\bar{q}_2u_{2,\bar{y}}-\frac{h_2}{4}\bar{q}_2\left(T_t\frac{\partial u_2}{\partial x}\right)_{\bar{y}},\quad x=a_2,$$

$$\mu_i=(T_xT_yr_i)u_i-T_xT_yT_t(r_iu_i),\quad x\in\omega_{i,h_i},\ y\in\omega_k,\ t\in\omega_\tau,\ i=1,2,$$

$$\mu_1=(T_x^{-}T_yr_1)u_1-T_x^{-}T_yT_t(r_1u_1)-\frac{h_1}{4}(T_x^{-}T_yr_1)\left(T_yT_t\frac{\partial u_1}{\partial x}\right),\quad x=b_1,$$

$$\mu_2=(T_x^{+}T_yr_2)u_2-T_x^{+}T_yT_t(r_2u_2)+\frac{h_2}{4}(T_x^{+}T_yr_2)\left(T_yT_t\frac{\partial u_2}{\partial x}\right),\quad x=a_2,$$

$$\nu_1=\alpha_1u_1(b_1,\cdot)-T_yT_t\big(\alpha_1u_1(b_1,\cdot)\big)+\int_c^d T_y\beta(y,y')T_tu_2(a_2,y',t)\mathrm{d}y'$$
$$-k\sum_{y'\in\omega_k}\beta(y,y')u_2(a_2,y',t)+\frac{h_1^2}{8}\left\{\left(T_y\frac{\partial u_1}{\partial x}\right)_{\bar{t}t}-\left[\bar{q}_1\left(T_t\frac{\partial u_1}{\partial x}\right)_{\bar{y}}\right]_y+(T_x^{-}T_yr_1)\left(T_yT_t\frac{\partial u_1}{\partial x}\right)\right\}_{(b_1,\cdot)},\quad y\in\omega_k,\ t\in\omega_\tau,$$

$$\nu_2=\alpha_2u_2(a_2,\cdot)-T_yT_t\big(\alpha_2u_2(a_2,\cdot)\big)+\int_c^d T_y\beta(y,y')T_tu_1(b_1,y',t)\mathrm{d}y'-k\sum_{y'\in\omega_k}\beta(y,y')u_1(b_1,y',t)$$
$$-\frac{h_2^2}{8}\left\{\left(T_y\frac{\partial u_2}{\partial x}\right)_{\bar{t}t}-\left[\bar{q}_2\left(T_t\frac{\partial u_2}{\partial x}\right)_{\bar{y}}\right]_y+(T_x^{+}T_yr_2)\left(T_yT_t\frac{\partial u_2}{\partial x}\right)\right\}_{(a_2,\cdot)},\quad y\in\omega_k,\ t\in\omega_\tau,$$

$$\chi_i=\tau\tilde{\xi}_i+\frac{\tau^2}{2}\big(\tilde{\eta}_{i,x}^{+}+\tilde{\zeta}_{i,y}^{+}+\mu_i^{+}\big),\quad x\in\omega_{i,h_i},\ y\in\omega_k,\ t=0,\ i=1,2,$$

$$\chi_1=\tau\tilde{\xi}_1+\frac{\tau^2}{2}\left[\frac{2}{h_1}(-\tilde{\eta}_1^{+}+\nu_1^{+})+\tilde{\zeta}_{1,y}^{+}+\mu_1^{+}\right],\quad x=b_1,\ y\in\omega_k,\ t=0,$$

$$\chi_2=\tau\tilde{\xi}_2+\frac{\tau^2}{2}\left[\frac{2}{h_2}\big(\tilde{\eta}_2^{+}(x+h_2,\cdot)+\nu_2^{+}\big)+\tilde{\zeta}_{2,y}^{+}+\mu_2^{+}\right],\quad x=a_2,\ y\in\omega_k,\ t=0,$$

$$\tilde{\eta}_i^+ = T_y T_t^+\Big(p_i \frac{\partial u_i}{\partial x}\Big)\Big(x - \frac{h_i}{2}, \cdot\Big) - \bar{p}_i u_{i,\bar{x}}, \quad x \in \omega_{i,h_i}^+,\ y \in \omega_k,\ t = 0, i = 1, 2,$$

$$\tilde{\xi}_i^+ = T_x T_t^+\Big(q_i \frac{\partial u_i}{\partial y}\Big)\Big(\cdot, y - \frac{k}{2}, \cdot\Big) - \bar{q}_i u_{i,\bar{y}}, \quad x \in \omega_{i,h_i},\ y \in \omega_k^+,\ t = 0,\ i = 1, 2,$$

$$\tilde{\xi}_1^+ = T_x^- T_t^+\Big(q_1 \frac{\partial u_1}{\partial y}\Big)\Big(\cdot, y - \frac{k}{2}, \cdot\Big) - \bar{q}_1 u_{1,\bar{y}}, \quad x = b_1,\ y \in \omega_k^+,\ t = 0,$$

$$\tilde{\xi}_2^+ = T_x^+ T_t^+\Big(q_2 \frac{\partial u_2}{\partial y}\Big)\Big(\cdot, y - \frac{k}{2}, \cdot\Big) - \bar{q}_2 u_{2,\bar{y}}, \quad x = a_2,\ y \in \omega_k^+,\ t = 0,$$

$$\mu_i^+ = (T_x T_y r_i)u_i - T_x T_y T_t^+(r_i u_i), \quad x \in \omega_{i,h_i},\ y \in \omega_k,\ t = 0,\ i = 1, 2,$$

$$\mu_1^+ = (T_x^- T_y r_1)u_1 - T_x^- T_y T_t^+(r_1 u_1), \quad x = b_1,\ y \in \omega_k,\ t = 0,$$

$$\mu_2^+ = (T_x^+ T_y r_2)u_2 - T_x^+ T_y T_t^+(r_2 u_2), \quad x = a_2,\ y \in \omega_k,\ t = 0,$$

$$\begin{aligned}\nu_1^+ \ =\ & \alpha_1(y)u_1(b_1, y, 0) - T_y T_t^+\big(\alpha_1(y)u_1(b_1, y, 0)\big) + \frac{h_1^2}{4\tau}\Big(T_y \frac{\partial u_1}{\partial x}\Big)_t (b_1, y, 0) \\ & + \int_c^d T_y \beta(y, y') T_t^+ u_2(a_2, y', 0)\mathrm{d}y' - k \sum_{y' \in \omega_k} \beta(y, y') u_2(a_2, y', 0), \quad y \in \omega_k,\end{aligned}$$

$$\begin{aligned}\nu_2^+ \ =\ & \alpha_2(y)u_2(a_2, y, 0) - T_y T_t^+\big(\alpha_2(y)u_2(a_2, y, 0)\big) - \frac{h_2^2}{4\tau}\Big(T_y \frac{\partial u_2}{\partial x}\Big)_t (a_2, y, 0) \\ & + \int_c^d T_y \beta(y, y') T_t^+ u_1(b_1, y', 0)\mathrm{d}y' - k \sum_{y' \in \omega_k} \beta(y, y') u_1(b_1, y', 0), \quad y \in \omega_k.\end{aligned}$$

From the Theorem 5, under assumptions (21), we immediately obtain the following a priori estimate for the error $z = u - v$:

$$\begin{aligned}\|\widehat{z}\|_{L_\infty(\omega_\tau^-, L_h)} \le\ & C_1 \tau\Big(\|\tilde{\eta}^+\|_{L_{h'}} + \|\tilde{\xi}^+\|_{L_{h''}} + \|\mu^+\|_{L_h} + \|\nu^+\|_{L_2(\omega_k)}\Big) \\ & + C_2\Big(\|\tilde{\xi}\|_{L_2(\omega_\tau^-, L_h)} + \|\tilde{\eta}\|_{L_2(\omega_\tau, L_{h'})} + \|\tilde{\xi}\|_{L_2(\omega_\tau, L_{h''})} + \|\mu\|_{L_2(\omega_\tau, L_h)} + \|\nu\|_{L_2(\omega_\tau, L_2(\omega_k))}\Big).\end{aligned} \tag{50}$$

Therefore, in order to determine the convergence rate of the FDS (22)–(27), (48) and (49) it is enough to estimate the right-hand side terms in the inequality (50).

The value of $\tilde{\xi}_i$ in the node $(x, y, t) \in \omega_{i,h_i} \times \omega_k \times \omega_\tau$ is a bounded linear functional of $u_i \in H^s(\tilde{e}_i)$, where $\tilde{e}_i = (x - h_i/2, x + h_i/2) \times (y - k/2, y + k/2) \times (t, t + \tau)$ and $s > 1.5$, which vanishes on polynomials of degree ≤ 2. Using the Bramble–Hilbert lemma [23,24] we obtain:

$$|\tilde{\xi}_i(x, y, t)| \le C h^{s-2.5} |u_i|_{H^s(\tilde{e}_i)}, \quad 1.5 < s \le 3,$$

where $h = \max\{h_1, h_2, k, \tau\}$. A similar result holds for $x = b_1$:

$$|\tilde{\xi}_1(b_1, y, t)| \le C h^{s-2.5} |u_1|_{H^s(\tilde{e}_1^-)}, \quad 2.5 < s \le 3,$$

where denoted $\tilde{e}_1^- = (b_1 - h_1/2, b_1) \times (y - k/2, y + k/2) \times (t, t + \tau)$, and for $x = a_2$. Summation over the mesh yields

$$\|\tilde{\xi}\|_{L_2(\omega_\tau^-, L_h)} \le C h^{s-1} \|u\|_{H^s(Q)}, \quad 2.5 < s \le 3. \tag{51}$$

Similarly one obtains:

$$\|\tilde{\eta}\|_{L_2(\omega_\tau, L_{h'})} \le C h^{s-1} \max_i \|p_i\|_{H^{s-1}(\Omega_i)} \|u\|_{H^s(Q)}, \quad 2 < s \le 3, \tag{52}$$

$$\|\tilde{\xi}\|_{L_2(\omega_\tau, L_{h''})} \le C h^{s-1} \max_i \|q_i\|_{H^{s-1}(\Omega_i)} \|u\|_{H^s(Q)}, \quad 2.5 < s \le 3, \tag{53}$$

and

$$\|\mu\|_{L_2(\omega_\tau, L_h)} \le C h^{s-1} \max_i \|r_i\|_{H^{s-2}(\Omega_i)} \|u\|_{H^s(Q)}, \quad 2 < s \le 3. \tag{54}$$

We decompose term ν_1 in the following way:

$$\nu_1 = \nu_{11} + \nu_{12} + \nu_{13} + \nu_{14} + \nu_{15}, \quad \text{where}$$

$$\nu_{11} = \alpha_1 u_1(b_1, \cdot) - T_y T_t\big(\alpha_1 u_1(b_1, \cdot)\big),$$

$$\nu_{12} = \int_c^d T_y \beta(y, y') T_t u_2(a_2, y', t)\mathrm{d}y' - k \sum_{y' \in \omega_k} \beta(y, y') u_2(a_2, y', t),$$

$$\nu_{13} = \frac{h_1^2}{8}\Big(T_y \frac{\partial u_1}{\partial x}\Big)_{\bar{t}t}(b_1,\cdot),$$
$$\nu_{14} = -\frac{h_1^2}{8}\Big[\bar{q}_1\Big(T_t \frac{\partial u_1}{\partial x}\Big)_{\bar{y}}\Big]_y(b_1,\cdot)$$
$$\nu_{15} = \frac{h_1^2}{6}(T_x^- T_y r_1(b_1,\cdot))\Big(T_y T_t \frac{\partial u_1}{\partial x}(b_1,\cdot)\Big).$$

Using the Bramble–Hilbert lemma and the trace theorem for Sobolev spaces we obtain

$$\Big(k\sum_{y\in\omega_k}\tau\sum_{t\in\omega_\tau}\nu_{11}^2\Big)^{1/2} \le Ch^{s-1}\|\alpha_1\|_{H^{s-1}(c,d)}\|u_1(b_1,\cdot)\|_{H^{s-1}((c,d)\times(0,T))}$$
$$\le Ch^{s-1}\|\alpha_1\|_{H^{s-1}(c,d)}\|u_1\|_{H^s(Q_1)}, \quad 2<s\le 3. \tag{55}$$

Analogously

$$\Big(k\sum_{y\in\omega_k}\tau\sum_{t\in\omega_\tau}\nu_{12}^2\Big)^{1/2} \le Ch^{s-1}\|\beta\|_{H^{s-1}((c,d)^2)}\|u_2\|_{H^s(Q_2)}, \quad 2<s\le 3. \tag{56}$$

Using boundary condition (3) we immediately obtain

$$\Big(k\sum_{y\in\omega_k}\tau\sum_{t\in\omega_\tau}\nu_{13}^2\Big)^{1/2} \le Ch^2\Big(\|\alpha_1\|_{H^{s-1}(c,d)}\|u_1\|_{H^s(Q_1)} + \|\beta\|_{H^{s-1}((c,d)^2)}\|u_2\|_{H^s(Q_2)}\Big), \quad s>2.5 \tag{57}$$

and

$$\Big(k\sum_{y\in\omega_k}\tau\sum_{t\in\omega_\tau}\nu_{14}^2\Big)^{1/2} \le Ch^{s-1}\Big[\|\tilde{\beta}_1\|_{H^{s-1}((c,d)^2)}\|u_2\|_{H^s(Q_2)} + \Big(\|q_1\|_{H^{s-1+\varepsilon(s)}(\Omega_1)} + \|\tilde{\alpha}_1\|_{H^{s-1}(c,d)}\Big)\|u_1\|_{H^s(Q_1)}\Big],$$
$$2.5<s\le 3, \tag{58}$$

where we denoted

$$\varepsilon(s)\begin{cases}=0, & s<3,\\ >0, & s=3,\end{cases} \qquad \tilde{\alpha}_1(y) = \frac{\alpha_1(y)}{p_1(b_1,y)}, \qquad \tilde{\beta}_1(y,y') = \frac{\beta(y,y')}{p_1(b_1,y)}.$$

Term ν_{15} can be estimated directly, using the inequality (42):

$$\Big(k\sum_{y\in\omega_k}\tau\sum_{t\in\omega_\tau}\nu_{15}^2\Big)^{1/2} \le Ch^{s-1}\|r_1\|_{H^{s-2}(\Omega_1)}\|u_1\|_{H^s(Q_1)}, \quad 2<s\le 3. \tag{59}$$

Finally, from (55)–(59) and analogous estimates for ν_2 ve obtain:

$$\|\nu\|_{L_2(\omega_\tau,L_2(\omega_k))} \le Ch^{s-1}\Big(\max_i\|\alpha_i\|_{H^{s-1}(c,d)} + \|\beta\|_{H^{s-1}((c,d)^2)}$$
$$+ \max_i\|\tilde{\alpha}_i\|_{H^{s-1}(c,d)} + \max_i\|\tilde{\beta}_i\|_{H^{s-1}((c,d)^2)} + \max_i\|q_i\|_{H^{s-1+\varepsilon(s)}(\Omega_i)} + \max_i\|r_i\|_{H^{s-2}(\Omega_i)}\Big)\|u\|_{H^s(Q)}, \quad 2.5<s\le 3. \tag{60}$$

Terms $\tilde{\eta}^+$, $\tilde{\xi}^+$, μ^+ and ν^+ can be estimated analogously as $\tilde{\eta}$, $\tilde{\xi}$, μ and ν:

$$\|\tilde{\eta}^+\|_{L_{h'}} \le Ch\max_i\|p_i\|_{H^{s-1}(\Omega_i)}\|u\|_{H^s}(Q), \quad s\ge 2.5, \tag{61}$$

$$\|\tilde{\xi}^+\|_{L_{h''}} \le Ch\max_i\|q_i\|_{H^{s-1}(\Omega_i)}\|u\|_{H^s}(Q), \quad s\ge 2.5, \tag{62}$$

$$\|\mu^+\|_{L_h} \le Ch\max_i\|r_i\|_{H^{s-2}(\Omega_i)}\|u\|_{H^s}(Q), \quad s\ge 2.5, \tag{63}$$

$$\|\nu^+\|_{L_2(\omega_k)} \le Ch\Big(\max_i\|\alpha_i\|_{H^{s-1}(c,d)} + \|\beta\|_{H^{s-1}((c,d)^2)}\Big)\|u\|_{H^s(Q)}, \quad s\ge 2.5. \tag{64}$$

Finally, from (50)–(54) and (60)–(64) one obtains the next assertion.

Theorem 7. *Let $u\in H^s(Q)$, $s>2.5$, $p_i\in H^{s-1}(\Omega_i)$, $q_i\in H^{s-1+\varepsilon(s)}(\Omega_i)$, $r_i\in H^{s-2}(\Omega_i)$, $\alpha_i,\tilde{\alpha}_i\in H^{s-1}(c,d)$, $\beta,\tilde{\beta}_i\in H^{s-1}((c,d)^2)$, $i=1,2$, and let the assumptions (9) and (31) hold. Then the solution v of* FDS (22)–(27), (48) *and* (49) *converges to the solution u of* IBVP (1)–(7) *and the following convergence rate estimate holds:*

$$\|\hat{u} - \hat{v}\|_{L_\infty(\omega_\tau^-,L_h)} \le Ch^{s-1}\|u\|_{H^s(Q)}, \quad 2.5<s\le 3. \tag{65}$$

Here C is a computable constant depending on the norms of input data.

Remark 2. Note that obtained convergence rate estimates (47) and (65) are not optimal. In the case of second-order hyperbolic equations with input data from special anisotropic functional spaces such convergence rate estimates for finite element and finite difference methods are obtained in [27,28]. In the present paper our goal was to obtain a simple convergence rate estimate for the FDS approximating the IBVP in disjoint domains and we left the question of optimality for future research.

6. Factorized finite difference scheme

Substituting in (30) $A_h v$ with $A_h\left[\sigma v(\cdot,t+\tau)+(1-2\sigma)v(\cdot,t)+\sigma v(\cdot,t-\tau)\right]$ one obtains a weighted FDS for the IBVP (1)–(7). The weighted FDS is unconditionally stable for $\sigma \geq 1/4$ (i.e. without the assumptions (31)) and also satisfies the convergence rate estimates (47) and (65). Unfortunately, the weighted FDS is numerically inefficient, because on each time level we need to solve an elliptic difference problem.

Let us now consider the following factorized finite difference scheme

$$B_h v_{\bar{t}t} + A_h v = \bar{f}, \tag{66}$$

where difference operator A_h and the right hand side $\bar{f}$ are same as in (30) while the difference operator B_h is defined in the following manner:

$$\begin{aligned}
&B_h v = (B_{h,1}v_1, B_{h,2}v_2),\\
&B_{h,i}v_i = (I+\sigma_i\tau^2 C_{h,i})(I+\sigma_i\tau^2 D_{h,i}),\\
&C_{h,i}v_i = -v_{i,\bar{x}x}, \quad x\in\omega_{i,h_i},\ y\in\omega_k,\\
&C_{h,1}v_1 = \frac{2}{h_1}v_{1,\bar{x}}, \quad x=b_1,\ y\in\omega_k,\\
&C_{h,2}v_2 = -\frac{2}{h_2}v_{2,x}, \quad x=a_2,\ y\in\omega_k\\
&D_{h,i}v_i = -v_{i,\bar{y}y}, \quad x\in\omega^{\pm}_{i,h_i},\ y\in\omega_k.
\end{aligned}$$

In such a way, in expanded form FDS (66) reduces to

$$\begin{aligned}
&v_{1,\bar{t}t} - \sigma_1\tau^2 v_{1,\bar{t}t\bar{x}x} - \sigma_1\tau^2 v_{1,\bar{t}t\bar{y}y} + \sigma_1^2\tau^4 v_{1,\bar{t}t\bar{x}\bar{y}y} - (\bar{p}_1 v_{1,\bar{x}})_x - (\bar{q}_1 v_{1,\bar{y}})_y + \bar{r}_1 v_1 = \bar{f}_1, \quad x\in\omega_{1,h_1},\ y\in\omega_k,\ t\in\omega_\tau,\\
&v_{1,\bar{t}t} + \frac{2\sigma_1\tau^2}{h_1}v_{1,\bar{t}t\bar{x}} - \sigma_1\tau^2 v_{1,\bar{t}t\bar{y}y} - \frac{2\sigma_1^2\tau^4}{h_1}v_{1,\bar{t}t\bar{x}\bar{y}y} + \frac{2}{h_1}\left[\bar{p}_1 v_{1,\bar{x}} + \alpha_1 v_1 - k\sum_{y'\in\omega_k}\beta(y,y')v_2(a_2,y',\cdot)\right]\\
&\quad -(\bar{q}_1 v_{1,\bar{y}})_y + \bar{r}_1 v_1 = \bar{f}_1, \quad x=b_1,\ y\in\omega_k,\ t\in\omega_\tau,\\
&v_{2,\bar{t}t} - \sigma_2\tau^2 v_{2,\bar{t}t\bar{x}x} - \sigma_2\tau^2 v_{2,\bar{t}t\bar{y}y} + \sigma_2^2\tau^4 v_{2,\bar{t}t\bar{x}\bar{y}y} - (\bar{p}_2 v_{2,\bar{x}})_x - (\bar{q}_2 v_{2,\bar{y}})_y + \bar{r}_2 v_2 = \bar{f}_2, \quad x\in\omega_{2,h_2}\ y\in\omega_k,\ t\in\omega_\tau,\\
&v_{2,\bar{t}t} - \frac{2\sigma_2\tau^2}{h_2}v_{2,\bar{t}t\bar{x}} - \sigma_2\tau^2 v_{2,\bar{t}t\bar{y}y} + \frac{2\sigma_2^2\tau^4}{h_2}v_{2,\bar{t}t\bar{x}\bar{y}y} - \frac{2}{h_2}\left[\bar{p}_2(x+h_2,\cdot)v_{2,x} - \alpha_2 v_2 + k\sum_{y'\in\omega_k}\beta(y,y')v_1(b_1,y',\cdot)\right]\\
&\quad -(\bar{q}_2 v_{2,\bar{y}})_y + \bar{r}_2 v_2 = \bar{f}_2, \quad x=a_2,\ y\in\omega_k,\ t\in\omega_\tau,
\end{aligned}$$

subject to initial and boundary conditions (26)–(28).

For sufficiently large σ_1 and σ_2 from the general theory of difference schemes [22] follows that the FDS (66) is unconditionally stable. This FDS is computationally efficient, because on each time level it can be resolved by two application of the Thomas algorithm. In such a manner, factorized finite difference scheme combine the good properties of explicit and weighted schemes.

Let the coefficients of FDS (66) be defined by (29). Then the error $z = u - v$ satisfies an a priori estimate analogous to (36):

$$\begin{aligned}
&\|\widehat{z}\,\|_{L_\infty(\omega_\tau^-,H_h^1)} + \|z_t\|_{L_\infty(\omega_\tau^-,L_h)}\\
&\quad \leq C(T)\left(\|\chi\|_{H_h^1} + \frac{1}{\tau}\|\chi\|_{L_h} + \|\xi\|_{L_2(\omega_\tau,L_h)} + \|\eta\|_{L_2(\omega_\tau,L_h)} + \|\zeta\|_{L_2(\omega_\tau,L_h)} + \|\rho\|_{L_2(\omega_\tau,L_h)}\right),
\end{aligned} \tag{67}$$

where ξ, η, ζ and χ are as in the previous section and

$$\begin{aligned}
&\rho_i = -\sigma_i\tau^2 u_{i,\bar{t}t\bar{x}x} - \sigma_i\tau^2 v_{i,\bar{t}t\bar{y}y} + \sigma_i^2\tau^4 u_{i,\bar{t}t\bar{x}\bar{y}y}, \quad x\in\omega_{i,h_i},\ y\in\omega_k,\ t\in\omega_\tau,\ i=1,2,\\
&\rho_1 = \frac{2\sigma_1\tau^2}{h_1}u_{1,\bar{t}t\bar{x}} - \sigma_1\tau^2 u_{1,\bar{t}t\bar{y}y} - \frac{2\sigma_1^2\tau^4}{h_1}u_{1,\bar{t}t\bar{x}\bar{y}y}, \quad x=b_1,\ y\in\omega_k,\ t\in\omega_\tau,\\
&\rho_2 = -\frac{2\sigma_2\tau^2}{h_2}u_{2,\bar{t}t\bar{x}} - \sigma_2\tau^2 u_{2,\bar{t}t\bar{y}y} + \frac{2\sigma_2^2\tau^4}{h_2}u_{2,\bar{t}t\bar{x}\bar{y}y}, \quad x=a_2,\ y\in\omega_k,\ t\in\omega_\tau.
\end{aligned}$$

Using the Bramble–Hilbert lemma and inequality (41) we easily obtain

$$\|\rho\|_{L_2(\omega_\tau,L_h)} \le Ch^{3/2}\|u\|_{H^s(Q)}, \quad s > 3.5. \tag{68}$$

From (37), (38), (40), (42), (46), (67) and (68) we conclude that the factorized FDS (66) and (26)–(29) satisfies the convergence rate estimate (47).

In the case when the coefficients of FDS (66) are defined by (48) the error $z = u - v$ satisfies the a priori estimate

$$\begin{aligned}\|\widehat{z}\|_{L_\infty(\omega_\tau^-,L_h)} \le{}& C_1\tau\left(\|\tilde{\eta}^+\|_{L_{h'}} + \|\tilde{\zeta}^+\|_{L_{h''}} + \|\mu^+\|_{L_h} + \|\nu^+\|_{L_2(\omega_k)}\right)\\ &+ C_2\Big(\|\tilde{\xi}\|_{L_2(\omega_\tau^-,L_h)} + \|\tilde{\eta}\|_{L_2(\omega_\tau,L_{h'})} + \|\tilde{\zeta}\|_{L_2(\omega_\tau,L_{h''})} + \|\theta\|_{L_2(\omega_\tau,L_{h'})}\\ &+ \|\upsilon\|_{L_2(\omega_\tau,L_{h''})} + \|\mu\|_{L_2(\omega_\tau,L_h)} + \|\nu\|_{L_2(\omega_\tau,L_2(\omega_k))}\Big),\end{aligned} \tag{69}$$

where $\tilde{\xi}$, $\tilde{\eta}$, $\tilde{\zeta}$, η^+, ζ^+, μ, ν, μ^+ and ν^+ are as in (50) and

$$\begin{aligned}&\theta_i = -\sigma_i\tau^2 u_{i,\bar{t}t\bar{x}} + \sigma_i^2\tau^4 u_{i,\bar{t}t\bar{x}\bar{y}y}, && x \in \omega^+_{i,h_i},\ y \in \omega_k,\ t \in \omega_\tau,\\ &\upsilon_i = -\sigma_i\tau^2 v_{i,\bar{t}t\bar{y}}, && x \in \omega^\pm_{i,h_i},\ y \in \omega^+_k,\ t \in \omega_\tau,\end{aligned} \qquad i = 1, 2.$$

Using the Bramble–Hilbert lemma we easily obtain

$$\|\theta\|_{L_2(\omega_\tau,L_{h'})}, \|\upsilon\|_{L_2(\omega_\tau,L_{h''})} \le Ch^{s-1}\|u\|_{H^s(Q)}, \quad 1.5 < s \le 3. \tag{70}$$

From (51)–(54), (60)–(64), (69) and (70) we conclude that the factorized FDS (66), (26)–(27), (48) and (49) satisfies the convergence rate estimate (65).

For further reading

[29]

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